N-wave soliton solution on a generic background for KPI equation

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We try to generalize the IST for KPI equation to the case of potentials with "ray" type behavior, that is non-decaying along a finite number of directions in the plane. We present here the special but rather wide subclass of such potentials obtained by applying recursively N binary Bäcklund transformations to a decaying potential. we start with a regular rapidly decaying potential for which all elements of the direct and inverse problem are given. We introduce an exact recursion procedure for an arbitrary number of binary Bäcklund transformations and corresponding Darboux transformations for Jost solutions and solutions of the discrete spectrum. We show that Jost solutions obey modified integral equations and present their analytical properties . We formulate conditions of reality and regularity of the potentials constructed by these means and derive spectral data of the transformed Jost solutions. Finally we solve the recursion procedure getting a solution which describes N solitons superimposed to a generic background.

INTRODUCTION

We are interested in generalizing the direct and inverse scattering transform for the nonstationary Schrödinger operator

$$L = i\partial_{x_2} + \partial_{x_1}^2 - u(x_1, x_2) \tag{1}$$

to the case of real potential with "ray" type behavior, that is rapidly decaying on the plane with the exception of some finite number of directions, where it has finite and nontrivial limits

$$u_{n,\pm}(x_1) = \lim_{x_2\to\pm\infty} u(x_1 - 2\mu_n x_2, x_2), \qquad n = 1, \ldots, N \tag{2}$$

for N real constants μ_n.

Spectral theory of operator L with potentials of this class is interesting *per se* and because it is associated to the Kadomtsev-Petviashvili equation in its version called KPI

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = 3u_{x_2x_2}. \tag{3}$$

In the case of "ray" type potentials spectral theory is essentially more involved than the standard case of rapidly decaying potentials.

In the standard case one defines the Jost solution $\Phi(x,k)$ as the solution of Eq. (1) that is analytic in the complex plane of spectral parameter k, $k_\Im \neq 0$, and normalized at infinity by the condition that for

$$\chi(x,k) = e^{ikx_1 + ik^2 x_2}\Phi(x,k), \tag{4}$$

one has

$$\lim_{k\to\infty,\ k_\Im\neq 0} \chi(x,k) = 1. \tag{5}$$

This function can be given as the solution of the integral equation

$$\chi(x,k) = 1 + \int dx' G_0(x-x',k)u(x')\chi(x',k), \tag{6}$$

where $G_0(x,k)$ is the Green's function of spectral problem (1).

$$G_0(x,k) = \frac{\mathrm{sign} x_2}{2\pi i}\int d\alpha \theta(\alpha k_\Im x_2) e^{i\alpha x_1 - i\alpha(\alpha - 2k)x_2}. \tag{7}$$

The Jost solution is an analytic function of k in the complex plane, $k_\Im \neq 0$ with finite limits on the real axis

$$\Phi^{\pm}(x,k) = lim_{k_\Im\to\pm 0}\Phi(x,k). \tag{8}$$

The following definition of spectral data

$$\mathcal{F}(k,p) = \frac{1}{2\pi}\int dx_1' \overline{\Phi^+(x_1',x_2,k)}\Phi^+(x_1',x_2,p) - \delta(p-k) \tag{9}$$

allows to formulate the inverse problem as the nonlocal Riemann-Hilbert problem of construction of the function $\Phi(x,k)$ analytic in upper and bottom half planes with proper normalization and discontinuity at the real axis given by

$$\Phi^+(x,k) = \Phi^-(x,k) + \int dp \Phi^-(x,p)\mathcal{F}(p,k). \tag{10}$$

However, the standard integral equation cannot be applied in the case of a potential $u(x)$ not vanishing in all directions at large distances as the Green's function is slowly decaying at space infinity.

We suggest [1] the following modification of this integral equation:

$$\chi(x,k) = 1 + \int\limits_{-k_\Im\infty}^{x_1} dy_1 \int dx' \partial_{y_1} G_0(y_1 - x_1', x_2 - x_2', k)u(x')\chi(x',k) \tag{11}$$

where the order of operations is explicitly prescribed. Applying for instance modified integral equation to the simplest case of a potential of type (2), i.e. to the case $u(x) = u(x_1)$ one recovers the standard one dimensional equation for the Jost solution. The full description of the solutions of the modified integral equation with potentials of the class (2) is absent till now. Up to now only multiple pure soliton solutions were constructed.

Potentials of "ray" type present new interesting features. For instance, it was shown [1] that the modified integral equation may have solutions with completely unexpected spectral properties, namely an additional cut in the plane of complex spectral parameter. For this reason, we are studying here the special but rather wide subclass of ray potentials that is obtained by applying recursively the so-called binary Bäcklund transformations [3], with complex parametr, to a decaying potential. as we are interested in the spectral properties of potential u, we study also the corresponding Darboux transformations furnishing the Jost solutions of the transformed potentials and their analytical proerties, as well as the transformation of spectral data.

Binary Bäcklund Transformations

Let u be a rapidly decaying potential. A new potential u' and the corresponding nonstationary Schödinger operator L' can be generated using a gauge operator B such that

$$L'B = BL. \tag{12}$$

The particularly simple gauge of the form

$$B = \partial_{x_1} - (\partial_{x_1} \log \varphi(x)), \tag{13}$$

where φ is a solution of the original spectral problem for potential u,

$$\left(i\partial_{x_2} + \partial_{x_1}^2 - u(x)\right)\varphi(x) = 0 \tag{14}$$

is called an elementary Bäcklund gauge and the corresponding potential is

$$u'(x) = u(x) - 2\partial_{x_1}^2 \log \varphi(x). \tag{15}$$

It is convenient to choose $\varphi(x) = \Phi(x, \lambda)$ where $\Phi(x, \lambda)$ is the Jost solution of the original spectral problem computed at $k = \lambda$ ($\lambda_\Im \neq 0$).

One can easily check that, given ϕ and ψ solutions of the original spectral problem and of its dual, respectively, then ϕ' and ψ' defined by

$$\phi'(x,k) = B\phi(x,k), \tag{16}$$
$$B_d\psi'(x,k) = \psi(x,k) \tag{17}$$

solve the dual equations

$$L'\phi' = 0, \tag{18}$$
$$B_d L'_d \psi' = 0 \tag{19}$$

where

$$B_d = \partial_{x_1} + (\partial_{x_1} \log \varphi) \tag{20}$$

and

$$L'_d = -i\partial_{x_2} + \partial^2_{x_1} - u'. \tag{21}$$

The condition that both potentials u and u' are real reduces drastically the class of potentials to which the elementary BT can be applied, indeed only one dimensional potentials are allowed. In order to get a truly bidimensional potential we have to consider an elementary BT and its dual with different parameters and compose them getting a so-called binary Bäcklund transformation. So we consider

$$\widetilde{L}_d = -i\partial_{x_2} + \partial^2_{x_1} - \widetilde{u} \tag{22}$$

given by

$$\widetilde{L}_d\widetilde{B}_d = \widetilde{B}_d L'_d \tag{23}$$

with the gauge

$$\widetilde{B}_d = \partial_{x_1} + (\partial_{x_1}\log\widetilde{\varphi}). \tag{24}$$

The choice $\widetilde{\varphi}^{-1}(x) = \psi'(x, \lambda')$, gives a transformed potential

$$\widetilde{u}(x) = u(x) - 2\partial^2_{x_1}\log\Delta(x) \tag{25}$$

where

$$\Delta(x) = C + \int_{(\lambda'_\Im - \lambda_\Im)\infty}^{x_1} dx'_1 \Psi(x'_1, x_2, \lambda')\Phi(x'_1, x_2, \lambda). \tag{26}$$

If we choose $\lambda' = \overline{\lambda}$ and use that due to reality of potential u Jost solutions obey $\overline{\Psi(x, \overline{k})} = \Phi(x, k)$, we get

$$\Delta(x) = C + \int_{-\lambda_\Im\infty}^{x_1} dx'_1 \left|\Phi(x'_1, x_2, \lambda)\right|^2 \tag{27}$$

and so we see that in order to get a real and regular potential $\widetilde{u}$, we have only to impose the following conditions

$$C = \overline{C}, \quad \lambda_\Im C > 0 \tag{28}$$

The Darboux transform of ϕ', $\widetilde{B}\widetilde{\phi} = \phi'$, or explicitly

$$\widetilde{\phi}(x, k) = \Phi(x, k) - \frac{\Phi(x, \lambda)}{\Delta(x)}\left[\widetilde{C}(k) + \int_{-(k_\Im + \lambda_\Im)\infty}^{x_1} dx'_1 \overline{\Phi(x'_1, x_2, \overline{\lambda})}\Phi(x'_1, x_2, k)\right] \tag{29}$$

will furnish a solution of $\left(i\partial_{x_2} + \partial^2_{x_1} - \widetilde{u}\right)\widetilde{\phi} = 0$ in terms of the Jost solution of the original spectral problem and the expression of the dual Jost solution can be omitted since it can be reconstructed from $\widetilde{\phi}$. This transformation for complex parameter λ supplies the simplest nontrivial example of "ray" type potential: one soliton superimposed to a generic background.

Recursion Procedure

Applying recursively the previous procedure we can compose an arbitrary number of binary Bäcklund transformations. Indeed, if $\phi_n(x,k)$ solves

$$i\partial_{x_2}\phi_n(x,k) + \partial^2_{x_1}\phi_n(x,k) = u_n(x)\phi_n(x,k), \tag{30}$$

we specify the equation for ϕ_{n+1} in the following way:

$$\begin{aligned}\phi_{n+1}(x,k) &= \phi_n(x,k) - \frac{\phi_n(x,\lambda_{n+1})}{\Delta_{n+1}(x)} \times \\ &\times \left[B'_{n+1}(k) + \int_{-(k_\Im+\lambda_{n+1\Im})\infty}^{x_1} dx'_1\, \overline{\phi_n(x'_1,x_2,\lambda_{n+1})}\phi_n(x'_1,x_2,k)\right]\end{aligned}$$

where

$$\Delta_{n+1}(x) = c_{n+1} + \int_{-\lambda_{n+1\Im}\infty}^{x_1} dx'_1 \left|\phi_n(x'_1,x_2,\lambda_{n+1})\right|^2 \tag{31}$$

and c_{n+1} and $B'_{n+1}(k)$ are some x-independent constant and function of k, respectively. This ϕ_{n+1} has to be a solution of the spectral problem with potential

$$u_{n+1}(x) = u_n(x) - 2\partial^2_{x_1}\log\Delta_{n+1}(x). \tag{32}$$

It is convenient to write all ϕ_n as sums of two solutions $\phi_n(x,k) = F_n(x,k) + f_n(x,k)$ that are given by the recursion relations

$$\begin{aligned}F_{n+1}(x,k) &= F_n(x,k) - \frac{\phi_{n+1}(x,\lambda_{n+1})}{\Delta_{n+1}(x)} \times \\ &\int\limits_{-(k_\Im+\lambda_{n+1\Im})\infty}^{x_1} dx'_1\overline{\phi_n(x'_1,x_2,\lambda_{n+1})}F_n(x'_1,x_2,k)\end{aligned} \tag{33}$$

$$\begin{aligned}f_{n+1}(x,k) &= f_n(x,k) - \frac{\phi_{n+1}(x,\lambda_{n+1})}{\Delta_{n+1}(x)} \times \\ &\left[B_{n+1}(k) + \int\limits_{-\lambda_{n+1\Im}\infty}^{x_1} dx'_1\overline{\phi_n(x'_1,x_2,\lambda_{n+1})}f_n(x'_1,x_2,k)\right].\end{aligned} \tag{34}$$

We also put $u_0(x) = u(x)$, $F_0(x,k) = \Phi(x,k)$, $f_0(x,k) = 0$, so that we start with the standard, real, regular, and rapidly decaying at space infinity potential u.

Properties of all these objects are given in the following

Theorem Let the potential $u(x)$ be real and rapidly decaying at space infinity and let $\Phi(x,k)$ be the corresponding Jost solution. Let also the sets of complex constants λ_1, λ_2, ... and real nonzero constants c_1, c_2, ... obey the following conditions:

$$\lambda_{n\Im} \neq 0, \qquad \lambda_{n\Im}c_n > 0 \qquad n = 1,2,\ldots, \tag{35}$$

$$|\lambda_{1\Im}| > |\lambda_{2\Im}| > \ldots, \tag{36}$$

and let be given some functions $B_1(k)$, $B_2(k)$, ... of the complex parameter k. Then one has the following results:

1. Integrals in (31), (34), and (34) converge and define regular functions of x and k, for $k_{\Im} \neq 0$, $k_{\Im} + \lambda_{j\Im} \neq 0$ $(j = 1, 2, \ldots, n)$. Functions $\Delta_{n+1}(x)$ have no zeroes.
2. Functions $F_n(x,k)$, $f_n(x,k)$, and $\phi_n(x,k)$ are solutions of the nonstationary Schrödinger equation with potential

$$u_n(x) = u(x) - 2\partial_{x_1}^2 \log \prod_{j=1}^{n} \Delta_j(x). \tag{37}$$

3. There exist limits

$$\lim_{x_1 \to \pm k_{\Im}\infty} e^{ikx_1 + ik^2 x_2} F_n(x,k) = A_n(\pm, k). \tag{38}$$

They are independent on x_2 and obey the recursion relation

$$\begin{aligned} A_0(\pm,k) &= 1, \\ A_{n+1}(\pm,k) &= A_n(\pm,k) \times \left[1 + \theta(\pm k_{\Im}\lambda_{n+1\Im}) \frac{2i\lambda_{n+1\Im}}{\overline{\lambda}_{n+1} - k}\right] \end{aligned} \tag{39}$$

4. Functions $\Phi_n(x,k)$ defined as

$$\Phi_n(x,k) = \frac{F_n(x,k)}{A_n(-,k)}, \qquad n = 1, 2, \ldots, \tag{40}$$

are the Jost solutions of the spectral problem with potential u_n, i.e. functions

$$\chi_n(x,k) = e^{ikx_1 + ik^2 x_2} \Phi_n(x,k) \tag{41}$$

obey the modified integral equation with potential $u_n(x)$.

Moreover one has the following

Corollary $F_n(x,k)$ is a sectionally meromorphic function of k with discontinuity at the real axis and poles at points $k = \overline{\lambda}_j$, $j = 1, \ldots, n$.

Corollary Because of (39)

$$A_n(\pm,k) = \prod_{j=1}^{n} \left(\frac{k - \lambda_j}{k - \overline{\lambda}_j}\right)^{\theta(\pm k_{\Im}\lambda_{j\Im})}. \tag{42}$$

Thus$A_n(-,k)$ is a meromorphic function of k discontinuous at $k_{\Im} = 0$ and with simple poles at $k = \bar{\lambda}_j$, $j = 1, \ldots, n$.

Corollary $\Phi_n(x,k)$ is an analytic function in the complex plane of k with possible discontinuity at the real axis. For limiting values of the Jost solution at the real axis we have relation

$$\frac{\Phi_n^+(x,k)}{a_n^+(k)} = \Phi_n^-(x,k) + \int dp\, \Phi_n^-(x,p)\mathcal{F}_n(p,k) \tag{43}$$

where

$$a_n(k) = \frac{A_n(+,k)}{A_n(-,k)} \tag{44}$$

and continuous part of the spectral data is given by

$$\mathcal{F}_n(k,p) = \mathcal{F}(k,p)\prod_{j=1}^{n}\left(\frac{(k-\lambda_j)(p-\overline{\lambda}_j)}{(k-\overline{\lambda}_j)(p-\lambda_j)}\right)^{\theta(\lambda_{j\Im})} \quad . \tag{45}$$

Resolution of the recursion relations

The recursion procedure formulated above is solved in terms of the original potential $u(x)$ and its Jost solution $\Phi(x,k)$. In this way we get a solution depending on $n(n+1)$ complex parameters describing n solitons superimposed to a generic background. Let us introduce for $l,m=1,2,\ldots,n$

$$B_{l,m}(x) = \int\limits_{-(\lambda_{l\Im}+\lambda_{m\Im})\infty}^{x_1} dx_1'\,\overline{\Phi(x_1',x_2,\lambda_l)}\Phi(x_1',x_2,\lambda_m), \tag{46}$$

$$\beta_l(x,k) = \int\limits_{-(k_\Im+\lambda_{l\Im})\infty}^{x_1} dx_1'\,\overline{\Phi(x_1',x_2,\lambda_l)}\Phi(x_1',x_2,k). \tag{47}$$

Let us introduce the $n\times n$ matrix $B_n(x)$

$$B_n(x) = \|B_{l,m}(x)\|_{l,m=1,\ldots,n} \tag{48}$$

and rows and column

$$\Phi(x) = (\Phi(x,\lambda_1),\ldots,\Phi(x,\lambda_n)), \tag{49}$$

$$\beta(x,k) = (\beta_1(x,k),\ldots,\beta_n(x,k))^{\mathrm{T}}, \tag{50}$$

$$\gamma(x,k) = (\gamma_1(k),\ldots,\gamma_n(k))^{\mathrm{T}}, \tag{51}$$

where superscript T means transposition and $\gamma_n(k)$ are some given functions such that matrix

$$C_n = \|c_{l,m}\|_{l,m=1,\ldots,n}, \qquad c_{l,m} = \gamma_l(\lambda_m), \tag{52}$$

is Hermitian. Let A_n denote

$$A_n(x) = C_n + B_n(x). \tag{53}$$

In order to formulate conditions of regularity of the potential $u_n(x)$ we introduce also matrices

$$C_n(\pm) = \|\{c_{l,m}|\ \pm\lambda_{l\Im} > 0,\ \pm\lambda_{m\Im} > 0\}\|_{l,m=1,\ldots,n}\ . \tag{54}$$

These matrices are Hermitian also and if all rows and columns are removed, then by definition we put $\det C_n(\pm) = 1$. We proved the following

Theorem Let be given $\lambda_1, \ldots, \lambda_n$ with $\lambda_{j\Im} \neq 0$ for $j = 1, \ldots, n$ and $|\lambda_{1\Im}| > |\lambda_{2\Im}| > \ldots$. Let us also assume that constants $c_{l,m}$ are such that matrices $C_n(\pm)$ satisfy the following condition

$$\pm C_n(\pm) > 0. \tag{55}$$

Then for any x and $n = 1, 2, \ldots$

$$\det A_0(x) = 1, \tag{56}$$

$$\det A_n(x) \prod_{l=1}^{n} \lambda_{l\Im} > 0, \tag{57}$$

and the solutions of the recursion equations are given by means of the following relations:

$$F_n(x,k) = \frac{1}{\det A_n(x)} \begin{vmatrix} A_n(x) & \beta(x,k) \\ \Phi(x) & \Phi(x,k) \end{vmatrix} \tag{58}$$

$$f_n(x,k) = \frac{1}{\det A_n(x)} \begin{vmatrix} A_n(x) & \gamma(k) \\ \Phi(x) & 0 \end{vmatrix} \tag{59}$$

$$u_n(x) = u(x) - 2\partial_{x_1}^2 \log\det A_n(x). \tag{60}$$

Corollary

$$\Phi_n(x, \lambda_m) = \sum_{l=1}^{n} d_{l,m} \Phi_n(x, \overline{\lambda}_l), \tag{61}$$

where we introduced the constants

$$d_{l,m} = 2c_{l,m}\lambda_{l\Im} \prod_{j=1, j\neq l}^{n} \left(\frac{\overline{\lambda}_l - \lambda_j}{\overline{\lambda}_l - \overline{\lambda}_j}\right)^{\theta(\lambda_{l\Im}\lambda_{j\Im})} \times \prod_{j=1}^{n} \left(\frac{\lambda_m - \overline{\lambda}_j}{\lambda_m - \lambda_j}\right)^{\theta(-\lambda_{m\Im}\lambda_{j\Im})}. \tag{62}$$

Corollary Condition $|\lambda_{1\Im}| > |\lambda_{2\Im}| > \ldots$ for potentials and Jost solutions, can be omitted. Indeed, this condition was relevant only for the proof that these formulas obey the recursion procedure, while final formulas for potential and solutions f_n and F_n are invariant under any permutation of λ_j's.

Properties of potentials

We proved that potentials obtained above are real and regular and these properties are equivalent to Hermiticity of the matrix C_n and conditions (55). In order to demonstrate that they are "ray" potentials we have to study their asymptotic behavior when x_1 is replaced with $x_1 - 2\mu x_2$ and $x_2 \to \infty$ (with fixed x_1). Real parameter μ determines the direction of asymptotics on the plane. It is easy to prove that the leading term of the potential is decaying for all $\mu \neq \lambda_{1\Re}, \ldots, \lambda_{n\Re}$. Taking into account that the original potential is rapidly decaying we have from (60) for this limit

$$\lim_{x_2 \to \pm\infty} u_n(x_1 - 2\lambda_{j\Re}x_2, x_2 = -\frac{2\lambda_{j\Im}^2}{\cosh^2(\lambda_{j\Im}x_1 + \varepsilon_{j,\pm})} \tag{63}$$

which proves that the potentials constructed by means of the binary Bäcklund transformations indeed give nontrivial examples of the class (2) since they are not decaying along directions $x_1 + 2\lambda_{j\Re}x_2$ =const $j = 1, \ldots, n$. The two rays belonging to each direction are mutually shifted and this shift has been explicitly computed [2]

$$e^{\varepsilon_{j,+} - \varepsilon_{j,-}} = \frac{\det \widehat{C}(j,+) \det C(j,-)}{\det \widehat{C}(j,-) \det C(j,+)} \times$$
$$\times \prod_{l=1,\ l\neq j}^{n} \left| \frac{|\lambda_{j\Re} - \lambda_{l\Re}| - i(\lambda_{j\Im}\mathrm{sign}(\lambda_{l\Re} - \lambda_{j\Re}) - |\lambda_{l\Im}|)}{|\lambda_{j\Re} - \lambda_{l\Re}| - i(\lambda_{j\Im}\mathrm{sign}(\lambda_{l\Re} - \lambda_{j\Re}) + |\lambda_{l\Im=}|)} \right|^2, \tag{64}$$

where matrices $\widehat{C}(j, \pm)$ are obtained from C_n by removing all rows and columns with numbers $l \neq j$ such that $\pm\lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re}) < 0$.

Note that if $\lambda_{j\Re} = \lambda_{m\Re}$ the limit of potential at large x_2 along the direction $x_1 + \lambda_{j\Re}x_2 = const$ does not exist for a generic matrix C_n (that is non diagonal) as there are oscillating terms.

In a more detailed investigation performed for the case of one soliton on a background, it was shown that the asymptotic behavior gets some rational corrections. Moreover, these corrections do not depend on smallness of background in whatever norm and depend on the signs of imaginary parts of λ_j's in a non trivial way.

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