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An integrable discretization of KdV at large times

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Abstract
An 'exact discretization' of the Schrödinger operator is considered and its direct and inverse scattering problems are solved. It is shown that a differential-difference nonlinear evolution equation depending on two arbitrary constants can be solved by using this spectral transform and that for a special choice of the constants it can be considered an integrable discretization of the KdV equation at large times. An integrable difference-difference equation is also obtained.

1. Introduction

We are interested in discrete dynamical systems that in the continuous limit give the KdV equation and that are integrable by means of the inverse scattering technique.

It is a long time since the inverse scattering method was extended to solve differential-difference and difference-difference nonlinear equations. See in particular the pioneer works of Flaschka [1], Manakov [2], Ablowitz and Ladik [3, 4] and Kac and van Moerbeke [5]. In these papers and in subsequent papers (among other discrete equations) different versions of the discrete KdV equation were introduced and studied. See [6–9], the review paper [10] and references quoted therein.

The various already considered discrete KdV equations were related to a reduced form of the Ablowitz–Ladik spectral problem or to different discretized versions of the Schrödinger spectral problem. See, in addition, for the spectral theory of these discrete Schrödinger operators [11–13] and [14, 15] for the case of cellular automata.

Here, we use the discretization of the Schrödinger equation

$$\psi_{n+2} = g_n \psi_{n+1} + \lambda \psi_n \tag{1.1}$$

that was recently obtained by Shabat [16] iterating Darboux transformations of the continuous Schrödinger equation and that, for this reason, was called an 'exact discretization'. The interested reader can find in [16] its use in the context of the so-called dressing chains and lattices with specific applications to the Painlevé transcendents.

We show that its direct and inverse scattering problems, in contrast with the other discretizations considered in the literature, closely mimic the same problems for the continuous

Schrödinger equation. More precisely, we completely solve the direct and inverse scattering problems in the case of a potential g_n satisfying

$$\sum_{n=-\infty}^{+\infty}(1+|n|)|g_{n-1}-2|<\infty. \tag{1.2}$$

Then, we associate with the discrete spectral problem (1.1) the differential-difference equation (p and q are arbitrary constants)

$$(g_n g_{n-1})_t = p\left(\frac{g_{n-1}}{g_{n+1}}-\frac{g_n}{g_{n-2}}\right)+q\,(g_n-g_{n-1})\,g_n g_{n-1} \tag{1.3}$$

and we solve its initial-value Cauchy problem in the standard way by using the inverse scattering method.

Different possible continuous limits remain to be explored. In the special case

$$p=2q \tag{1.4}$$

if we let

$$g_n = 2+h^2u(nh) \tag{1.5}$$

$$x=nh, \qquad T=h^3\frac{p}{8}t \tag{1.6}$$

in the limit $h\to 0$ we obtain the KdV equation

$$u_T=-u_{xxx}+6u_xu. \tag{1.7}$$

Therefore the equation (1.3), in the case $p=2q$, can be considered an integrable discretization at large times of the KdV equation.

Finally, we consider also a double discrete equation, i.e. discrete both in space and in time, associated with the double discrete Schrödinger equation

$$\psi_{n+2,m}=g_{n,m}\psi_{n+1,m}+\lambda\psi_{n,m}, \tag{1.8}$$

that in the limit of continuous time reduces to (1.3).

2. Differential-difference equation

In the following it is sometimes convenient to use, for any discrete function $f(n)$, the notation

$$f_i=f(n+i) \tag{2.1}$$

when n is considered generic but fixed.

Then the discrete Schrödinger equation (1.1) reads

$$\psi_2=g_0\psi_1+\lambda\psi_0. \tag{2.2}$$

We consider the auxiliary spectral problem

$$\psi_{1t}=(A_0+\lambda B_0)\,\psi_0+(C_1+\lambda D_1)\psi_1, \tag{2.3}$$

and we impose compatibility in order to determine the coefficients in the rhs and the nonlinear differential-difference evolution equation which can be solved by using the spectral transform related to (2.2).

Precisely, let us denote by E the shift operator, $Ev_n=v_{n+1}$. Then, the compatibility condition between (2.2) and (2.3), that is $E\psi_{1t}=\partial_t\psi_2$, gives

$$\begin{aligned}(A_1+\lambda B_1)\,\psi_1+\lambda\,(C_2+\lambda D_2)\,\psi_0+(C_2+\lambda D_2)\,g_0\psi_1\\ =(A_{-1}+\lambda B_{-1})\,\psi_1-(A_{-1}+\lambda B_{-1})g_{-1}\psi_0+\lambda(C_0+\lambda D_0)\psi_0\\ +g_{0t}\psi_1+(A_0+\lambda B_0)g_0\psi_0+(C_1+\lambda D_1)g_0\psi_1.\end{aligned}$$

Assuming ψ_1 and ψ_0 to be linearly independent and equating the coefficients of the different powers in λ we obtain the following relations:

$$D_2 = D_0 \tag{2.4}$$

$$C_2 - C_0 = g_0 B_0 - g_{-1} B_{-1} \tag{2.5}$$

$$A_0 g_0 = A_{-1} g_{-1} \tag{2.6}$$

$$B_1 - B_{-1} = g_0(D_1 - D_2) \tag{2.7}$$

$$g_{0t} = A_1 - A_{-1} + g_0 (C_2 - C_1) . \tag{2.8}$$

From (2.4), (2.6) and (2.7) we immediately have

$$D(n) = d \tag{2.9}$$

$$A(n)g(n) = p \tag{2.10}$$

$$B(n) = q \tag{2.11}$$

where p, q and d are constants, and then we are left with

$$C_2 - C_0 = q\,(g_0 - g_{-1}) \tag{2.12}$$

and

$$g_{0t} = p\left(\frac{1}{g_1} - \frac{1}{g_{-1}}\right) + g_0(C_2 - C_1). \tag{2.13}$$

We write (2.13) for the shifted indices

$$g_{-1t} = p\left(\frac{1}{g_0} - \frac{1}{g_{-2}}\right) + g_{-1}(C_1 - C_0) \tag{2.14}$$

and sum it to (2.13). Using (2.12) we finally obtain the searched for differential-difference equation

$$(g_0 g_{-1})_t = p\left(\frac{g_{-1}}{g_1} - \frac{g_0}{g_{-2}}\right) + q\,(g_0 - g_{-1})\,g_0 g_{-1}. \tag{2.15}$$

The coefficients of the auxiliary spectral problem are given by

$$A(n) = \frac{p}{g(n)} \tag{2.16}$$

$$B(n) = q \tag{2.17}$$

$$C(n) = q \sum_{j=1}^{\infty} (g(n-2j) - g(n-2j-1)) \tag{2.18}$$

$$D(n) = 0, \tag{2.19}$$

where for convenience we choose equal to zero the integration constant of C and D. In the following we need the limit of these coefficients as $n \to \pm\infty$. For $n \to -\infty$ we have $C(n) \to 0$, but for $n \to +\infty$ it is necessary to consider separately the cases odd n and even n. We have

$$C(2m) = q \sum_{r=-\infty}^{m-1} (g(2r) - g(2r-1)) \tag{2.20}$$

$$C(2m+1) = q \sum_{r=-\infty}^{m-1} (g(2r+1) - g(2r)) . \tag{2.21}$$

The limits

$$c_e = \lim_{m\to+\infty} C(2m) \tag{2.22}$$

$$c_d = \lim_{m\to+\infty} C(2m+1), \tag{2.23}$$

thanks to (1.2), exist and satisfy

$$c_e + c_d = 0. \tag{2.24}$$

3. Spectral transform

3.1. Direct problem

The spectral problem (1.1) by using notations

$$\lambda = -1 - k^2 \tag{3.1}$$

$$g(n) = 2 + u(n) \tag{3.2}$$

$$\psi(n;k) = (1+\mathrm{i}k)^n \chi(n;k) \tag{3.3}$$

can be more conveniently rewritten as

$$(1+\mathrm{i}k)\chi(n+2;k) - 2\chi(n+1;k) + (1-\mathrm{i}k)\chi(n;k) = u(n)\chi(n+1;k). \tag{3.4}$$

The Jost solutions can be defined via the following discrete integral equations:

$$\mu^+(n;k) = 1 - \frac{1}{2\mathrm{i}k}\sum_{j=n+1}^{+\infty}\left[1-\left(\frac{1+\mathrm{i}k}{1-\mathrm{i}k}\right)^{j-n}\right]u(j-1)\mu^+(j;k) \tag{3.5}$$

$$\mu^-(n;k) = 1 + \frac{1}{2\mathrm{i}k}\sum_{j=-\infty}^{n}\left[1-\left(\frac{1+\mathrm{i}k}{1-\mathrm{i}k}\right)^{j-n}\right]u(j-1)\mu^-(j;k). \tag{3.6}$$

We show in the next section that for a potential $u(n)$ decaying sufficiently rapidly to zero at large n the Jost solutions μ^+ and μ^- are, respectively, analytic functions of k in the upper and in the lower half-plane and have a continuous limit on the real k-axis. Then, it is natural to introduce the following spectral data (defined on the real axis $k_{\mathrm{Im}} = 0$):

$$a^\pm(k) = 1 \mp \frac{1}{2\mathrm{i}k}\sum_{j=-\infty}^{+\infty} u(j-1)\mu^\pm(j;k) \tag{3.7}$$

$$b^\pm(k) = \pm\frac{1}{2\mathrm{i}k}\sum_{j=-\infty}^{+\infty}\left(\frac{1+\mathrm{i}k}{1-\mathrm{i}k}\right)^{j} u(j-1)\mu^\pm(j;k) \tag{3.8}$$

$$\rho^\pm(k) = \frac{b^\pm(k)}{a^\pm(k)}. \tag{3.9}$$

Due to the above-mentioned properties of analyticity of $\mu^\pm$, it is clear that $a^\pm$ can be analytically extended, respectively, to the upper half-plane and to the lower half-plane of the complex spectral parameter. The function $1/a$ plays the role of transmission coefficient and ρ is the reflection coefficient.

By using the definitions of $\mu^\pm$ in (3.5) and (3.6) one can check directly that on the real k-axis, where both Jost solutions are defined,

$$\left[\frac{\mu^+(n;k)}{a^+(k)} - \mu^-(n;k)\right]\left(\frac{1+\mathrm{i}k}{1-\mathrm{i}k}\right)^n = \rho^+(k) - \frac{1}{2\mathrm{i}k}$$
$$\times\sum_{j=-\infty}^{n}\left[1-\left(\frac{1+\mathrm{i}k}{1-\mathrm{i}k}\right)^{n-j}\right]u(j-1)\left(\frac{1+\mathrm{i}k}{1-\mathrm{i}k}\right)^{j}\left[\frac{\mu^+(j;k)}{a^+(k)} - \mu^-(j;k)\right]$$

so that, assuming that the 'integral equation' (3.6) is uniquely solvable, we obtain

$$\frac{\mu^+(n;k)}{a^+(k)} = \mu^-(n;k) + \left(\frac{1-\mathrm{i}k}{1+\mathrm{i}k}\right)^n \rho^+(k)\mu^-(n;-k), \qquad k = k_{\mathrm{Re}}. \tag{3.10}$$

Analogously we obtain

$$\frac{\mu^-(n;k)}{a^-(k)} = \mu^+(n;k) + \left(\frac{1-\mathrm{i}k}{1+\mathrm{i}k}\right)^n \rho^-(k)\mu^+(n;-k), \qquad k = k_{\mathrm{Re}}. \tag{3.11}$$

Taking into account the integral equations for $\mu^\pm$, we see that

$$\lim_{n\to+\infty} \mu^+(n;k) = 1, \qquad k_{\mathrm{Im}} \geqslant 0, \tag{3.12}$$

$$\lim_{n\to-\infty} \mu^-(n;k) = 1, \qquad k_{\mathrm{Im}} \leqslant 0 \tag{3.13}$$

while for $k_{\mathrm{Im}} = 0$ we have

$$\mu^\pm(n;k) \simeq a^\pm(k) + \left(\frac{1-\mathrm{i}k}{1+\mathrm{i}k}\right)^n b^\pm(k), \qquad \text{for} \quad n \to \mp\infty. \tag{3.14}$$

Note that $\left(\frac{1-\mathrm{i}k}{1+\mathrm{i}k}\right)^n$ gives rise to an oscillating term, since $\left|\frac{1-\mathrm{i}k}{1+\mathrm{i}k}\right| = 1$ for $k = k_{\mathrm{Re}}$. Then, from the asymptotics at large n of (3.10) and (3.11) we obtain the unitarity relations ($k = k_{\mathrm{Re}}$)

$$a^+(k)a^-(k) + b^+(k)b^-(-k) = 1 \tag{3.15}$$

$$b^+(k)b^-(-k) = b^+(-k)b^-(k) \tag{3.16}$$

$$a^+(k)b^-(k) + b^+(k)a^-(-k) = 0. \tag{3.17}$$

It is convenient to also introduce

$$\nu^+(n;k) = \left(\frac{1-\mathrm{i}k}{1+\mathrm{i}k}\right)^n \mu^-(n;-k) \tag{3.18}$$

$$\nu^-(n;k) = \left(\frac{1-\mathrm{i}k}{1+\mathrm{i}k}\right)^n \mu^+(n;-k) \tag{3.19}$$

which are solutions of the spectral problem (3.4) and, up to the kinematical factor $\left(\frac{1-\mathrm{i}k}{1+\mathrm{i}k}\right)^n$, are, respectively, analytic in the upper and lower half-plane of the spectral parameter.

If one defines the discrete Wronskian as

$$W(\chi,\varphi)(n) = \left(\frac{1+\mathrm{i}k}{1-\mathrm{i}k}\right)^n [\chi(n+1;k)\varphi(n;k) - \chi(n;k)\varphi(n+1;k)] \tag{3.20}$$

it is easy to verify that for two solutions χ and φ of (3.4) the Wronskian is independent of n. Then, we can evaluate $W(\mu^+,\nu^+)$ using the asymptotic limits of $\mu^+(n;k)$ and $\mu^-(n;-k)$ for $n \to -\infty$ in (3.12)–(3.14), obtaining

$$W(\mu^+,\nu^+)(k) = \frac{2\mathrm{i}k}{1+\mathrm{i}k} a^+(k). \tag{3.21}$$

Of special interest are the zeros of $a^+(k)$, say at $k = k_m^+$ ($m = 1, 2, \ldots, N^+$), which are located in the upper half-plane and that we expect to be related to the soliton solutions of the evolution equations related to the spectral problem (3.4). From (3.21) we deduce that $\mu^+(n;k_m^+)$ and $\nu^+(n;k_m^+)$ are proportional, i.e.

$$\mu^+(n;k_m^+) = D_m^+ \nu^+(n;k_m^+). \tag{3.22}$$

The constant D_m^+ can be deduced by rewriting the integral equation (3.5) for μ^+ at $k = k_m^+$ as follows:

$$\left(\frac{1+\mathrm{i}k_m^+}{1-\mathrm{i}k_m^+}\right)^n \mu^+(n;k_m^+) = \frac{1}{2\mathrm{i}k_m^+}\sum_{j=-\infty}^{+\infty}\left(\frac{1+\mathrm{i}k_m^+}{1-\mathrm{i}k_m^+}\right)^j u(j-1)\mu^+(j;k_m^+)$$
$$-\frac{1}{2\mathrm{i}k_m^+}\sum_{j=-\infty}^{n}\left[1-\left(\frac{1-\mathrm{i}k_m^+}{1+\mathrm{i}k_m^+}\right)^{j-n}\right]u(j-1)\left(\frac{1+\mathrm{i}k_m^+}{1-\mathrm{i}k_m^+}\right)^j \mu^+(j;k_m^+).$$

Comparing with the integral equation (3.6) defining μ^- we obtain

$$D_m^+ = \frac{1}{2\mathrm{i}k_m^+}\sum_{j=-\infty}^{+\infty}\left(\frac{1+\mathrm{i}k_m^+}{1-\mathrm{i}k_m^+}\right)^j u(j-1)\mu^+(j;k_m^+). \tag{3.23}$$

Notice that the convergence of the series is guaranteed by the proportionality between $\mu^+(n;k_m^+)$ and $\nu^+(n;k_m^+)$ that have, respectively, good behaviour for $n \to +\infty$ and $-\infty$.

3.2. Neumann series for $\mu^\pm$

We consider the series

$$\mu^+(n;k) = \sum_{l=0}^{+\infty}\mu^{(l)}(n;k) \tag{3.24}$$

whose coefficients are defined through the iterative algorithm

$$\mu^{(l+1)}(n;k) = \sum_{j=n+1}^{+\infty} M(n,j;k)u(j-1)\mu^{(l)}(j;k) \tag{3.25}$$

starting at $\mu^{(0)}(n;k) = 1$, where

$$M(n,j;k) = -\frac{1}{2\mathrm{i}k}\left[1-\left(\frac{1+\mathrm{i}k}{1-\mathrm{i}k}\right)^{j-n}\right]. \tag{3.26}$$

If the series is uniformly convergent it solves the discrete integral equation (3.5) defining the Jost solution μ^+. This study is most conveniently performed following the procedure outlined in the continuous case in [17].

We have for $j \geqslant n+1$ and $k_{\mathrm{Im}} \geqslant 0$, at the same time,

$$|M(n,j;k)| \leqslant \frac{1}{2|k|}\left[1+\left|\frac{1+\mathrm{i}k}{1-\mathrm{i}k}\right|^{j-n}\right] \leqslant \frac{1}{|k|}$$
$$|M(n,j;k)| \leqslant \left|\frac{1}{1-\mathrm{i}k}\right|\sum_{r=0}^{j-n-1}\left|\frac{1+\mathrm{i}k}{1-\mathrm{i}k}\right|^r \leqslant (j-n).$$

Now setting

$$I(x) = 1 + |x|\theta(-x) \tag{3.27}$$

and noticing that for $(j-n) \geqslant 0$

$$(j-n) \leqslant I(n)I(-j) \tag{3.28}$$

we can write

$$|M(n,j;k)u(j-1)| \leqslant \alpha(n)\beta(j) \tag{3.29}$$

where

$$\alpha(n) = \begin{cases} |k|^{-1} \\ I(n) \end{cases} \tag{3.30}$$

$$\beta(j) = |u(j-1)| \begin{cases} 1 \\ I(-j). \end{cases} \tag{3.31}$$

Notice that

$$\alpha(j)\beta(j) = |u(j-1)| \begin{cases} |k|^{-1} \\ (1+|j|). \end{cases} \tag{3.32}$$

Using these bounds in (3.25) we obtain as majorant $\hat{\mu}^{(l)}$ of $|\mu^{(l)}|$ the corresponding term of a series expansion of the solution of

$$\hat{\mu}(n;k) = 1 + \sum_{j=n+1}^{+\infty} \alpha(n)\beta(j)\hat{\mu}(j;k). \tag{3.33}$$

One can prove by induction that for $l \geqslant 0$

$$\hat{\mu}^{(l+1)}(n) \leqslant \sum_{j=n+l+1}^{+\infty} \alpha(n)\beta(j) \frac{Q^l(n+1, j-1)}{l!} \tag{3.34}$$

where

$$Q(n,j) = \sum_{s=n}^{j} \alpha(s)\beta(s). \tag{3.35}$$

In fact, if (3.34) is true for l, we have

$$\hat{\mu}^{(l+1)}(n) \leqslant \sum_{j=n+1}^{+\infty} \alpha(n)\beta(j) \sum_{r=j+l}^{+\infty} \alpha(j)\beta(r) \frac{Q^{l-1}(j+1, r-1)}{(l-1)!}.$$

Changing the order of the sums we obtain

$$\hat{\mu}^{(l+1)}(n) \leqslant \sum_{r=n+l+1}^{+\infty} \alpha(n)\beta(r) \sum_{j=n+1}^{r-l} \alpha(j)\beta(j) \frac{Q^{l-1}(j+1, r-1)}{(l-1)!}$$

and then, remarking that, since $Q(j, r-1) \geqslant Q(j+1, r-1)$,

$$\begin{aligned} Q^l(j, r-1) &\geqslant Q^l(j, r-1) - Q^l(j+1, r-1) \\ &= (Q(j, r-1) - Q(j+1, r-1)) \sum_{s=0}^{l-1} Q^{l-1-s}(j, r-1) Q^s(j+1, r-1) \\ &\geqslant l\alpha(j)\beta(j) Q^{l-1}(j+1, r-1) \end{aligned}$$

we derive (3.34).

Then, we change the summation limit in (3.34) as follows:

$$\hat{\mu}^{(l+1)}(n) \leqslant \sum_{j=n}^{+\infty} \alpha(n)\beta(j) \frac{Q^l(n+1, j-1)}{l!} \tag{3.36}$$

and sum up over l, obtaining

$$\hat{\mu}(n) \leqslant 1 + \sum_{j=n}^{+\infty} \alpha(n)\beta(j) \exp\left[Q(n+1, j-1)\right] \tag{3.37}$$

and finally

$$\hat{\mu}(n) \leqslant 1 + C \sum_{j=n}^{+\infty} |u(j-1)| \begin{cases} |k|^{-1} \\ I(-j)I(n), \end{cases} \tag{3.38}$$

where

$$C = \exp\left(\sum_{j=-\infty}^{+\infty} (1+|j|)|u(j-1)|\right). \tag{3.39}$$

Thus for $k_{\mathrm{Im}} \geqslant 0$ and for a potential u satisfying

$$\sum_{j=-\infty}^{+\infty} (1+|j|)|u(j-1)| < \infty \tag{3.40}$$

the series in the rhs of (3.24) is uniformly bounded and each iterated term $\mu^{(l)}$ is absolutely bounded by the lth term of a convergent series. Since the $\mu^{(l)}(n;k)$'s are analytic functions of k for $k_{\mathrm{Im}} > 0$, we conclude that $\mu^+(n;k)$ is analytic for $k_{\mathrm{Im}} > 0$ and has a continuous limit as $k_{\mathrm{Im}} \to 0$. A quite similar analysis shows that $\mu^-(n;k)$ is analytic for $k_{\mathrm{Im}} < 0$ and has a continuous limit as $k_{\mathrm{Im}} \to 0$.

3.3. Inverse problem

Let us suppose that a^+ has simple zeros at k_j^+ and let us denote by c_j^+ the residuum of $1/a^+$, that is

$$c_j^+ = \lim_{k\to k_j^+} \frac{(k-k_j^+)}{a^+(k)} \qquad j = 1, \ldots, N^+. \tag{3.41}$$

Then we can write (3.10) in the form ($k = k_{\mathrm{Re}}$)

$$\left[\frac{\mu^+(n;k)}{a^+(k)} - \sum_{j=1}^{N^+} c_j^+ \frac{\mu^+(n;k_j^+)}{k-k_j^+}\right] - \left[\mu^-(n;k) - \sum_{j=1}^{N^+} c_j^+ \frac{\mu^+(n;k_j^+)}{k-k_j^+}\right]$$
$$= \mu^-(n;-k)\rho^+(k)\left(\frac{1-\mathrm{i}k}{1+\mathrm{i}k}\right)^n.$$

The terms on the left-hand side are the limiting values at the real axis of a sectionally holomorphic function; then, taking into account the asymptotic behaviour of $\mu^\pm$ at large k and recalling (3.22), we can use the Cauchy–Green formula and obtain for $k_{\mathrm{Im}} \geqslant 0$

$$\begin{aligned}\frac{\mu^+(n;k)}{a^+(k)} = 1 &+ \sum_{j=1}^{N^+} C_j^+ \frac{\mu^-(n;-k_j^+)}{k-k_j^+}\left(\frac{1-\mathrm{i}k_j^+}{1+\mathrm{i}k_j^+}\right)^n \\ &+ \frac{1}{2\mathrm{i}\pi}\int_{-\infty}^{+\infty} \frac{\mu^-(n;-s)\rho^+(s)}{s-k-\mathrm{i}0}\left(\frac{1-\mathrm{i}s}{1+\mathrm{i}s}\right)^n \mathrm{d}s\end{aligned} \tag{3.42}$$

and for $k_{\mathrm{Im}} \leqslant 0$

$$\begin{aligned}\mu^-(n;k) = 1 &+ \sum_{j=1}^{N^+} C_j^+ \frac{\mu^-(n;-k_j^+)}{k-k_j^+}\left(\frac{1-\mathrm{i}k_j^+}{1+\mathrm{i}k_j^+}\right)^n \\ &+ \frac{1}{2\mathrm{i}\pi}\int_{-\infty}^{+\infty} \frac{\mu^-(n;-s)\rho^+(s)}{s-k+\mathrm{i}0}\left(\frac{1-\mathrm{i}s}{1+\mathrm{i}s}\right)^n \mathrm{d}s\end{aligned} \tag{3.43}$$

where

$$C_j^+ = c_j^+ D_j^+. \tag{3.44}$$

Equations (3.42) and (3.43) allow us to reconstruct the Jost solutions, given the spectral data. In order to complete the formulation of the inverse problem, we have to reconstruct the potential u. By inserting the asymptotic behaviour of μ^- at large k

$$\mu^-(n;k) = 1 + \frac{\mu^-_{(-1)}(n)}{k} + o(k^{-1}) \tag{3.45}$$

into (3.4) we obtain

$$u(n) = \mathrm{i}\big[\mu^-_{(-1)}(n+2) - \mu^-_{(-1)}(n)\big]. \tag{3.46}$$

Therefore from (3.43) we have

$$u(n) = \mathrm{i}\sum_{j=1}^{N^+}\left[\mu^-(n+2;-k_j^+)\left(\frac{1-\mathrm{i}k_j^+}{1+\mathrm{i}k_j^+}\right)^2 - \mu^-(n;-k_j^+)\right]\left(\frac{1-\mathrm{i}k_j^+}{1+\mathrm{i}k_j^+}\right)^n C_j^+$$
$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\mu^-(n+2;-s)\left(\frac{1-\mathrm{i}s}{1+\mathrm{i}s}\right)^2 - \mu^-(n;-s)\right]\left(\frac{1-\mathrm{i}s}{1+\mathrm{i}s}\right)^n \rho^+(s)\,\mathrm{d}s. \tag{3.47}$$

3.4. Time evolution of spectral data

Now we want to find the time evolution of the spectral data. First, by using (3.1) and (3.3) we rewrite the auxiliary spectral problem in terms of χ as follows:

$$(1+\mathrm{i}k)\chi_{1t} = \big(A_0 - B_0 - k^2B_0\big)\chi_0 + (1+\mathrm{i}k)\big(C_1 - D_1 - k^2D_1\big)\chi_1. \tag{3.48}$$

Since the $\mu^\pm$ have at large n functionally different asymptotic behaviours there exists a $\chi(t,n;k)$ satisfying (3.4) and (3.48), and an $\Omega(t;k)$ such that

$$\chi(t,n;k) = \Omega(t;k)\mu^+(t,n;k), \tag{3.49}$$

that is satisfying

$$(1+\mathrm{i}k)\big(\Omega_t\Omega^{-1}\mu_1^+ + \mu_{1t}^+\big) = \big(A_0 - B_0 - k^2B_0\big)\mu_0^+ + (1+\mathrm{i}k)\big(C_1 - D_1 - k^2D_1\big)\mu_1^+. \tag{3.50}$$

We evaluate (3.50) separately for $n = 2m \to +\infty$ and for $n = 2m+1 \to +\infty$ using (3.12) and (2.16)–(2.19) and recalling that $g(n) \to 2$ as $n \to \infty$. Then, we sum up the two limits taking into account (2.24), obtaining

$$\Omega_t\Omega^{-1} = \frac{p}{2(1+\mathrm{i}k)} - q(1-\mathrm{i}k) \tag{3.51}$$

and, consequently, the following time dependence for Ω:

$$\Omega(t;k) = \Omega_0(k)\ \exp\left[\frac{p}{2(1+\mathrm{i}k)} - q(1-\mathrm{i}k)\right]t. \tag{3.52}$$

Finally, inserting (3.51) into (3.50) and taking the limit $n \to -\infty$ we derive the evolution of spectral data. Indeed, evaluating (3.50) on the real axis and taking into account (3.14) and (2.16)–(2.19) we have

$$\frac{p}{2}\left(\frac{1-\mathrm{i}k}{1+\mathrm{i}k}\right)^{n+1} b^+(k) - q\frac{(1-\mathrm{i}k)^{n+2}}{(1+\mathrm{i}k)^n}b^+(k) + (1+\mathrm{i}k)a_t^+(k)$$
$$+\frac{(1-\mathrm{i}k)^{n+1}}{(1+\mathrm{i}k)^n}b_t^+(k) = \frac{p}{2}\left(\frac{1-\mathrm{i}k}{1+\mathrm{i}k}\right)^n b^+(k) - q\frac{(1-\mathrm{i}k)^{n+1}}{(1+\mathrm{i}k)^{n-1}}b^+(k).$$

Therefore

$$a^+(t;k) = a^+(0;k) \tag{3.53}$$

$$b^+(t;k) = b^+(0;k)\exp\left[\mathrm{i}k\frac{p-2q(1+k^2)}{1+k^2}t\right]. \tag{3.54}$$

Analogously one obtains for the evolution of the normalization coefficients C_j^+

$$C_j^+(t) = C_j^+(0)\exp\left[\mathrm{i}k_j^+\frac{p-2q(1+k_j^{+2})}{1+k_j^{+2}}t\right]. \tag{3.55}$$

If $p = 2q$ the evolution for $b^+(t;k)$ and $C_j^+(t)$ reduces to

$$b^+(t;k) = b^+(0;k)\exp\left[-2\mathrm{i}q\frac{k^3}{1+k^2}t\right] \tag{3.56}$$

$$C_j^+(t) = C_j^+(0)\exp\left[-2\mathrm{i}q\frac{k_j^{+3}}{1+k_j^{+2}}t\right] \tag{3.57}$$

which is consistent with the expected time evolution of KdV in the continuous limit.

4. Difference-difference equation

We discretize now also the time and we consider the following Lax pair:

$$\psi_{20} = \lambda\psi_{00} + g_{00}\psi_{10} \tag{4.1}$$

$$\psi_{11} = (A_{00} + \lambda B_{00})\,\psi_{00} + (C_{10} + \lambda D_{10})\,\psi_{10}. \tag{4.2}$$

If we denote by E and F the shift operators $Ef_{ij} = f_{i+1j}$ and $Ff_{ij} = f_{ij+1}$ and impose on (4.1) and (4.2) the compatibility condition $E\psi_{11} = F\psi_{20}$ we obtain

$$\begin{aligned}(A_{10} + \lambda B_{10})\,\psi_{10} &+ (C_{20} + \lambda D_{20})\,[\lambda\psi_{00} + g_{00}\psi_{10}]\\
&= (A_{-10} + \lambda B_{-10})\left[\psi_{10} - g_{-10}\psi_{00}\right] + \lambda\,(C_{00} + \lambda D_{00})\,\psi_{00}\\
&\quad + g_{01}\,[(A_{00} + \lambda B_{00})\,\psi_{00} + (C_{10} + \lambda D_{10})\,\psi_{10}]\end{aligned}$$

and, then, equating the coefficients of ψ_{00}, ψ_{10} and of the different powers of λ we obtain

$$D_{20} = D_{00} \tag{4.3}$$

$$C_{20} - C_{00} = g_{01}B_{00} - g_{-10}B_{-10} \tag{4.4}$$

$$g_{01}A_{00} = g_{-10}A_{-10} \tag{4.5}$$

$$B_{10} - B_{-10} = g_{01}D_{10} - g_{00}D_{20} \tag{4.6}$$

$$-A_{10} + A_{-10} = g_{00}C_{20} - g_{01}C_{10}. \tag{4.7}$$

We choose

$$D_{ij} = 0 \tag{4.8}$$

$$B_{ij} = b \tag{4.9}$$

and, then, this system of nonlinear recursion relations can be explicitly solved. It is convenient to introduce the following notations:

$$\gamma_{00} = \frac{g_{01}}{g_{00}}\frac{g_{-11}}{g_{-10}} - 1 \tag{4.10}$$

$$G_{00} = (g_{01} - g_{-10}) \tag{4.11}$$

$$\beta_{00} = \frac{1}{g_{-20}g_{-11}}\left(\frac{g_{-20}g_{-10}}{g_{11}g_{01}} - 1\right). \tag{4.12}$$

We obtain for the coefficients of the auxiliary spectral problem

$$\tau_{00}A_{00} + b\sigma_{00} = 0 \tag{4.13}$$

$$\tau_{00}\gamma_{00}C_{00} + b\alpha_{00}\sigma_{00} = b\tau_{00}G_{00} \tag{4.14}$$

where

$$\tau_{00} = (g_{00})^2\beta_{20}\gamma_{10} - (g_{01})^2\beta_{10}\gamma_{20} \tag{4.15}$$

$$\sigma_{00} = g_{00}\gamma_{10}G_{20} - g_{01}\gamma_{20}G_{10} \tag{4.16}$$

$$\alpha_{00} = \frac{1}{g_{11}} - \frac{g_{01}}{g_{-10}g_{-20}}(1 + \gamma_{00}) \tag{4.17}$$

and the nonlinear double discrete equation for g

$$\begin{aligned} g_{00}g_{01}\big[\,(g_{-10})^2\,\beta_{10}\gamma_{00} - (g_{-11})^2\,\beta_{00}\gamma_{10}\big]G_{20} \\ +(g_{01})^2\big[\,(g_{-11})^2\,\beta_{00}\gamma_{20} - (g_{-10}/g_{01})^2\,(g_{00})^2\,\beta_{20}\gamma_{00}\big]G_{10} \\ +g_{-10}g_{-11}\big[\,(g_{00})^2\,\beta_{20}\gamma_{10} - (g_{01})^2\,\beta_{10}\gamma_{20}\big]G_{00} = 0. \end{aligned} \tag{4.18}$$

If we let

$$g(n, m) = g_n(mh) \tag{4.19}$$

$$t = mh, \tag{4.20}$$

taking into account that in the limit $h \to 0$

$$\frac{\gamma(n, m)}{h} \to \frac{(g_n g_{n-1})_t}{g_n g_{n-1}} \tag{4.21}$$

$$\beta(n, m) \to \frac{1}{g_{n+1}g_n} - \frac{1}{g_{n-1}g_{n-2}} \tag{4.22}$$

$$G(n, m) \to g_n - g_{n-1}, \tag{4.23}$$

it is easy to check that this equation in the limit of continuous time is satisfied by the differential-difference equation (1.3), which contains two arbitrary constants p and q. We expect that also in the general case the equation (4.18) could be integrated to a lower-order equation containing two arbitrary constants.

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