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Determinant and Pfaffian solutions of the strong coupling limit of integrable discrete NLS systems

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Abstract
The strong coupling limits of the integrable semi-discrete and fully discrete nonlinear Schrödinger systems are studied by using the Hirota bilinear method. The determinant solutions (in both infinite and finite lattice cases) for the strong coupling limits of semi-discrete and fully discrete nonlinear Schrödinger systems are obtained using a determinant technique. The vector generalizations of the strong coupling limits of semi-discrete and fully discrete nonlinear Schrödinger systems are also presented. The Pfaffian solutions for vector systems are obtained using the Pfaffian technique.

1. Introduction

Semi-discrete systems, i.e., systems described by differential-difference equations (discrete in space and continuous in time), or fully discrete systems with both space and time taking values on a lattice, have received considerable attention recently [1]. Especially, semi-discrete integrable systems are important from a physical point of view. One of the most important semi-discrete integrable systems is the integrable discrete nonlinear Schrödinger (IDNLS) equation

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} = u_{n+1} + u_{n-1} + a|u_n|^2(u_{n+1} + u_{n-1}) \qquad n \in \mathbb{Z}, \tag{1.1}$$

which is a semi-discrete analog of the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t = u_{xx} + 2|u|^2u. \tag{1.2}$$

The IDNLS equation was originally derived by using the inverse scattering method [2, 3], as an $O(h^2)$ finite-difference approximation of NLS, with $x = nh$ and $t \to t/h^2$, with h being the lattice spacing. Besides being used as a basis for numerical schemes for its continuous counterpart, the IDNLS equation has also numerous physical applications, related to the dynamics of anharmonic lattices [4], self-trapping on a dimer [5], Heisenberg spin chains [6, 7], etc.

The IDNLS equation can be bilinearized into

$$\mathrm{i}D_t G_n \bullet F_n = \delta(G_{n+1}F_{n-1} + F_{n+1}G_{n-1}), \tag{1.3a}$$

$$F_n^2 + aG_nG_n^* = \delta F_{n+1}F_{n-1} \tag{1.3b}$$

through the dependent variable transformation $u_n = G_n/F_n$, where D_t, called Hirota D operator, is defined by

$$D_t f \bullet g = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right) f(t)g(t')\bigg|_{t'=t}.$$

With these bilinear forms the IDNLS equation can be solved by the Hirota bilinear method. In [8] the relationship of IDNLS with the two-component Toda lattice hierarchy was established in the framework of Sato theory, developed by the Kyoto group.

In the case of strong nonlinearity, one can neglect the diffraction term in equation (1.1) and obtain the strong coupling limit of the IDNLS (sc-IDNLS) equation [9]

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} = \alpha_r |u_n|^2(u_{n+1} + u_{n-1}), \tag{1.4}$$

where $\alpha_r \in \mathbb{R}$. The sc-IDNLS equation can be generalized to

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} = \alpha_r |u_n|^2(u_{n+1} + u_{n-1}) + \mathrm{i}\alpha_i |u_n|^2(u_{n+1} - u_{n-1}), \tag{1.5}$$

(where $\alpha_r, \alpha_i \in \mathbb{R}$) which can be rewritten as

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} = |u_n|^2(\alpha u_{n+1} + \alpha^* u_{n-1}), \tag{1.6}$$

with $\alpha = \alpha_r + \mathrm{i}\alpha_i$. Here and in the following '$*$' denotes complex conjugate. Equation (1.6) is the strong coupling limit of the discrete Hirota equation [10]

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} = \alpha_r(1 + |u_n|^2)(u_{n+1} + u_{n-1}) + \mathrm{i}\alpha_i(1 + |u_n|^2)(u_{n+1} - u_{n-1}), \tag{1.7}$$

which is a discrete version of the Hirota equation [11]

$$\mathrm{i}u_t = u_{xx} + 2|u|^2u + \mathrm{i}a|u|^2u_x + \mathrm{i}bu_{xxx}. \tag{1.8}$$

Equation (1.6) can be derived from the equation of motion

$$\dot{u}_n = \{H, u_n\}, \tag{1.9}$$

where the Hamiltonian is

$$H = \sum_k (\alpha^* u_k u_{k+1}^* + \alpha u_k^* u_{k+1}), \tag{1.10}$$

with the deformed Poisson brackets

$$\{A, B\} = \mathrm{i}\sum_k |u_k|^2 \left(\frac{\partial A}{\partial u_k}\frac{\partial B}{\partial u_k^*} - \frac{\partial B}{\partial u_k}\frac{\partial A}{\partial u_k^*}\right). \tag{1.11}$$

In this paper, we study the strong coupling limits of the integrable semi-discrete and fully discrete NLS systems by using the Hirota bilinear method.

This paper is organized as follows. In section 2, the determinant solutions (in both infinite and finite lattice cases) for the strong coupling limit of the semi-discrete NLS system are obtained using a determinant technique. In section 3, the Pfaffian solutions for a vector generalization of the strong coupling limits of the semi-discrete NLS system are also presented. The Pfaffian solutions for vector systems are obtained using the Wronski-type Pfaffian technique. In section 4, the Pfaffian solutions for a vector generalization of the strong coupling limit of the fully discrete NLS system are presented.

2. Determinant solutions

After substitution of $u_n = v_n \exp(\mathrm{i}\pi n/2)$ into equation (1.6), we get

$$\frac{\mathrm{d}v_n}{\mathrm{d}t} = |v_n|^2(\alpha v_{n+1} - \alpha^* v_{n-1}). \tag{2.1}$$

If v_n and α are real, equation (2.1) is reduced to

$$\frac{\mathrm{d}v_n}{\mathrm{d}t} = \alpha v_n^2(v_{n+1} - v_{n-1}), \tag{2.2}$$

which is the discrete KdV equation [12]. Substituting $v_n = \tau_{n+1}\tau_{n-1}/\tau_n^2$ into equation (2.2), we obtain a bilinear form

$$\dot{\tau}_{n+1}\tau_n - \tau_{n+1}\dot{\tau}_n = \alpha\tau_{n+2}\tau_{n-1} - \alpha\tau_{n+1}\tau_n. \tag{2.3}$$

If $v_n = \mathrm{i} + w_n$ with real w_n and α is real, equation (2.1) is reduced to

$$\frac{\mathrm{d}w_n}{\mathrm{d}t} = \alpha\left(1 + w_n^2\right)(w_{n+1} - w_{n-1}), \tag{2.4}$$

which is the discrete mKdV equation [2, 3].

An interesting problem is represented by the case in which v_n attains complex values. We consider two kinds of solutions: soliton solutions and molecule solutions (solutions in finite lattices).

Soliton solution. Let us consider equation (1.4). We transform equation (1.4) with real α_r into the bilinear forms

$$\dot{g}_{n+1}g_n - g_{n+1}\dot{g}_n = \alpha_r g_{n+2}g_{n-1} - \alpha_r g_{n+1}g_n, \tag{2.5a}$$

$$\dot{g}_{n+1}^* g_n^* - g_{n+1}^*\dot{g}_n^* = \alpha_r g_{n+2}^* g_{n-1}^* - \alpha_r g_{n+1}^* g_n^*, \tag{2.5b}$$

through the dependent variable transformation $u_n = \frac{g_{n+1}g_{n-1}^*}{g_n g_n^*}\exp(\mathrm{i}\pi n/2)$. We can remove α_r from the bilinear forms (2.5*a*) and (2.5*b*) by rescaling $t' = \alpha_r t$, obtaining

$$\dot{g}_{n+1}g_n - g_{n+1}\dot{g}_n = g_{n+2}g_{n-1} - g_{n+1}g_n, \tag{2.6a}$$

$$\dot{g}_{n+1}^* g_n^* - g_{n+1}^*\dot{g}_n^* = g_{n+2}^* g_{n-1}^* - g_{n+1}^* g_n^*. \tag{2.6b}$$

Remark 2.1. Each of the bilinear equations (2.6*a*) and (2.6*b*) corresponds to a bilinear equation of the Volterra lattice. Thus we can construct an *N*-soliton solution of the strong coupling limit of the semi-discrete NLS system from an *N*-soliton solution of the Volterra lattice. This result was found by Narita [13].

Remark 2.2. Equation (1.6) is transformed into

$$\dot{v_n} = |v_n|^2(v_{n+1} - v_{n-1}), \tag{2.7}$$

through $u_n = B\,\mathrm{e}^{\mathrm{i}An}v_n$ where A and B satisfy $\alpha_r \cos A - \alpha_i \sin A = 0$ and $\alpha_r \sin A + \alpha_i \cos A = 1/B^2$. Using the dependent variable transformation $v_n = \frac{g_{n+1}g_{n-1}^*}{g_n g_n^*}$, we will obtain equations (2.6*a*) and (2.6*b*) [10]. Thus, the following theorem also gives an *N*-soliton solution of equation (1.6).

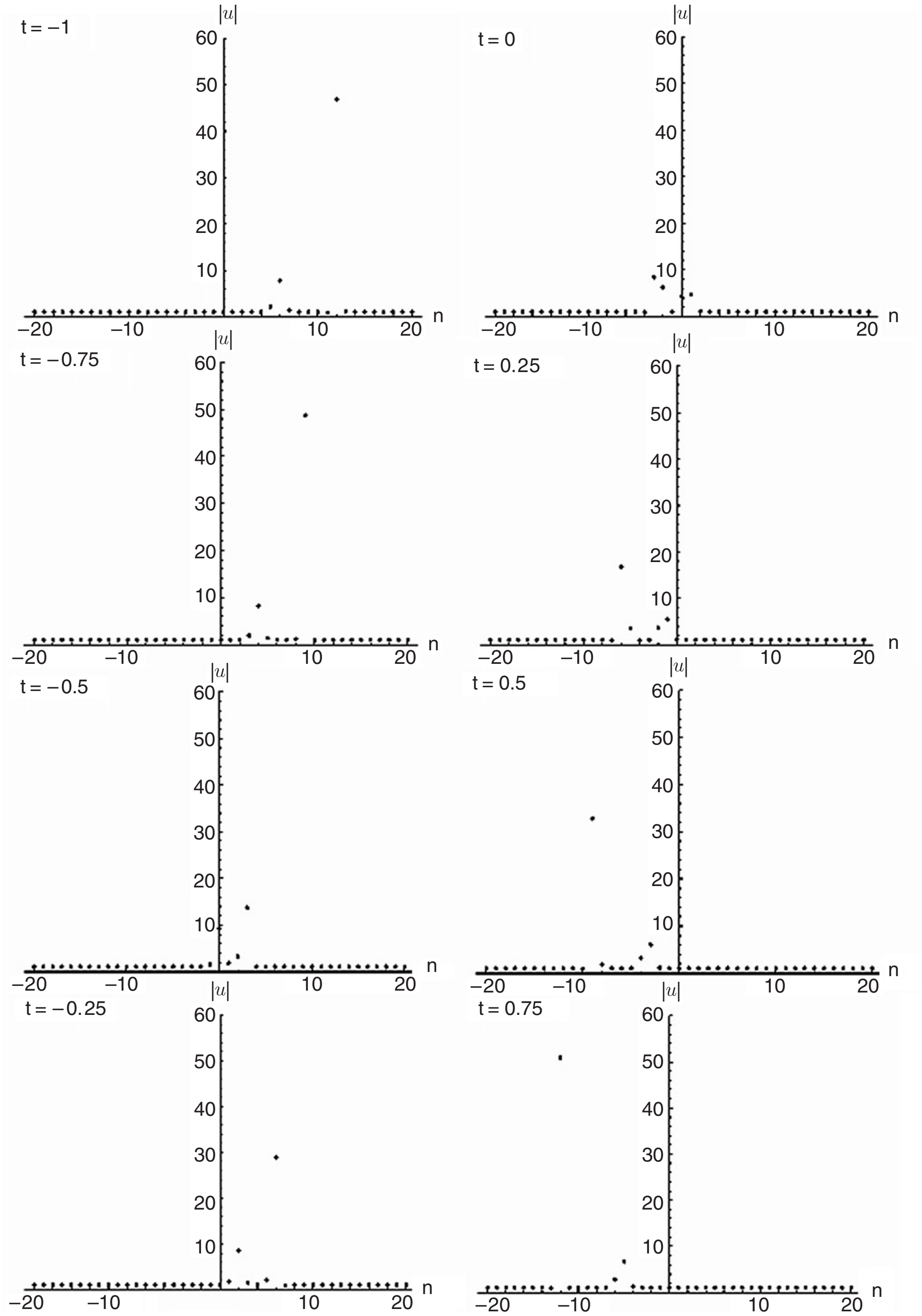

Figure 1. Time evolution of 2-soliton solution (plots of $|u|$) : $p_1 = 5, p_2 = 7, \phi_1^{(1)} = 1, \phi_1^{(0)} = 2, \phi_2^{(1)} = 5, \phi_2^{(0)} = 3, \psi_1^{(1)} = 2, \psi_1^{(0)} = 1, \psi_2^{(1)} = 3, \psi_2^{(0)} = 4$.

Theorem 2.3. *The bilinear forms (2.6a) and (2.6b) have the following determinant solution:*

$$g_n = \begin{vmatrix} \phi_1(n) & \phi_1(n+2) & \cdots & \phi_1(n+2N-2) \\ \phi_2(n) & \phi_2(n+2) & \cdots & \phi_2(n+2N-2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+2) & \cdots & \phi_N(n+2N-2) \end{vmatrix}, \tag{2.8a}$$

$$g_n^* = \begin{vmatrix} \phi_1^*(n) & \phi_1^*(n+2) & \cdots & \phi_1^*(n+2N-2) \\ \phi_2^*(n) & \phi_2^*(n+2) & \cdots & \phi_2^*(n+2N-2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^*(n) & \phi_N^*(n+2) & \cdots & \phi_N^*(n+2N-2) \end{vmatrix}, \tag{2.8b}$$

where the linear dispersion relation is

$$\partial\phi_i(n)/\partial t = \phi_i(n+2), \qquad \partial\phi_i^*(n)/\partial t = \phi_i^*(n+2), \tag{2.9}$$

and

$$\phi_j(n+1) + \phi_j(n-1) = \beta_j\phi_j(n), \qquad \phi_j^*(n+1) + \phi_j^*(n-1) = \beta_j^*\phi_j^*(n). \tag{2.10}$$

Proof. See appendix B. □

For example, the 1-soliton solution of (1.4) is given by

$$u_n = \frac{g_{n+1}g_{n-1}^*}{g_n g_n^*}\exp(\mathrm{i}\pi n/2) \tag{2.11}$$

where

$$g_n = \exp(\alpha_r p^2 t + \phi^{(1)} + \mathrm{i}\phi^{(0)})p^n + \exp\left(\frac{\alpha_r}{p^2}t + \psi^{(1)} + \mathrm{i}\psi^{(0)}\right)\left(\frac{1}{p}\right)^n \tag{2.12}$$

and g_n^* is its complex conjugate.

If we take

$$\phi_j(n) = \exp\left(\alpha_r p_j^2 t + \phi_j^{(1)} + \mathrm{i}\phi_j^{(0)}\right){p_j}^n + \exp\left(\frac{\alpha_r}{p_j^2}t + \psi_j^{(1)} + \mathrm{i}\psi_j^{(0)}\right)\left(\frac{1}{p_j}\right)^n,$$

$$\phi_j^*(n) = \exp\left(\alpha_r p_j^2 t + \phi_j^{(1)} - \mathrm{i}\phi_j^{(0)}\right){p_j}^n + \exp\left(\frac{\alpha_r}{p_j^2}t + \psi_j^{(1)} - \mathrm{i}\psi_j^{(0)}\right)\left(\frac{1}{p_j}\right)^n,$$

we have an N-soliton solution of equations (2.5a) and (2.5b). From this we obtain an N-soliton solution of the sc-IDNLS equation (1.4). Plots of a 2-soliton solution are presented as an example in figure 1. When one takes complex-valued wave numbers, soliton solutions of the sc-IDNLS equation (1.4) exhibit singularities. An interesting open problem is whether non-singular solutions with complex-valued wave numbers exist.

Molecule solution. Let us now consider equation (1.6) with $\alpha_r, \alpha_i \neq 0$. We can transform equation (2.1) into the bilinear forms

$$\dot{g}_{n+1}g_n - g_{n+1}\dot{g}_n = \alpha g_{n+2}g_{n-1}, \tag{2.13a}$$

$$\dot{g}_{n+1}^*g_n^* - g_{n+1}^*\dot{g}_n^* = \alpha^* g_{n+2}^*g_{n-1}^*, \tag{2.13b}$$

through the dependent variable transformation $v_n = \frac{g_{n+1}g_{n-1}^*}{g_n g_n^*}$.

Performing the change of variable $T = \alpha t$ and $T^* = \alpha^* t$, we obtain

$$\frac{dg_{n+1}}{dT} g_n - g_{n+1}\frac{dg_n}{dT} = g_{n+2}g_{n-1}, \tag{2.14a}$$

$$\frac{dg^*_{n+1}}{dT^*} g^*_n - g^*_{n+1}\frac{dg^*_n}{dT^*} = g^*_{n+2}g^*_{n-1}. \tag{2.14b}$$

We use the following boundary conditions for g_n and g^*_n:

$$g_{-1} = 0, \qquad g^*_{-1} = 0, \tag{2.15a}$$

$$g_0 = 1, \qquad g^*_0 = 1, \tag{2.15b}$$

$$g_1 = 1, \qquad g^*_1 = 1, \tag{2.15c}$$

$$g_2 = h(T, 0), \qquad g^*_2 = h^*(T^*, 0), \tag{2.15d}$$

where $h(T, m)$ and $h^*(T^*, m)$ are arbitrary functions of T and satisfy the equations

$$\frac{d}{dT} h(T, m) = h(T, m + 1), \qquad \frac{d}{dT^*} h^*(T^*, m) = h^*(T^*, m + 1), \tag{2.16}$$

for non-negative integers m.

We then solve equations (2.14a) and (2.14b) iteratively starting with $n = 1$. For $n = 1$, equation (2.14a) becomes

$$\frac{dg_2}{dT} g_1 - g_2\frac{dg_1}{dT} = g_3 g_0. \tag{2.17}$$

Using the boundary conditions, we obtain g_3:

$$g_3 = \frac{d}{dT} h(T, 0) = h(T, 1). \tag{2.18}$$

Equation (2.14b) for $n = 1$

$$\frac{dg^*_2}{dT^*} g_1 - g_2\frac{d{g_1}^*}{dT^*} = g^*_3 g^*_0 \tag{2.19}$$

can then be solved using the boundary conditions to obtain g^*_3:

$$\begin{aligned} g^*_3 &= \frac{d}{dT^*} h^*(T^*, 0) \\ &= h^*(T^*, 1). \end{aligned} \tag{2.20}$$

For $n = 2$, equation (2.14a) becomes

$$\frac{dg_3}{dT} g_2 - g_3\frac{dg_2}{dT} = g_4 g_1. \tag{2.21}$$

Substituting equation (2.18) into equation (2.21), we obtain

$$\begin{aligned} g_4 &= \frac{dh(T, 1)}{dT} h(T, 0) - h(T, 1)\frac{dh(T, 0)}{dT} \\ &= h(T, 2)h(T, 0) - h(T, 1)^2 \\ &= \begin{vmatrix} h(T, 0) & h(T, 1) \\ h(T, 1) & h(T, 2) \end{vmatrix}. \end{aligned} \tag{2.22}$$

Then we consider equation (2.14b) for $n = 2$

$$\frac{dg^*_3}{dT^*} g^*_2 - g^*_3\frac{dg^*_2}{dT^*} = g^*_4 g^*_1 \tag{2.23}$$

which is solved for g_4^*:

$$\begin{aligned} g_4^* &= \frac{\mathrm{d}h^*(T^*,1)}{\mathrm{d}T^*}h^*(T^*,0) - h^*(T^*,1)\frac{\mathrm{d}h^*(T^*,0)}{\mathrm{d}T^*} \\ &= h^*(T^*,2)h^*(T^*,0) - h^*(T^*,1)^2 \\ &= \begin{vmatrix} h^*(T^*,0) & h^*(T^*,1) \\ h^*(T^*,1) & h^*(T^*,2) \end{vmatrix}. \end{aligned} \tag{2.24}$$

For $n = 3$, equation (2.14*a*) becomes

$$\frac{\mathrm{d}g_4}{\mathrm{d}T}g_3 - g_4\frac{\mathrm{d}g_3}{\mathrm{d}T} = g_5 g_2. \tag{2.25}$$

Substituting equations (2.18) and (2.21) into equation (2.25), we obtain

$$\begin{aligned} g_5 &= h(T,3)h(T,1) - h(T,2)^2 \\ &= \begin{vmatrix} h(T,1) & h(T,2) \\ h(T,2) & h(T,3) \end{vmatrix}. \end{aligned} \tag{2.26}$$

Finally, we have equation (2.14*b*) for $n = 3$

$$\frac{\mathrm{d}g_4^*}{\mathrm{d}T^*}g_3^* - g_4^*\frac{\mathrm{d}g_3^*}{\mathrm{d}T^*} = g_5^* g_2^* \tag{2.27}$$

which is solved for g_5^*:

$$\begin{aligned} g_5^* &= h^*(T^*,3)h^*(T^*,1) - h^*(T^*,2)^2 \\ &= \begin{vmatrix} h^*(T^*,1) & h^*(T^*,2) \\ h^*(T^*,2) & h^*(T^*,3) \end{vmatrix}. \end{aligned} \tag{2.28}$$

Theorem 2.4. *Solutions of the bilinear forms (2.14a) and (2.14b) with the boundary conditions (2.15a), (2.15b), (2.15c) and (2.15d) are*

$$g_{2N} = \begin{vmatrix} h(T,0) & h(T,1) & \cdots & h(T,N-1) \\ h(T,1) & h(T,2) & \cdots & h(T,N) \\ \vdots & \vdots & \ddots & \vdots \\ h(T,N-1) & h(T,N) & \cdots & h(T,2N-2) \end{vmatrix}, \tag{2.29a}$$

$$g_{2N}^* = \begin{vmatrix} h^*(T^*,0) & h^*(T^*,1) & \cdots & h^*(T^*,N-1) \\ h^*(T^*,1) & h^*(T^*,2) & \cdots & h^*(T^*,N) \\ \vdots & \vdots & \ddots & \vdots \\ h^*(T^*,N-1) & h^*(T^*,N) & \cdots & h^*(T^*,2N-2) \end{vmatrix}, \tag{2.29b}$$

$$g_{2N+1} = \begin{vmatrix} h(T,1) & h(T,2) & \cdots & h(T,N) \\ h(T,2) & h(T,3) & \cdots & h(T,N+1) \\ \vdots & \vdots & \ddots & \vdots \\ h(T,N) & h(T,N+1) & \cdots & h(T,2N-1) \end{vmatrix}, \tag{2.29c}$$

$$g_{2N+1}^* = \begin{vmatrix} h^*(T^*,1) & h^*(T^*,2) & \cdots & h^*(T^*,N) \\ h^*(T^*,2) & h^*(T^*,3) & \cdots & h^*(T^*,N+1) \\ \vdots & \vdots & \ddots & \vdots \\ h^*(T^*,N) & h^*(T^*,N+1) & \cdots & h^*(T^*,2N-1) \end{vmatrix}, \tag{2.29d}$$

$$N \geqslant 2. \tag{2.29e}$$

Proof. See appendix C. □

From theorem 2.2, we obtain the solutions of equations (2.13*a*) and (2.13*b*):

$$g_{2N} = \begin{vmatrix} h(\alpha t, 0) & h(\alpha t, 1) & \cdots & h(\alpha t, N-1) \\ h(\alpha t, 1) & h(\alpha t, 2) & \cdots & h(\alpha t, N) \\ \vdots & \vdots & \ddots & \vdots \\ h(\alpha t, N-1) & h(\alpha t, N) & \cdots & h(\alpha t, 2N-2) \end{vmatrix}, \tag{2.30a}$$

$$g^*_{2N} = \begin{vmatrix} h^*(\alpha^* t, 0) & h^*(\alpha^* t, 1) & \cdots & h^*(\alpha^* t, N-1) \\ h^*(\alpha^* t, 1) & h^*(\alpha^* t, 2) & \cdots & h^*(\alpha^* t, N) \\ \vdots & \vdots & \ddots & \vdots \\ h^*(\alpha^* t, N-1) & h^*(\alpha^* t, N) & \cdots & h^*(\alpha^* t, 2N-2) \end{vmatrix}, \tag{2.30b}$$

$$g_{2N+1} = \begin{vmatrix} h(\alpha t, 1) & h(\alpha t, 2) & \cdots & h(\alpha t, N) \\ h(\alpha t, 2) & h(\alpha t, 3) & \cdots & h(\alpha t, N+1) \\ \vdots & \vdots & \ddots & \vdots \\ h(\alpha t, N) & h(\alpha t, N+1) & \cdots & h(\alpha t, 2N-1) \end{vmatrix}, \tag{2.30c}$$

$$g^*_{2N+1} = \begin{vmatrix} h^*(\alpha^* t, 1) & h^*(\alpha^* t, 2) & \cdots & h^*(\alpha^* t, N) \\ h^*(\alpha^* t, 2) & h^*(\alpha^* t, 3) & \cdots & h^*(\alpha^* t, N+1) \\ \vdots & \vdots & \ddots & \vdots \\ h^*(\alpha^* t, N) & h^*(\alpha^* t, N+1) & \cdots & h^*(\alpha^* t, 2N-1) \end{vmatrix}, \tag{2.30d}$$

with the boundary conditions

$$\begin{aligned} &g_{-1} = 0, && g^*_{-1} = 0, \\ &g_0 = 1, && g^*_0 = 1, \\ &g_1 = 1, && g^*_1 = 1, \\ &g_2 = h(\alpha t, 0), && g^*_2 = h^*(\alpha^* t, 0). \end{aligned} \tag{2.31a}$$

Thus we obtain the molecule solutions of equation (2.1) with the boundary conditions

$$v_0 = 0, \qquad v^*_0 = 0, \tag{2.32a}$$

$$v_1 = h(\alpha t, 0), \qquad v^*_1 = h^*(\alpha^* t, 0). \tag{2.32b}$$

If we take $\alpha_i = 0$, we have the molecule solution of equation (1.4).

3. Strong coupling limit of the integrable discrete vector NLS equation

3.1. Molecule solutions

In this section, we consider the strong coupling limit of the integrable discrete vector NLS (sc-IDVNLS) equation

$$\mathrm{i}\frac{\mathrm{d}u_n^{(j)}}{\mathrm{d}t} = \sum_{k=1}^{N} \left|u_n^{(k)}\right|^2 \left(u_{n+1}^{(j)} + u_{n-1}^{(j)}\right). \tag{3.1}$$

Here we present the 'molecule solution' which satisfies the boundary conditions $u_n^{(j)} = 0$ at $n = 0$, $j = 1, 2, \ldots, N$. We transform equation (3.1) into the bilinear forms

$$D_t g_n^{(j)} \bullet f_n - g_{n+1}^{(j)} f_{n-1} + g_{n-1}^{(j)} f_{n+1} = 0, \tag{3.2a}$$

$$D_t g_n^{*(j)} \bullet f_n - g_{n+1}^{*(j)} f_{n-1} + g_{n-1}^{*(j)} f_{n+1} = 0, \tag{3.2b}$$

$$f_{n+1} f_{n-1} = \sum_k^N g_n^{(k)} g_n^{*(k)}, \tag{3.2c}$$

through the dependent variable transformation

$$u_n^{(j)} = \frac{g_n^{(j)}}{f_n} \exp\left(\mathrm{i}\frac{n}{2}\pi\right), \qquad u_n^{*(j)} = \frac{g_n^{*(j)}}{f_n} \exp\left(-\mathrm{i}\frac{n}{2}\pi\right), \qquad j = 1, 2, \ldots, N. \tag{3.3}$$

We use the following boundary conditions for f_n, $g_n^{(j)}$ and $g_n^{*(j)}$ for $j = 1, 2, \ldots, N$:

$$f_{-1} = 0, \qquad g_{-1}^{(j)} = 0, \qquad g_{-1}^{*(j)} = 0, \tag{3.4a}$$

$$f_0 = 1, \qquad g_0^{(j)} = 0, \qquad g_0^{*(j)} = 0, \tag{3.4b}$$

$$f_1 = 1, \qquad g_1^{(j)} = h^{(j)}(t, 0), \qquad g_1^{*(j)} = h^{*(j)}(t, 0), \tag{3.4c}$$

where $h^{(j)}(t, m)$ and $h^{*(j)}(t, m)$, for $j = 1, 2, \ldots, N$, are arbitrary functions of t and satisfy the equations

$$\frac{\mathrm{d}}{\mathrm{d}t} h^{(j)}(t, m) = h^{(j)}(t, m + 1), \qquad \frac{\mathrm{d}}{\mathrm{d}t} h^{*(j)}(t, m) = h^{*(j)}(t, m + 1), \tag{3.5}$$

for non-negative integers m.

We solve equations (3.2*a*) and (3.2*b*) iteratively starting with $n = 1$. For $n = 1$, equation (3.2*c*) becomes

$$f_2 f_0 = \sum_{k=1}^N g_1^{(k)} g_1^{*(k)}. \tag{3.6}$$

Using the boundary conditions, we obtain f_2:

$$f_2 = \sum_{k=1}^N h^{(k)}(t, 0) h^{*(k)}(t, 0). \tag{3.7}$$

Then we consider equation (3.2*a*) for $n = 1$:

$$D_t g_1^{(j)} \bullet f_1 - g_2^{(j)} f_0 + g_0^{(j)} f_2 = 0. \tag{3.8}$$

Solving the equation under the boundary conditions, we obtain

$$g_2^{(j)} = h^{(j)}(t, 1). \tag{3.9}$$

Similarly, equation (3.2*b*) for $n = 1$:

$$D_t g_1^{*(j)} \bullet f_1 - g_2^{*(j)} f_0 + g_0^{*(j)} f_2 = 0. \tag{3.10}$$

Solving the equation under the initial conditions yields

$$g_2^{*(j)} = h^{*(j)}(t, 1). \tag{3.11}$$

Equation (3.2c) for $n = 2$ gives

$$f_3 f_1 = \sum_{k=1}^{N} g_2^{(k)} g_2^{*(k)} \tag{3.12}$$

and substituting equations (3.9) and (3.11) into equation (3.12) we obtain

$$f_3 = \sum_{k=1}^{N} h^{(k)}(t,1) h^{*(k)}(t,1). \tag{3.13}$$

We have equation (3.2a) for $n = 2$,

$$D_t g_2^{(j)} \bullet f_2 - g_3^{(j)} f_1 + g_1^{(j)} f_3 = 0, \tag{3.14}$$

which is solved for $g_3^{(j)}$:

$$g_3^{(j)} = h^{(j)}(t,2) f_2 - h^{(j)}(t,1)\frac{\mathrm{d}f_2}{\mathrm{d}t} + h^{(j)}(t,0) f_3. \tag{3.15}$$

and similarly equation (3.2b) for $n = 2$

$$D_t g_2^{*(j)} \bullet f_2 - g_3^{*(j)} f_1 + g_1^{*(j)} f_3 = 0, \tag{3.16}$$

is solved for $g_3^{*(j)}$:

$$g_3^{*(j)} = h^{*(j)}(t,2) f_2 - h^{*(j)}(t,1)\frac{df_2}{\mathrm{d}t} + h^{*(j)}(t,0) f_3. \tag{3.17}$$

Here we introduce the Wronski-type Pfaffians whose elements are defined as

$$\mathrm{pf}(p_j, m) = h^{(j)}(t,m), \tag{3.18a}$$

$$\mathrm{pf}(p_j^*, m) = h^{*(j)}(t,m), \tag{3.18b}$$

$$\frac{1}{2}[\mathrm{pf}(l, m+1) - \mathrm{pf}(l+1, m)] = \sum_{k=1}^{N} h^{(k)}(t,l) h^{*(k)}(t,m), \tag{3.18c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{pf}(p_j, m) = \mathrm{pf}(p_j, m+1), \tag{3.18d}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{pf}(p_j^*, m) = \mathrm{pf}(p_j^*, m+1), \tag{3.18e}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{pf}(l, m) = \mathrm{pf}(l+1, m) + \mathrm{pf}(l, m+1), \tag{3.18f}$$

for $j = 1, 2, \ldots, N$ and for non-negative integers l and m. Then we find

$$g_1^{(j)} = \mathrm{pf}(p_j, 0), \tag{3.19a}$$

$$g_1^{*(j)} = \mathrm{pf}(p_j^*, 0), \tag{3.19b}$$

$$f_2 = \mathrm{pf}(0, 1), \tag{3.19c}$$

$$g_2^{(j)} = \mathrm{pf}(p_j, 1), \tag{3.19d}$$

$$g_2^{*(j)} = \mathrm{pf}(p_j^*, 1), \tag{3.19e}$$

$$f_3 = \mathrm{pf}(1, 2), \tag{3.19f}$$

$$g_3^{(j)} = \mathrm{pf}(p_j, 0, 1, 2), \tag{3.19g}$$

$$g_3^{*(j)} = \mathrm{pf}(p_j^*, 0, 1, 2). \tag{3.19h}$$

Theorem 3.1. *Solutions of the bilinear forms (3.2a), (3.2b) and (3.2c) with the boundary conditions (3.4a), (3.4b) and (3.4c) are*

$$f_{2n} = \mathrm{pf}(0, 1, 2, \ldots, 2n-1), \tag{3.20a}$$

$$g_{2n}^{(j)} = \mathrm{pf}(p_j, 1, 2, \ldots, 2n-1), \tag{3.20b}$$

$$g_{2n}^{*(j)} = \mathrm{pf}(p_j^*, 1, 2, \ldots, 2n-1), \tag{3.20c}$$

$$f_{2n+1} = \mathrm{pf}(1, 2, 3, \ldots, 2n), \tag{3.20d}$$

$$g_{2n+1}^{(j)} = \mathrm{pf}(p_j, 0, 1, \ldots, 2n), \tag{3.20e}$$

$$g_{2n+1}^{*(j)} = \mathrm{pf}(p_j^*, 0, 1, \ldots, 2n). \tag{3.20f}$$

Proof. The proof is similar to the proof of molecule solutions of the coupled discrete modified KdV equation [14].

First, we present the differential rules of f_n, $g_n^{(j)}$ and $g_n^{*(j)}$ by using the differential rules given by equations (3.18*d*), (3.18*e*) and (3.18*f*) and a differential formula for the Wronski-type Pfaffian [15–17]:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f_{2n} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{pf}(0, 1, 2, \ldots, 2n-2, 2n-1)\\ &= \mathrm{pf}(0, 1, 2, \ldots, 2n-2, 2n),\end{aligned} \tag{3.21a}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f_{2n+1} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{pf}(1, 2, 3, \ldots, 2n-1, 2n)\\ &= \mathrm{pf}(1, 2, 3, \ldots, 2n-1, 2n+1),\end{aligned} \tag{3.21b}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} g_{2n}^{(j)} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{pf}(p_j, 1, 2, 3, \ldots, 2n-2, 2n-1)\\ &= \mathrm{pf}(p_j, 1, 2, 3, \ldots, 2n-2, 2n),\end{aligned} \tag{3.21c}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} g_{2n+1}^{(j)} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{pf}(p_j, 0, 1, 2, \ldots, 2n-1, 2n)\\ &= \mathrm{pf}(p_j, 0, 1, 2, \ldots, 2n-1, 2n+1),\end{aligned} \tag{3.21d}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} g_{2n}^{*(j)} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{pf}(p_j^*, 1, 2, 3, \ldots, 2n-2, 2n-1)\\ &= \mathrm{pf}(p_j^*, 1, 2, 3, \ldots, 2n-2, 2n),\end{aligned} \tag{3.21e}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} g_{2n+1}^{*(j)} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{pf}(p_j^*, 0, 1, 2, \ldots, 2n-1, 2n)\\ &= \mathrm{pf}(p_j^*, 0, 1, 2, \ldots, 2n-1, 2n+1).\end{aligned} \tag{3.21f}$$

Substituting these formulae into the lhs of equation (3.2*a*), we obtain for the odd subscripts $2n+1$

$$\begin{aligned}D_t g_{2n+1}^{(j)} \bullet f_{2n+1} - g_{2n+2}^{(j)} f_{2n} + g_{2n}^{(j)} f_{2n+2}\\ &= \mathrm{pf}(p_j, 0, 1, 2, \ldots, 2n-1, 2n+1)\,\mathrm{pf}(1, 2, 3, \ldots, 2n-1, 2n)\\ &\quad - \mathrm{pf}(p_j, 0, 1, 2, \ldots, 2n-1, 2n)\,\mathrm{pf}(1, 2, 3, \ldots, 2n-1, 2n+1)\\ &\quad - \mathrm{pf}(p_j, 1, 2, 3, \ldots, 2n-1, 2n, 2n+1)\,\mathrm{pf}(0, 1, 2, \ldots, 2n-1)\\ &\quad + \mathrm{pf}(p_j, 1, 2, 3, \ldots, 2n-1)\,\mathrm{pf}(0, 1, 2, \ldots, 2n-1, 2n, 2n+1)\\ &= \mathrm{pf}(p_j, 0, *, 2n+1)\,\mathrm{pf}(*, 2n) - \mathrm{pf}(p_j, 0, *, 2n)\,\mathrm{pf}(*, 2n+1)\\ &\quad - \mathrm{pf}(p_j, *, 2n, 2n+1)\,\mathrm{pf}(0, *) + \mathrm{pf}(p_j, *)\,\mathrm{pf}(0, *, 2n, 2n+1),\end{aligned} \tag{3.22}$$

where we replaced the list $\{1, 2, \ldots, 2n-1\}$ by $*$. The right-hand side of (3.22) vanishes because of the identity for the Pfaffians [15]:

$$\begin{aligned}\mathrm{pf}(a_1, a_2, a_3, *)\,\mathrm{pf}(*, 2n) &= \mathrm{pf}(a_1, *)\,\mathrm{pf}(a_2, a_3, *, 2n)\\ &\quad - \mathrm{pf}(a_2, *)\,\mathrm{pf}(a_1, a_3, *, 2n) + \mathrm{pf}(a_3, *)\,\mathrm{pf}(a_1, a_2, *, 2n).\end{aligned} \tag{3.23}$$

Thus, we have proved equation (3.2*a*) for the odd subscripts. Equation (3.2*b*) for the odd subscripts is proved by the same procedure.

Equation (3.2*c*) for the odd subscripts is proved as follows. We express the rhs of equation (3.2*c*) for the odd subscripts $2n+1$ by the Pfaffians

$$\sum_{k=1}^{N} g_{2n+1}^{(k)} g_{2n+1}^{*(k)} = \sum_{k=1}^{N} \mathrm{pf}(p_k, 0, 1, 2, \ldots, 2n)\,\mathrm{pf}(p_k^*, 0, 1, 2, \ldots, 2n). \tag{3.24}$$

Expanding the Pfaffians and using equation (3.18*c*), we obtain

$$\begin{aligned}\sum_{k=1}^{N} g_{2n+1}^{(k)} g_{2n+1}^{*(k)}\\ &= \sum_{l,m=0}^{2n} \sum_{k=1}^{N} \mathrm{pf}(p_k, l)(-1)^l\,\mathrm{pf}(0, 1, 2, \ldots, \hat{l}, \ldots, 2n)\\ &\quad \times \mathrm{pf}(p_k^*, m)(-1)^m\,\mathrm{pf}(0, 1, 2, \ldots, \hat{m}, \ldots, 2n)\\ &= \sum_{l,m=0}^{2n} \frac{1}{2}[\mathrm{pf}(l, m+1) - \mathrm{pf}(l+1, m)]\\ &\quad \times (-1)^l\,\mathrm{pf}(0, 1, 2, \ldots, \hat{l}, \ldots, 2n)(-1)^m\,\mathrm{pf}(0, 1, 2, \ldots, \hat{m}, \ldots, 2n),\end{aligned} \tag{3.25}$$

where $\hat{x}$ indicates that the letter x is missing. Using the expansion rule of the Pfaffian

$$\sum_{m=0}^{2n} \mathrm{pf}(l+1, m)(-1)^m\,\mathrm{pf}(0, 1, 2, \ldots, \hat{m}, \ldots, 2n) = \mathrm{pf}(l+1, 0, 1, 2, \ldots, 2n) \tag{3.26}$$

we have

$$\sum_{k=1}^{N} g_{2n+1}^{(k)} g_{2n+1}^{*(k)} = -\sum_{l=0}^{2n} (-1)^l\,\mathrm{pf}(0, 1, 2, \ldots, \hat{l}, \ldots, 2n)\,\mathrm{pf}(l+1, 0, 1, 2, \ldots, 2n), \tag{3.27}$$

where the sum over l vanishes except for $l = 2n$. Thus we obtain

$$\begin{aligned}\sum_{k=1}^{N} g_{2n+1}^{(k)} g_{2n+1}^{*(k)} &= -\mathrm{pf}(0, 1, 2, \ldots, 2n-1)\,\mathrm{pf}(2n+1, 0, 1, 2, \ldots, 2n)\\ &= f_{2n} f_{2n+2},\end{aligned} \tag{3.28}$$

which is the lhs of equation (3.2c) for the odd subscript $2n+1$. Thus, we have proved equation (3.2c) for the odd subscripts.

Equations (3.2a), (3.2b) and (3.2c) for the even subscripts are proved by the same procedure. Thus we have proved that f_n, $g_n^{(j)}$ and $g_n^{*(j)}$, for $j = 1, 2, \ldots, N$, expressed by equations (3.20a)–(3.20f) are solutions of equations (3.2a), (3.2b) and (3.2c). □

We note that

$$\mathrm{i}\frac{\mathrm{d}u_n^{(j)}}{\mathrm{d}t} = \sum_{k=1}^{M} C_k \big|u_n^{(k)}\big|^2 \big(u_{n+1}^{(j)} + u_{n-1}^{(j)}\big) \tag{3.29}$$

also has a Pfaffian-type molecule solution. The proof is similar to that given above.

3.2. Soliton solutions

We consider the case in which the sc-IDVNLS equation has soliton type solutions.

Here we consider the infinite lattice with the conditions $u_n^{(1)} = \text{const.} \times \exp(\mathrm{i}\pi n/2)$.

Two-component case

$$\mathrm{i}\frac{\mathrm{d}u_n^{(1)}}{\mathrm{d}t} = \big(\big|u_n^{(1)}\big|^2 + \big|u_n^{(2)}\big|^2\big)\big(u_{n+1}^{(1)} + u_{n-1}^{(1)}\big), \tag{3.30a}$$

$$\mathrm{i}\frac{\mathrm{d}u_n^{(2)}}{\mathrm{d}t} = \big(\big|u_n^{(1)}\big|^2 + \big|u_n^{(2)}\big|^2\big)\big(u_{n+1}^{(2)} + u_{n-1}^{(2)}\big). \tag{3.30b}$$

Suppose $u_n^{(1)} = \exp(\mathrm{i}\pi n/2)$. Then we obtain

$$\mathrm{i}\frac{\mathrm{d}u_n^{(2)}}{\mathrm{d}t} = \big(1 + \big|u_n^{(2)}\big|^2\big)\big(u_{n+1}^{(2)} + u_{n-1}^{(2)}\big). \tag{3.31}$$

This is the IDNLS equation. Thus this equation has an N-soliton solution.

$N(\geqslant 3)$*-component case*

$$\mathrm{i}\frac{\mathrm{d}u_n^{(1)}}{\mathrm{d}t} = \sum_{k=1}^{N} \big|u_n^{(k)}\big|^2 \big(u_{n+1}^{(1)} + u_{n-1}^{(1)}\big), \tag{3.32a}$$

$$\mathrm{i}\frac{\mathrm{d}u_n^{(j)}}{\mathrm{d}t} = \sum_{k=1}^{N} \big|u_n^{(k)}\big|^2 \big(u_{n+1}^{(j)} + u_{n-1}^{(j)}\big), \qquad j = 2, 3, \ldots, N. \tag{3.32b}$$

Suppose $u_n^{(1)} = \exp(\mathrm{i}\pi n/2)$. Then we obtain

$$\mathrm{i}\frac{\mathrm{d}u_n^{(j)}}{\mathrm{d}t} = \left(1 + \sum_{k=2}^{N} \big|u_n^{(k)}\big|^2\right) \big(u_{n+1}^{(j)} + u_{n-1}^{(j)}\big), \qquad j = 2, 3, \ldots, N. \tag{3.33}$$

This is the discrete vector NLS equation, whose soliton solutions are known [18, 19].

4. The strong coupling limit of the fully discrete vector NLS equation

4.1. Molecule solutions

In this section, we consider the strong coupling limit of the integrable fully discrete vector NLS (sc-IFDVNLS) equation [20, 21]:

$$\begin{aligned} &\mathrm{i}\big(u_n^{(j),t+1}-u_n^{(j),t}\big)-\delta\left[\sum_{k=1}^{N}c_k\big|u_n^{(k),t}\big|^2\right]\Gamma_n^t\big[u_{n+1}^{(j),t}+u_{n-1}^{(j),t+1}\big]=0,\\ &\Gamma_{n+1}^t=\left[\sum_{k=1}^{N}c_k\big|u_n^{(k),t}\big|^2\right]\Gamma_n^t\Bigg/\left[\sum_{k=1}^{N}c_k\big|u_n^{(k),t+1}\big|^2\right]. \end{aligned}\tag{4.1}$$

Here we present the 'molecule solution' which satisfies the boundary conditions $u_n^{(j),t}=0$ at $n=0$, $j=1,2,\ldots,N$. We transform equation (4.1) into the bilinear forms

$$g_n^{(j),t+1}f_n^t-g_n^{(j),t}f_n^{t+1}-\delta\big[g_{n+1}^{(j),t}f_{n-1}^{t+1}-g_{n-1}^{(j),t+1}f_{n+1}^t\big]=0,\tag{4.2a}$$

$$g_n^{*(j),t+1}f_n^t-g^{*(j),t}_{\ \,n}f_n^{t+1}-\delta\big[g_{n+1}^{*(j),t}f_{n-1}^{t+1}-g_{n-1}^{*(j),t+1}f_{n+1}^t\big]=0,\tag{4.2b}$$

$$f_{n+1}^tf_{n-1}^t=\sum_{j=1}^{N}c_jg_n^{(j),t}g_n^{*(j),t},\tag{4.2c}$$

through the dependent variable transformation

$$\begin{aligned} &u_n^{(j),t}=\frac{g_n^{(j),t}}{f_n^t}\exp\left(\mathrm{i}\frac{n}{2}\pi\right),\qquad u_n^{*(j)}=\frac{g_n^{*(j),t}}{f_n^t}\exp\left(-\mathrm{i}\frac{n}{2}\pi\right),\qquad j=1,2,\ldots,N,\\ &\Gamma_n^t=\frac{f_n^tf_{n-1}^{t+1}}{f_n^{t+1}f_{n-1}^t}. \end{aligned}\tag{4.3}$$

We use the following boundary conditions for f_n^t, $g_n^{(j),t}$ and $g_n^{*(j),t}$ for $j=1,2,\ldots,N$:

$$f_{-1}^t=0,\qquad g_{-1}^{(j),t}=0,\qquad g^{*(j),t}_{\ \,-1}=0,\tag{4.4a}$$

$$f_0^t=1,\qquad g_0^{(j),t}=0,\qquad g^{*(j),t}_{\ \,0}=0,\tag{4.4b}$$

$$f_1^t=1,\qquad g_1^{(j),t}=h^{(j)}(t,0),\qquad g^{*(j),t}_{\ \,1}=h^{*(j)}(t,0),\tag{4.4c}$$

where $h^{(j)}(t,m)$ and $h^{*(j)}(t,m)$ for $j=1,2,\ldots,N$ are arbitrary functions of t and satisfy the equations

$$\frac{\Delta}{\Delta t}h^{(j)}(t,m)=h^{(j)}(t,m+1),\qquad \frac{\Delta}{\Delta t}h^{*(j)}(t,m)=h^{*(j)}(t,m+1),\tag{4.5}$$

where

$$\frac{\Delta}{\Delta t}h(t,m)\equiv\frac{h(t+1,m)-h(t,m)}{\delta},$$

for non-negative integers m.

We solve equations (4.2a) and (4.2b) iteratively starting with $n=1$. For $n=1$, equation (4.2c) becomes

$$f_2^tf_0^t=\sum_{k=1}^{N}c_kg_1^{(k),t}g_1^{*(k),t}.\tag{4.6}$$

Using the boundary conditions, we obtain f_2^t:

$$f_2^t = \sum_{k=1}^{N} c_k h^{(k)}(t,0) h^{*(k)}(t,0). \tag{4.7}$$

We have equation (4.2*a*) for $n = 1$:

$$g_1^{(j),t+1} f_1^t - g_1^{(j),t} f_1^{t+1} - \delta\left[g_2^{(j),t} f_0^{t+1} - g_0^{(j),t+1} f_2^t\right] = 0. \tag{4.8}$$

Solving the equation under the boundary conditions, we obtain

$$g_2^{(j),t} = h^{(j)}(t,1). \tag{4.9}$$

We have equation (4.2*b*) for $n = 1$:

$$g_1^{*(j),t+1} f_1^t - g_1^{*(j),t} f_1^{t+1} - \delta\left[g_2^{*(j),t} f_0^{t+1} - g_0^{*(j),t+1} f_2^t\right] = 0. \tag{4.10}$$

Solving the equation under the initial conditions, we obtain

$$g_2^{*(j),t} = h^{*(j)}(t,1). \tag{4.11}$$

For $n = 2$, equation (4.2*c*) becomes

$$f_3^t f_1^t = \sum_{k=1}^{N} c_k g_2^{(k)} g_2^{*(k)}. \tag{4.12}$$

Substituting equations (4.9) and (4.11) into equation (4.12), we obtain

$$f_3^t = \sum_{k=1}^{N} c_k h^{(k)}(t,1) h^{*(k)}(t,1). \tag{4.13}$$

We have equation (4.2*a*) for $n = 2$,

$$g_2^{(j),t+1} f_2^t - g_2^{(j),t} f_2^{t+1} - \delta\left[g_3^{(j),t} f_1^{t+1} - g_1^{(j),t+1} f_3^t\right] = 0, \tag{4.14}$$

which is solved for $g_3^{(j),t}$:

$$g_3^{(j),t} = h^{(j)}(t,2) f_2^t - h^{(j)}(t,1) \frac{\Delta f_2^t}{\Delta t} + h^{(j)}(t+1,0) f_3^t. \tag{4.15}$$

We have equation (4.2*b*) for $n = 2$,

$$g_2^{*(j),t+1} f_2^t - g^{*(j),t}_2 f_2^{t+1} - \delta\left[g_3^{*(j),t} f_1^{t+1} - g^{*(j),t+1}_1 f_3^t\right] = 0, \tag{4.16}$$

which is solved for $g^{*(j),t}_3$:

$$g^{*(j),t}_3 = h^{*(j)}(t,2) f_2^t - h^{*(j)}(t,1) \frac{\Delta f_2^t}{\Delta t} + h^{*(j)}(t+1,0) f_3^t. \tag{4.17}$$

Here we introduce the Pfaffians whose elements are defined as

$$\mathrm{pf}(p_j, m) = h^{(j)}(t,m), \tag{4.18a}$$

$$\mathrm{pf}(p_j^*, m) = h^{*(j)}(t,m), \tag{4.18b}$$

$$\frac{1}{2}[\mathrm{pf}(l, m+1) - \mathrm{pf}(l+1, m)] = \sum_{k=1}^{N} c_k h^{(k)}(t,l) h^{*(k)}(t,m), \tag{4.18c}$$

$$\frac{\Delta}{\Delta t}\,\mathrm{pf}(p_j, m) = \mathrm{pf}(p_j, m+1), \tag{4.18d}$$

$$\frac{\Delta}{\Delta t}\,\mathrm{pf}(p_j^*, m) = \mathrm{pf}(p_j^*, m+1), \tag{4.18e}$$

$$\frac{\Delta}{\Delta t}\,\mathrm{pf}(l, m) = \mathrm{pf}(l+1, m) + \mathrm{pf}(l, m+1), \tag{4.18f}$$

for $j = 1, 2, \ldots, N$ and for non-negative integers l and m. Then we find

$$g_1^{(j),t} = \mathrm{pf}(p_j, 0), \tag{4.19a}$$

$$g_1^{*(j),t} = \mathrm{pf}(p_j^*, 0), \tag{4.19b}$$

$$f_2^t = \mathrm{pf}(0, 1), \tag{4.19c}$$

$$g_2^{(j),t} = \mathrm{pf}(p_j, 1), \tag{4.19d}$$

$$g_2^{*(j),t} = \mathrm{pf}(p_j^*, 1), \tag{4.19e}$$

$$f_3^t = \mathrm{pf}(1, 2), \tag{4.19f}$$

$$g_3^{(j),t} = \mathrm{pf}(p_j, 0, 1, 2), \tag{4.19g}$$

$$g_3^{*(j),t} = \mathrm{pf}(p_j^*, 0, 1, 2). \tag{4.19h}$$

Theorem 4.1. *Solutions of the bilinear forms (4.2a), (4.2b) and (4.2c) with the boundary conditions (4.4a), (4.4b) and (4.4c) are*

$$f_{2n}^t = \mathrm{pf}(0, 1, 2, \ldots, 2n-1), \tag{4.20a}$$

$$g_{2n}^{(j),t} = \mathrm{pf}(p_j, 1, 2, \ldots, 2n-1), \tag{4.20b}$$

$$g_{2n}^{*(j),t} = \mathrm{pf}(p_j^*, 1, 2, \ldots, 2n-1), \tag{4.20c}$$

$$f_{2n+1}^t = \mathrm{pf}(1, 2, 3, \ldots, 2n), \tag{4.20d}$$

$$g_{2n+1}^{(j),t} = \mathrm{pf}(p_j, 0, 1, \ldots, 2n), \tag{4.20e}$$

$$g_{2n+1}^{*(j),t} = \mathrm{pf}(p_j^*, 0, 1, \ldots, 2n). \tag{4.20f}$$

Proof. First, we present the difference rules of f_n^t, $g_n^{(j),t}$ and $g_n^{*(j),t}$ by using the difference rules given by equations (4.18d), (4.18e) and (4.18f) and a difference formula for the Wronski-type Pfaffian:

$$\begin{aligned}\frac{\Delta}{\Delta t} f_{2n} &= \frac{\Delta}{\Delta t}\,\mathrm{pf}(0, 1, 2, \ldots, 2n-2, 2n-1)\\ &= \mathrm{pf}(0, 1, 2, \ldots, 2n-2, 2n),\end{aligned} \tag{4.21a}$$

$$\begin{aligned}\frac{\Delta}{\Delta t} f_{2n+1} &= \frac{\Delta}{\Delta t}\,\mathrm{pf}(1, 2, 3, \ldots, 2n-1, 2n)\\ &= \mathrm{pf}(1, 2, 3, \ldots, 2n-1, 2n+1),\end{aligned} \tag{4.21b}$$

$$\begin{aligned}\frac{\Delta}{\Delta t} g_{2n}^{(j)} &= \frac{\Delta}{\Delta t}\,\mathrm{pf}(p_j, 1, 2, 3, \ldots, 2n-2, 2n-1)\\ &= \mathrm{pf}(p_j, 1, 2, 3, \ldots, 2n-2, 2n),\end{aligned} \tag{4.21c}$$

$$\frac{\Delta}{\Delta t} g_{2n+1}^{(j)} = \frac{\Delta}{\Delta t} \mathrm{pf}(p_j, 0, 1, 2, \ldots, 2n-1, 2n)$$
$$= \mathrm{pf}(p_j, 0, 1, 2, \ldots, 2n-1, 2n+1), \tag{4.21d}$$

$$\frac{\Delta}{\Delta t} g_{2n}^{*(j)} = \frac{\Delta}{\Delta t} \mathrm{pf}(p_j^*, 1, 2, 3, \ldots, 2n-2, 2n-1)$$
$$= \mathrm{pf}(p_j^*, 1, 2, 3, \ldots, 2n-2, 2n), \tag{4.21e}$$

$$\frac{\Delta}{\Delta t} g_{2n+1}^{*(j)} = \frac{\Delta}{\Delta t} \mathrm{pf}(p_j^*, 0, 1, 2, \ldots, 2n-1, 2n)$$
$$= \mathrm{pf}(p_j^*, 0, 1, 2, \ldots, 2n-1, 2n+1). \tag{4.21f}$$

Substituting these formulae into the lhs of equation (4.2*a*), we obtain for the odd subscripts $2n+1$

$$\begin{aligned}
&\left[g_{2n+1}^{(j),t+1} f_{2n+1}^{t} - g_{2n+1}^{(j),t} f_{2n+1}^{t+1}\right]/\delta - g_{2n+2}^{(j),t} f_{2n}^{t+1} + g_{2n}^{(j),t+1} f_{2n+2}^{t} = 0 \\
&= \mathrm{pf}(p_j, 0, 1, 2, \ldots, 2n-1, 2n+1)\,\mathrm{pf}(1, 2, 3, \ldots, 2n-1, 2n) \\
&\quad - \mathrm{pf}(p_j, 0, 1, 2, \ldots, 2n-1, 2n)\,\mathrm{pf}(1, 2, 3, \ldots, 2n-1, 2n+1) \\
&\quad - \mathrm{pf}(p_j, 1, 2, 3, \ldots, 2n-1, 2n, 2n+1)\,\mathrm{pf}(0, 1, 2, \ldots, 2n-1) \\
&\quad + \mathrm{pf}(p_j, 1, 2, 3, \ldots, 2n-1)\,\mathrm{pf}(0, 1, 2, \ldots, 2n-1, 2n, 2n+1) \\
&= \mathrm{pf}(p_j, 0, *, 2n+1)\,\mathrm{pf}(*, 2n) - \mathrm{pf}(p_j, 0, *, 2n)\,\mathrm{pf}(*, 2n+1) \\
&\quad - \mathrm{pf}(p_j, *, 2n, 2n+1)\,\mathrm{pf}(0, *) + \mathrm{pf}(p_j, *)\,\mathrm{pf}(0, *, 2n, 2n+1),
\end{aligned} \tag{4.22}$$

where we replaced the list $\{1, 2, \ldots, 2n-1\}$ by $*$. This vanishes because of the identity of the Pfaffians [15]:

$$\begin{aligned}
\mathrm{pf}(a_1, a_2, a_3, *)\,\mathrm{pf}(*, 2n) &= \mathrm{pf}(a_1, *)\,\mathrm{pf}(a_2, a_3, *, 2n) \\
&\quad - \mathrm{pf}(a_2, *)\,\mathrm{pf}(a_1, a_3, *, 2n) + \mathrm{pf}(a_3, *)\,\mathrm{pf}(a_1, a_2, *, 2n).
\end{aligned} \tag{4.23}$$

Thus we have proved equation (4.2*a*) for the odd subscripts. Equation (4.2*b*) for the odd subscripts is proved by the same procedure.

Equation (4.2*c*) for the odd subscripts is proved in the same way as the proof of theorem 4.1. Equations (4.2*a*), (4.2*b*) and (4.2*c*) for the even subscripts are proved by the same procedure. Thus we have proved that f_n^t, $g_n^{(j),t}$ and $g_n^{*(j),t}$, for $j = 1, 2, \ldots, N$, expressed by equations (4.20*a*)–(4.20*f*) are solutions of equations (4.2*a*), (4.2*b*) and (4.2*c*). □

4.2. Soliton solutions

Let us now investigate the case in which the sc-IFDVNLS equation has soliton-type solutions. Here we consider the infinite lattice with the conditions $u_n^{(1),t} = \mathrm{const} \times \exp(\mathrm{i}\pi n/2)$.

Two-component case

$$\begin{aligned}
&\mathrm{i}\left(u_n^{(1),t+1} - u_n^{(1),t}\right) - \delta\left[\sum_{k=1}^{2} c_k \left|u_n^{(k),t}\right|^2\right]\Gamma_n^t\left[u_{n+1}^{(1),t} + u_{n-1}^{(1),t+1}\right] = 0, \\
&\mathrm{i}\left(u_n^{(2),t+1} - u_n^{(2),t}\right) - \delta\left[\sum_{k=1}^{2} c_k \left|u_n^{(k),t}\right|^2\right]\Gamma_n^t\left[u_{n+1}^{(2),t} + u_{n-1}^{(2),t+1}\right] = 0, \\
&\Gamma_{n+1}^t = \left[\sum_{k=1}^{2} c_k \left|u_n^{(k),t}\right|^2\right]\Gamma_n^t \Bigg/ \left[\sum_{k=1}^{2} c_k \left|u_n^{(k),t+1}\right|^2\right].
\end{aligned} \tag{4.24}$$

Suppose $u_n^{(1),t} = \exp(\mathrm{i}\pi n/2)$. Then we obtain

$$\begin{aligned} &\mathrm{i}\left(u_n^{(2),t+1} - u_n^{(2),t}\right) - \delta\left[1 + c_2\left|u_n^{(2),t}\right|^2\right]\Gamma_n^t\left[u_{n+1}^{(2),t} + u_{n-1}^{(2),t+1}\right] = 0, \\ &\Gamma_{n+1}^t = \left[1 + c_2\left|u_n^{(2),t}\right|^2\right]\Gamma_n^t \Big/ \left[1 + c_2\left|u_n^{(2),t+1}\right|^2\right]. \end{aligned} \tag{4.25}$$

This is the fully discrete NLS equation. Thus this equation has an N-soliton solution.

$N(\geqslant 3)$*-component case*

$$\begin{aligned} &\mathrm{i}\left(u_n^{(1),t+1} - u_n^{(1),t}\right) - \delta\left[\sum_{k=1}^{N} c_k\left|u_n^{(k),t}\right|^2\right]\Gamma_n^t\left[u_{n+1}^{(1),t} + u_{n-1}^{(1),t+1}\right] = 0, \\ &\mathrm{i}\left(u_n^{(j),t+1} - u_n^{(j),t}\right) - \delta\left[\sum_{k=1}^{2} c_k\left|u_n^{(k),t}\right|^2\right]\Gamma_n^t\left[u_{n+1}^{(j),t} + u_{n-1}^{(j),t+1}\right] = 0, \\ &\Gamma_{n+1}^t = \left[\sum_{k=1}^{N} c_k\left|u_n^{(k),t}\right|^2\right]\Gamma_n^t \Bigg/ \left[\sum_{k=1}^{N} c_k\left|u_n^{(k),t+1}\right|^2\right], \\ &j = 2, 3, \ldots, N. \end{aligned} \tag{4.26}$$

Suppose $u_n^{(1)} = \exp(\mathrm{i}\pi n/2)$. Then we obtain

$$\begin{aligned} &\mathrm{i}\left(u_n^{(j),t+1} - u_n^{(j),t}\right) - \delta\left[1 + \sum_{k=2}^{N} c_k\left|u_n^{(k),t}\right|^2\right]\Gamma_n^t\left[u_{n+1}^{(j),t} + u_{n-1}^{(j),t+1}\right] = 0, \\ &\Gamma_{n+1}^t = \left[1 + \sum_{k=2}^{N} c_k\left|u_n^{(k),t}\right|^2\right]\Gamma_n^t \Bigg/ \left[1 + \sum_{k=2}^{N} c_k\left|u_n^{(k),t+1}\right|^2\right], \\ &j = 2, 3, \ldots, N. \end{aligned} \tag{4.27}$$

This is the fully discrete vector NLS equation, whose soliton solutions are known [20, 21].

5. Conclusions

We have analyzed the strong coupling limits of the integrable discrete NLS, discrete vector NLS and fully discrete vector NLS equations, and we have found determinant and Pfaffian solutions. We have also studied the strong coupling limit of the discrete Hirota equation.

It is worth noting that when one takes complex-valued wave numbers, soliton solutions of the sc-IDNLS equation (1.4) exhibit singularities. An interesting open problem is whether non-singular solutions with complex-valued wave numbers exist.

Consideration of physical interpretations of molecule solutions (i.e., solutions for the finite lattice) is also an important problem. We note that molecule solutions (i.e., the Wronski-type Pfaffian) in vector cases are similar to solutions of the Pfaff lattice [17, 22, 23]. Thus we expect that there is a connection with random matrix theory.

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Appendix A

Equation (1.6), i.e.,

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} = |u_n|^2(\alpha u_{n+1} + \alpha^* u_{n-1})$$

can be considered as a reduction of the following system of differential-difference equations

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} = -u_n v_n(\alpha u_{n+1} + \alpha^* u_{n-1})$$
$$-\mathrm{i}\frac{\mathrm{d}v_n}{\mathrm{d}t} = -u_n v_n(\alpha v_{n-1} + \alpha^* v_{n+1})$$

when $v_n = -u_n^*$. Then, it can be treated as in the appendix in [9], i.e., consider

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}(u_n v_{n-1}) = -\alpha u_n v_{n-1}(v_n u_{n+1} - u_{n-1} v_{n-2})$$
$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}(u_{n-1} v_n) = -\alpha^* u_{n-1} v_n(u_n v_{n+1} - u_{n-2} v_{n-1})$$

and in terms of the new dependent variables

$$s_n = \log(u_n v_{n-1}), \qquad \tilde{s}_n = \log(u_{n-1} v_n)$$

the previous system becomes

$$\mathrm{i}\frac{\mathrm{d}s_n}{\mathrm{d}t} = \alpha(\mathrm{e}^{s_{n-1}} - \mathrm{e}^{s_{n+1}})$$
$$\mathrm{i}\frac{\mathrm{d}\tilde{s}_n}{\mathrm{d}t} = \alpha^*(\mathrm{e}^{\tilde{s}_{n+1}} - \mathrm{e}^{\tilde{s}_{n-1}}).$$

Finally, performing the change of independent variables $T = \mathrm{i}\alpha t$ in the first equation and $T^* = -\mathrm{i}\alpha^* t$ in the second one, we obtain

$$\frac{\mathrm{d}s_n}{\mathrm{d}T} = \mathrm{e}^{s_{n+1}} - \mathrm{e}^{s_{n-1}}$$
$$\frac{\mathrm{d}\tilde{s}_n}{\mathrm{d}T^*} = \mathrm{e}^{\tilde{s}_{n+1}} - \mathrm{e}^{\tilde{s}_{n-1}},$$

i.e., a system of decoupled nonlinear ladder network equations, each possessing a well-known Lax pair. The original dependent variables are then reconstructed by means of the relations

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} = -u_n(\alpha \mathrm{e}^{s_{n+1}} + \alpha^* \mathrm{e}^{\tilde{s}_n})$$
$$-\mathrm{i}\frac{\mathrm{d}v_n}{\mathrm{d}t} = -v_n(\alpha \mathrm{e}^{s_n} + \alpha^* \mathrm{e}^{\tilde{s}_{n+1}}).$$

Appendix B

Proof of theorem 2.3.

Proof. Assuming that $\Phi(n) = (\phi_1(n), \ldots, \phi_N(n))^T$, we adopt the notations

$$\begin{aligned}
&(i_1, i_2, \ldots, i_N) := \det(\Phi(n+i_1), \Phi(n+i_2), \ldots, \Phi(n+i_N)),\\
&(2\widehat{(N-k)}) := (0, 2, \ldots, 2(N-k)),\\
&(2\widetilde{(N-k)}) := (2, 4, \ldots, 2(N-k)),
\end{aligned} \tag{B.1}$$

where $i_1, i_2, \ldots, i_N$ are arbitrary integers. Therefore, for example, we have

$$(2\widehat{(N-1)}) = (2\widehat{(N-2)}, 2(N-1)) = (0, 2\widetilde{(N-1)})$$
$$= \begin{vmatrix} \phi_1(n) & \phi_1(n+2) & \cdots & \phi_1(n+2N-2) \\ \phi_2(n) & \phi_2(n+2) & \cdots & \phi_2(n+2N-2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+2) & \cdots & \phi_N(n+2N-2) \end{vmatrix}.$$

From (2.9), it is easy to obtain the determinant identities:

$$\dot{g}_n = (2\widehat{(N-2)}, 2N). \tag{B.2}$$

From (2.10), we can also obtain the determinant identities [24]:

$$\begin{aligned} \gamma g_{n+1} &= \sum_{k=0}^{N} (\widehat{2N})_k \\ &= \sum_{k=0}^{N-2} (2\widehat{(N-2)}, 2(N-1), 2N)_k + (2\widehat{(N-2)}, 2N) + (2\widehat{(N-1)}), \end{aligned} \tag{B.3}$$

$$\begin{aligned} \gamma g_{n-1} &= \sum_{k=-1}^{N-1} (-2, 2\widehat{(N-1)})_k \\ &= \sum_{k=0}^{N-2} (-2, 2\widehat{(N-2)}, 2(N-1))_k + (-2, 2\widehat{(N-2)}) + (2\widehat{(N-1)}), \end{aligned} \tag{B.4}$$

$$\begin{aligned} &\gamma(\dot{g}_{n-1} - g_{n-1}) \\ &\quad = \sum_{k=0}^{N-2} (-2, 2\widehat{(N-2)}, 2N)_k - (-2, 2\widehat{(N-2)}) + (2\widehat{(N-2)}, 2N), \end{aligned} \tag{B.5}$$

where $\gamma = \prod_{i=1}^{N} \beta_i$ and $(i_1, i_2, \ldots, i_N)_k = \det(\Phi(n+i_1), \Phi(n+i_2), \ldots, \Phi(n+i_{k-1}), \Phi(n+i_{k+1}), \ldots, \Phi(n+i_N))$.

Now, if we substitute our expression for g_n into (2.6*a*), and make use of (B.2), (B.3), (B.4) and (B.5), we find that the left-hand side of (2.6*a*) gives the terms

$$\begin{aligned} &\gamma(\dot{g}_n g_{n-1} - g_n \dot{g}_{n-1} - g_{n+1} g_{n-2} + g_n g_{n-1}) \\ &\quad = \dot{g}_n(\gamma g_{n-1}) - g_n(\gamma \dot{g}_{n-1} - \gamma g_{n-1}) - (\gamma g_{n+1}) g_{n-2} \\ &\quad = (2\widehat{(N-2)}, 2N) \left(\sum_{k=0}^{N-2} (-2, 2\widehat{(N-2)}, 2(N-1))_k + (-2, 2\widehat{(N-2)}) + (2\widehat{(N-1)}) \right) \\ &\qquad - (2\widehat{(N-1)}) \left(\sum_{k=0}^{N-2} (-2, 2\widehat{(N-2)}, 2N)_k - (-2, 2\widehat{(N-2)}) + (2\widehat{(N-2)}, 2N) \right) \\ &\qquad - \left(\sum_{k=0}^{N-2} (2\widehat{(N-2)}, 2(N-1), 2N)_k + (2\widehat{(N-2)}, 2N) + (2\widehat{(N-1)}) \right) (-2, 2\widehat{(N-2)}) \\ &\quad = \left(\sum_{k=0}^{N-2} (-2, 2\widehat{(N-2)}, 2(N-1))_k \right) (2\widehat{(N-2)}, 2N) \end{aligned}$$

$$-\left(\sum_{k=0}^{N-2}(-2, 2\widehat{(N-2)}, 2N)_k\right)(2\widehat{(N-2)}, 2(N-1))$$
$$-\left(\sum_{k=0}^{N-2}(2\widehat{(N-2)}, 2(N-1), 2N)_k\right)(-2, 2\widehat{(N-2)}). \tag{B.6}$$

This is the Laplace expansion by $N \times N$ minors of the $2N \times 2N$ determinant

$$(-1)^{N-1}\sum_{k=0}^{N-2}\begin{vmatrix} -2 & (2\widehat{(N-2)})_k & \oslash & 2(N-1) & 2N \\ -2 & \oslash & (2\widehat{(N-2)}) & 2(N-1) & 2N \end{vmatrix}$$

(where $\oslash$ indicates the zero matrix), which can be shown to be identically zero. Therefore, the solution is verified. Thus we have proved equation (2.6*a*). Equation (2.6*b*) is just the complex conjugate of (2.6*a*). □

Appendix C

Proof of theorem 2.4.

Proof. We introduce symbols D, $D\binom{i}{j}$ and $D\binom{i,k}{j,l}$ which are the determinants of the $(N+1)\times(N+1)$, $N\times N$ and $(N-1)\times(N-1)$ matrices with the respective definitions:

$$D = \begin{vmatrix} h(T,0) & h(T,1) & \cdots & h(T,N-1) & h(T,N) \\ h(T,1) & h(T,2) & \cdots & h(T,N) & h(T,N+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h(T,N-1) & h(T,N) & \cdots & h(T,2N-2) & h(T,2N-1) \\ h(T,N) & h(T,N+1) & \cdots & h(T,2N-1) & h(T,2N) \end{vmatrix}$$

$D\binom{i}{j}$ = same as D except that the ith row and the jth column are removed from it,

$D\binom{i,k}{j,l}$ = same as D except that the ith and kth rows and the jth and lth columns are removed from it.

Equation (2.14*a*) for the even $n = 2N$ is proved as follows. With the above notations we find that equation (2.14*a*) is expressed as

$$D\binom{1}{N}D\binom{N+1}{N+1} - D\binom{1}{N+1}D\binom{N+1}{N} = D\cdot D\binom{1, N+1}{N, N+1}, \tag{C.1}$$

which is nothing but the Jacobi formula for the determinant.

Equation (2.14*a*) for the odd $n = 2N+1$ is proved as follows. We introduce the notation

$$\begin{vmatrix} h(T,m) & h(T,m+1) & \cdots & h(T,m+N) \\ h(T,m+1) & h(T,m+2) & \cdots & h(T,m+N+1) \\ \vdots & \vdots & \ddots & \vdots \\ h(T,m+N) & h(T,m+N+1) & \cdots & h(T,m+2N) \end{vmatrix}$$
$$=: [m, m+1, \ldots, m+N]_{N+1}. \tag{C.2}$$

Then equation (2.14*a*) can be written for $n = 2N+1$ as follows:

$$\frac{\mathrm{d}g_{n+1}}{\mathrm{d}T}g_n - g_{n+1}\frac{\mathrm{d}g_n}{\mathrm{d}T} - g_{n+2}g_{n-1}$$
$$= [0, 1, \ldots, N-1, N+1]_{N+1}[1, \ldots, N-1, N]_N$$

$$
\begin{aligned}
&- [0, 1, \ldots, N-1, N]_{N+1}[1, \ldots, N-1, N+1]_N \\
&- [1, \ldots, N-1, N, N+1]_{N+1}[0, \ldots, N-1]_N \\
= &\begin{vmatrix} 0 & 1 & \cdots & N-1 & \oslash & N & N+1 \\ 0 & \oslash & 1 & \cdots & N-1 & N & N+1 \end{vmatrix} \\
= &\ 0,
\end{aligned} \tag{C.3}
$$

where the upper elements in the $(2N+1)\times(2N+1)$ determinant (C.3) have $(N+1)$ components and its lower elements have N components. Thus, we have proved equation (2.14*a*). Equation (2.14*b*) is nothing but the complex conjugate of (2.14*a*).

This completes the proof that g_m and g_m^* expressed by equations (2.29*a*)–(2.29*d*) are solutions of equations (2.14*a*) and (2.14*b*). □

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