Inverse Scattering Transform for the Focusing Nonlinear Schrödinger Equation with a One-Sided Non-Zero Boundary Condition

B. Prinari and F. Vitale

Abstract. The inverse scattering transform (IST) as a tool to solve the initial-value problem for the focusing nonlinear Schrödinger (NLS) equation with one-sided non-zero boundary value $q_r(t) \equiv A_r e^{-2iA_r^2 t + i\theta_r}$, $A_r \geq 0$, $0 \leq \theta_r < 2\pi$, as $x \to +\infty$ is presented. The direct problem is shown to be well-defined for NLS solutions $q(x,t)$ such that $[q(x,t) - q_r(t)\vartheta(x)] \in L^{1,1}(\mathbb{R})$ [here and in the following $\vartheta(x)$ denotes the Heaviside function] with respect to $x \in \mathbb{R}$ for all $t \geq 0$, for which analyticity properties of eigenfunctions and scattering data are established. The inverse scattering problem is formulated both via (left and right) Marchenko integral equations and as a Riemann-Hilbert problem on a single sheet of the scattering variables $\lambda_r = \sqrt{k^2 + A_r^2}$, where k is the usual complex scattering parameter in the IST. The direct and inverse problems are also formulated in terms of a suitable uniformization variable that maps the two-sheeted Riemann surface for k into a single copy of the complex plane. The time evolution of the scattering coefficients is then derived, showing that, unlike the case of solutions with the same amplitude as $x \to \pm\infty$, here both reflection and transmission coefficients have a nontrivial (although explicit) time dependence. The results presented in this paper will be instrumental for the investigation of the long-time asymptotic behavior of physically relevant NLS solutions with nontrivial boundary conditions, either via the nonlinear steepest descent method on the Riemann-Hilbert problem, or via matched asymptotic expansions on the Marchenko integral equations.

1. Introduction

Nonlinear Schrödinger (NLS) systems have been extensively investigated both mathematically and physically for almost sixty years, and remarkably continue to offer interesting research problems and new venues for applications. Equations of NLS-type have proven over the years to be fundamental for modelling nonlinear wave phenomena in such diverse fields as deep water waves [**4, 38**], plasma physics [**34**], nonlinear fiber optics [**22**], magnetic spin waves [**16, 40**], low temperature physics and Bose-Einstein condensates [**29**], just to mention a few. Mathematically, the scalar NLS equation is particularly relevant in view of its universal nature, since most dispersive energy preserving systems reduce to it in appropriate limits. All this clearly explains the keen interest in NLS equations as prototypical

2010 *Mathematics Subject Classification.* Primary 35Q55, 37K15, 47J35, 35C08.

integrable systems, and motivates the effort put into advancing our mathematical understanding of this equation.

The inverse scattering transform (IST) as a method to solve the initial-value problem for the scalar NLS equation:

$$iq_t = q_{xx} - 2\sigma\,|q|^2 q \tag{1.1}$$

(subscripts x and t denote partial differentiation throughout) has been amply studied in the literature, both in the focusing ($\sigma = -1$) and in the defocusing ($\sigma = 1$) dispersion regimes; see, for instance, [**2**–**4**, **15**, **27**, **38**] for detailed accounts of the IST in the case of solutions $q(x,t)$ rapidly decaying as $x \to \pm\infty$. The situation is quite different when one is interested in solutions that do not decay at space infinity. As a matter of fact, even though the IST for the focusing NLS equation with rapidly decaying potentials was first proposed more than 40 years ago, and has been subsequently the subject of a vast amount of studies and applications, not nearly as much is available in the literature in the case of nontrivial boundary conditions. The reason for this deficiency is twofold: on one hand, the technical difficulties resulting from the non-zero boundary conditions (NZBCs) significantly complicate the formulation of the IST; on the other hand, the onset of modulational instability, also known as the Benjamin-Feir instability [**10**, **11**] in the context of water waves, was believed to be an obstacle to the development of the IST, or at least to its validity. Nonetheless, direct methods have been extensively used over the years to derive a large number of exact solutions to the focusing NLS equation with NZBCs, known as Peregrine and multi-Peregrine solutions, Akhmediev breathers, and more general solitonic solutions [**6**–**9**, **23**–**26**, **30**, **33**, **35**]. Lately these solutions have been the subject of a renewed interest, due to the fact that the development of modulation instability in the governing equation has been recently suggested as a mechanism for the formation of "extreme" (rogue, freak) waves, for which energy density exceeds the mean level by an order of magnitude (see, for instance, [**28**, **36**, **37**] in relation to water waves, and [**32**] regarding the observation of "rogue" waves in optical systems).

In view of these recent developments, it is natural to wonder about the role that soliton solutions play in the nonlinear development of the modulation instability, which makes the study of the long-time asymptotics of NLS solutions of great practical importance, crucial for developing a consistent theory for rogue waves in the ocean, and for extreme events in optical fibers. Likewise, also the investigation of the IST for the focusing case with NZBCs [Eq. (1.1) with $\sigma = -1$], i.e.

$$iq_t = q_{xx} + 2|q|^2 q\,, \tag{1.2}$$

as a means to provide the time evolution of a fairly general initial one-dimensional pulse/wave profile over a nontrivial background, has been receiving a greater deal of attention, since it allows the study of the long-time asymptotic behavior via the nonlinear steepest descent method [**14**, **17**], matched asymptotic expansions [**1**, **5**], or other germane techniques.

The IST for the defocusing NLS equation with NZBCs was first studied in 1973 [**39**]; the problem was subsequently clarified and generalized in various works (see [**18**, **20**] and references therein). On the other hand, to the best of our knowledge until recently the only results on the IST for the focusing NLS with NZBCs available in the literature could be found in [**21**, **25**], which only partially address the problem since the study was limited to the case of completely symmetric boundary conditions with $\lim_{x\to+\infty} q(x,t) = \lim_{x\to-\infty} q(x,t)$, i.e., only the case in which the potential exhibits no asymptotic phase difference and no amplitude difference is treated. In [**12**], Biondini & Kovačić have contributed filling this gap by developing the IST for potentials with an arbitrary asymptotic phase difference, although assuming the same amplitude at both space infinities. They also discuss the general behavior of the soliton solutions, as well as the reductions to all special solutions previously known in the literature and mentioned above. The IST for focusing NLS with fully asymmetric NZBCs has been developed in [**19**], where different asymptotic amplitudes and phases are considered: $q(x,t) \to A_{l/r} e^{-2iA_{l/r}^2 t + i\theta_{l/r}}$ as $x \to \pm\infty$, with $A_r \geq A_l > 0$. This is a nontrivial generalization of [**12**], and it involves dealing with additional technical difficulties, the most important of which being the fact that when the amplitudes of the NLS solutions as $x \to \pm\infty$ are different, in the spectral domain one cannot introduce a uniformization variable [**20**] that allows mapping the multiply sheeted Riemann surface for the scattering parameter to a single complex plane, which provides a remarkable simplification in the study of both direct and inverse problems.

In this work we will develop the IST for the scalar focusing NLS (1.2) with the following one-sided NZBCs:

$$q(x,t) \to 0 \quad \text{as} \quad x \to -\infty, \tag{1.3a}$$

$$q(x,t) \to q_r(t) = A_r\, e^{-2iA_r^2 t + i\theta_r} \quad \text{as} \quad x \to +\infty, \tag{1.3b}$$

where $A_r > 0$ and $0 \leq \theta_r < 2\pi$ are arbitrary constants. Obviously, the case of a nontrivial boundary condition only as $x \to -\infty$ can be treated in a similar fashion.

Such kind of boundary conditions are obviously outside the class considered in [**12**], where the amplitudes of the background field are taken to be the same at both space infinities. The problem with one-sided non-zero boundary conditions clearly has a physical relevance on its own, and, unlike what happens, for instance, for the Korteweg-de Vries (KdV) equation, for NLS one cannot set one of boundary conditions to zero without loss of generality, by performing suitable rescalings of the field. At the same time, the mathematical motivation for the present work is twofold. On one hand, in [**19**] we assumed $A_l > 0$, and the limit $A_l \to 0$ is a singular limit, which makes recovering the corresponding results from the fully asymmetric case far from straightforward. On the other hand, the case of one-sided NZBCs presents some specific features that make it deserving a separate investigation. In fact, unlike the case of fully asymmetric boundary conditions and similarly to the same-amplitude case dealt with in [**12**], with boundary conditions such as the ones in (1.3) it is still possible to introduce a uniformization variable [**20**] that allows mapping the multiply sheeted Riemann surface for the scattering parameter into a single complex plane. Yet, important differences with respect to the same-amplitude case arise both in the direct and in the inverse problems, and they will be properly highlighted in this work. From the point of view of physical applications, such a work would be particularly significant for the theoretical investigation of rogue waves and perturbed soliton solutions in microstructured fiber optical systems with different background amplitudes enforced at either end of the fiber. This work would also be relevant in clarifying the role that soliton solutions play in the nonlinear development of modulation instability in such systems.

The plan of the paper is outlined below. Sec. 2 is devoted to the study of the direct scattering problem on a single sheet of the scattering variables k, $\lambda_r = \sqrt{k^2 + A_r^2}$, where k is the usual complex scattering parameter in the IST. We will show that the direct problem is well defined for potentials $q(x,t)$ such that $[q(x,t) - \vartheta(x)q_r(t)] \in L^{1,1}(\mathbb{R})$ with respect to $x \in \mathbb{R}$ and for all $t \geq 0$, where $L^{1,s}(\mathbb{R})$ is the complex Banach space of all measurable functions $f(x)$ for which $(1+|x|)^s f(x)$ is integrable and $\vartheta(x)$ is the Heaviside function [i.e., $\vartheta(x) = 1$ for $x \geq 0$, and zero otherwise]. We will then establish analyticity of eigenfunctions and scattering data in k, and obtain integral representations for the latter for potentials in this class. In Sec. 3 we will formulate the inverse problem both in terms of Marchenko integral equations, and as a Riemann-Hilbert (RH) problem on a single sheet of the scattering variables $(k, \lambda_r(k))$. Important differences with respect to the symmetric case also arise in the inverse problem, where, in addition to solitons (corresponding to the discrete eigenvalues of the scattering problem), and to radiation (corresponding to the continuous spectrum of the scattering operator, and represented in the inverse problem by the reflection coefficients for $k \in \mathbb{R}$), one also has a nontrivial contribution from additional spectral data for $k \in (-iA_r, iA_r)$, which appears in both formulations of the inverse problem. In particular, this implies that no pure soliton solutions exist, and solitons are always accompanied by a radiative contribution of some sort. As a consequence, unlike the equal-amplitude case dealt with in [**12**], here no explicit solution can be obtained by simply reducing the inverse problem to a set of algebraic equations. In view of this, the present study provides a very powerful tool for the asymptotic investigation of NLS solutions that cannot be obtained by direct methods. Specifically, the RH formulation of the inverse problem makes it amenable to the study of the long-time asymptotic behavior via the nonlinear steepest descent method, as shown, for instance, in [**17**] for the modified KdV equation, or in [**14**] for the focusing NLS with initial condition $q(x,0) = A\, e^{i\mu|x|}$, A and μ being positive constants. The Marchenko integral equations provide an alternative setup for the study of the long-time behavior of the solutions by means of matched asymptotics, as was recently done for KdV in [**1**]. Sec. 4 deals with the time evolution of eigenfunctions and scattering coefficients. In Sec. 5 we develop the direct scattering problem in terms of the uniform variable $z = k + \lambda_r$, and formulate the inverse problem as a Riemann-Hilbert problem in $z \in \mathbb{C}$. Finally, Sec. 6 is devoted to some concluding remarks.

2. Direct problem

It is well-known that the focusing NLS equation (1.2) can be associated to the following Lax pair:

$$\frac{\partial v}{\partial x} = (-ik\sigma_3 + Q)\, v\,, \tag{2.1a}$$

$$\frac{\partial v}{\partial t} = \left[i(2k^2 - |q|^2 + Q_x)\sigma_3 - 2kQ\right] v\,, \tag{2.1b}$$

where $v(x,k,t)$ is a two component vector, $k \in \mathbb{C}$ the scattering parameter, and

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -q^*(x,t) & 0 \end{pmatrix}. \tag{2.2}$$

[Here and in the following the asterisk indicate complex conjugates; σ_2 is given for future reference.] We will consider potentials $q(x,t)$ with one-sided NZBC as in (1.3), where we assume that the limits exist in the standard sense, with asymptotic amplitude A_r positive and time-independent, and asymptotic phase given by $\theta_r(t) = -2A_r^2 t + \theta_r$, to ensure compatibility with the NLS evolution. Furthermore, we assume the following integrability condition

$$(2.3) \qquad (\boldsymbol{H}_s): \qquad \int_{-\infty}^{\infty} dx\,(1+|x|)^s\,|q(x,t) - q_r(t)\vartheta(x)| < \infty\,,$$

for all $t \geq 0$, where $s = 0, 1$ depending on the situation.

For later convenience, we denote by $Q_r(t)$ the limit of $Q(x,t)$ as $x \to +\infty$ [obviously, $Q(x,t) \to 0_{2\times 2}$ as $x \to -\infty$, according to (1.3)]. We also introduce the "free" potential matrix $Q_f(x,t)$ as follows:

$$(2.4) \qquad Q_f(x,t) = Q_r(t)\,\vartheta(x).$$

In the formulation of the direct problem we will omit to explicitly specify the time-dependence for brevity. It will be clear from the context whether one is considering $t = 0$ or an arbitrary $t > 0$.

It is convenient to introduce the asymptotic scattering operator corresponding to the NZBC as $x \to +\infty$, namely:

$$(2.5a) \qquad \Lambda_r(k) = -ik\sigma_3 + Q_r\,,$$

as well as

$$(2.5b) \qquad \Lambda(x,k) = -ik\sigma_3 + Q_f(x) = -ik\sigma_3 + \vartheta(x)\Lambda_r(k)\,,$$

and attempt to define the *fundamental eigensolution* $\tilde{\Psi}(x,k)$ and the eigenfunction $\Phi(x,k)$ as those 2×2 matrix solutions to (2.1a) which satisfy the asymptotic conditions

$$(2.6a) \qquad \tilde{\Psi}(x,k) = e^{x\,\Lambda_r(k)}[I_2 + o(1)] \qquad x \to +\infty\,,$$

$$(2.6b) \qquad \Phi(x,k) = e^{-ikx\sigma_3}[I_2 + o(1)] \qquad x \to -\infty\,,$$

[here and in the following I_2 denotes the 2×2 identity matrix]. Note that because of the choice of boundary conditions (1.3), $\Phi(x,k)$ coincides with the usual pair of Jost solutions from the left for the scattering problem (cf. Sec. 2.1). On the other hand, as far as the eigensolution from the right is concerned, $e^{x\,\Lambda_r(k)}$ is a bounded group for all $x \in \mathbb{R}$ iff $\Lambda_r(k)$ has only zero or purely imaginary eigenvalues and is diagonalizable, i.e., iff $k \in \mathbb{R} \cup (-iA_r, iA_r)$. For $k = \pm iA_r$ the norm of the group $e^{x\,\Lambda_r(k)}$ grows linearly in x as $x \to +\infty$. Then the following result can be established.

Proposition 2.1. *Let the potential satisfy* $(\boldsymbol{H}_0)$. *Then for $k \in \mathbb{R}$ the eigenfunction $\Phi(x,k)$ is given by the unique solution to the integral equation*

$$(2.7a) \qquad \Phi(x,k) = e^{-ikx\sigma_3} + \int_{-\infty}^{x} dy\, e^{-ik(x-y)\sigma_3} Q(y)\Phi(y,k)\,,$$

continuous for $x \in \mathbb{R}$, and for all $k \in \mathbb{R}$. For $k \in \mathbb{R} \cup (-iA_r, iA_r)$ the fundamental eigensolution $\tilde{\Psi}(x,k)$ with asymptotic behavior (2.6a) *can be obtained as the unique solution to the integral equation*

$$(2.7b) \qquad \tilde{\Psi}(x,k) = e^{x\,\Lambda_r(k)} - \int_{x}^{\infty} dy\, e^{(x-y)\Lambda_r(k)}[Q(y) - Q_r]\tilde{\Psi}(y,k)\,.$$

Moreover, $\tilde{\Psi}(x,k)$ is continuous for $x_0 \le x$ for any finite x_0[1]*, and, as a function of k, for all $k \in \mathbb{R} \cup (-iA_r, iA_r)$. In addition, if the potential satisfies ($\boldsymbol{H}_1$), then* (2.7b) *has a unique, continuous solution for $k \in [-iA_r, iA_r]$ (i.e., the continuity result can be extended to include the branch points $k = \pm iA_r$).*

The result for $\Phi(x,k)$ follows from standard iteration arguments for decaying potentials in L^1, and for $\tilde{\Psi}(x,k)$ it can be proved as in [**18**] and [**19**].

Assuming ($\boldsymbol{H}_1$), one can replace the integral equations (2.7) by different ones. To this aim, let us introduce the fundamental matrix $\mathcal{G}(x,y;k)$ as follows:

$$\begin{aligned}\mathcal{G}(x,y;k) = {} & \vartheta(x)\vartheta(y)\, e^{(x-y)\Lambda_r(k)} + \vartheta(-x)\vartheta(-y)\, e^{-ik\sigma_3(x-y)} \\ & + \vartheta(x)\vartheta(-y)\, e^{x\Lambda_r(k)}\, e^{ik\sigma_3 y} + \vartheta(-x)\vartheta(y)\, e^{-ik\sigma_3 x}\, e^{-y\Lambda_r(k)} \,.\end{aligned} \tag{2.8}$$

$\mathcal{G}(x,y;k)$ is a continuous matrix function of $(x,y,k) \in \mathbb{R}^2 \times \mathbb{C}$ which satisfies the initial value problems:

$$\frac{\partial \mathcal{G}(x,y;k)}{\partial x} = \Lambda(x,k)\mathcal{G}(x,y,k)\,, \quad \mathcal{G}(y,y;k) = I_2\,, \tag{2.9a}$$

$$\frac{\partial \mathcal{G}(x,y;k)}{\partial y} = -\,\mathcal{G}(x,y,k)\Lambda(y,k)\,, \quad \mathcal{G}(x,x;k) = I_2\,, \tag{2.9b}$$

where $\Lambda(x,k)$ is given by (2.5b). For further details on the fundamental matrix we refer to [**18**, App A], where the analogous problem is considered for the defocusing NLS equation. Then using (2.9) one can easily check that the eigenfunction $\Phi(x,k)$ and the fundamental eigensolution $\tilde{\Psi}(x,k)$ also satisfy the integral equations

$$\Phi(x,k) = \mathcal{G}(x,0;k) + \int_{-\infty}^{x} dy\, \mathcal{G}(x,y;k)[Q(y) - Q_f(y)]\Phi(y,k)\,, \tag{2.10a}$$

$$\tilde{\Psi}(x,k) = \mathcal{G}(x,0;k) - \int_{x}^{\infty} dy\, \mathcal{G}(x,y;k)[Q(y) - Q_f(y)]\tilde{\Psi}(y,k)\,, \tag{2.10b}$$

where $\mathcal{G}(x,0;k) = \vartheta(x)e^{x\Lambda_r(k)} + \vartheta(-x)e^{-ik\sigma_3 x}$, according to (2.8), and $Q_f(x)$ is given by (2.4). Note that (2.10a) coincides with (2.7a) for $x \le 0$, and (2.10b) coincides with (2.7b) for $x \ge 0$. On the other hand, using (2.8) we get

$$\Phi(x,k) = e^{x\Lambda_r(k)} \left[I_2 + \int_{-\infty}^{x} dy\, \mathcal{G}(0,y;k)[Q(y) - Q_f(y)]\Phi(y,k) \right], \quad x \ge 0\,, \tag{2.11a}$$

$$\tilde{\Psi}(x,k) = e^{-ik\sigma_3 x} \left[I_2 - \int_{x}^{\infty} dy\, \mathcal{G}(0,y;k)[Q(y) - Q_f(y)]\tilde{\Psi}(y,k) \right], \quad x \le 0\,. \tag{2.11b}$$

Note that in (2.11b) the integral in the right-hand side converges absolutely as $x \to -\infty$, unlike (2.7b).

2.1. Jost solutions.

Since the asymptotic scattering operator $\Lambda_r(k)$ is traceless, and such that $\Lambda_r^2(k) = -(k^2 + A_r^2)I_2$, we consider the two-sheeted Riemann surface associated with $\lambda_r^2 = k^2 + A_r^2$ by introducing appropriate local polar coordinates, with $r_j \ge 0$ and $-\pi/2 \le \theta_j < 3\pi/2$ for $j = 1,2$, and define:

$$\lambda_r = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2} \quad \text{on Sheet I}\,, \qquad \lambda_r = -\sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2} \quad \text{on Sheet II}\,. \tag{2.12}$$

[1] Note that the integral in (2.7b) does not converge absolutely for $x \to -\infty$.

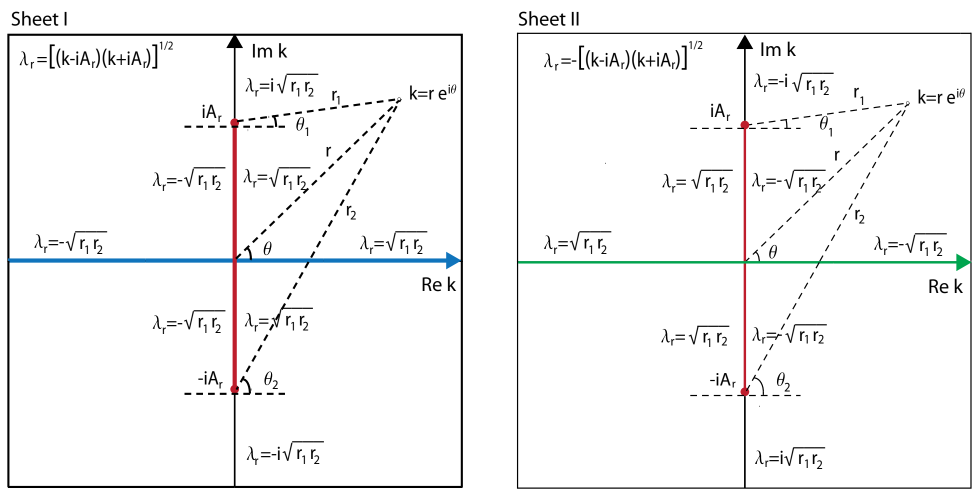

FIGURE 1. The branch cut on the two-sheeted Riemann surface associated with $\lambda_r^2 = k^2 + A_r^2$: we define $\lambda_r = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$ on Sheet I, and $\lambda_r = -\sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$ on Sheet II, with $r_1 = |k - iA_r|$, $r_2 = |k + iA_r|$ and angles $-\pi/2 \le \theta_1, \theta_2 < 3\pi/2$ for $j = 1, 2$.

The branch cut is along the imaginary segment $\Sigma_r = [-iA_r, iA_r]$. The Riemann surface is then obtained by gluing together the two copies of the complex plane along the cut Σ_r [see Fig. 1].

For the purpose of Secs. 2-4, we will consider a single sheet (Sheet I) of the complex plane for k, and denote by $\mathbb{K}_r$ the plane cut along the segment Σ_r on the imaginary axis. $\mathbb{C}^\pm$ will denote the open upper/lower complex half planes, and $\mathbb{K}_r^\pm$ the open upper/lower complex half-planes, respectively, cut along Σ_r.
It is easy to show that λ_r provides one-to-one correspondences between the following sets:

- $k \in \mathbb{K}_r^+ \equiv \mathbb{C}^+ \setminus (0, iA_r]$ and $\lambda_r \in \mathbb{C}^+$
- $k \in \partial\mathbb{K}_r^+ \equiv \mathbb{R} \cup \{is - 0^+ : 0 < s < A_r\} \cup \{iA_r\} \cup \{is + 0^+ : 0 < s < A_r\}$ and $\lambda_r \in \mathbb{R}$
- $k \in \mathbb{K}_r^- \equiv \mathbb{C}^- \setminus [-iA_r, 0)$ and $\lambda_r \in \mathbb{C}^-$
- $k \in \partial\mathbb{K}_r^- \equiv \mathbb{R} \cup \{is - 0^+ : -A_r < s < 0\} \cup \{-iA_r\} \cup \{is + 0^+ : -A_r < s < 0\}$ and $\lambda_r \in \mathbb{R}$.

Note that with this choice for the branch cut one has $\lambda_r \sim k$ as $k \to \infty$ in the entire Sheet I, while $\lambda_r \sim -k$ as $k \to \infty$ on Sheet II. In the following, $\lambda_r^\pm(k)$ will denote the boundary values taken by $\lambda_r(k)$ for $k \in \Sigma_r$ from the right/left edge of the cut on Sheet I, with

$$\lambda_r^\pm(k) = \pm\sqrt{A_r^2 - |k|^2}\,, \qquad k = is \pm 0^+, \quad |s| \le A_r \tag{2.13}$$

on the right/left edge (cf. Fig. 1).

The eigenvalues of $\Lambda_r(k)$ are $\pm i\lambda_r$, and the eigenvector matrix $W_r(k)$, such that

$$\Lambda_r(k)W_r(k) = -i\lambda_r W_r(k)\sigma_3\,, \tag{2.14}$$

can be conveniently chosen as follows:

$$W_r(k) = I_2 - \frac{i}{\lambda_r + k}\sigma_3 Q_r\,. \tag{2.15}$$

Note that $\det W_r(k) = \frac{2\lambda_r}{\lambda_r+k}$, and $W_r(k)$ is a nonsingular matrix on either sheet because $A_r > 0$ holds strictly ($\lambda_r + k$ can only vanish on Sheet II, and only in the limit $k \to \infty$).

We can then define the "right" Jost solutions $\Psi(x,k) = \big(\bar\psi(x,k) \;\; \psi(x,k)\big)$ in terms of the fundamental eigensolution $\tilde\Psi(x,k)$ as:

$$\Psi(x,k) = \big(\bar\psi(x,k) \;\; \psi(x,k)\big) := \tilde\Psi(x,k)W_r(k)\,, \tag{2.16}$$

which then satisfies the following boundary condition:

$$\Psi(x,k) \sim W_r(k)\, e^{-i\lambda_r x\sigma_3}\,, \qquad x \to +\infty\,. \tag{2.17}$$

The Jost solutions from the right $\bar\psi(x,k)$, $\psi(x,k)$ are then defined via a customary asymptotic plane wave behavior (cf. (2.17)) when $\lambda_r \in \mathbb{R}$, i.e., for $k \in \partial\mathbb{K}_r^+ \cup \partial\mathbb{K}_r^-$, and when $k = is \in [-iA_r, iA_r]$ we will denote with a superscript $^\pm$ the values on the right/left edge of the cut in both half-planes, i.e.:

$$\Psi^\pm(x,is) \equiv \big(\bar\psi^\pm(x,is) \;\; \psi^\pm(x,is)\big) := \tilde\Psi(x,is)\, W_r(is \pm 0^+) \qquad |s| \le A_r\,, \tag{2.18}$$

since $\tilde\Psi(x,k)$ is single-valued across the cut, and $W_r(k)$ has right/left limits defined by (2.13).

On the other hand, the "left" Jost solutions $\Phi(x,k) = \big(\phi(x,k) \;\; \bar\phi(x,k)\big)$ are defined as asymptotic "plane waves" (cf. (2.6b)) for $k \in \mathbb{R}$.

Taking into account (2.7) and (2.16), the Jost solutions can be represented in terms of the following integral equations:

$$e^{i\lambda_r x}\bar\psi(x,k) = W_{r,1}(k) - \int_x^\infty dy\, \Xi_r^-(y-x,k)[Q(y)-Q_r]e^{i\lambda_r y}\bar\psi(y,k), \tag{2.19a}$$

$$e^{-i\lambda_r x}\psi(x,k) = W_{r,2}(k) - \int_x^\infty dy\, \Xi_r^+(y-x,k)[Q(y)-Q_r]e^{-i\lambda_r y}\psi(y,k), \tag{2.19b}$$

$$e^{ikx}\phi(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^x dy\, \Xi_l^+(x-y,k)\, Q(y)\, e^{iky}\phi(y,k), \tag{2.19c}$$

$$e^{-ikx}\bar\phi(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^x dy\, \Xi_l^-(x-y,k)\, Q(y)\, e^{-iky}\bar\phi(y,k), \tag{2.19d}$$

where the subscripts $j = 1, 2$ in the matrix $W_r(k)$ denote its j-th column, and

$$\Xi_r^-(x,k) = \begin{pmatrix} 1 + \frac{\lambda_r - k}{2\lambda_r}\left[e^{-2i\lambda_r x} - 1\right] & -\frac{iq_r}{2\lambda_r}\left[e^{-2i\lambda_r x} - 1\right] \\ \frac{iq_r^*}{2\lambda_r}\left[e^{-2i\lambda_r x} - 1\right] & e^{-2i\lambda_r x} - \frac{\lambda_r - k}{2\lambda_r}\left[e^{-2i\lambda_r x} - 1\right] \end{pmatrix}, \tag{2.20a}$$

$$\Xi_r^+(x,k) = \begin{pmatrix} e^{2i\lambda_r x} - \frac{\lambda_r - k}{2\lambda_r}\left[e^{2i\lambda_r x} - 1\right] & \frac{iq_r}{2\lambda_r}\left[e^{2i\lambda_r x} - 1\right] \\ -\frac{iq_r^*}{2\lambda_r}\left[e^{2i\lambda_r x} - 1\right] & 1 + \frac{\lambda_r - k}{2\lambda_r}\left[e^{2i\lambda_r x} - 1\right] \end{pmatrix}, \tag{2.20b}$$

$$\Xi_l^+(x,k) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2ikx} \end{pmatrix}, \tag{2.20c}$$

$$\Xi_l^-(x,k) = \begin{pmatrix} e^{-2ikx} & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.20d}$$

Note that the integral equations (2.19a) and (2.19b) for the "right" Jost solutions are the same as in the fully asymmetric case [**19**], while the integral equations (2.19c) and (2.19d) for the "left" Jost solutions are the same as in the vanishing case [**3**].

The following result then establishes the analyticity properties of the Jost solutions as functions of the scattering parameter k.

Proposition 2.2. *Suppose* $(\boldsymbol{H}_1)$ *holds.*[2] *Then, for every* $x \in \mathbb{R}$, *the Jost solution* $\psi(x,k)$ *[resp.* $\bar{\psi}(x,k)$*] extends to a function that is analytic for* $k \in \mathbb{K}_r^+$ *[resp.* $k \in \mathbb{K}_r^-$*], and continuous for* $k \in \mathbb{K}_r^+ \cup \partial\mathbb{K}_r^+ \cup \partial\mathbb{K}_r^-$ *[resp.* $k \in \mathbb{K}_r^- \cup \partial\mathbb{K}_r^- \cup \partial\mathbb{K}_r^+$*]. On the other hand, the Jost solution* $\phi(x,k)$ *[resp.* $\bar{\phi}(x,k)$*] extends to a function that is continuous for* $k \in \mathbb{C}^+ \cup \mathbb{R}$ *[resp.* $k \in \mathbb{C}^- \cup \mathbb{R}$*] and analytic for* $k \in \mathbb{C}^+$ *[resp.* $k \in \mathbb{C}^-$*].*

Note that $\mathbb{K}_r^\pm$ are intended as analytic manifolds, and continuity of the Jost solutions across the cuts is intended as the existence of right/left continuous limits only in the domains that have the branch cut as part of their boundary as an analytic manifold, i.e. $\mathbb{K}_r^+$ for $\psi(x,k)$ and $\mathbb{K}_r^-$ for $\bar{\psi}(x,k)$. In the half-planes where locally there is no analytic continuation off the branch cut, the functions $\psi^\pm(x,k)$, $\bar{\psi}^\pm(x,k)$ are as given in (2.18) with the two choices of $\lambda_r^\pm$, and can be obtained as the unique solutions of the corresponding Volterra integral equations (2.19).
The proof of Prop. 2.2 for $\psi(x,k)$ and $\bar{\psi}(x,k)$ can be carried out as in [**18, 19**], while the result for $\phi(x,k)$ and $\bar{\phi}(x,k)$ can be established following the standard approach for sufficiently rapidly decaying potentials.

In conclusion, for potentials satisfying $(\boldsymbol{H}_1)$ all four Jost solutions are in general simultaneously defined only for $k \in \mathbb{R}$. Note that $\phi^+(x,k) = \phi^-(x,k)$ for $k \in [0, iA_r]$, and $\bar{\phi}^+(x,k) = \bar{\phi}^-(x,k)$ for $k \in [-iA_r, 0]$, where the superscript $^\pm$ denotes again the values on the right/left edge of the cut in both half-planes. On the contrary, $\bar{\phi}(x,k)$ is in general not defined for $k \in (0, iA_r]$, and $\phi(x,k)$ is not defined for $k \in [-iA_r, 0)$.

2.2. Scattering coefficients. From the integral equations (2.11) for $\Phi(x,k)$ and $\tilde{\Psi}(x,k)$, one can easily find

$$\tilde{\Psi}(x,k) = e^{-ik\sigma_3 x}[B_r(k) + o(1)], \qquad x \to -\infty, \tag{2.21a}$$

$$\Phi(x,k) = e^{x\Lambda_r(k)}[B_l(k) + o(1)], \qquad x \to +\infty, \tag{2.21b}$$

[2] In fact, since the potential decays as $x \to -\infty$, it would be enough to assume $[q(x,t) - \vartheta(x)q_r(t)] \in L^1(\mathbb{R}^-) \cap L^{1,1}(\mathbb{R}^+)$ with respect to x for all $t \geq 0$.

where the coupling matrices

$$B_r(k) = I_2 - \int_{-\infty}^{\infty} dy\, \mathcal{G}(0,y;k)[Q(y) - Q_f(y)]\tilde{\Psi}(y,k), \tag{2.22a}$$

$$B_l(k) = I_2 + \int_{-\infty}^{\infty} dy\, \mathcal{G}(0,y;k)[Q(y) - Q_f(y)]\Phi(y,k), \tag{2.22b}$$

are each other's inverse. Under the assumptions of Prop. 2.1, in Eqs. (2.21) and (2.22) one needs to take $k \in \mathbb{R}$, where all eigenfunctions (2.10) and (2.11) are simultaneously defined.

Using (2.16) and (2.21) to obtain the asymptotic behavior of the Jost solutions as $x \to \pm\infty$, for $k \in \mathbb{R}$ we can then express each set of Jost solutions as a linear combination of the other set, i.e.,

$$\begin{pmatrix}\phi(x,k) & \bar{\phi}(x,k)\end{pmatrix} = \begin{pmatrix}\bar{\psi}(x,k) & \psi(x,k)\end{pmatrix} S(k)\,, \tag{2.23a}$$

$$\begin{pmatrix}\bar{\psi}(x,k) & \psi(x,k)\end{pmatrix} = \begin{pmatrix}\phi(x,k) & \bar{\phi}(x,k)\end{pmatrix} \bar{S}(k)\,, \tag{2.23b}$$

where the *scattering matrices* $S(k)$ and $\bar{S}(k)$ are obviously each other inverses, and they are given by

$$S(k) = W_r^{-1}(k)B_l(k)\,, \qquad \bar{S}(k) = B_r(k)W_r(k). \tag{2.24}$$

For later convenience we write

$$S(k) = \begin{pmatrix} a(k) & \bar{b}(k) \\ b(k) & \bar{a}(k)\end{pmatrix}, \qquad \bar{S}(k) = \begin{pmatrix} \bar{c}(k) & d(k) \\ \bar{d}(k) & c(k)\end{pmatrix}, \tag{2.25}$$

where the entries of the scattering matrices are usually referred to as *scattering coefficients*, and at this stage they are all in general defined only for $k \in \mathbb{R}$. Moreover, since $\det \Phi(x,k)$ and $\det \Psi(x,k)$ are independent of x, from (2.6b) and (2.17) it follows that

$$\det \Phi(x,k) = 1\,, \qquad \det \Psi(x,k) = \frac{2\lambda_r}{\lambda_r + k}\,, \tag{2.26}$$

and consequently we obtain for $k \in \mathbb{R}$:

$$\det S(k) = \frac{1}{\det W_r(k)} = \frac{\lambda_r + k}{2\lambda_r}, \quad \det \bar{S}(k) = \det W_r(k) = \frac{2\lambda_r}{\lambda_r + k}\,. \tag{2.27}$$

If we now denote by $\mathrm{Wr}(v_1, v_2) \overset{\text{def}}{=} \det \begin{pmatrix} v_1 & v_2\end{pmatrix}$ the Wronskian of any two vector solutions v_1, v_2 of the scattering problem (2.1a), then Eqs. (2.23) yield the following "Wronskian" representations for the scattering coefficients in (2.25) for $k \in \mathbb{R}$:

(2.28a)

$$a(k) = \frac{\mathrm{Wr}(\phi,\psi)}{\mathrm{Wr}(\bar{\psi},\psi)} = \frac{\lambda_r + k}{2\lambda_r}\,\mathrm{Wr}(\phi,\psi), \qquad \bar{a}(k) = \frac{\mathrm{Wr}(\bar{\psi},\bar{\phi})}{\mathrm{Wr}(\bar{\psi},\psi)} = \frac{\lambda_r + k}{2\lambda_r}\,\mathrm{Wr}(\bar{\psi},\bar{\phi}),$$

(2.28b)

$$b(k) = \frac{\mathrm{Wr}(\bar{\psi},\phi)}{\mathrm{Wr}(\bar{\psi},\psi)} = \frac{\lambda_r + k}{2\lambda_r}\,\mathrm{Wr}(\bar{\psi},\phi), \qquad \bar{b}(k) = \frac{\mathrm{Wr}(\bar{\phi},\psi)}{\mathrm{Wr}(\bar{\psi},\psi)} = \frac{\lambda_r + k}{2\lambda_r}\,\mathrm{Wr}(\bar{\phi},\psi),$$

and

$$c(k) = \frac{\mathrm{Wr}(\phi,\psi)}{\mathrm{Wr}(\phi,\bar{\phi})} = \frac{2\lambda_r}{\lambda_r + k}a(k), \qquad \bar{c}(k) = \frac{\mathrm{Wr}(\bar{\psi},\bar{\phi})}{\mathrm{Wr}(\phi,\bar{\phi})} = \frac{2\lambda_r}{\lambda_r + k}\bar{a}(k)\,, \tag{2.28c}$$

$$d(k) = \frac{\mathrm{Wr}(\psi,\bar{\phi})}{\mathrm{Wr}(\phi,\bar{\phi})} = -\frac{2\lambda_r}{\lambda_r + k}\bar{b}(k), \qquad \bar{d}(k) = \frac{\mathrm{Wr}(\phi,\bar{\psi})}{\mathrm{Wr}(\phi,\bar{\phi})} = -\frac{2\lambda_r}{\lambda_r + k}b(k)\,, \tag{2.28d}$$

where the arguments (x,k) of the Jost solutions have been omitted for brevity, and the second set of identities in (2.28c) and (2.28d) are obtained from $\bar{S}(k)=S^{-1}(k)$.

These Wronskian representations can then be used to define the values of the scattering coefficients from the right/left edge of the cut Σ_r, consistently with (2.18). Explicitly, taking into account that $\lambda_r^{\pm}(k)=\pm\sqrt{A_r^2-|k|^2}$ on the right/left edge of the cut for $k\in\Sigma_r$ (cf. Fig. 1), one has

$$a^{\pm}(k)=\frac{\lambda_r^{\pm}(k)+k}{2\lambda_r^{\pm}(k)}\,\mathrm{Wr}(\phi(x,k),\psi^{\pm}(x,k))\qquad k\in[0,iA_r]\,,\tag{2.29a}$$

$$\bar{a}^{\pm}(k)=\frac{\lambda_r^{\pm}(k)+k}{2\lambda_r^{\pm}(k)}\,\mathrm{Wr}(\bar{\psi}^{\pm}(x,k),\bar{\phi}(x,k))\qquad k\in[-iA_r,0]\,,\tag{2.29b}$$

$$b^{\pm}(k)=\frac{\lambda_r^{\pm}(k)+k}{2\lambda_r^{\pm}(k)}\,\mathrm{Wr}(\bar{\psi}^{\pm}(x,k),\phi(x,k))\qquad k\in[0,iA_r]\,,\tag{2.29c}$$

$$\bar{b}^{\pm}(k)=\frac{\lambda_r^{\pm}(k)+k}{2\lambda_r^{\pm}(k)}\,\mathrm{Wr}(\bar{\phi}(x,k),\psi^{\pm}(x,k))\qquad k\in[-iA_r,0]\,,\tag{2.29d}$$

and similarly for the scattering coefficients from the left defined by (2.28c)-(2.28d).

Eqs. (2.28) allow one to analytically continue some of the scattering coefficients off the real k-axis under the assumption $(\boldsymbol{H}_1)$. In fact, (2.28) and Prop. 2.2 imply:

- $a(k)$ is analytic in $k\in\mathbb{K}_r^+$, and continuous for $k\in\overline{\mathbb{K}_r^+}\setminus\{iA_r\}$ [with values across the cut $a^{\pm}(k)$ as in (2.29a)]; also

$$a(k)\sim\frac{iA_r}{2\lambda_r}\,\mathrm{Wr}(\phi(x,iA_r),\psi(x,iA_r)),\quad k\to iA_r.\tag{2.30a}$$

- $\bar{a}(k)$ is analytic in $k\in\mathbb{K}_r^-$, and continuous in $k\in\overline{\mathbb{K}_r^-}\setminus\{-iA_r\}$ [with values across the cut $\bar{a}^{\pm}(k)$ as in (2.29b)]; also

$$\bar{a}(k)\sim\frac{-iA_r}{2\lambda_r}\,\mathrm{Wr}(\bar{\psi}(x,-iA_r),\bar{\phi}(x,-iA_r)),\quad k\to -iA_r.\tag{2.30b}$$

- $b(k)$ is continuous for $\partial\mathbb{K}_r^+\setminus\{iA_r\}$ [with values across the cut $b^{\pm}(k)$ as in (2.29c)], and $\bar{b}(k)$ is continuous for $k\in\partial\mathbb{K}_r^-+\setminus\{-iA_r\}$ [with values across the cut $\bar{b}^{\pm}(k)$ as in (2.29d)]; at the branch points

$$b^{\pm}(k)\sim\frac{iA_r}{2\lambda_r}\,\mathrm{Wr}(\bar{\psi}(x,iA_r),\phi(x,iA_r)),\qquad k\to iA_r\,,\tag{2.30c}$$

$$\bar{b}^{\pm}(k)\sim\frac{-iA_r}{2\lambda_r}\,\mathrm{Wr}(\bar{\phi}(x,-iA_r),\psi(x,-iA_r)),\quad k\to -iA_r\,.\tag{2.30d}$$

Similar results can be derived for the four scattering coefficients from the left, although the corresponding properties can also be obtained from those above using (2.28c) and (2.28d). Note, in particular, that $a(k)$ [resp., $\bar{a}(k)$] has a branch point singularity at $k=iA_r$ [resp., $k=-iA_r$], where $\lambda_r=0$, while $c(k)$ [resp., $\bar{c}(k)$] is well-defined there, consistently with (2.28c).

For future convenience we also define the *reflection coefficients* from the right as follows:

$$\rho(k)=\frac{b(k)}{a(k)}\qquad\text{for } k\in\mathbb{R},\qquad \rho^{\pm}(k)=\frac{b^{\pm}(k)}{a^{\pm}(k)}\quad\text{for } k\in[0,iA_r),\tag{2.31a}$$

$$\bar{\rho}(k)=\frac{\bar{b}(k)}{\bar{a}(k)}\qquad\text{for } k\in\mathbb{R},\qquad \bar{\rho}^{\pm}(k)=\frac{\bar{b}^{\pm}(k)}{\bar{a}^{\pm}(k)}\quad\text{for } k\in(-iA_r,0],\tag{2.31b}$$

and the *reflection coefficients* from the left as:

$$r(k) = \frac{d(k)}{c(k)} = -\frac{\bar{b}(k)}{a(k)}, \qquad \bar{r}(k) = \frac{\bar{d}(k)}{\bar{c}(k)} = -\frac{b(k)}{\bar{a}(k)}, \quad k \in \mathbb{R}, \tag{2.31c}$$

where in the last two expressions we have used that $S(k) = \bar{S}^{-1}(k)$. The reciprocals $1/a(k)$, $1/\bar{a}(k)$, $1/c(k)$, and $1/\bar{c}(k)$ are usually referred to as (right and left) *transmission coefficients*.

Furthermore, as shown in [**19**] for the case of fully asymmetric NZBCs, from (2.24) and (2.22), using (2.14) and (2.16) we also obtain the following integral representation for the scattering matrix:

$$\begin{aligned} S(k) = & \int_0^\infty dy\, e^{i\lambda_r y\sigma_3} W_r^{-1}(k)[Q(y) - Q_r]\Phi(y,k) \\ & + W_r^{-1}(k) \left[I_2 + \int_{-\infty}^0 dy\, e^{iky\sigma_3}[Q(y) - Q_l]\Phi(y,k) \right], \end{aligned} \tag{2.32}$$

which could serve as an alternative to the Wronskian representations to establish the analytic continuation in the appropriate half planes of the scattering coefficients $a(k)$ and $\bar{a}(k)$.

2.3. Symmetries of eigenfunctions and scattering data. The scattering problem (2.1a) admits two involutions: $(k, \lambda_r) \to (k^*, \lambda_r^*)$ and $(k, \lambda_r) \to (k, -\lambda_r)$. Correspondingly, eigenfunctions and scattering data satisfy two sets of symmetry relations.

First symmetry. Using the asymptotic behaviors (2.6b) and (2.17), the symmetries for the Jost solutions are given by:

(2.33a)
$$\bar{\psi}^*(x,k^*) = i\sigma_2\psi(x,k) \text{ for } k \in \mathbb{K}_r^+ \cup \mathbb{R}\,, \quad \psi^*(x,k^*) = -i\sigma_2\bar{\psi}(x,k) \text{ for } k \in \mathbb{K}_r^- \cup \mathbb{R}\,,$$
(2.33b)
$$\big(\bar{\psi}^\pm(x,k^*)\big)^* = i\sigma_2\psi^\pm(x,k)\,, \quad \big(\psi^\pm(x,k^*)\big)^* = -i\sigma_2\bar{\psi}^\pm(x,k) \text{ for } k \in [-iA_r, iA_r]\,,$$
(2.33c)
$$\phi^*(x,k^*) = i\sigma_2\bar{\phi}(x,k) \text{ for } k \in \mathbb{C}^- \cup \mathbb{R}\,, \quad \bar{\phi}^*(x,k^*) = -i\sigma_2\phi(x,k) \text{ for } k \in \mathbb{C}^+ \cup \mathbb{R}\,,$$

where σ_2 is the second Pauli matrix introduced in (2.2). From (2.23) we then obtain $S^*(k^*) = \sigma_2 S(k)\sigma_2$ on the continuous spectrum $k \in \mathbb{R}$, and wherever all entries in the scattering matrix are simultaneously defined. Under the assumption ($\boldsymbol{H}_0$) for the potential, the symmetry relations for the scattering coefficients can be written as

$$\bar{a}^*(k^*) = a(k) \text{ for } k \in \mathbb{K}_r^+ \cup \mathbb{R}\,, \quad \big(\bar{a}^\pm(k^*)\big)^* = a^\pm(k) \text{ for } k \in [0, iA_r]\,, \tag{2.34a}$$
$$\bar{b}^*(k) = -\,b(k) \quad \text{for } k \in \mathbb{R}\,, \qquad \big(\bar{b}^\pm(k^*)\big)^* = -b^\pm(k) \quad \text{for } k \in [0, iA_r]\,. \tag{2.34b}$$

We note that the above symmetries relate the values of the scattering coefficients in the upper/lower half plane of k, and from the same side of the cut. Taking into account (2.31), one can easily establish the symmetry relations satisfied by the reflection coefficients:

$$\bar{\rho}^*(k) = -\,\rho(k) \quad \text{for } k \in \mathbb{R}\,, \qquad \big(\bar{\rho}^\pm(k^*)\big)^* = -\rho^\pm(k) \quad \text{for } k \in [0, iA_r]\,, \tag{2.35a}$$
$$\bar{r}^*(k) = -\,r(k) \quad \text{for } k \in \mathbb{R}\,. \tag{2.35b}$$

Second symmetry. When using a single sheet for the Riemann surface of the function $\lambda_r^2 = k^2 + A_r^2$, the involution $(k, \lambda_r) \to (k, -\lambda_r)$ can only be considered across the cut. So this second involution relates values of eigenfunctions and scattering coefficients for the same value of k from either side of the cut. On the cut one has

$$\bar{\psi}^{\mp}(x,k) = \frac{\lambda_r^{\pm}(k) + k}{-iq_r} \psi^{\pm}(x,k)\,, \qquad \text{for } k \in [-iA_r, iA_r]\,, \tag{2.36}$$

while $\phi^+(x,k) = \phi^-(x,k)$ for $k \in [0, iA_r]$, and $\bar{\phi}^+(x,k) = \bar{\phi}^-(x,k)$ for $k \in [-iA_r, 0]$. Using these symmetries in the Wronskian representations for the scattering coefficients (2.28), one obtains:

$$a^{\pm}(k) = \frac{\lambda_r^{\mp}(k) - k}{iq_r^*} b^{\mp}(k) \qquad \text{for } k \in [0, iA_r]\,, \tag{2.37a}$$

$$\bar{a}^{\mp}(k) = \frac{\lambda_r^{\mp}(k) + k}{-iq_r} \bar{b}^{\pm}(k) \qquad \text{for } k \in [-iA_r, 0]\,. \tag{2.37b}$$

From (2.31a), we then have the following symmetries for the reflection coefficients from the right:

$$\rho^{\pm}(k) = \frac{iq_r^*}{\lambda_r^{\pm}(k) - k} \frac{a^{\mp}(k)}{a^{\pm}(k)} \qquad \text{for } k \in [0, iA_r]\,, \tag{2.38a}$$

$$\bar{\rho}^{\pm}(k) = \frac{-iq_r}{\lambda_r^{\mp}(k) + k} \frac{\bar{a}^{\mp}(k)}{\bar{a}^{\pm}(k)} \qquad \text{for } k \in [-iA_r, 0]\,. \tag{2.38b}$$

Note that the above relationships imply that

$$\rho^+(k)\rho^-(k) = q_r^*/q_r \quad \text{for } k \in [0, iA_r]\,, \quad \bar{\rho}^+(k)\bar{\rho}^-(k) = q_r/q_r^* \quad \text{for } k \in [-iA_r, 0]\,. \tag{2.39}$$

Using (2.36) in (2.28c) and (2.28d), for $k \in \Sigma_r$ the symmetry relations for the scattering coefficients from the left are given by:

$$c^{\pm}(k) = \frac{-iq_r}{\lambda_r^{\pm}(k) + k} \bar{d}^{\mp}(k) \qquad \text{for } k \in [0, iA_r]\,, \tag{2.40a}$$

$$\bar{c}^{\pm}(k) = \frac{\lambda_r^{\mp}(k) + k}{-iq_r} d^{\mp}(k) \qquad \text{for } k \in [-iA_r, 0]\,. \tag{2.40b}$$

2.4. Discrete eigenvalues. A *discrete eigenvalue* is a value of $k \in \mathbb{K}_r^+ \cup \mathbb{K}_r^-$ (corresponding to $\lambda_r \in \mathbb{C} \setminus \mathbb{R}$) for which there exists a nontrivial solution v to (1.2) with entries in $L^2(\mathbb{R})$. These eigenvalues occur for $k \in \mathbb{K}_r^+$ iff the functions $\phi(x,k)$ and $\psi(x,k)$ are linearly dependent (i.e., iff $a(k) = 0$), and for $k \in \mathbb{K}_r^-$ iff the functions $\bar{\psi}(x,k)$ and $\bar{\phi}(x,k)$ are linearly dependent (i.e., iff $\bar{a}(k) = 0$). Equations (2.6b) and (2.17) imply that the corresponding eigenfunctions are exponentially decaying as $x \to \pm\infty$. The conjugation symmetry (2.34a) then ensures that the discrete eigenvalues occur in complex conjugate pairs. The algebraic multiplicity of each discrete eigenvalue coincides with the multiplicity of the corresponding zero of $a(k)$ [for $k \in \mathbb{K}_r^+$], or $\bar{a}(k)$ [for $k \in \mathbb{K}_r^-$].

In this work we assume that discrete eigenvalues are simple, and finite in number. Also, we assume that there are no spectral singularities, i.e., zeros of the scattering coefficients $a(k)$ and $\bar{a}(k)$ for $k \in \mathbb{R} \cup \Sigma_r$. Establishing conditions on the asymptotic amplitudes and phases that guarantee absence of spectral singularities is an interesting problem, but is beyond the scope of this paper and will be the subject of future investigation. In any event, we mention that spectral singularities can be incorporated in the inverse problem with slight modifications of the approach presented in Sec. 3.

2.5. Large k behavior of eigenfunctions and scattering data. In order to properly pose the inverse scattering problem, one has to determine the asymptotic behavior of eigenfunctions and scattering data as $k \to \infty$. Assuming $\partial_x q \in L^1$, integration by parts on the integral equations (2.19) yields for the asymptotic behaviors of the eigenfunctions as $|k| \to \infty$ in the appropriate half planes on Sheet I:

$$\Psi_d(x,k)e^{i\lambda_r\sigma_3 x} = I_2 + o(1)\,, \qquad \Psi_o(x,k)e^{i\lambda_r\sigma_3 x} = \frac{iQ(x)\sigma_3}{2k} + o(1/k)\,, \tag{2.41a}$$

$$\Phi_d(x,k)e^{ik\sigma_3 x} = I_2 + o(1)\,, \qquad \Phi_o(x,k)e^{ik\sigma_3 x} = \frac{iQ(x)\sigma_3}{2k} + o(1/k)\,, \tag{2.41b}$$

where subscripts $_d$ and $_o$ denote the diagonal and off-diagonal parts, respectively, of the corresponding matrix Jost solutions $\Psi(x,k)$ and $\Phi(x,k)$. From the Wronskian representations (2.28) for the scattering coefficients, and taking into account that $\lambda_r \sim k$ as $k \to \infty$, we then obtain the asymptotic behavior of the scattering coefficients:

$$a(k) = \frac{\lambda_r + k}{2\lambda_r}\mathrm{Wr}\begin{pmatrix}\phi(x,k) & \psi(x,k)\end{pmatrix} \sim 1 \quad \text{as} \quad |k| \to \infty,\ k \in \mathbb{K}_r^+ \cup \mathbb{R} \tag{2.42a}$$

$$\bar{a}(k) = -\frac{\lambda_r + k}{2\lambda_r}\mathrm{Wr}\begin{pmatrix}\bar{\phi}(x,k) & \bar{\psi}(x,k)\end{pmatrix} \sim 1 \quad \text{as} \quad |k| \to \infty,\ k \in \mathbb{K}_r^- \cup \mathbb{R} \tag{2.42b}$$

while

$$b(k) = O(1/k^2)\,, \qquad \bar{b}(k) = O(1/k^2) \quad \text{as} \quad |k| \to \infty,\ k \in \mathbb{R}\,.$$

Taking into account (2.31), the above also imply that

$$\rho(k) = O(1/k^2)\,, \qquad \bar{\rho}(k) = O(1/k^2) \quad \text{as} \quad |k| \to \infty,\ k \in \mathbb{R}\,, \tag{2.42c}$$

$$r(k) = O(1/k^2)\,, \qquad \bar{r}(k) = O(1/k^2) \quad \text{as} \quad |k| \to \infty,\ k \in \mathbb{R}\,. \tag{2.42d}$$

3. Inverse scattering problem

3.1. Triangular representations for the eigenfunctions. We introduce the following two triangular representations for the fundamental eigenfunctions:

$$\tilde{\Psi}(x,k)e^{-x\Lambda_r(k)} = I_2 + \int_x^\infty ds\, K(x,s)e^{(s-x)\Lambda_r(k)}\,, \tag{3.1a}$$

$$\Phi(x,k)e^{ik\sigma_3 x} = I_2 + \int_{-\infty}^x ds\, J(x,s)e^{-ik\sigma_3(s-x)}\,, \tag{3.1b}$$

where the kernels $K(x,s) = [K_{ij}(x,s)]_{i,j=1,2}$ and $J(x,s) = [J_{ij}(x,s)]_{i,j=1,2}$ are "triangular" kernels, i.e., such that $K(x,y) \equiv 0$ for $x > y$ and $J(x,y) \equiv 0$ for $x < y$. The above ansatz for the triangular representations is standard for the Jost solutions in the rapidly decaying case (see, for instance [**4**, **38**]). In the NZBC case, the issue of existence of triangular representations such as the above has been addressed in [**19**]. Here we omit the details for brevity.

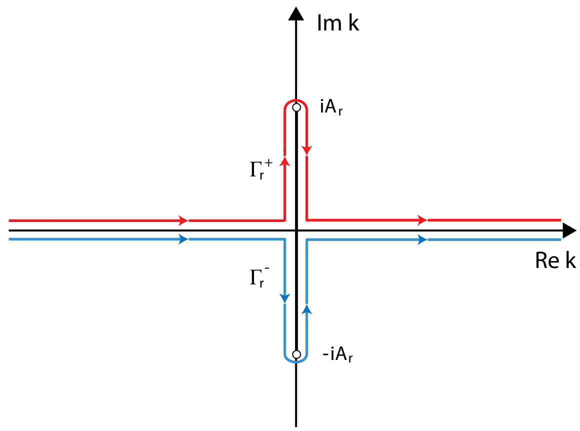

FIGURE 2. The oriented contours $\Gamma_r^{\pm}$.

We note that (3.1) yield the corresponding triangular representations for the Jost solutions (2.6b) and (2.17):

$$\Psi(x,k) = W_r(k)e^{-i\lambda_r\sigma_3 x} + \int_x^{\infty} ds\, K(x,s)W_r(k)\, e^{-i\lambda_r\sigma_3 s}\,, \tag{3.2a}$$

$$\Phi(x,k) = e^{-ik\sigma_3 x} + \int_{-\infty}^{x} ds\, J(x,s)\, e^{-ik\sigma_3 s}\,. \tag{3.2b}$$

Eq. (3.2b) is the standard triangular representation for the Jost solutions in the decaying case, while (3.2a) provides the appropriate generalization to the NZBC case. Inserting the representations (3.2) into the scattering problem (2.1a), and matching terms with the same k-dependence, one obtains the reconstruction of the potential $q(x)$ in terms of the entries of the kernels $K(x,y)$ and $J(x,y)$:

$$q(x) = q_r - 2K_{12}(x,x) = 2J_{12}(x,x)\,. \tag{3.3}$$

In [**19**] it is shown that in the fully asymmetric case a sufficient condition for the triangular representations and reconstruction formulas to hold is that the potential $q(x)$ satisfies $(\boldsymbol{H}_2)$ and $\partial_x q \in L^1(\mathbb{R})$. The analog obviously holds here, although less strict integrability requirements are necessary for $x \in \mathbb{R}^-$.

Note that here we have omitted the time dependence for brevity. If all the above assumptions on the potential hold for all $t \geq 0$, then inserting the time dependence on the Jost and fundamental eigenfunctions (see Sec. 4 for details) yields a parametric t-dependence for the Marchenko kernels, and the reconstruction formulas for the potential read

$$q(x,t) = q_r(t) - 2K_{12}(x,x;t) = 2J_{12}(x,x;t)\,. \tag{3.4}$$

3.2. Marchenko equations. In this subsection we formulate the inverse scattering problem in terms of right and left Marchenko integral equations.

3.2.1. *Right Marchenko equations.* Let us write (2.23a) explicitly as:

$$\frac{\phi(x,k)}{a(k)} = \bar{\psi}(x,k) + \rho(k)\psi(x,k) \qquad k \in \mathbb{R}\,, \tag{3.5a}$$

$$\frac{\phi^{\pm}(x,k)}{a^{\pm}(k)} = \bar{\psi}^{\pm}(x,k) + \rho^{\pm}(k)\psi^{\pm}(x,k) \qquad k \in [0, iA_r)\,, \tag{3.5b}$$

$$\frac{\bar{\phi}(x,k)}{\bar{a}(k)} = \psi(x,k) + \bar{\rho}(k)\bar{\psi}(x,k) \qquad k \in \mathbb{R}\,, \tag{3.5c}$$

$$\frac{\bar{\phi}^{\pm}(x,k)}{\bar{a}^{\pm}(k)} = \psi^{\pm}(x,k) + \bar{\rho}^{\pm}(k)\bar{\psi}^{\pm}(x,k) \qquad k \in (-iA_r, 0]\,, \tag{3.5d}$$

where $\rho(k)$, $\rho^{\pm}(k)$ and $\bar{\rho}(k)$, $\bar{\rho}(k)^{\pm}$ are given by (2.31a) and (2.31b), respectively. Multiplying (3.5a) by $e^{i\lambda_r y}$ for $y > x$, and substituting the triangular representation (3.2a) we obtain

$$\begin{aligned}\left[\frac{e^{i\lambda_r x}\phi(x,k)}{a(k)} - W_{r,1}(k)\right] e^{i\lambda_r(y-x)} &= \int_x^{\infty} ds\, K(x,s) W_{r,1}(k)\, e^{i\lambda_r(y-s)} \\ &+ \rho(k)\left[e^{i\lambda_r(x+y)} W_{r,2}(k) + \int_x^{\infty} ds\, K(x,s) W_{r,2}(k)\, e^{i\lambda_r(s+y)}\right],\end{aligned} \tag{3.6}$$

where $W_{r,j}(k)$ denotes the j-th column of the eigenvector matrix $W_r(k)$ in (2.15). We remark that $\lambda_r \sim k$ as $|k| \to \infty$, so that the term in the left-hand side decays as $|k| \to \infty$ in $\mathbb{K}_r^+ \cup \mathbb{R}$. For the purpose of this section, it will be convenient to consider the eigenfunctions as functions of λ_r, i.e., to use:

$$k = k(\lambda_r) \equiv \sqrt{\lambda_r^2 - A_r^2}\,.$$

Note that $\lambda_r \in \mathbb{R}$ is in one-to-one correspondence with either $k \in \Gamma_r^+$ or $k \in \Gamma_r^-$ (cf. Fig. 2). In the following we will assume $k \in \Gamma_r^+$ for the eigenfunction $\psi(x,k)$ [analytic for $k \in \mathbb{K}_r^+$], and $k \in \Gamma_r^-$ for $\bar{\psi}(x,k)$ [analytic for $k \in \mathbb{K}_r^-$]. We then formally integrate (3.6) with respect to $\lambda_r \in \mathbb{R}$, multiply by $1/2\pi$, exchange the order of integration, and evaluate

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda_r \begin{pmatrix} 1 \\ -iq_r^*/(\lambda_r + k) \end{pmatrix} e^{i\lambda_r(y-s)} = \begin{pmatrix} \delta(y-s) \\ 0 \end{pmatrix}$$

to obtain

$$\mathcal{I} = K(x,y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + F(x+y) + \int_x^{\infty} ds\, K(x,s) F(s+y),$$

where

$$\mathcal{I} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda_r \left[\frac{e^{i\lambda_r x}\phi(x,k)}{a(k)} - W_{r,1}(k)\right] e^{i\lambda_r(y-x)}\,,$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda_r\, \rho(k) W_{r,2}(k)\, e^{i\lambda_r x}\,.$$

As explained above, in the above integrals $k = k(\lambda_r)$ with $k \in \Gamma_r^+$. The procedure and the results are the same shown in [**19**]. We assume discrete eigenvalues $k_1, \ldots, k_N$ in $\mathbb{K}_r^+$ (which we assume here to be finite in number) are simple. Since the eigenfunctions $\phi(x,k_n)$ and $\psi(x,k_n)$ are proportional, i.e., there exists a complex constant b_n such that $\phi_n(x,k_n) = b_n\psi(x,k_n)$, then denoting by τ_n the residue of $1/a(k)$ at $\lambda_r = \lambda_r(k_n)$, we can write

$$\lim_{k\to k_n} (\lambda_r(k) - \lambda_r(k_n)) \frac{\phi(x,k)}{a(k)} = C_n\, \psi(x,k_n)\,, \qquad C_n = b_n \tau_n\,, \tag{3.7}$$

and C_n is referred to as the *norming constant* associated with the discrete eigenvalue k_n. The expression of $\mathcal{I}$ can be easily obtained using Residue Theorem and Jordan's Lemma, thus yielding the right Marchenko integral equation as:

$$K(x,y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \Omega_r(x+y) + \int_x^\infty ds\, K(x,s)\Omega_r(s+y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,, \tag{3.8}$$

where

$$\Omega_r(x) := F(x) - F_d(x)\,, \qquad F_d(x) = i \sum_{n=1}^N e^{i\lambda_r(k_n)x} C_n W_{r,2}(k_n)\,. \tag{3.9}$$

Next, let us multiply (3.5c) by $e^{-i\lambda_r y}$ for $y > x$, substitute (3.2a), and then formally integrate (3.6) with respect to $\lambda_r \in \mathbb{R}$ and multiply by $1/2\pi$. At the (simple) discrete eigenvalues $k_1^*, \ldots, k_N^*$ in $\mathbb{K}_r^-$ [necessarily finite in number, and the complex conjugates of the zeros of $a(k)$ in $\mathbb{K}_r^+$] the eigenfunctions $\bar\phi(x,k_n^*)$ and $\bar\psi(x,k_n^*)$ are proportional to each other, i.e., there exist complex constants $\bar b_n$ such that $\bar\phi(x,k_n^*) = \bar b_n \bar\psi(x,k_n^*)$. Then, denoting by $\bar\tau_n$ the residue of $1/\bar a(k)$ at $\lambda_r = \lambda_r(k_n^*)$, we can write

$$\lim_{k\to k_n^*} (\lambda_r - \lambda_r(k_n^*)) \frac{\bar\phi(x,k)}{\bar a(k)} = \bar C_n \bar\psi(x,k_n^*)\,, \qquad \bar C_n = \bar b_n \bar\tau_n\,, \tag{3.10}$$

and $\bar C_n$ is referred to as the *norming constant* associated with the discrete eigenvalue k_n^*. Proceeding as before, we obtain

$$K(x,y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \bar\Omega_r(x+y) + \int_x^\infty ds\, K(x,s)\bar\Omega_r(s+y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,, \tag{3.11}$$

where

$$\bar\Omega_r(x) := \bar F(x) - \bar F_d(x)\,, \tag{3.12}$$

with

$$\bar F(x) = \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda_r\, e^{-i\lambda_r x} \bar\rho(k) W_{r,1}(k)\,, \qquad \bar F_d(x) = -i \sum_{n=1}^N e^{-i\lambda_r(k_n^*)x} \bar C_n W_{r,1}(k_n^*)\,. \tag{3.13}$$

Note that in the integral in (3.13) $\lambda_r \in \mathbb{R}$, and $k = k(\lambda_r) \in \Gamma_r^-$ (see Fig. 2). Using the symmetry relations (2.33) and (2.34), and the definitions (3.7) and (3.10), we get $\bar\tau_n = \tau_n^*$, $\bar b_n = -b_n^*$ and $\bar C_n = -C_n^*$. As a result,

$$F^*(x) = i\sigma_2 \bar F(x), \quad F_d^*(x) = i\sigma_2 \bar F_d(x), \quad \Omega_r^*(x) = i\sigma_2 \bar\Omega_r(x). \tag{3.14}$$

In conclusion, we can write the Marchenko equations (3.8) and (3.11) and their kernels (3.9) and (3.12) as a single 2×2 Marchenko equation:

$$K(x,y) + \mathbf{\Omega}_r(x+y) + \int_x^\infty ds\, K(x,s)\mathbf{\Omega}_r(s+y) = 0_{2\times 2}\,, \tag{3.15}$$

where

$$\mathbf{\Omega}_r(x) = \begin{pmatrix} \Omega_r(x) & \bar\Omega_r(x) \end{pmatrix}\,. \tag{3.16}$$

$\Omega_r\,,\bar{\Omega}_r$ are given by (3.9) and (3.12), and satisfy $\bar{\Omega}_r(x) = i\sigma_2\Omega_r^*(x)$. Note that the 2×2 kernel $\mathbf{\Omega}_r(x)$ anticommutes with the Pauli matrix σ_3, and satisfies the conjugation symmetry relation

$$\mathbf{\Omega}_r^*(x) = \sigma_2\mathbf{\Omega}_r(x)\sigma_2. \tag{3.17}$$

3.2.2. *Left Marchenko equations.* In order to derive the left Marchenko equations, let us write (2.23b) explicitly as:

$$\frac{1}{\bar{c}(k)}\bar{\psi}(x,k) = \phi(x,k) + \bar{r}(k)\bar{\phi}(x,k) \quad k\in\mathbb{R}\,, \tag{3.18a}$$

$$\frac{1}{c(k)}\psi(x,k) = \bar{\phi}(x,k) + r(k)\phi(x,k) \quad k\in\mathbb{R}\,, \tag{3.18b}$$

where $r(k)$ and $\bar{r}(k)$ are given by (2.31c). Under the same assumptions as in Sec. 3.2.1 regarding the potential and the discrete spectrum, and considering in this case the eigenfunctions as functions of k, we multiply (3.18a) by e^{-iky} for $y<x$ and substitute the triangular representations (3.2b). Integrating (3.18a) with respect to $k\in\mathbb{R}$, multiplying by $1/2\pi$, and exchanging the order of integration, we have

$$\tilde{\mathcal{I}} = J(x,y)\begin{pmatrix}0\\1\end{pmatrix} + G(x+y) + \int_{-\infty}^{x} ds\, J(x,s)G(s+y)\,, \tag{3.19}$$

where

$$\tilde{\mathcal{I}} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\left[\frac{e^{-ikx}\psi(x,k)}{c(k)} - \begin{pmatrix}0\\1\end{pmatrix}\right]e^{ik(x-y)}\,,$$
$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, r(k)\, e^{-ikx}\begin{pmatrix}1\\0\end{pmatrix}.$$

In order to compute $\tilde{\mathcal{I}}$ so as to express it in terms of the Marchenko kernel $J(x,y)$, one needs to be able to close the contour at infinity in the upper half-plane of k. Unlike what happens for the Marchenko equations from the right, in this case closing the contour at infinity requires including the contribution of the branch cut that corresponds to $k\in[0,iA_r]$, i.e. Σ_r in the upper half-plane. To this end, let us consider, for $0<\varepsilon<R<+\infty$, the closed contour $\Gamma(R,\varepsilon)$ consisting of the following pieces, with the orientation specified in Fig. 3: (i) $[-R,-\varepsilon]$; (ii) $[-\varepsilon+i0,-\varepsilon+iA_r]$; (iii) the semicircle $\left\{iA_r+\varepsilon e^{i(\pi-\theta)}\,:\,0\le\theta\le\pi\right\}$ clockwise; (iv) $[\varepsilon+i0,\varepsilon+iA_r]$; (v) $[\varepsilon,R]$; (vi) $\{Re^{i\theta}\,:\,0\le\theta\le\pi\}$ counterclockwise. R is assumed large enough and ε small enough so that all of the finitely many discrete eigenvalues k_n $(n=1,2,\ldots,N)$ in $\mathbb{K}_r^+$ belong to the interior region of the contour. Since $\psi(x,k)$ and $1/c(k)$ have finite limits as $k\to iA_r$, the integral defining $\tilde{\mathcal{I}}$ with the integration confined to the semicircle around the branch point does not contribute as $\varepsilon\to 0^+$. Because of Jordan's Lemma, the integral defining $\tilde{\mathcal{I}}$ when confined to the large semicircle (vi) does not contribute either as $R\to+\infty$. Then one has $\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_1+\tilde{\mathcal{I}}_2$, the contribution $\tilde{\mathcal{I}}_1$ pertaining to the residues of the function under the integral sign at the poles $k_n\in\mathbb{K}_r^+$; and the contribution $\tilde{\mathcal{I}}_2$ pertaining to the integral around $k\in[0,iA_r]$ in the upper-half k-plane. We shall evaluate the two contributions separately.

Since we assumed that the discrete eigenvalues k_n in $\mathbb{K}_r^+$ are simple poles of $1/c(k)$, and the transmission coefficient is continuous for $k\in\partial\mathbb{K}_r^+$, taking into

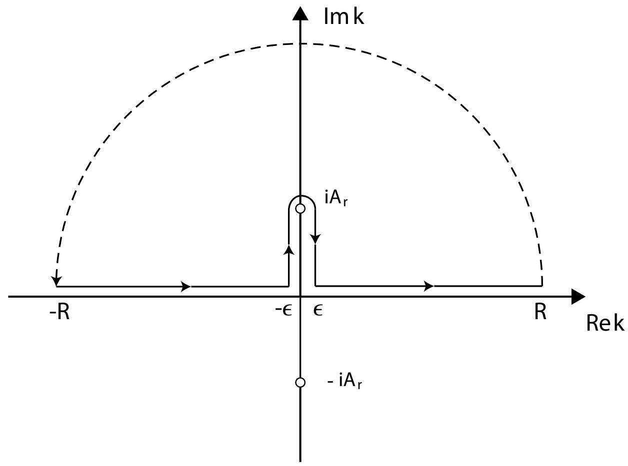

FIGURE 3. The contour $\Gamma(R,\varepsilon)$.

account that $\psi(x,k_n)=\phi(x,k_n)/b_n$, we obtain

$$\tilde{\mathcal{I}}_1 = i\sum_{n=1}^{N} e^{-ik_n y}\tilde{C}_n\phi(x,k_n)\,, \qquad \tilde{C}_n = \frac{\tilde{\tau}_n}{b_n}\,,$$

where $\tilde{\tau}_n$ is the residue of $1/c(k)$ at $k=k_n$, and $\tilde{C}_n$ is the associated norming constant. Therefore, we have

$$\tilde{\mathcal{I}}_1 = G_1(x+y) + \int_{-\infty}^{x} ds\, J(x,s)G_1(s+y)\,, \tag{3.20}$$

where

$$G_1(x) = i\sum_{n=1}^{N} e^{-ik_n x}\tilde{C}_n\begin{pmatrix}1\\0\end{pmatrix}\,.$$

Note that (2.28c) implies the residues $\tilde{\tau}_n$ and τ_n, and hence the norming constants $\tilde{C}_n$ and C_n, are related as follows:

$$\tilde{\tau}_n = \frac{\lambda_r(k_n)+k_n}{2\lambda_r(k_n)}\tau_n, \qquad \tilde{C}_n\, C_n = \tau_n^2\frac{\lambda_r(k_n)+k_n}{2\lambda_r(k_n)}\,. \tag{3.21}$$

Let us now look into the second contribution $\tilde{\mathcal{I}}_2$, which arises for $k\in[0,iA_r]$ on either side of the cut and $\lambda_r\in\mathbb{R}$. We have

$$\begin{aligned}\tilde{\mathcal{I}}_2 &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\left(\int_{i0-\epsilon}^{iA_r-\epsilon} - \int_{i0+\epsilon}^{iA_r+\epsilon}\right) dk\left[\frac{\psi(x,k)}{c(k)}\,e^{-ikx} - \begin{pmatrix}0\\1\end{pmatrix}\right]e^{ik(x-y)}\\ &= \frac{1}{2\pi}\int_0^{iA_r} dk\left[\frac{\psi^-(x,k)}{c^-(k)} - \frac{\psi^+(x,k)}{c^+(k)}\right]e^{-iky}\,,\end{aligned} \tag{3.22}$$

where, as usual, superscripts $^{\pm}$ denote the limiting values from the left/right edge of the cut, respectively. Using (2.28c) we can write $\tilde{\mathcal{I}}_2$ as

$$\tilde{\mathcal{I}}_2 = \frac{1}{2\pi}\int_0^{iA_r} dk\left[\frac{\lambda_r^+(k)-k}{2\lambda_r^+(k)}\frac{\psi^-(x,k)}{a^-(k)} - \frac{\lambda_r^+(k)+k}{2\lambda_r^+(k)}\frac{\psi^+(x,k)}{a^+(k)}\right]e^{-iky}\,,$$

and the symmetry relations (2.36) and (2.37a) allow to express $(\lambda_r^+(k)-k)\psi^-/a^- = -(\lambda_r^+(k)+k)\bar{\psi}^+/b^+$, so that:

$$\tilde{\mathcal{I}}_2 = -\frac{1}{4\pi}\int_0^{iA_r} dk \frac{\lambda_r^+(k)+k}{\lambda_r^+(k)}\left[\frac{\bar{\psi}^+(x,k)}{b^+(k)} + \frac{\psi^+(x,k)}{a^+(k)}\right] e^{-iky}\,. \tag{3.23}$$

Using first the scattering equation (3.5a) and then again the symmetry relation (2.37a), we finally have

$$\tilde{\mathcal{I}}_2 = -\frac{1}{4\pi}\int_0^{iA_r} dk\, \frac{\lambda_r^+(k)+k}{\lambda_r^+(k)}\frac{\phi^+(x,k)}{a^+(k)\,b^+(k)}\, e^{-iky} = \frac{iq_r}{4\pi}\int_0^{iA_r} \frac{dk}{\lambda_r^+(k)}\frac{\phi^+(x,k)}{a^-(k)a^+(k)} e^{-iky}\,. \tag{3.24}$$

We can now insert into the last expression the triangular representation (3.2b), and obtain

$$\tilde{\mathcal{I}}_2 = G_2(x+y) + \int_{-\infty}^{x} ds\, J(x,s)G_2(s+y)\,, \tag{3.25}$$

with

$$G_2(x) = \frac{iq_r}{4\pi}\int_0^{iA_r} \frac{dk}{\lambda_r^+(k)}\frac{e^{-ikx}}{a^+(k)a^-(k)}\begin{pmatrix}1\\0\end{pmatrix}.$$

If we now define

$$\begin{aligned}\Omega_l(x) =& G(x) - G_1(x) - G_2(x)\\ \equiv& \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, r(k)\, e^{-ikx} - i\sum_{n=1}^{N} e^{-ik_nx}\tilde{C}_n - \frac{iq_r}{4\pi}\int_0^{iA_r}\frac{dk}{\lambda_r^+(k)}\frac{e^{-ikx}}{a^+(k)a^-(k)}\right]\begin{pmatrix}1\\0\end{pmatrix},\end{aligned} \tag{3.26}$$

use (3.20) and (3.25) to compute $\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_1 + \tilde{\mathcal{I}}_2$ and introduce it into (3.19), we finally arrive at the left Marchenko integral equation

$$J(x,y)\begin{pmatrix}0\\1\end{pmatrix} + \Omega_l(x+y) + \int_{-\infty}^{x} ds\, J(x,s)\Omega_l(s+y) = \begin{pmatrix}0\\0\end{pmatrix}. \tag{3.27}$$

In a similar way, starting from (3.18b), one can derive the "adjoint" left Marchenko equation

$$J(x,y)\begin{pmatrix}1\\0\end{pmatrix} + \bar{\Omega}_l(x+y) + \int_{-\infty}^{x} ds\, J(x,s)\bar{\Omega}_l(s+y) = \begin{pmatrix}0\\0\end{pmatrix}, \tag{3.28}$$

where

$$\bar{\Omega}_l(x) = i\sigma_2\Omega_l^*(x)\,.$$

The two Marchenko equations can be written in a compact matrix form as follows:

$$J(x,y) + \mathbf{\Omega}_l(x+y) + \int_{-\infty}^{x} ds\, J(x,s)\mathbf{\Omega}_l(s+y) = 0_{2\times 2}\,, \tag{3.29}$$

where

$$\mathbf{\Omega}_l(x) = \begin{pmatrix}\bar{\Omega}_l(x) & \Omega_l(x)\end{pmatrix}, \qquad \mathbf{\Omega}_l^*(x) = \sigma_2\mathbf{\Omega}_l(x)\sigma_2\,. \tag{3.30}$$

The asymmetry between left/right Marchenko integral equations is due to the choice of the one-sided NZBC (1.3) with $A_r > 0$. Indeed, if one considered boundary conditions such that $q(x,t) \to q_l(t) = A_l\, e^{-2iA_l^2t+i\theta_l}$ as $x \to -\infty$, and $q(x,t) \to 0$ as $x \to +\infty$, with $A_l > 0$, $0 \le \theta_l < 2\pi$ arbitrary constants, the roles of the two integral equations would be reversed. As in the fully asymmetric case, in the Marchenko integral equations from the left, $\Omega_l(x)$ (cf. Eq. (3.26)) has three separate contributions: one from the discrete spectrum, one from the reflection coefficients from the left, $r(k)$ and $\bar{r}(k)$, integrated over values of k in the continuous spectrum, i.e., $k \in \mathbb{R}$, and a third contribution (sometimes referred to as the dispersive-shock wave, or DSW, contribution) which contains an integral over imaginary values of k where the product of transmission coefficients $1/[a^+(k)a^-(k)]$ appears. On the other hand, $\Omega_r(x)$ in the integral equations from the right (cf. Eq. (3.9)) has only two contributions: one from the discrete spectrum, and one from the reflection coefficients from the right, $\rho(k)$ and $\bar{\rho}(k)$. In (3.9), however, the reflection coefficients are integrated over all $\lambda_r \in \mathbb{R}$, which means that the integral includes, in addition to the continuous spectrum $k \in \mathbb{R}$, also a contribution from $k \in \Sigma_r$. Moreover, the integrand over Σ_r can never be set to be identically zero (due to the symmetries (2.38a) and (2.38b)), which implies that when $\Sigma_r \neq \emptyset$ (i.e., whenever one deals with one-sided boundary conditions (1.3) with $A_r \neq 0$), no pure soliton solutions exist, and solitons are always accompanied by a radiative contribution of some sort.

The Marchenko integral equations obtained here provide the necessary setup for the study of the long-time behavior of the solutions by means of matched asymptotics, as was recently done for KdV in [**1**].

3.3. Riemann-Hilbert problem formulation. The purpose of this section is to formulate the inverse problem as matrix Riemann-Hilbert problems from the left and from the right for a suitable set of sectionally analytic/meromorphic functions in the cut plane $\mathbb{K}_r$, with assigned jumps across $\mathbb{R} \cup \Sigma_r$.

3.3.1. *Riemann-Hilbert problem from the right.* For the formulation of the Riemann-Hilbert problem in terms of scattering data from the right, we consider the following matrix of eigenfunctions:

$$M(x,k) = \begin{cases} \begin{bmatrix} \dfrac{\phi(x,k)}{a(k)}\, e^{ikx} & \psi(x,k)\, e^{-i\lambda_r x} \end{bmatrix} & k \in \mathbb{K}_r^+ \\[2ex] \begin{bmatrix} \bar{\psi}(x,k)\, e^{i\lambda_r x} & \dfrac{\bar{\phi}(x,k)}{\bar{a}(k)}\, e^{-ikx} \end{bmatrix} & k \in \mathbb{K}_r^- \end{cases}, \tag{3.31}$$

such that $M(x,k) \to I_2$ as $k \to \infty$, and formulate the inverse problem as a Riemann-Hilbert problem for the sectionally meromorphic matrix $M(x,k)$ across $\partial\mathbb{K}_r^+ \cup \partial\mathbb{K}_r^-$. Explicitly, we determine three jump matrices as illustrated in Fig. 4: V_0 is the jump matrix across the real axis of the complex k-plane; V_1 across $\Sigma_r^+ = [0, iA_r]$, and V_2 across $\Sigma_r^- = [-iA_r, 0]$. All jump matrices depend on k along the appropriate contour in the complex plane, as well as, parametrically, on $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ [the x-dependence is explicit, while the time dependence is "hidden" in that of the corresponding reflection coefficients, see below, and will be omitted for brevity]. The RH problem across the real axis can be written in matrix form as: $M^+(x,k) = M^-(x,k)V_0(x,k)$, $k \in \mathbb{R}$, i.e.,

$$\begin{bmatrix} \dfrac{\phi^+(x,k)}{a^+(k)}\, e^{ikx} & \psi^+(x,k)\, e^{-i\lambda_r x} \end{bmatrix} = \begin{bmatrix} \bar{\psi}^-(x,k)\, e^{i\lambda_r x} & \dfrac{\bar{\phi}^-(x,k)}{\bar{a}^-(k)}\, e^{-ikx} \end{bmatrix} V_0(x,k)\,, \tag{3.32}$$

where in this case the superscripts $^\pm$ denote limiting values from the upper/lower complex plane, respectively. The jump matrix across the real axis can be easily

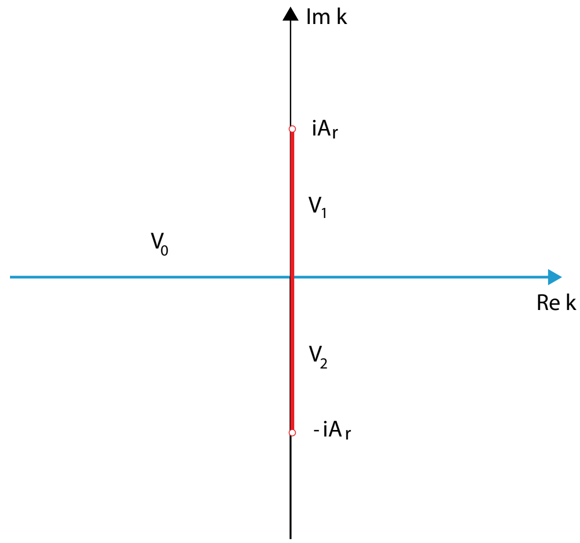

FIGURE 4. The jump matrices V_j, $j = 0, 1, 2$ of the RH problem across $\mathbb{R} \cup \Sigma_r^+ \cup \Sigma_r^-$.

computed from (2.23a), and it is given by:

$$V_0(x,k) = \begin{pmatrix} [1-\rho(k)\bar{\rho}(k)]\, e^{i(k-\lambda_r)x} & -\bar{\rho}(k)\, e^{-2i\lambda_r x} \\ \rho(k)\, e^{2ikx} & e^{i(k-\lambda_r)x} \end{pmatrix}. \tag{3.33}$$

We then write the RH problem across Σ_r^+ as: $M^+(x,k) = M^-(x,k)V_1(x,k)$, $k \in \mathbb{C}^+$, where now the superscripts $^\pm$ denote limiting values from the right/left edge of the cut across Σ_r^+ (Σ_r in the upper half plane). Taking into account that for $k \in \Sigma_r^+$ $\lambda_r^\pm(k) = \pm\sqrt{A_r^2 - |k|^2}$ from the right/left edge of the cut, while k and ϕ are continuous, we have:

$$\left[\frac{\phi^+(x,k)}{a^+(k)}\, e^{ikx} \quad \psi^+(x,k)\, e^{-i\lambda_r^+ x} \right] = \left[\frac{\phi^-(x,k)}{a^-(k)}\, e^{ikx} \quad \psi^-(x,k)\, e^{-i\lambda_r^- x} \right] V_1(x,k)\,. \tag{3.34}$$

In order to compute $V_1(x,k)$ for $k \in \mathbb{C}^+$, we note that using (2.29) we can write:

$$\phi^\pm(x,k) = a^\pm(k)\bar{\psi}^\pm(x,k) + b^\pm(k)\psi^\pm(x,k)\,, \quad k \in \Sigma_r^+\,, \tag{3.35}$$

and relate $\bar{\psi}^\pm(x,k)$ to $\psi^\mp(x,k)$ using the symmetry relations (2.36). One then finds

$$\frac{\phi^+(x,k)}{a^+(k)} = -\frac{iq_r^*}{\lambda_r^+(k)+k}\psi^-(x,k) + \rho^+(k)\psi^+(x,k)\,, \tag{3.36}$$

$$\psi^+(x,k) = -\frac{iq_r}{\lambda_r^+(k)+k}\left[\frac{\phi^-(x,k)}{a^-(k)} - \rho^-(k)\psi^-(x,k) \right], \tag{3.37}$$

and inserting (3.37) into (3.36), we obtain:

$$\frac{\phi^+(x,k)}{a^+(k)} = -\frac{iq_r}{\lambda_r^+(k)+k}\left[\frac{\phi^-(x,k)}{a^-(k)}\rho^-(k) + \left(\frac{q_r^*}{q_r} - \rho^+(k)\rho^-(k) \right)\psi^-(x,k) \right]. \tag{3.38}$$

Using (2.39) and comparing (3.38) and (3.37) to (3.34), the jump matrix $V_1(x,k)$ is then found to be:

$$V_1(x,k) = -\frac{iq_r}{\lambda_r^+(k)+k}\begin{pmatrix} \rho^+(k) & e^{-i(\lambda_r^+(k)+k)x} \\ 0 & -\rho^-(k)\,e^{-2i\lambda_r^+(k)x} \end{pmatrix}. \tag{3.39}$$

The RH problem across Σ_r^- will be written as $M^+(x,k) = M^-(x,k)V_2(x,k)$, $k \in \mathbb{C}^-$, with superscripts $^\pm$ denoting non-tangential limits from the right/left of the cut across Σ_r^-, i.e., Σ_r in the lower half plane. Explicitly, using (2.13) one has:

$$\left[\bar\psi^+(x,k)\,e^{i\lambda_r^+x} \quad \frac{\bar\phi^+(x,k)}{\bar a^+(k)}\,e^{-ikx}\right] = \left[\bar\psi^-(x,k)\,e^{i\lambda_r^-x} \quad \frac{\bar\phi^-(x,k)}{\bar a^-(k)}\,e^{-ikx}\right]V_2(x,k). \tag{3.40}$$

In order to compute $V_2(x,k)$ for $k \in \mathbb{C}^-$, we again use (2.29) to write:

$$\bar\phi^\pm(x,k) = \psi^\pm(x,k) + \bar\rho^\pm(k)\bar\psi^\pm(x,k)\,, \quad k \in \Sigma_r^-\,, \tag{3.41}$$

and relate the eigenfunctions $\psi^\pm(x,k)$ to $\bar\psi^\mp(x,k)$ using the symmetry relations (2.36). We the obtain the following expressions:

$$\frac{\bar\phi^+(x,k)}{\bar a^+(k)} = -\frac{iq_r}{\lambda_r^+(k)+k}\bar\psi^-(x,k) + \bar\rho^+(k)\bar\psi^+(x,k)\,, \tag{3.42}$$

$$\bar\psi^+(x,k) = -\frac{iq_r^*}{\lambda_r^+(k)+k}\left[\frac{\bar\phi^-(x,k)}{\bar a^-(k)} - \bar\rho^-(k)\bar\psi^-(x,k)\right]. \tag{3.43}$$

Inserting (3.43) into (3.42), one has:

$$\frac{\bar\phi^+(x,k)}{\bar a^+(k)} = -\frac{iq_r^*}{\lambda_r^+(k)+k}\left[\frac{\bar\phi^-(x,k)}{\bar a^-(k)}\bar\rho^+(k) + \left(\frac{q_r}{q_r^*} - \bar\rho^+(k)\bar\rho^-(k)\right)\bar\psi^-(x,k)\right]. \tag{3.44}$$

Using (2.39) and comparing (3.44) and (3.43) to (3.40), the jump matrix $V_2(x,k)$ is found to be:

$$V_2(x,k) = -\frac{iq_r^*}{\lambda_r^+(k)+k}\begin{pmatrix} -\bar\rho^-(k)\,e^{2i\lambda_r^+(k)x} & 0 \\ e^{i(\lambda_r^+(k)+k)x} & \bar\rho^+(k) \end{pmatrix}. \tag{3.45}$$

Note that the jump matrices satisfy the following upper/lower half plane symmetry:

$$V_2(x,k) = \sigma_2 V_1^*(x,k^*)\sigma_2\,.$$

Solving the inverse problem as a RH problem (with poles, corresponding to the zeros of $a(k)$ and $\bar a(k)$ in the upper/lower half planes) then amounts to computing the sectionally meromorphic matrix $M(x,k)$ with the given jumps, and normalized to the identity as $k \to \infty$. Specifically, we can write the problem as $M^+ = M^- + (V - I_2)M^-$, where $V(x,k) = V_j(x,k)$ for $j = 0,1,2$ depending on which piece of the contour is being considered, and superscripts $\pm$ denote non-tangential limits from either side of the contour. Then, subtracting the behavior as $k \to \infty$, and the residues of $M^\pm$ at the poles in $\mathbb{K}_r^\pm$ from both sides we obtain

$$\begin{aligned} &M^+ - I_2 - \sum_{n=1}^N \frac{1}{k-k_n}\mathrm{Res}_{k_n}M^+ - \sum_{n=1}^N \frac{1}{k-k_n^*}\mathrm{Res}_{k_n^*}M^- = \\ &M^- - I_2 - \sum_{n=1}^N \frac{1}{k-k_n}\mathrm{Res}_{k_n}M^+ - \sum_{n=1}^N \frac{1}{k-k_n^*}\mathrm{Res}_{k_n^*}M^- + (V-I_2)M^-\,. \end{aligned} \tag{3.46}$$

The left-hand side of the above equation is now analytic in $\mathbb{K}_r^+$, and it is $O(1/k)$ as $k \to \infty$ there, while the sum of all terms but the last one in the right-hand side is analytic in $\mathbb{K}_r^-$, and is $O(1/k)$ as $k \to \infty$ there. We then introduce projectors $P_\pm$ over $\Gamma_r^\pm \equiv \mathbb{R} \cup \Sigma_r^\pm$:

$$P_\pm[f](z) = \frac{1}{2\pi i} \int_{\Gamma_r^\pm} \frac{f(\xi)}{\xi - k}\, d\xi \,,$$

where $\int_{\Gamma_r^+}$ [resp. $\int_{\Gamma_r^-}$] denotes the integral along the oriented contours in Fig. 2, and when $k \in \Gamma_r^\pm \cap \mathbb{R}$ the limit is taken from the above/below. One can easily prove that if $f^\pm$ are analytic in $\mathbb{K}_r^\pm$ and are $O(1/k)$ as $k \to \infty$, the following holds: $P_\pm f^\pm = \pm f^\pm$ and $P_+ f^- = P_- f^+ = 0$. Then, applying $P_\pm$ to both sides of (3.46), we find for $k \in \mathbb{C}^\pm \setminus \Sigma_r$

$$\text{(3.47)}\quad M(k) = I_2 + \sum_{n=1}^{N} \frac{\mathrm{Res}_{k_n} M^+}{k - k_n} + \sum_{n=1}^{N} \frac{\mathrm{Res}_{k_n^*} M^-}{k - k_n^*} + \frac{1}{2\pi i} \int_{\Gamma_r^\pm} \frac{M^-(\xi)}{\xi - k} \left[V(\xi) - I_2\right] d\xi \,,$$

where the x-dependence in eigenfunctions and jump matrices has been omitted for brevity. Taking into account that the second column of $\mathrm{Res}_{k_n} M^+$ is zero for all n, while the first column is proportional to the second column of $M^+(x, k_n)$, and vice-versa the first column of $\mathrm{Res}_{k_n^*} M^-$ is zero for all n, while the second column is proportional to the second column of $M^-(x, k_n^*)$ according to (3.7), the above integral/algebraic system can be closed by evaluating it at each $k = k_n$ and $k = k_n^*$. The potential is then reconstructed by the large k expansion of the latter, since

$$M_o(x, k) = \frac{i}{2k} Q(x) \sigma_3 + o(1/k) \,,$$

where subscript $_o$ denotes the off-diagonal part of the matrix $M(x, k)$. Note that unlike what happens in the same-amplitude case, the above system cannot be reduced to a purely algebraic one: although the reflection coefficients can be chosen to be identically zero on the continuous spectrum, i.e., for $k \in \mathbb{R}$, the integrals appearing in the right-hand side of (3.47) always exhibit a non-zero contribution from the contours $\Sigma_r^\pm$. In particular, this implies that no pure soliton solutions exist, and solitons are always accompanied by a radiative contribution of some sort. One could nonetheless solve the system iteratively, assuming the reflection coefficients are small for $k \in \Sigma_r^\pm$, and thus obtaining NLS solutions comprising solitons superimposed to small radiation. Moreover, the RH problem formulated here provides the key setup for the investigation of the long-time asymptotic behavior by the Deift-Zhou steepest descent method [**13**, **14**, **17**]. The time dependence in the system is simply accounted for by the time dependence of the scattering coefficients, as described in Sec. 4. When one is interested only in capturing the leading order behavior of the solution for large t, the jumps across the contours illustrated in Fig. 4 and determined above can be simplified by suitable factorizations and contour deformations, and reduced to certain model problems for which "explicit" solutions (often expressed in terms of Riemann theta functions) can be sought for. This study obviously goes beyond the scope of the present paper, and will be the subject of future investigation.

3.3.2. *Riemann-Hilbert problem from the left.* The inverse problem can also be formulated as a RH problem from the left, i.e., written in terms of left scattering data, by introducing the sectionally meromorphic matrix of eigenfunctions

$$\tilde{M}(x,k) = \begin{cases} \left[\phi(x,k)\, e^{ikx} \quad \frac{\psi(x,k)}{c(k)}\, e^{-i\lambda_r x}\right], & k \in \mathbb{K}_r^+ \\ \left[\frac{\bar{\psi}(x,k)}{\bar{c}(k)}\, e^{i\lambda_r x} \quad \bar{\phi}(x,k)\, e^{-ikx}\right], & k \in \mathbb{K}_r^- \end{cases} .$$

The RH problem across the real k-axis can be written in matrix form as: $\tilde{M}^+(x,k) = \tilde{M}^-(x,k)\tilde{V}_0(x,k)$, $k \in \mathbb{R}$ i.e.,

(3.48)
$$\left[\phi^+(x,k)\, e^{ikx} \quad \frac{\psi^+(x,k)}{c^+(k)}\, e^{-i\lambda_r x}\right] = \left[\frac{\bar{\psi}^-(x,k)}{\bar{c}^-(k)}\, e^{i\lambda_r x} \quad \bar{\phi}^-(x,k)\, e^{-ikx}\right] \tilde{V}_0(x,k)\,,$$

where in this case the superscripts $^\pm$ denote limiting values from the upper/lower complex plane, respectively. The jump matrix across the real axis can be easily computed from (2.23b), and it is given by:

$$\tilde{V}_0(x,k) = \begin{pmatrix} e^{i(k-\lambda_r)x} & r(k)\, e^{-2i\lambda_r x} \\ -\bar{r}(k)\, e^{2ikx} & [1 - r(k)\bar{r}(k)]\, e^{i(k-\lambda_r)x} \end{pmatrix} . \tag{3.49}$$

In the RH problem across Σ_r^+, one has

$$\tilde{M}^+(x,k) = \left[\phi^+(x,k)\, e^{ikx} \quad \frac{\psi^+(x,k)}{c^+(k)}\, e^{-i\lambda_r^+ x}\right] ,$$
$$\tilde{M}^-(x,k) = \left[\phi^-(x,k)\, e^{ikx} \quad \frac{\psi^-(x,k)}{c^-(k)}\, e^{-i\lambda_r^- x}\right] .$$

Note, however, that unlike what happens in the RH problem from the right, here one cannot use (2.23b) to determine the jump. The same holds for the RH problem on Σ_r^-. In fact, in both equations (2.23b), i.e., $\bar{\psi}(x,k) = \bar{c}(k)\phi(x,k) + \bar{d}(k)\bar{\phi}(x,k)$ and $\psi(x,k) = d(k)\phi(x,k)+c(k)\bar{\phi}(x,k)$, the right-hand sides are only simultaneously defined for $k \in \mathbb{R}$, and cannot be extended on either Σ_r^+ or Σ_r^-. This is also evident from (2.31c), where it is clear that, unlike $\rho(k)$ and $\bar{\rho}(k)$, which can be respectively continued on Σ_r^+ and Σ_r^-, the reflection coefficients from the left $r(k)$ and $\bar{r}(k)$ are only generically defined on the continuous spectrum, i.e., for $k \in \mathbb{R}$.

In order to formulate the RH problem from the left on Σ_r, one has to consider both pieces of the cut Σ_r^+ and Σ_r^- simultaneously, and take into account that: (i) λ_r changes sign across Σ_r; (ii) $\phi^+(x,k) = \phi^-(x,k)$ for $k \in \Sigma_r^+$, and $\bar{\phi}^+(x,k) = \bar{\phi}^-(x,k)$ for $k \in \Sigma_r^-$; (iii) $\psi^\pm(x,k)/c^\pm(k)$ and $\bar{\psi}^\pm(x,k)/\bar{c}^\pm(x,k)$ are related to each other via the symmetry relations (2.36), (2.40a) and (2.40b), i.e.:

$$\frac{\psi^\pm(x,k)}{c^\pm(x,k)} = \frac{\bar{\psi}^\mp(x,k)}{\bar{d}^\mp(x,k)} \qquad k \in \Sigma_r^+\,, \tag{3.50a}$$

$$\frac{\bar{\psi}^\pm(x,k)}{\bar{c}^\pm(x,k)} = \frac{\psi^\mp(x,k)}{d^\mp(x,k)} \qquad k \in \Sigma_r^-\,. \tag{3.50b}$$

Solving the RH problem from the left (with poles, corresponding to the zeros of $c(k)$ and $\bar{c}(k)$ in the upper/lower half planes, which, by (2.28c) are the same as the ones from the right) amounts to computing the sectionally meromorphic matrix $\tilde{M}(x,k)$ with the given jumps, and normalized to the identity as $k \to \infty$. The potential is then reconstructed by the large k expansion of the latter, since

$$\tilde{M}_o(x,k) = \frac{i}{2k}Q(x)\sigma_3 + o(1/k)\,,$$

where, as before, subscript $_o$ is used to denote the off-diagonal part of the matrix. Once the time dependence of the scattering coefficients is included (see Sec. 4), the RH problems from the right and from the left formulated in this section provide the necessary set-up for the investigation of the long-time asymptotic behavior of fairly general NLS solutions with nontrivial boundary conditions via the nonlinear steepest descent method, as done, for instance, in [**14**] for special piecewise constant initial conditions.

4. Time evolution of the scattering data

The time evolution of the eigenfunctions is to be determined from (2.1b), which, taking into account that $q(x,t) \to 0$ as $x \to -\infty$, yields

$$v_t \sim 2ik^2\sigma_3\, v \quad \text{as } \; x \to -\infty\,. \tag{4.1}$$

The time-independent boundary condition (2.6b) for $\Phi = (\phi \;\; \bar{\phi})$, however, is not compatible with the above time evolution. Therefore we define time-dependent functions $\varphi(x,k,t) = e^{A_\infty t}\phi(x,k,t)$ and $\bar{\varphi}(x,k,t) = e^{-A_\infty t}\bar{\phi}(x,k,t)$, with $A_\infty = 2ik^2$, to be solutions of (2.1b). Then the time evolution equations for the Jost solutions $\Phi = (\phi \;\; \bar{\phi})$ are found to be:

$$\partial_t \Phi = \left[i(2k^2 - |q|^2 + Q_x)\sigma_3 - 2kQ\right]\Phi - 2ik^2\,\Phi\,\sigma_3\,. \tag{4.2}$$

Similarly, taking into account that $q(x,t) \to A_r\, e^{-2iA_r^2 t + i\theta_r}$ as $x \to +\infty$, one finds the time evolution of the Jost solutions $\Psi = (\bar{\psi} \;\; \psi)$ to be given by:

$$\partial_t \Psi = \left[i(2k^2 - |q|^2 + Q_x)\sigma_3 - 2kQ\right]\Psi - i(2k\lambda_r - A_r^2)\Psi\,\sigma_3\,. \tag{4.3}$$

Differentiating (2.23a) with respect to t and taking into account the time evolution of the Jost solutions (4.2)-(4.3), we obtain for the scattering matrix $S(k,t)$ the following ODE with respect to t:

$$\partial_t S = i(2k\lambda_r - A_r^2)\sigma_3\, S - 2ik^2 S\,\sigma_3\,. \tag{4.4}$$

As a consequence, the scattering coefficients from the right are such that:

$$a(k,t) = a(k,0)\, e^{i[2k(\lambda_r - k) - A_r^2]t}\,, \qquad \bar{a}(k,t) = \bar{a}(k,0)\, e^{i[2k(k-\lambda_r) + A_r^2]t}\,, \tag{4.5a}$$

$$b(k,t) = b(k,0)\, e^{i[-2k(\lambda_r + k) + A_r^2]t}\,, \qquad \bar{b}(k,t) = \bar{b}(k,0)\, e^{i[2k(\lambda_r + k) - A_r^2]t}\,, \tag{4.5b}$$

$$\rho(k,t) = \rho(k,0)\, e^{-2i(2k\lambda_r - A_r^2)t}\,, \qquad \bar{\rho}(k,t) = \bar{\rho}(k,0)\, e^{2i(2k\lambda_r - A_r^2)t}\,. \tag{4.5c}$$

In particular, Eq. (4.5a) shows that the discrete eigenvalues k_n are time independent, and given by the zeros of $a(k,0)$. Moreover, for the large k behavior of $a(k,t)$, taking into account that

$$\lambda_r - k = \frac{A_r^2}{2k}\left[1 + O(k^{-2})\right]\,,$$

one still finds from (4.5a) that, consistently with (2.42), $a(k,t) \sim 1$ as $|k| \to \infty$ for $k \in \mathbb{K}_r^+ \cup \mathbb{R}$ and for all $t \geq 0$.

Similarly, differentiating (2.23b) with respect to t and taking into account the time evolution of the Jost solutions (4.2) and (4.3), we get for the scattering matrix $\bar{S}(k,t)$ the following ODE:

$$\partial_t \bar{S} = -i(2k\lambda_r - A_r^2)\bar{S}\,\sigma_3 + 2ik^2\sigma_3\,\bar{S}\,. \tag{4.6}$$

Therefore, the time dependence of the scattering coefficients from the left is given by

$$c(k,t) = c(k,0)\, e^{i[2k(\lambda_r - k) - A_r^2]t}\,, \qquad \bar{c}(k,t) = \bar{c}(k,0)\, e^{i[2k(k-\lambda_r) + A_r^2]t}\,, \tag{4.7a}$$

$$d(k,t) = d(k,0)\, e^{i[2k(\lambda_r + k) - A_r^2]t}\,, \qquad \bar{d}(k,t) = \bar{d}(k,0)\, e^{i[-2k(\lambda_r + k) + A_r^2]t}\,, \tag{4.7b}$$

$$r(k,t) = r(k,0)\, e^{4ik^2 t}\,, \qquad \bar{r}(k,t) = \bar{r}(k,0)\, e^{-4ik^2 t}\,. \tag{4.7c}$$

Finally, we determine the time dependence of the norming constants. Differentiating $\phi(x,k_n) = b_n \psi(x,k_n)$ with respect to time and evaluating the first column of (4.2) and the second column of (4.3) at $k = k_n$, we get

$$b_n(t) = b_n(0) e^{-i[2k_n(k_n + \lambda_r(k_n)) - A_r^2]t}, \quad n = 1, \ldots, N.$$

Then from the definition of the norming constants in (3.7), we obtain

$$C_n(t) = C_n(0) e^{-2i[2k_n \lambda_r(k_n) - A_r^2]t}, \quad n = 1, \ldots, N. \tag{4.8}$$

The time dependence of the norming constants $\tilde{C}_n$ appearing in (3.26) can be found in a similar way, or simply taking into account the symmetry relation (3.21).

5. Direct and inverse scattering in the uniformization variable z

Unlike what happens when dealing with fully asymmetric boundary conditions [**19**], here we can introduce a uniformization variable z (cf. [**12**, **20**]) defined by the conformal mapping:

$$z = k + \lambda_r(k)\,, \tag{5.1}$$

with inverse mapping given:

$$k = \frac{1}{2}\left(z - \frac{A_r^2}{z}\right), \qquad \lambda_r = z - k = \frac{1}{2}\left(z + \frac{A_r^2}{z}\right).$$

The conformal transformation (5.1) maps the two-sheeted Riemann surface defined by $\lambda_r^2 = k^2 + A_r^2$ onto a single complex z-plane, as shown in Fig. 5. Specifically, one has:

- Sheet I [resp. Sheet II] of the Riemann surface is mapped onto the exterior [resp. interior] of the circle C_r of radius A_r;
- The branch cut Σ_r on either sheet is mapped onto C_r;
- The real k-axis on Sheet I [resp. Sheet II] is mapped onto $(-\infty, -A_r) \cup (A_r, +\infty)$ [resp. $(-A_r, A_r)$];
- $z(\pm iA_r) = \pm iA_r$ from either sheet, $z(0_{\mathrm{I}}^{\pm}) = \pm A_r$, and $z(0_{\mathrm{II}}^{\pm}) = \mp A_r$;
- $\operatorname{Im} k > 0$ [$\operatorname{Im} k < 0$] on either sheet is mapped into $\operatorname{Im} z > 0$ [$\operatorname{Im} z < 0$];
- The cut half-plane $\mathbb{K}_r^+$ [resp. $\mathbb{K}_r^-$] of Sheet I is mapped into the upper half z-plane [resp. lower half z-plane] outside the circle C_r;
- The cut half-plane $\mathbb{K}_r^+$ [resp. $\mathbb{K}_r^-$] of Sheet II is mapped into the upper half z-plane [resp. lower half z-plane] inside the circle C_r.

We introduce the following regions in the complex z-plane:

$$D_+ = \{z \in \mathbb{C} : (|z|^2 - A_r^2)\, \operatorname{Im} z > 0\}, \quad D_- = \{z \in \mathbb{C} : (|z|^2 - A_r^2)\, \operatorname{Im} z < 0\},$$

corresponding to $\operatorname{Im} \lambda_r > 0$ and $\operatorname{Im} \lambda_r < 0$, respectively, on either sheet. The complex z-plane is then partitioned into four regions: the upper/lower half z-plane

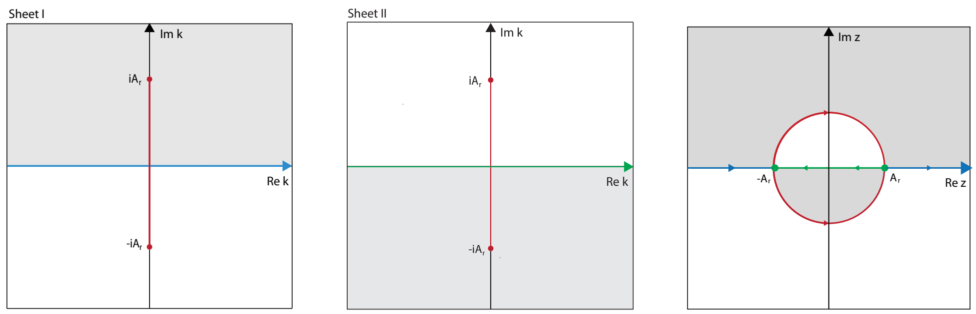

FIGURE 5. Left & Center: the two-sheets of the Riemann surface associated with $\lambda_r^2 = k^2 + A_r^2$. Right: the complex plane for the uniformization variable $z = k + \lambda_r$. The grey regions (D_+) correspond to $\operatorname{Im}\lambda_r > 0$, while the white regions (D_-) correspond to $\operatorname{Im}\lambda_r < 0$; the circle C_r (red) corresponds to the cut Σ_r on either sheet; $(-\infty, -A_r) \cup (A_r, +\infty)$ (blue) and $(-A_r, A_r)$ (green) correspond to the real k-axis on sheets I and II, respectively. The oriented contour in the complex z-plane for the Riemann-Hilbert problem is also shown.

outside the circle C_r, denoted as $D_\pm^{\mathrm{out}}$, respectively, and the upper/lower half z-plane inside the circle C_r denoted as $D_\pm^{\mathrm{in}}$, respectively. In the following, we will also denote with $C_r^\pm$ the upper and lower semicircles of radius A_r, respectively.

The asymptotic behaviors (2.6b) and (2.17), expressed in terms of z, read

$$\Phi(x,z) \sim I_2\, e^{-ik(z)x\sigma_3}\,, \qquad x \to -\infty\,, \tag{5.2a}$$

$$\Psi(x,z) \sim \left[I_2 - \frac{i}{z}\sigma_3 Q_r\right] e^{-i\lambda_r(z)x\sigma_3}\,, \qquad x \to +\infty\,, \tag{5.2b}$$

which then allows one to introduce the Jost solutions $\Psi(x,z) = \big(\bar\psi(x,z)\ \psi(x,z)\big)$ where $\lambda_r(z) \in \mathbb{R}$, i.e., for $z \in \mathbb{R} \cup C_r$, and $\Phi(x,z) = \big(\phi(x,z)\ \bar\phi(x,z)\big)$ where $k(z) \in \mathbb{R}$, i.e., for $z \in \mathbb{R}$. Consistently with Th. 2.2 and the analogous analyticity properties of the Jost solutions on sheet II, it then follows that if the potential satisfies ($\boldsymbol{H}_1$) $\psi(x,z)$ can be analytically continued in D_+, $\bar\psi(x,z)$ is analytic in D_-, while $\phi(x,z)$ and $\bar\phi(x,z)$ are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively.

Unlike what happens with same-amplitude NZBCs [**12**], here the continuous spectrum, where all four eigenfunctions are simultaneously defined, corresponds to $z \in \mathbb{R}$, while the circle C_r corresponds to the DSW region. In analogy to what discussed in Sec. 2.2, one can express the two sets of Jost solutions $\Phi(x,z)$ and $\Psi(x,z)$ as

$$\Phi(x,z) = \Psi(x,z)\, S(z)\,, \qquad S(z) = \begin{pmatrix} a(z) & \bar b(z) \\ b(z) & \bar a(z) \end{pmatrix}, \qquad z \in \mathbb{R}\,, \tag{5.3a}$$

$$\Psi(x,z) = \Phi(x,z)\, \bar S(z)\,, \qquad S(z) = \begin{pmatrix} \bar c(z) & d(z) \\ \bar d(z) & c(z) \end{pmatrix}, \qquad z \in \mathbb{R}\,. \tag{5.3b}$$

Then, expressing the scattering coefficients as Wronskians of the Jost solutions:

$$a(z) = \frac{z}{2\lambda_r(z)}\,\mathrm{Wr}(\phi(x,z),\psi(x,z)), \qquad \bar{a}(z) = \frac{z}{2\lambda_r(z)}\,\mathrm{Wr}(\bar{\psi}(x,z),\bar{\phi}(x,z)), \tag{5.4a}$$

$$b(k) = \frac{z}{2\lambda_r(z)}\,\mathrm{Wr}(\bar{\psi}(x,z),\phi(x,z)), \qquad \bar{b}(k) = \frac{z}{2\lambda_r(z)}\,\mathrm{Wr}(\bar{\phi}(x,z),\psi(x,z)), \tag{5.4b}$$

one can establish that:

- $a(z)$ is continuous in $\mathbb{R}\cup C_r^+\setminus\{iA_r\}$ and analytic in D_+^{out}
- $\bar{a}(z)$ is continuous in $\mathbb{R}\cup C_r^-\setminus\{-iA_r\}$ and analytic in D_-^{out}
- $b(z)$ is continuous in $\mathbb{R}\cup C_r^+\setminus\{iA_r\}$ and analytic in D_-^{in}
- $\bar{b}(z)$ is continuous in $\mathbb{R}\cup C_r^-\setminus\{-iA_r\}$ and analytic in D_-^{in}

Obviously, the above scattering coefficients are the same as those in (2.28a) and (2.28b), expressed in terms of the uniformization variable. The reflection coefficients are then given by:

$$\rho(z) = \frac{b(z)}{a(z)}\,, \qquad \bar{\rho}(z) = \frac{\bar{b}(z)}{\bar{a}(z)}\,, \qquad z\in\mathbb{R}\,. \tag{5.5a}$$

Note that $\rho(z)$ is also defined on C_r^+, and the corresponding values obviously coincide with the limiting values $\rho^{\pm}(k)$ on either edge of the cut Σ_r^+. Similarly, $\bar{\rho}(z)$ is defined on C_r^-, and the values coincides with the limiting values $\bar{\rho}^{\pm}(k)$ on either edge of the cut Σ_r^-.

Similar results can be derived for the scattering coefficients from the left i.e., the entries of $\bar{S}(z)$, using the analog of (2.28c) and (2.28d). In particular, the reflection coefficients from the left are given by:

$$r(z) = \frac{d(z)}{c(z)} \equiv -\frac{\bar{b}(z)}{a(z)}\,, \qquad \bar{r}(z) = \frac{\bar{d}(z)}{\bar{c}(z)} \equiv -\frac{b(z)}{\bar{a}(z)}\,, \qquad z\in\mathbb{R}\,, \tag{5.5b}$$

corresponding to (2.31c) expressed in terms of the uniformization variable. Unlike $\rho(z)$ and $\bar{\rho}(z)$, the reflection coefficients from the left are generically not defined on either C_r^+ or C_r^-.

5.1. Symmetries of eigenfunctions and scattering data. In terms of the uniformization variable z, since $\lambda_{r,\mathrm{II}} = -\lambda_{r,\mathrm{I}}$ when both sheets are considered, the two involutions $(k,\lambda_r)\to(k^*,\lambda_r^*)$ (i.e., upper/lower half k-plane), and $(k,\lambda_r)\to(k,-\lambda_r)$ (i.e., opposite sheets) are: $z\to z^*$ (i.e., upper/lower half z-plane) and $z\to -A_r^2/z$ (outside/inside the circle C_r).

First symmetry. The asymptotic behavior (5.2) yields the following symmetries for the Jost solutions:

$$\bar{\psi}^*(x,z^*) = i\sigma_2\psi(x,z) \quad \text{for } z\in D_+\cup C_r\cup\mathbb{R}\,, \tag{5.6a}$$

$$\psi^*(x,z^*) = -\,i\sigma_2\bar{\psi}(x,z) \quad \text{for } z\in D_-\cup C_r\cup\mathbb{R}\,, \tag{5.6b}$$

$$\phi^*(x,z^*) = i\sigma_2\bar{\phi}(x,z) \quad \text{for } \mathbb{C}^+\cup\mathbb{R}\,, \qquad \bar{\phi}^*(x,z^*) = -i\sigma_2\phi(x,z) \quad \text{for } \mathbb{C}^-\cup\mathbb{R}\,. \tag{5.6c}$$

Consequently, the symmetry relations for the scattering coefficients can be written as

$$\bar{a}^*(z^*) = a(z) \quad \text{for } z\in D_+^{\mathrm{out}}\cup C_r^+\cup\mathbb{R}\,, \qquad \bar{b}^*(z^*) = -b(z) \quad \text{for } z\in D_-^{\mathrm{in}}\cup C_r^-\cup\mathbb{R}\,. \tag{5.7}$$

Taking into account (5.5a), the reflection coefficients then satisfy the following symmetry relation:

$$\bar{\rho}^*(z^*) = -\rho(z) \quad \text{for } z \in \mathbb{R} \cup C_r^+ \,. \tag{5.8}$$

Second symmetry. Since $\lambda_r(-A_r^2/z) = -\lambda_r(z)$ and $k(-A_r^2/z) = k(z)$, taking into account the boundary conditions (5.2), one can easily establish the following additional symmetry relations for the Jost solutions:

$$\bar{\psi}(x,z) = \frac{-iq_r^*}{z}\psi(x,-A_r^2/z) \quad \text{for } z \in D_- \cup C_r \,, \tag{5.9a}$$

$$\psi(x,z) = \frac{-iq_r}{z}\bar{\psi}(x,-A_r^2/z) \quad \text{for } z \in D_+ \cup C_r \,, \tag{5.9b}$$

(5.9c)
$$\phi(x,z) = \phi(x,-A_r^2/z) \quad \text{for } z \in \mathbb{R} \cup C_r^+ \,, \qquad \bar{\phi}(x,z) = \bar{\phi}(x,-A_r^2/z) \quad \text{for } z \in \mathbb{R} \cup C_r^- \,.$$

Using the above symmetries in the Wronskian representations for the scattering coefficients (5.4) one obtains:

$$a(z) = \frac{z}{-iq_r^*} b(-A_r^2/z) \quad \text{for } z \in D_+^{\text{out}} \cup \mathbb{R} \cup C_r^+ \,, \tag{5.10a}$$

$$\bar{a}(z) = \frac{z}{-iq_r} \bar{b}(-A_r^2/z) \quad \text{for } z \in D_-^{\text{out}} \cup \mathbb{R} \cup C_r^- \,. \tag{5.10b}$$

Similarly, the symmetry relations for the scattering coefficients from the left are given by:

$$c(z) = \frac{-iq_r}{z} \bar{d}(-A_r^2/z) \quad \text{for } z \in D_+^{\text{out}} \cup \mathbb{R} \cup C_r^+ \,, \tag{5.10c}$$

$$\bar{c}(z) = \frac{-iq_r^*}{z} d(-A_r^2/z) \quad \text{for } z \in D_-^{\text{out}} \cup \mathbb{R} \cup C_r^- \,. \tag{5.10d}$$

5.2. Discrete eigenvalues. A *discrete eigenvalue* is a value of $z \in D_+ \cup D_-$ for which there exists a nontrivial solution v to (1.2) with entries in $L^2(\mathbb{R})$. These eigenvalues occur for $z \in D_+^{\text{out}}$ iff the functions $\phi(x,z)$ and $\psi(x,z)$ are linearly dependent (i.e., iff $a(z) = 0$), for $z \in D_-^{\text{out}}$ iff the functions $\bar{\psi}(x,z)$ and $\bar{\phi}(x,z)$ are linearly dependent (i.e., iff $\bar{a}(z) = 0$), for $z \in D_-^{\text{in}}$ iff the functions $\phi(x,z)$ and $\bar{\psi}(x,z)$ are linearly dependent (i.e., iff $b(z) = 0$), and finally for $z \in D_+^{\text{in}}$ iff the functions $\psi(x,z)$ and $\bar{\phi}(x,z)$ are linearly dependent (i.e., iff $\bar{b}(z) = 0$). The conjugation symmetry (5.7) and the second symmetry (5.10) then imply that the discrete eigenvalues occur in quartets: $\left\{z_n \,, z_n^* \,, -A_r^2/z_n \,, -A_r^2/z_n^*\right\}_{n=1}^N$.

5.3. Asymptotic behavior as $z \to \infty$ and $z \to 0$. The asymptotic properties of the eigenfunctions and the scattering coefficients are needed to properly pose the inverse problem. Note that the limit $|k| \to \infty$ corresponds to $z \to \infty$ in Sheet I, and $z \to 0$ in Sheet II. Standard Wentzel-Kramers-Brillouin (WKB) expansions in the scattering problem (2.1a) rewritten in terms of z yield the following asymptotic behaviors for the eigenfunctions:

(5.11a)

as $\quad z \to \infty:$

$$\Psi_d(x,z)\, e^{i\lambda_r(z)\sigma_3 x} = I_2 + o(1)\,, \quad \Psi_o(x,z)\, e^{i\lambda_r(z)\sigma_3 x} = \frac{i}{z} Q(x)\sigma_3 + o(1/z)\,,$$

(5.11b)
as $\quad z \to 0:$

$$\Psi_o(x,z)\, e^{i\lambda_r(z)\sigma_3 x} = \frac{i}{z} Q_r\sigma_3 + O(1)\,, \qquad \Psi_d(x,z)\, e^{i\lambda_r(z)\sigma_3 x} = -Q(x)\sigma_3 Q_r^{-1}\sigma_3 + o(1)\,,$$

and

(5.12a)

as $\quad z \to \infty:$

$$\Phi_d(x,z)\, e^{ik(z)\sigma_3 x} = I_2 + o(1)\,, \qquad \Phi_o(x,z)\, e^{ik(z)\sigma_3 x} = \frac{i}{z} Q(x)\sigma_3 + o(1/z)\,,$$

(5.12b)
as $\quad z \to 0:$

$$\Phi_d(x,z)\, e^{ik(z)\sigma_3 x} = I_2 + O(z)\,, \qquad \Phi_o(x,z)\, e^{ik(z)\sigma_3 x} = -\frac{iz}{A_r^2} Q(x)\sigma_3 + o(z)\,,$$

where, as before, subscripts $_d$ and $_o$ denote diagonal and off-diagonal part of the matrix. Under the assumption $(\boldsymbol{H}_1)$ for the potential, from the Wronskian representations (5.4) for the scattering coefficients and the above asymptotic behavior of the eigenfunctions, we then obtain the following asymptotic behavior for the scattering coefficients at large z:

(5.13a) $$\lim_{z\to\infty} a(z) = 1 \quad \text{for } z \in D_+^{\text{out}} \cup \mathbb{R}\,, \quad \lim_{z\to\infty} \bar{a}(z) = 1 \quad \text{for } z \in D_-^{\text{out}} \cup \mathbb{R}\,,$$

and

(5.13b) $$\lim_{z\to\infty} z\, b(z) = 0\,, \quad \lim_{z\to\infty} z\, \bar{b}(z) = 0 \quad \text{for } z \in \mathbb{R}\,.$$

Similarly, the asymptotic behavior for the scattering coefficients as $z \to 0$ is as follows:

(5.14a) $$b(z) = \frac{i}{q_r} z + O(z^2) \quad \text{for } z \in D_-^{\text{in}} \cup (-A_r, A_r)\,,$$

(5.14b) $$\bar{b}(z) = \frac{i}{q_r^*} z + O(z^2) \quad \text{for } z \in D_+^{\text{in}} \cup (-A_r, A_r)\,,$$

and

(5.14c) $$\lim_{z\to 0} \frac{a(z)}{z^2} = 0\,, \qquad \lim_{z\to 0} \frac{\bar{a}(z)}{z^2} = 0 \quad \text{for } z \in (-A_r, A_r)\,.$$

5.4. Riemann-Hilbert problem. In this section we formulate the inverse scattering problem as matrix Riemann-Hilbert problems from the right and from the left for a suitable set of sectionally analytic/meromorphic functions in $D_+ \cup D_-$, with assigned jumps across $\mathbb{R} \cup C_r$, i.e., the oriented contour in the complex z-plane indicated in Fig. 5.

5.4.1. *Riemann-Hilbert problem from the right.* For the formulation of the Riemann-Hilbert problem in terms of scattering data from the right, we introduce the following matrix of eigenfunctions:

$$M(x,k)=\begin{cases}\left[\dfrac{\phi(x,z)}{a(z)}\,e^{ik(z)x}\quad \psi(x,z)\,e^{-i\lambda_r(z)x}\right] & z\in D_+^{\text{out}}\,,\\ \left[\dfrac{\phi(x,z)}{b(z)}\,e^{ik(z)x}\quad \bar{\psi}(x,z)\,e^{i\lambda_r(z)x}\right] & z\in D_-^{\text{in}}\,,\\ \left[\psi(x,z)\,e^{-i\lambda_r(z)x}\quad \dfrac{\bar{\phi}(x,z)}{\bar{b}(z)}\,e^{-ik(z)x}\right] & z\in D_+^{\text{in}}\,,\\ \left[\bar{\psi}(x,z)\,e^{i\lambda_r(z)x}\quad \dfrac{\bar{\phi}(x,z)}{\bar{a}(z)}\,e^{-ik(z)x}\right] & z\in D_-^{\text{out}}\,.\end{cases} \tag{5.15}$$

The asymptotic behavior of the eigenfunctions and scattering coefficients (cf. Sec. 5.3) establishes that for $z\in D_\pm^{\text{out}}$

$$M(x,z)=I_2+O\left(\frac{1}{z}\right)\,,\quad \text{as } z\to\infty\,, \tag{5.16}$$

and for $z\in D_\pm^{\text{in}}$

$$M(x,z)=-\frac{i}{z}\sigma_3 Q_r\sigma_1+O(1)\,,\quad \text{as } z\to 0\,, \tag{5.17}$$

where σ_1 is the first Pauli matrix given by $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

One has to determine four jump matrices: $V_0(x,z)$ across $z\in(-\infty,-A_r)\cup(A_r,+\infty)$; $V_1(x,z)$ across $z\in(-A_r,A_r)$; $V_2(x,z)$ across the semicircle C_r^+ in the upper half z-plane; and $V_3(x,z)$ across the semicircle C_r^- in the lower half z-plane.

The RH problem across $z\in(-\infty,-A_r)\cup(A_r,+\infty)$ can be written in matrix form as $M^+(x,z)=M^-(x,z)\,V_0(x,z)$, i.e.,

$$\left[\frac{\phi^+(x,z)}{a^+(z)}\,e^{ikx}\quad \psi^+(x,z)\,e^{-i\lambda_r x}\right]=\left[\bar{\psi}^-(x,z)\,e^{i\lambda_r x}\quad \frac{\bar{\phi}^-(x,z)}{\bar{a}^-(z)}\,e^{-ikx}\right]V_0(x,z)\,, \tag{5.18}$$

where superscripts $^\pm$ denote limiting values from the upper/lower complex z-plane, respectively. Using (5.3a), the jump matrix is found to be:

$$V_0(x,z)=\begin{pmatrix}[1-\rho(z)\bar{\rho}(z)]\,e^{-i(A_r^2/z)x} & -\bar{\rho}(z)\,e^{-2i\lambda_r(z)x}\\ \rho(z)\,e^{2ik(z)x} & e^{-i(A_r^2/z)x}\end{pmatrix}. \tag{5.19}$$

We can write the RH problem across $z\in(-A_r,A_r)$ as $M^+(x,z)=M^-(x,z)\,V_1(x,z)$, where the superscripts $^\pm$ again denote limiting values from the upper/lower complex z-plane, respectively. Explicitly, from (5.3a) we have

$$\left[\frac{\phi^+(x,z)}{b^+(z)}\,e^{ik(z)x}\quad \bar{\psi}^+(x,z)\,e^{i\lambda_r(z)x}\right]\left[\psi^-(x,z)\,e^{-i\lambda_r(z)x}\quad \frac{\bar{\phi}^-(x,z)}{\bar{b}^-(z)}\,e^{-ik(z)x}\right]V_1(x,z)\,, \tag{5.20}$$

and (5.3a) yields the following expression for the jump matrix $V_1(x,z)$:

$$V_1(x,z)=\begin{pmatrix}\left[1-\dfrac{1}{\rho(z)\bar{\rho}(z)}\right]e^{izx} & -\dfrac{1}{\bar{\rho}(z)}\,e^{2i\lambda_r(z)x}\\ \dfrac{1}{\rho(z)}\,e^{2ik(z)x} & e^{izx}\end{pmatrix}. \tag{5.21}$$

The RH problem across the semicircle C_r^+ in the upper half z-plane is written as $M^+(x,z) = M^-(x,z)V_2(x,z)$, i.e.,

(5.22)
$$\left[\frac{\phi^+(x,z)}{a^+(z)}\,e^{ik(z)x} \quad \psi^+(x,z)\,e^{-i\lambda_r^+(z)x}\right] = \left[\frac{\phi^-(x,z)}{b^-(z)}\,e^{ik(z)x} \quad \bar{\psi}^-(x,z)\,e^{i\lambda_r^-(z)x}\right] V_2(x,z)\,,$$

where the superscripts $\pm$ denote limiting values from the exterior/interior of C_r^+, respectively. Taking into account that $\phi(x,z)$ is continuous across C_r^+, i.e., $\phi^+(x,z) = \phi^-(x,z)$, and that from (5.4) it follows $\phi(x,z) = a(z)\bar{\psi}(x,z) + b(z)\psi(x,z)$ for $z \in C_r^+$, the jump matrix $V_2(x,z)$ is then found to be:

$$V_2(x,z) = \begin{pmatrix} \rho(z) & e^{-izx} \\ 0 & -\dfrac{1}{\rho(z)}e^{-2i\lambda_r(z)x} \end{pmatrix}. \tag{5.23}$$

Finally, the RH problem across the semicircle C_r^- in the lower half z-plane is $M^+(x,z) = M^-(x,z)V_3(x,z)$, and explicitly as

(5.24)
$$\left[\bar{\psi}^+(x,z)\,e^{i\lambda_r^+(z)x} \quad \frac{\bar{\phi}^+(x,z)}{\bar{a}^+(z)}\,e^{-ik(z)x}\right] = \left[\psi^-(x,z)\,e^{-i\lambda_r^-(z)x} \quad \frac{\bar{\phi}^-(x,z)}{\bar{b}^-(z)}\,e^{-ik(z)x}\right] V_3(x,z)\,,$$

where the superscripts $\pm$ denote limiting values from the exterior/interior of C_r^-, respectively. Clearly $\bar{\phi}(x,z)$ is continuous across C_r^-, i.e., $\bar{\phi}^+(x,z) = \bar{\phi}^-(x,z)$, and since, according to (5.4), the relationship $\bar{\phi}(x,z) = \bar{a}(z)\psi(x,z) + \bar{b}(z) + \bar{\psi}(x,z)$ still holds on C_r^-, the jump matrix $V_3(x,z,t)$ is found to be:

$$V_3(x,z) = \begin{pmatrix} -\dfrac{1}{\bar{\rho}(z)}e^{2i\lambda_r(z)x} & 0 \\ e^{izx} & \bar{\rho}(z) \end{pmatrix}. \tag{5.25}$$

It is worth noting that the RH problems across $z \in (-\infty, -A_r) \cup (A_r, +\infty)$ and $z \in (-A_r, A_r)$ correspond to the RH problems across the real axis in Sheet I and in Sheet II, respectively.

Solving the inverse problem as a RH problem [with poles, corresponding to the zeros of $a(z)$ and $\bar{a}(z)$ in $D_\pm^{\text{out}}$, and to the zeros of $b(z)$ and $\bar{b}(z)$ in $D_\mp^{\text{in}}$] then amounts to computing the sectionally meromorphic matrix $M(x,z)$ with the given jumps, and asymptotic behaviors (5.16) as $z \to \infty$, and (5.17) as $z \to 0$. As in defocusing NLS equation with NZBCs [**18**], in addition to the behavior at $z = \infty$ and the poles from the discrete spectrum one also needs to subtract the pole at $z = 0$ in order to obtain a regular RH problem.

Once the parametric time dependence of the scattering coefficients is taken into account in the jump matrices, which can be easily obtained from the results of Sec. 4:

$$a(z,t) = a(z,0)\,e^{-i(A_r^4/z^2)t}\,, \qquad \bar{a}(z,t) = \bar{a}(z,0)\,e^{i(A_r^4/z^2)t}\,, \tag{5.26a}$$

$$b(z,t) = b(z,0)\,e^{i(2A_r^2-z^2)t}\,, \qquad \bar{b}(z,t) = \bar{b}(z,0)\,e^{i(z^2-2A_r^2)t}\,, \tag{5.26b}$$

$$\rho(z,t) = \rho(z,0)\,e^{i(2A_r^2-z^2-A_r^2/z^2)t}\,, \qquad \bar{\rho}(z,t) = \bar{\rho}(z,0)\,e^{i(z^2-2A_r^2+A_r^2/z^2)t}\,, \tag{5.26c}$$

the potential is then reconstructed by the large-z expansion of the matrix $M(x,z,t)$:

$$M_o(x,z,t) = \frac{i}{z}Q(x,t)\sigma_3 + o\left(1/z\right)\,, \tag{5.27}$$

and by the asymptotic behavior as $z \to 0$ of $M(x,z,t)$:

$$M_o(x,z,t) = -\frac{i}{z}\sigma_3 Q_r(t)\sigma_1 + Q(x,t)Q_r^{-1}(t)\sigma_1 + o(1)\,, \tag{5.28}$$

with subscript $_o$ denoting the off-diagonal part.

5.4.2. *Riemann-Hilbert from the left.* The inverse problem can also be formulated as a Riemann-Hilbert from the left, considering the following eigenfunctions matrix

$$\tilde{M}(x,z) = \begin{cases} \begin{bmatrix} \phi(x,z)\,e^{ik(z)x} & \dfrac{\psi(x,z)}{c(z)}\,e^{-i\lambda_r(z)x} \end{bmatrix} & z \in D_+^{\text{out}}\,, \\ \begin{bmatrix} \phi(x,z)\,e^{ik(z)x} & \dfrac{\bar\psi(x,z)}{\bar d(z)}\,e^{i\lambda_r(z)x} \end{bmatrix} & z \in D_-^{\text{in}}\,, \\ \begin{bmatrix} \dfrac{\psi(x,z)}{d(z)}\,e^{-i\lambda_r(z)x} & \bar\phi(x,z)\,e^{-ik(z)x} \end{bmatrix} & z \in D_+^{\text{in}}\,, \\ \begin{bmatrix} \dfrac{\bar\psi(x,z)}{\bar c(z)}\,e^{i\lambda_r(z)x} & \bar\phi(x,z)\,e^{-ik(z)x} \end{bmatrix} & z \in D_-^{\text{out}}\,, \end{cases} \tag{5.29}$$

such that $\tilde{M}(x,z) \to I_2$ both as $z \to \infty$ for $z \in D_\pm^{\text{out}}$, and as $z \to 0$ for $z \in D_\pm^{\text{in}}$.

The RH problem across $z \in (-\infty, -A_r) \cup (A_r, +\infty)$ is written in matrix form as $\tilde{M}^+(x,z) = \tilde{M}^-(x,z)\,\tilde{V}_0(x,z)$, i.e.,

$$\begin{bmatrix} \phi^+(x,z)\,e^{ik(z)x} & \dfrac{\psi^+(x,z)}{c^+(z)}\,e^{-i\lambda_r(z)x} \end{bmatrix} = \begin{bmatrix} \dfrac{\bar\psi^-(x,z)}{\bar c^-(z)}\,e^{i\lambda_r(z)x} & \bar\phi^-(x,z)\,e^{-ik(z)x} \end{bmatrix} \tilde{V}_0(x,z)\,, \tag{5.30}$$

where superscripts $^\pm$ denote limiting values from the upper/lower complex z-plane, respectively. Using (5.3b), the jump matrix is determined as follows

$$\tilde{V}_0(x,z) = \begin{pmatrix} e^{-i(A_r^2/z)x} & r(z)\,e^{-2i\lambda_r(z)x} \\ -\bar r(z)\,e^{2ik(z)x} & [1 - r(z)\bar r(z)]\,e^{-i(A_r^2/z)x} \end{pmatrix}. \tag{5.31}$$

We can write the RH problem across $z \in (-A_r, A_r)$ as

$$\tilde{M}^+(x,z) = \tilde{M}^-(x,z)\,\tilde{V}_1(x,z),$$

where the superscripts $^\pm$ again denote limiting values from the upper/lower complex z-plane, respectively. Explicitly, one has

$$\begin{bmatrix} \phi^+(x,z)\,e^{ik(z)x} & \dfrac{\bar\psi^+(x,z)}{\bar d^+(z)}\,e^{i\lambda_r(z)x} \end{bmatrix} = \begin{bmatrix} \dfrac{\psi^-(x,z)}{d^-(z)}\,e^{-i\lambda_r(z)x} & \bar\phi^-(x,z)\,e^{-ik(z)x} \end{bmatrix} \tilde{V}_1(x,z)\,, \tag{5.32}$$

and again (5.3b) yields the following expression for the jump matrix $\tilde{V}_1(x,z)$:

$$\tilde{V}_1(x,z) = \begin{pmatrix} e^{izx} & \dfrac{1}{\bar r(z)}\,e^{2i\lambda_r(z)x} \\ -\dfrac{1}{r(z)}\,e^{2ik(z)x} & \left[1 - \dfrac{1}{r(z)\bar r(z)}\right] e^{izx} \end{pmatrix}. \tag{5.33}$$

In formulating the RH problems from the left across the semicircles, we should first of all notice that on C_r^+ one has $\tilde{M}^+(x,z) = \begin{bmatrix} \phi^+(x,z)\,e^{ik(z)x} & \frac{\psi^+(x,z)}{c^+(z)}\,e^{-i\lambda_r^+(z)x} \end{bmatrix}$ and $\tilde{M}^-(x,z) = \begin{bmatrix} \phi^-(x,z)\,e^{ik(z)x} & \frac{\bar\psi^-(x,z)}{\bar d^-(z)}\,e^{i\lambda_r^-(z)x} \end{bmatrix}$, while on C_r^- it is $\tilde{M}^+(x,z) = \begin{bmatrix} \frac{\bar\psi^+(x,z)}{\bar c^+(z)}\,e^{i\lambda_r^+x} & \bar\phi^+(x,z)\,e^{-ikx} \end{bmatrix}$ and $\tilde{M}^-(x,z) = \begin{bmatrix} \frac{\psi^-(x,z)}{d^-(z)}\,e^{-i\lambda_r^-x} & \bar\phi^-(x,z)\,e^{-ikx} \end{bmatrix}$. Clearly, $\phi(x,z)$ and $\bar\phi(x,z)$ are continuous across C_r^+ and C_r^-, respectively. Moreover, since $k(z) = k(-A_r^2/z)$, then one also has $\phi(x,z) = \phi(x,-A_r^2/z)$ and $\bar\phi(x,z) =$

$\bar\phi(x,-A_r^2/z)$, consistently with the fact that the eigenfunctions $\phi(x,k)$, $\bar\phi(x,k)$ are continuous across Σ_r^+ and Σ_r^-, respectively. The jump conditions are then provided by the symmetry relationships (5.9b), (5.10c) and (5.10d), which yield:

$$\frac{\psi(x,z)}{c(z)}=\frac{\bar\psi(x,-A_r^2/z)}{\bar d(-A_r^2/z)}\,,\qquad \phi(x,z)=\phi(x,-A_r^2/z)\,,\qquad z\in C_r^+\,, \tag{5.34}$$

$$\frac{\psi(x,z)}{d(z)}=\frac{\bar\psi(x,-A_r^2/z)}{\bar c(-A_r^2/z)}\,,\qquad \bar\phi(x,z)=\bar\phi(x,-A_r^2/z)\,,\qquad z\in C_r^-\,. \tag{5.35}$$

Note that the RH problem in this case is clearly posed as a nonlocal one, with the jumps relating values of the meromorphic eigenfunctions at symmetric points z and $-A_r^2/z$ on the semicircles $C_r^\pm$.

Solving the inverse problem as a RH problem [with poles, corresponding to the zeros of $c(z)$ and $\bar c(z)$ in $D_\pm^{\text{out}}$, and to the zeros of $d(z)$ and $\bar d(z)$ in $D_\pm^{\text{in}}$] then amounts to computing the sectionally meromorphic matrix $\tilde M(x,z)$ with the given jumps, and normalized to identity as $z\to\infty$ for $z\in D_\pm^{\text{out}}$ and as $z\to 0$ for $z\in D_\pm^{\text{in}}$. Once the parametric time dependence of the scattering coefficients is taken into account in the jump matrices (cf. Sec. 4):

$$c(z,t)=c(z,0)\,e^{-i(A_r^4/z^2)t}\,,\qquad \bar c(z,t)=\bar c(z,0)\,e^{i(A_r^4/z^2)t}\,, \tag{5.36a}$$

$$d(z,t)=d(z,0)\,e^{i(z^2-2A_r^2)t}\,,\qquad \bar d(z,t)=\bar d(z,0)\,e^{i(2A_r^2-z^2)t}\,, \tag{5.36b}$$

$$r(z,t)=r(z,0)\,e^{4ik^2(z)t}\,,\qquad \bar r(z,t)=\bar r(z,0)\,e^{-4ik^2(z)t}\,, \tag{5.36c}$$

the potential is then reconstructed by the large k expansion of the matrix $\tilde M(x,z,t)$:

$$\tilde M_o(x,z,t)=\frac{i}{z}Q(x,t)\sigma_3+o\left(1/z\right)\,, \tag{5.37}$$

and by the asymptotic behavior as $z\to 0$ of $\tilde M(x,z)$:

$$\tilde M_o(x,z,t)=-\frac{iz}{A_r^2}Q(x,t)\sigma_3+o(z)\,. \tag{5.38}$$

The RH problems from the right and from the left formulated in terms of the uniformization variable z provide an alternative, and possibly more advantageous set-up for the investigation of the long-time asymptotic behavior of NLS solutions with one-sided nontrivial boundary conditions via the nonlinear steepest descent method (see for instance [**14**]).

6. Conclusions

We have developed the IST for the focusing NLS equation with a (one-sided) nonzero boundary condition as $x\to+\infty$. Such kind of boundary conditions are obviously outside the class considered in [**12**], where the amplitudes of the background field are taken to be the same at both space infinities. One should notice, though, that unlike the case of fully asymmetric boundary conditions, i.e., when the amplitudes of the NLS solutions as $x\to\pm\infty$ are different, and similarly to what happens in the same-amplitude case, here one can still introduce a uniformization variable that allows mapping the multiply sheeted Riemann surface for the scattering parameter into a single complex plane. Nonetheless, important differences with respect to the same-amplitude case still arise both in the direct and in the inverse problem. In particular, in addition to solitons (corresponding to the discrete eigenvalues of the scattering problem), and to radiation (corresponding to the continuous spectrum of the scattering operator, and represented in the inverse problem by the reflection coefficients for $k \in \mathbb{R}$), one also has a nontrivial contribution from the transmission coefficients for $k \in \Sigma_r$, as shown by the last term in (3.26), contributing to the left Marchenko equations. Correspondingly, (2.38a) and (2.38b) show that in the right Marchenko equations one always has a nontrivial contribution from the integral terms in (3.9) and (3.12), since $\rho(k)$ [resp. $\bar{\rho}(k)$] cannot vanish for $k \in \Sigma_r^+$ [resp. $k \in \Sigma_r^-$]. In particular, this implies that no pure soliton solutions exist, and solitons are always accompanied by a radiative contribution of some sort. As a consequence, unlike the symmetric case, here no explicit solution can be obtained by simply reducing the inverse problem to a set of algebraic equations.

The results presented in this paper will pave the way for the investigation of the long-time asymptotic behavior of fairly general NLS solutions with nontrivial boundary conditions via the nonlinear steepest descent method, in analogy to what was done, for instance, in [**17**] for the modified KdV equation, or in [**14**] for the focusing NLS with initial condition $q(x,0) = Ae^{i\mu|x|}$, with A, μ positive constants. Moreover, the Marchenko integral equations obtained here will provide an alternative setup for the study of the long-time behavior of the solutions by means of matched asymptotics, as was recently done for KdV in [**1**].

The study of the long-time asymptotics, as well as the derivation of solutions describing solitons superimposed to small radiation, will be the subject of future investigation.

Acknowledgments. The authors would like to acknowledge Mark Ablowitz, Gino Biondini, Francesco Demontis and Cornelis van der Mee for many valuable discussions. Also, we are grateful to the anonymous referee for a large number of useful comments that have help us improve the manuscript. This research is partially supported by NSF, under grant No. DMS-1311883, by INFN, under INFN IS-CSN4 "Mathematical Methods of Nonlinear Physics", and by MIUR, under 2011 PRIN "Geometric and analytic theory of Hamiltonian systems in finite and infinite dimensions".

References

[1] M. J. Ablowitz and D. E. Baldwin, *Interactions and asymptotics of dispersive shock waves—Korteweg-de Vries equation*, Phys. Lett. A **377** (2013), no. 7, 555–559, DOI 10.1016/j.physleta.2012.12.040. MR3019830

[2] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Studies in Appl. Math. **53** (1974), no. 4, 249–315. MR0450815 (56 #9108)

[3] M. J. Ablowitz, B. Prinari, and A. D. Trubatch, *Discrete and continuous nonlinear Schrödinger systems*, London Mathematical Society Lecture Note Series, vol. 302, Cambridge University Press, Cambridge, 2004. MR2040621 (2005c:37117)

[4] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, SIAM Studies in Applied Mathematics, vol. 4, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1981. MR642018 (84a:35251)

[5] M. J. Ablowitz and H. Segur, *Asymptotic solutions of the Korteweg-deVries equation*, Studies in Appl. Math. **57** (1976/77), no. 1, 13–44. MR0481656 (58 #1757)

[6] N.N. Akhmediev, V.M. Eleonskii, and N.E. Kulagin, *Generation of a periodic sequence of picosecond pulses in an optical fiber. Exact solutions*, Sov. Phys. JETP **89** , 1542-1551 (1985) [in Russian].

[7] N.N. Akhmediev, A. Ankiewicz and M. Taki, *Waves that appear from nowhere and disappear without a trace*, Phys. Lett. A, **373**, 675–678 (2009). doi:10.1016/j.physleta.2008.12.036

[8] N.N. Akhmediev, A. Ankiewicz and J.M. Soto-Crespo, *Rogue waves and rational solutions of the nonlinear Schrödinger equation*, Phys. Rev. E, **80**, 026601 (2009). DOI: http://dx.doi.org/10.1103/PhysRevE.80.026601

[9] N. N. Akhmediev and V. I. Korneev, *Modulation instability and periodic solutions of the nonlinear Schrödinger equation* (Russian, with English summary), Teoret. Mat. Fiz. **69** (1986), no. 2, 189–194. MR884491 (88b:35165)

[10] T.B. Benjamin, *Instability of periodic wavetrains in nonlinear dispersive systems*, Proc. Roy. Soc. A **299** 59–75 (1967). DOI: 10.1098/rspa.1967.0123

[11] T.B. Benjamin and J.E. Feir, *The disintegration of wavetrains in deep water. Part I*, J. Fluid Mech. **27**, 417–430 (1967). DOI: http://dx.doi.org/10.1017/S002211206700045X

[12] G. Biondini and G. Kovačič, *Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions*, J. Math. Phys. **55** (2014), no. 3, 031506, 22, DOI 10.1063/1.4868483. MR3221240

[13] A. Boutet de Monvel, V. P. Kotlyarov, and D. Shepelsky, *Focusing NLS equation: long-time dynamics of step-like initial data*, Int. Math. Res. Not. IMRN **7** (2011), 1613–1653. MR2806517 (2012f:35498)

[14] R. Buckingham and S. Venakides, *Long-time asymptotics of the nonlinear Schrödinger equation shock problem*, Comm. Pure Appl. Math. **60** (2007), no. 9, 1349–1414, DOI 10.1002/cpa.20179. MR2337507 (2008j:35167)

[15] F. Calogero and A. Degasperis, *Spectral transform and solitons. Vol. I*, Studies in Mathematics and its Applications, vol. 13, North-Holland Publishing Co., Amsterdam-New York, 1982. Tools to solve and investigate nonlinear evolution equations; Lecture Notes in Computer Science, 144. MR680040 (84f:35121)

[16] M. Chen, M.A. Tsankov, J.M. Nash, and C.E. Patton, *Backward-volume-wave microwave-envelope solitons in yttrium iron garnet films*, Phys. Rev. B **49**, 12773–12790 (1994). DOI: http://dx.doi.org/10.1103/PhysRevB.49.12773

[17] P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation*, Ann. of Math. (2) **137** (1993), no. 2, 295–368, DOI 10.2307/2946540. MR1207209 (94d:35143)

[18] F. Demontis, B. Prinari, C. van der Mee, and F. Vitale, *The inverse scattering transform for the defocusing nonlinear Schrödinger equations with nonzero boundary conditions*, Stud. Appl. Math. **131** (2013), no. 1, 1–40, DOI 10.1111/j.1467-9590.2012.00572.x. MR3081238

[19] F. Demontis, B. Prinari, C. van der Mee, and F. Vitale, *The inverse scattering transform for the defocusing nonlinear Schrödinger equations with nonzero boundary conditions*, Stud. Appl. Math. **131** (2013), no. 1, 1–40, DOI 10.1111/j.1467-9590.2012.00572.x. MR3081238

[20] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1987. Translated from the Russian by A. G. Reyman [A. G. Reĭman]. MR905674 (89m:58103)

[21] J. Garnier and K. Kalimeris, *Inverse scattering perturbation theory for the nonlinear Schrödinger equation with non-vanishing background*, J. Phys. A **45** (2012), no. 3, 035202, 13, DOI 10.1088/1751-8113/45/3/035202. MR2871428

[22] A. Hasegawa, and F. Tappert, *Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion* and *II. Normal dispersion*, App. Phys. Lett. **23**(3) 142–144 and 171–172 (1973). http://dx.doi.org/10.1063/1.1654836

[23] A. R. It·s, A. V. Rybin, and M. A. Sall′, *On the exact integration of the nonlinear Schrödinger equation* (Russian, with English summary), Teoret. Mat. Fiz. **74** (1988), no. 1, 29–45, DOI 10.1007/BF01018207; English transl., Theoret. and Math. Phys. **74** (1988), no. 1, 20–32. MR940459 (89h:35295)

[24] E.A. Kuznetsov, *Solitons in a parametrically unstable plasma*, Sov. Phys. Dokl. (Engl. Transl.) 22, 507–508 (1977).

[25] Y. C. Ma, *The perturbed plane-wave solutions of the cubic Schrödinger equation*, Stud. Appl. Math. **60** (1979), no. 1, 43–58. MR517602 (80e:81012)

[26] D. Mihalache, F. Lederer and D-M Baboiu, *Two-parameter family of exact solutions of the nonlinear Schrödinger equation describing optical soliton propagation*, Phys. Rev. A **47**, 3285-3290 (1993). http://dx.doi.org/10.1103/PhysRevA.47.3285

[27] S. Novikov, S. V. Manakov, L. P. Pitaevskiĭ, and V. E. Zakharov, *Theory of solitons: The inverse scattering method*, Contemporary Soviet Mathematics, Consultants Bureau [Plenum], New York, 1984. Translated from the Russian. MR779467 (86k:35142)

[28] M. Onorato, A.R. Osborne and M. Serio, *Modulational instability in crossing sea states: A possible mechanism for the formation of freak waves*, Phys. Rev. Lett. **96**, 014503 (2006). http://dx.doi.org/10.1103/PhysRevLett.96.014503

[29] C.J. Pethick, and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, Cambridge University Press, Cambridge, 2002.

[30] D. H. Peregrine, *Water waves, nonlinear Schrödinger equations and their solutions*, J. Austral. Math. Soc. Ser. B **25** (1983), no. 1, 16–43, DOI 10.1017/S0334270000003891. MR702914 (84k:76033)

[31] B. Prinari, M. J. Ablowitz, and G. Biondini, *Inverse scattering transform for the vector nonlinear Schrödinger equation with nonvanishing boundary conditions*, J. Math. Phys. **47** (2006), no. 6, 063508, 33, DOI 10.1063/1.2209169. MR2239983 (2007f:37118)

[32] D.R. Solli, C. Ropers, P. Koonath and B. Jalali, *Optical rogue waves*, Nature **450**, 1054–1057 (2007). doi:10.1038/nature06402

[33] M. Tajiri and Y. Watanabe, *Breather solutions to the focusing nonlinear Schrödinger equation*, Phys. Rev. E (3) **57** (1998), no. 3, 3510–3519, DOI 10.1103/PhysRevE.57.3510. MR1611665 (98k:35183)

[34] V. E. Zakharov, *Hamilton formalism for hydrodynamic plasma models* (Russian, with English summary), Ž. Èksper. Teoret. Fiz. **60** (1971), 1714–1726; English transl., Soviet Physics JETP **33** (1971), 927–932. MR0312857 (47 #1412)

[35] V.E. Zakharov and A.A. Gelash, *Soliton on unstable condensate*, arXiv:nlin.si 1109.0620 (2011). arXiv:1109.0620 [nlin.SI]

[36] V.E. Zakharov and A.A. Gelash, *Nonlinear stage of modulational instability*, Phys. Rev. Lett. **111**, 054101 (2013). http://dx.doi.org/10.1103/PhysRevLett.111.054101

[37] V. E. Zakharov and L. A. Ostrovsky, *Modulation instability: The beginning*, Phys. D **238** (2009), no. 5, 540–548, DOI 10.1016/j.physd.2008.12.002. MR2591296 (2010h:35037)

[38] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media* (Russian, with English summary), Ž. Èksper. Teoret. Fiz. **61** (1971), no. 1, 118–134; English transl., Soviet Physics JETP **34** (1972), no. 1, 62–69. MR0406174 (53 #9966)

[39] V.E. Zakharov, and A.B. Shabat, *Interaction between solitons in a stable medium*, Sov. Phys. JETP **37**, 823–828 (1973).

[40] A.K. Zvezdin, and A.F. Popkov, *Contribution to the nonlinear theory of magnetostatic spin waves*, Sov. Phys. JETP **57**, 350–355 (1983).

Department of Mathematics, University of Colorado Colorado Springs, Colorado Springs - CO, USA, and Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento and Sezione INFN, Lecce - Italy

E-mail address: `bprinari@uccs.edu`

Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento and Sezione INFN, Lecce - Italy

E-mail address: `federica.vitale@le.infn.it`