Extended resolvent and inverse scattering with an application to KPI

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We present in detail an extended resolvent approach for investigating linear problems associated to $2+1$ dimensional integrable equations. Our presentation is based as an example on the nonstationary Schrödinger equation with potential being a perturbation of the one-soliton potential by means of a decaying two-dimensional function. Modification of the inverse scattering theory as well as properties of the Jost solutions and spectral data as follows from the resolvent approach are given. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1587874]

I. INTRODUCTION

The extended resolvent approach to the study of the spectral theory of differential operators was developed in Refs. 1–8 as a method that from one side unifies all known approaches of the inverse scattering theory, such as dressing transformation, nonlocal Riemann–Hilbert problem, d-bar method and, from the other side, enables considering operators of more generic type, say, with nontrivial asymptotic behavior at space infinity. Our presentation is based on the study of the well known differential operator

$$\mathcal{L}(x,\partial_x)=i\partial_{x_2}+\partial_{x_1}^2-u(x),\quad x=(x_1,x_2),\tag{1.1}$$

i.e., the nonstationary Schrödinger equation $\mathcal{L}(x,\partial_x)\Phi(x,k)=0$ which is the linear problem associated to the Kadomtsev–Petviashvili I equation (KPI):[9]

$$(u_t-6uu_{x_1}+u_{x_1x_1x_1})_{x_1}=3u_{x_2x_2}.\tag{1.2}$$

This equation has been known to be integrable for about three decades.[10,11] The Cauchy problem for this equation with rapidly decaying initial data was studied by using the inverse scattering method in Refs. 12–15. At the same time, it is well known that (1.2) is a $(2+1)$-dimensional generalization of the famous Korteweg–de Vries (KdV) equation. Indeed, if $u_1(t,x_1)$ obeys KdV, then

$$u(t,x_1,x_2)=u_1(t,x_1+\mu x_2+3\mu^2 t)\tag{1.3}$$

solves (1.2) for an arbitrary constant $\mu\in\mathbb{R}$. Thus, it is natural to consider solutions of (1.2) that are not decaying in all directions at space infinity but have one-dimensional rays with behavior of

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the type (1.3). The scattering theory for the simplest example of such potentials in (1.1) was developed in Ref. 16. There, the Cauchy problem for the KPI equation was considered with initial data

$$u(x)=u_1(x_1)+u_2(x), \tag{1.4}$$

where $u_1(x_1)$ is the zero time value of the one-soliton solution of the KdV equation and $u_2(x)$ is a smooth real function that decays rapidly enough on the (x_1,x_2)-plane. We consider here a simplified case of (1.3) in which $\mu=0$. The generic case is reconstructed by means of the Galileo invariance of (1.2), that is, if $u(t,x)$ is a solution of KPI, then

$$\tilde{u}(t,x_1,x_2)=u(t,x_1+\mu x_2+3\mu^2 t,x_2+6\mu t) \tag{1.5}$$

also obeys this equation.

In Ref. 1 it was mentioned that the standard approach to the spectral theory of the operator (1.1) based on the integral equation for the Jost solution fails for potentials of the above type. In Ref. 16 the direct problem was studied by using a modified integral equation for the Jost solution, i.e., an equation that involves as background solution the Jost solution of the one-soliton potential and corresponding Green's function. An appropriate Jost solution was introduced and it was shown that, in addition to a standard jump across the real k-axis, where k is the spectral parameter, it also has a jump across a segment of the imaginary axis of the complex k-plane. In the modified formulation of the direct and inverse problems given in Ref. 16 some essential properties of the Jost solutions and relations between spectral data were stated without proof. The reason for this was that a technique based on the Jost solutions themselves is not enough for the study of these properties and especially for investigating relations among spectral data. This gap can be filled effectively in the framework of the extended resolvent approach that we present in detail in this article. The outline of the article is as follows. In Sec. II we give the basic notions of the resolvent approach. In particular, we demonstrate that in this framework the known results for a decaying potential in (1.1) follow readily. In Sec. III we consider the theory of the two-dimensional operator (1.1) with the pure one-dimensional $[u_2(x)\equiv 0$ in (1.4)] one-soliton potential $u_1(x_1)$. In Sec. IV we apply the resolvent approach to the operator (1.1) with a potential $u(x)$ that is a perturbation of this one-dimensional potential. We introduce and study the properties of the corresponding resolvent and Green's functions, Jost and auxiliary Jost solutions and advanced/retarded solutions. In Sec. IV E we investigate the properties of the spectral data and derive their characterization equations. In Sec. V we summarize the main aspects of the spectral theory developed in the framework of the extended resolvent approach.

II. BASIC OBJECTS OF THE RESOLVENT APPROACH

A. Extension of differential operators and resolvent

Let $\mathcal{A}=\mathcal{A}(x,\partial_x)$ denote a differential operator with kernel

$$A(x,x')=\mathcal{A}(x,\partial_x)\,\delta(x-x'), \quad x=(x_1,x_2), \tag{2.1}$$

$\delta(x)=\delta(x_1)\,\delta(x_2)$ being the two-dimensional δ-function. Here we consider the two-dimensional situation, but the whole procedure can be applied in any dimension. In what follows we consider differential operators whose kernels $A(x,x')$ belong to the space $\mathcal{S}'$ of tempered distributions of the four real variables x and x'. Differential operators are polynomials with respect to the derivatives. In order to exploit this property we introduce the **extension** of differential operators, i.e., to any differential operator $\mathcal{A}$ we associate the differential operator $A(s)$ with kernel

$$A(x,x';s)\equiv e^{-s(x-x')}A(x,x')=\mathcal{A}(x,\partial_x+s)\,\delta(x-x'), \tag{2.2}$$

where all variables x, x', $s\in\mathbb{R}^2$. Under the above condition kernels of the extended operators form a subclass in the space $\mathcal{S}'$ of tempered distributions $A(x,x';s)$ of six real variables. We consider generic elements $A(x,x';s)$, $B(x,x';s)$, etc., of this space $\mathcal{S}'$ as operators (not necessarily differentiable) $A(s)$, $B(s)$, etc., with the standard composition rule

$$(AB)(x,x';s)=\int dx''\, A(x,x'';s)\, B(x'',x';s). \tag{2.3}$$

Since the kernels are distributions this composition is neither necessarily defined for all pairs of operators nor necessarily associative. On the space of these distributions we define the operation of Hermitian conjugation $A^{\dagger}$ as

$$A^{\dagger}(x,x';s)=\overline{A(x',x;-s)}, \tag{2.4}$$

where bar denotes complex conjugation and in the case of matrix operators the matrix on the r.h.s. must be transposed.

The kernels of the extended differential operators depend polynomially on the variables $s=(s_1,s_2)$ so that the analytic continuation of the kernels with respect to these variables is admissible. In what follows we denote such kernels as $A(x,x';\mathbf{s})$ using boldface font for the complex variable $\mathbf{s}$ which has s as real part. So, for the variable $\mathbf{s}$ we write

$$\mathbf{s}=\mathbf{s}_{\mathfrak{R}}+i\mathbf{s}_{\mathfrak{I}}=(\mathbf{s}_1,\mathbf{s}_2)\in\mathbb{C}^2, \quad \mathbf{s}_{\mathfrak{R}}\equiv s\in\mathbb{R}^2. \tag{2.5}$$

In the following we will use the same notation $\mathbf{k}=\mathbf{k}_{\mathfrak{R}}+i\mathbf{k}_{\mathfrak{I}}\equiv k+i\mathbf{k}_{\mathfrak{I}}$ for the other complex spectral parameter $\mathbf{k}\in\mathbb{C}$ we introduce.

Thanks to (2.2) continuation (2.5) (analytic for the differential operators) is given by

$$A(x,x';\mathbf{s})=e^{-i\mathbf{s}_{\mathfrak{I}}(x-x')}A(x,x';\mathbf{s}_{\mathfrak{R}}). \tag{2.6}$$

Since this formula is well defined in the space $\mathcal{S}'$ for any operator $A(s)$, it can be used as the definition of the continuation (in general not analytic) of a generic $A(s)$ in the complex domain, $\mathbf{s}\in\mathbb{C}^2$.

An operator A can have an inverse in the sense of the composition law (2.3), say $AA^{-1}=I$ or $A^{-1}A=I$ (in general left and right inverse can be different), where I is the unity operator in $\mathcal{S}'$,

$$I(x,x';s)=\delta(x-x'). \tag{2.7}$$

Thanks to (2.6) continuation to the complex domain of the operator inverse to (extended) differential operator $A(s)$ equals to the inverse of continued differential operator $A(\mathbf{s})$. Therefore, we can impose to the inverse the additional condition that the product

$$A(\mathbf{s}+\mathbf{s}')^{-1}A(\mathbf{s})^{-1}, \quad \mathbf{s},\mathbf{s}'\in\mathbb{C}, \tag{2.8}$$

is a bounded function of $\mathbf{s}'$ in a neighborhood of $\mathbf{s}'=0$. Then, if the inversion $A^{-1}(s)$ exists, it is unique and, therefore, in the following in order to guarantee the uniqueness of the inverse we will always require that this condition is fulfilled.

In order to clarify the meaning of the introduced extension of differential operators it is convenient to consider the "shifted" Fourier transform

$$A(p;\mathbf{s})=\frac{1}{(2\pi)^2}\int dx\int dx'\, e^{i(p-\mathbf{s}_{\mathfrak{I}})x+i\mathbf{s}_{\mathfrak{I}}x'}A(x,x';\mathbf{s}_{\mathfrak{R}}), \tag{2.9}$$

where $p\in\mathbb{R}^2$, $px=p_1x_1+p_2x_2$, and $\mathbf{s}\in\mathbb{C}^2$ and to consider it as the kernel of the operator $A(s)$ in the transformed p space. In this space the extension of a differential operator $\mathcal{A}$ with constant coefficients is particularly simple. Indeed, we have $A(p;\mathbf{s})=a(\mathbf{s})\delta(p)$, where $a(\mathbf{s})$ is a polynomial in $\mathbf{s}$. For the kernel of the inverse operator we get trivially $A^{-1}(p;\mathbf{s})=\delta(p)/a(\mathbf{s})$ and condition (2.8) means that possible terms proportional to $\delta(a(\mathbf{s}))$ must be omitted.

Let D_j denote the extension of the differential operator ∂_{x_j} $(j=1,2)$, i.e., according to (2.2) let

$$D_j(x,x';s)=(\partial_{x_j}+s_j)\,\delta(x-x'),\quad j=1,2. \tag{2.10}$$

In terms of the Fourier transform (2.9) $D_j(p;\mathbf{s})=\mathbf{s}_j\,\delta(p)$ and then $D_j^{-1}(p;\mathbf{s})=\mathbf{s}_j^{-1}\,\delta(p)$, so that inverting (2.9) we get

$$D_j^{-1}(x,x';s)=e^{-s(x-x')}[\theta(x_j-x_j')-\theta(-s_j)]\,\delta(x_{j+1}-x_{j+1}'), \tag{2.11}$$

where $j+1\equiv(j+1)\mathrm{mod}2$. These operators are just the standard resolvents of the operators ∂_{x_j}. This observation motivates the name extended resolvent for the mathematical object we introduced. In fact, we added a complex (spectral) parameter not to the operator itself but to the operators ∂_{x_j}. Below we discuss the relevance of this modification and compare our approach to the standard spectral theory of differential operators.

By (2.10) we have that $D_j(x,x';\mathbf{s})=(\partial_{x_j}+\mathbf{s}_j)\,\delta(x-x')$ that is, of course, an analytic (polynomial) function of $\mathbf{s}$. But for the continuation of the inverse operator we get by (2.11) and (2.6)

$$D_j^{-1}(x,x';\mathbf{s})=e^{-\mathbf{s}(x-x')}[\theta(x_j-x_j')-\theta(-\mathbf{s}_{j\mathfrak{R}})]\,\delta(x_{j+1}-x_{j+1}'), \tag{2.12}$$

that is not analytic with respect to $\mathbf{s}_j$. Moreover, let us perform the operation inverse to imbedding (2.2), i.e., to any operator $A(s)$ with kernel $A(x,x';s)$ we associate its "hat-kernel"

$$\hat{A}(x,x';s)=e^{s(x-x')}A(x,x';s). \tag{2.13}$$

It is clear that these objects (in the generic case) do not belong to the space $\mathcal{S}'(\mathbb{R}^6)$ of distributions, while we still can use their composition relation

$$(\widehat{AB})(x,x';s)=\int dx''\,\hat{A}(x,x'';s)\,\hat{B}(x'',x';s) \tag{2.14}$$

derived from (2.3). Let now $A(s)$ be the extension of a differential operator $\mathcal{A}$. Then of course $\hat{A}(x,x';s)=A(x,x')$, i.e., the kernel of the original differential operator. In particular this means that $\hat{A}(x,x';s)$ is independent of the variables s. At the same time, as demonstrated by example (2.11), for an operator inverse to a differential operator, the dependence on s of the kernel $\widehat{A^{-1}}(x,x';s)$ is in general nontrivial, for instance,

$$\widehat{D_j^{-1}}(x,x';s)=[\theta(x_j-x_j')-\theta(-s_j)]\,\delta(x_{j+1}-x_{j+1}'). \tag{2.15}$$

Using the "hat-operation," for any differential operator $\mathcal{A}(x,\partial_x)$, equalities $AA^{-1}=A^{-1}A=I$ take the form

$$\mathcal{A}(x,\partial_x)\widehat{A^{-1}}(x,x';s)=\mathcal{A}^{\mathrm{d}}(x',\partial_{x'})\widehat{A^{-1}}(x,x';s)=\delta(x-x'), \tag{2.16}$$

where $\mathcal{A}^{\mathrm{d}}$ is the operator dual to $\mathcal{A}$. Thus the hat-kernel (2.13) of the operator inverse to an extended differential operator gives a two-parametric ($s\in\mathbb{R}^2$) family of Green's functions of the operator $\mathcal{A}$. In what follows for equalities of the type (2.16) we use the following special notation,

$$\vec{\mathcal{A}}\,\widehat{A^{-1}}(s)=\widehat{A^{-1}}(s)\overleftarrow{\mathcal{A}}=I, \tag{2.17}$$

where $\vec{\mathcal{A}}$ denotes the operator $\mathcal{A}$ applied to the x-variable of the function $\widehat{A^{-1}}(x,x';s)$ and $\overleftarrow{\mathcal{A}}$ denotes the operator dual to $\mathcal{A}$ applied to the x'-variable of the same kernel. We also see that any discontinuity of the kernel $\widehat{A^{-1}}(x,x';s)$ with respect to the s-variables belongs to the null space of the operator $\mathcal{A}$ and its dual. The same is valid for the derivatives of such kernels with respect to the s-variables. Let us mention that for kernels continued by means of (2.6) we have by (2.13) the relation

$$A(x,x';\mathbf{s}) = e^{-\mathbf{s}(x-x')}\hat{A}(x,x';\mathbf{s}_{\mathfrak{R}}), \tag{2.18}$$

and for the inversion $A^{-1}(\mathbf{s})$ of a differential operator $A(\mathbf{s})$

$$A(\mathbf{s})\frac{\partial A^{-1}(\mathbf{s})}{\partial \overline{\mathbf{s}_j}} = \frac{\partial A^{-1}(\mathbf{s})}{\partial \overline{\mathbf{s}_j}}A(\mathbf{s}) = 0. \tag{2.19}$$

Thus the d-bar derivatives of the inverse operator also annulate the differential operator and its dual, while by (2.13)

$$\left(\frac{\partial A^{-1}(\mathbf{s})}{\partial \overline{\mathbf{s}_j}}\right)^{\widehat{}}\Bigg|_{\mathbf{s}_{\mathfrak{I}}=0} = \frac{1}{2}\frac{\partial \widehat{A^{-1}}(s)}{\partial s_j}, \quad s = \mathbf{s}_{\mathfrak{R}}, \tag{2.20}$$

where both derivatives are considered in the sense of distributions.

B. Resolvent approach in the case of rapidly decaying potential

1. Definition of the resolvent

The extension of operator $\mathcal{L}(x,\partial_x)$ in (1.1) is given by

$$L(s) = L_0(s) - U, \tag{2.21}$$

where

$$L_0 = iD_2 + D_1^2 \tag{2.22}$$

has kernel

$$L_0(x,x';s) = [i(\partial_{x_2}+s_2) + (\partial_{x_1}+s_1)^2]\delta(x-x'), \tag{2.23}$$

and U, which is called the potential operator, has kernel

$$U(x,x';s) = u(x)\delta(x-x'), \tag{2.24}$$

which is independent of s. Below we always suppose that $u(x)$ is real, which by (2.4) is equivalent to

$$L^{\dagger} = L. \tag{2.25}$$

The main object of our approach is the (extended) resolvent $M(s)$ of the operator $L(s)$, which is defined as the inverse of the operator L, that is, M satisfies

$$LM = ML = I, \tag{2.26}$$

and obeys condition (2.8). The main advantage of using the extended resolvent is that its derivatives with respect to s are given in terms of its reduced values that are just the Jost solutions (and their generalizations) of the linear problem under consideration. In their turn discontinuities and derivatives of these solutions lead us to the spectral data, again given as ''further'' reductions of the resolvent. Let us consider first the resolvent M_0 of the bare operator L_0. In terms of the Fourier transformation (2.9) and thanks to condition (2.8) we get for the kernel of this operator $M_0(p;\mathbf{s}) = \delta(p)(i\mathbf{s}_2 + \mathbf{s}_1^2)^{-1}$. Inverting transformation (2.9) we get for the hat-kernel

$$\hat{M}_0(x,x';s)=\frac{1}{2\pi i}\int_{\mathbf{k}_{\mathfrak{I}}=s_1} d\mathbf{k}_{\mathfrak{R}}[\theta(x_2-x_2')-\theta(2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}}-s_2)]\,\Phi_0(x,\mathbf{k})\Psi_0(x',\mathbf{k}), \tag{2.27}$$

where we introduced the functions

$$\Phi_0(x,\mathbf{k})=e^{-i\mathbf{k}x_1-i\mathbf{k}^2x_2}, \quad \Psi_0(x,\mathbf{k})=e^{i\mathbf{k}x_1+i\mathbf{k}^2x_2}, \tag{2.28}$$

where $\mathbf{k}=\mathbf{k}_{\mathfrak{R}}+i\mathbf{k}_{\mathfrak{I}}\in\mathbb{C}$ is a complex (one-dimensional) spectral parameter for which we again use boldface font and also, in order to simplify notation, standard font for its real part, i.e., $\mathbf{k}_{\mathfrak{R}}\equiv k$. The functions $\Phi_0(x,\mathbf{k})$ and $\Psi_0(x,\mathbf{k})$ which naturally appeared in (2.27) obey the differential equations

$$\mathcal{L}_0(x,\partial_x)\Phi_0(x,\mathbf{k})\equiv(i\partial_{x_2}+\partial_{x_1}^2)\Phi_0(x,\mathbf{k})=0, \tag{2.29}$$

$$\mathcal{L}_0^{\mathrm{d}}(x,\partial_x)\Psi_0(x,\mathbf{k})\equiv(-i\partial_{x_2}+\partial_{x_1}^2)\Psi_0(x,\mathbf{k})=0, \tag{2.30}$$

i.e., solve the nonstationary Schrödinger equation and its dual in the case of zero potential. Thus they can be considered as the Jost solutions for this trivial case. Notice also that they obey the conjugation property

$$\overline{\Phi_0(x,\mathbf{k})}=\Psi_0(x,\overline{\mathbf{k}}). \tag{2.31}$$

By using notation (2.17) we can write

$$\vec{\mathcal{L}}_0\hat{M}_0(s)=\hat{M}_0(s)\overleftarrow{\mathcal{L}}_0=I, \tag{2.32}$$

showing that the hat version of the extended resolvent $\hat{M}_0(s)$ is a two-parametric set of Green's functions of operator (1.1) and its dual. In analogy with these notations we write

$$\vec{\mathcal{L}}_0\Phi_0(\mathbf{k})=0, \quad \Psi_0(\mathbf{k})\overleftarrow{\mathcal{L}}_0=0, \tag{2.33}$$

considering $\Phi_0(\mathbf{k})$ and $\Psi_0(\mathbf{k})$ as ''vector'' and ''covector'' with ''components'' labeled by the variable x.

The resolvent M_0 given in (2.27) obeys the following properties. It is a continuous function of s when $s\neq 0$, it is bounded but discontinuous at $s=0$, it is self-adjoint [cf. (2.25)], that is $M_0^\dagger=M_0$, and from (2.27) we have

$$\frac{\partial\hat{M}_0(s)}{\partial s_1}=\frac{i}{\pi}\int_{\mathbf{k}_{\mathfrak{I}}=s_1} d\mathbf{k}_{\mathfrak{R}}\,\overline{\mathbf{k}}\,\delta(2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}}-s_2)\,\Phi_0(\mathbf{k})\otimes\Psi_0(\mathbf{k}), \tag{2.34}$$

$$\frac{\partial\hat{M}_0(s)}{\partial s_2}=\frac{1}{2\pi i}\int_{\mathbf{k}_{\mathfrak{I}}=s_1} d\mathbf{k}_{\mathfrak{R}}\,\delta(2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}}-s_2)\,\Phi_0(\mathbf{k})\otimes\Psi_0(\mathbf{k}), \tag{2.35}$$

where, in correspondence with the ''vector'' interpretation of the Jost solutions, the direct product in (2.34), (2.35) is defined in the standard way as an operator with kernel

$$(\Phi_0(\mathbf{k})\otimes\Psi_0(\mathbf{k}))(x,x')=\Phi_0(x,\mathbf{k})\Psi_0(x',\mathbf{k}). \tag{2.36}$$

Derivatives in (2.34) and (2.35) can be considered in the standard sense when $s_1\neq 0$, while in vicinity of $s_1=0$ they must be understood in the distributional sense. It is easy to see that the integral in (2.27) is exponentially divergent when, say, $s_2\to\infty$. Thus, indeed, $\hat{M}_0(x,x';s)\notin\mathcal{S}'(\mathbb{R}^6)$.

2. The resolvent and Hilbert identity

The resolvent of the operator L can also be defined as the solution of the integral equations

$$M = M_0 + M_0 U M, \quad M = M_0 + M U M_0. \tag{2.37}$$

Under a small norm assumption on the potential we expect that the solution M exists and is unique (the same for both integral equations) if the condition (2.8) is required. Taking into account the properties of M_0 given above, we get that thanks to the reality of the potential $u(x)$ the resolvent is self-adjoint,

$$M^{\dagger} = M, \tag{2.38}$$

$M(s)$ is a continuous function of s when $s \neq 0$ and discontinuous at $s = 0$, and by (2.26) has the asymptotic expansion

$$M(s_1, s_2 + \mathbf{s}_2') = \frac{I}{i\mathbf{s}_2'} + \frac{L(s)}{\mathbf{s}_2'^2} + \cdots, \quad \mathbf{s}_2' \in \mathbb{C}, \quad \mathbf{s}_2' \to \infty. \tag{2.39}$$

As we mentioned above essential information on the spectrum of the operator $\mathcal{L}$ can be derived from the d-bar derivatives of the resolvent $M(\mathbf{s})$ continued in the complex domain. In order to study these derivatives we use the following analog of the well known Hilbert identity,

$$M' - M = -M'(L' - L)M, \tag{2.40}$$

where L' is another operator of the type (1.1) and M' is its resolvent. Let us choose $U' = U$ and let $L = L(\mathbf{s})$ and $L' = L(\mathbf{s}')$ and the same for their resolvents. Then (2.40) can be written in the form

$$M(\mathbf{s}') - M(\mathbf{s}) = -M(\mathbf{s}')(L_0(\mathbf{s}') - L_0(\mathbf{s}))M(\mathbf{s}), \tag{2.41}$$

or by (2.26) as

$$M(\mathbf{s}') - M(\mathbf{s}) = M(\mathbf{s}')L_0(\mathbf{s}')(M_0(\mathbf{s}') - M_0(\mathbf{s}))L_0(\mathbf{s})M(\mathbf{s}), \tag{2.42}$$

that gives

$$\frac{\partial M(\mathbf{s})}{\partial \overline{\mathbf{s}}_j} = M(\mathbf{s})L_0(\mathbf{s}) \frac{\partial M_0(\mathbf{s})}{\partial \overline{\mathbf{s}}_j} L_0(\mathbf{s})M(\mathbf{s}). \tag{2.43}$$

In terms of hat-kernels as defined in (2.13), thanks to (2.20), we get

$$\frac{\partial \hat{M}(s)}{\partial s_j} = \hat{M}(s)\overleftarrow{\mathcal{L}}_0 \frac{\partial \hat{M}_0(s)}{\partial s_j} \vec{\mathcal{L}}_0 \hat{M}(s), \quad j = 1,2, \tag{2.44}$$

and then by (2.34) and (2.35) for $s_1 \neq 0$

$$\frac{\partial \hat{M}(s)}{\partial s_1} = \frac{i}{\pi} \int_{\mathbf{k}_{\mathfrak{I}} = s_1} d\mathbf{k}_{\mathfrak{R}} \overline{\mathbf{k}}\, \delta(2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}} - s_2)\, \Phi(\mathbf{k}) \otimes \Psi(\mathbf{k}), \tag{2.45}$$

$$\frac{\partial \hat{M}(s)}{\partial s_2} = \frac{1}{2\pi i} \int_{\mathbf{k}_{\mathfrak{I}} = s_1} d\mathbf{k}_{\mathfrak{R}}\, \delta(2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}} - s_2)\, \Phi(\mathbf{k}) \otimes \Psi(\mathbf{k}), \tag{2.46}$$

where we introduced the functions

$$\Phi(x,\mathbf{k})=\int dx'\,(\mathcal{L}_0^{\mathrm{d}}(x',\partial_{x'})G(x,x',\mathbf{k}))\Phi_0(x',\mathbf{k}), \tag{2.47}$$

$$\Psi(x',\mathbf{k})=\int dx\,\Psi_0(x,\mathbf{k})\,\mathcal{L}_0(x,\partial_x)\,G(x,x',\mathbf{k}), \tag{2.48}$$

with $G(x,x',\mathbf{k})$ defined as a specific value of the resolvent itself

$$G(x,x',\mathbf{k})=\hat{M}(x,x';\mathbf{k}_{\mathfrak{I}},2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}}). \tag{2.49}$$

In what follows we consider the function $G(x,x',\mathbf{k})$ as the kernel of the operator $G(\mathbf{k})$ and the functions $\Phi(x,\mathbf{k})$ and $\Psi(x',\mathbf{k})$ as vector $\Phi(\mathbf{k})$ and covector $\Psi(\mathbf{k})$. For shortness we write equations of the type (2.47)–(2.49) as

$$\Phi(\mathbf{k})=G(\mathbf{k})\overleftarrow{\mathcal{L}}_0\Phi_0(\mathbf{k}),\quad \Psi(\mathbf{k})=\Psi_0(\mathbf{k})\vec{\mathcal{L}}_0G(\mathbf{k}), \tag{2.50}$$

$$G(\mathbf{k})=\hat{M}(s)\big|_{s=(\mathbf{k}_{\mathfrak{I}},2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}})}. \tag{2.51}$$

Relation (2.49) shows that the function $G(x,x',\mathbf{k})$ is in one-to-one correspondence with the resolvent $\hat{M}(x,x';s)$ iff $\mathbf{k}_{\mathfrak{I}}\neq 0$, that is, iff the points along the above mentioned discontinuity of the resolvent at $s=0$ are excluded. In order to study this discontinuity we introduce the following notation for the specific limits of the resolvent at this point of discontinuity,

$$G_{\pm}(x,x')=\lim_{s_2\to\pm 0}\ \lim_{s_1\to 0}\hat{M}(x,x';s), \tag{2.52}$$

where the limit $s_1\to 0$ is independent of the sign. Now we choose in (2.42) both s and s' real, $s_1'=s_1=0$ and consider the limits $s_2'\to\pm 0$, $s_2\to\mp 0$. We get

$$G_+-G_-=G_{\pm}\overleftarrow{\mathcal{L}}_0(G_{0,+}-G_{0,-})\vec{\mathcal{L}}_0G_{\mp}, \tag{2.53}$$

where $G_{0,\pm}$ is defined in terms of the bare resolvent $\hat{M}_0$ like in (2.52), so that by (2.27) we have

$$G_{0,\pm}(x,x')=\frac{\pm\theta(\pm(x_2-x_2'))}{2\pi i}\int dk\,\Phi_0(x,k)\Psi_0(x',k), \tag{2.54}$$

and

$$G_{0,+}-G_{0,-}=\frac{1}{2\pi i}\int dk\,\Phi_0(k)\otimes\Psi_0(k). \tag{2.55}$$

Inserting this relation in the r.h.s. of (2.53) we derive the equality

$$G_+-G_-=\frac{1}{2\pi i}\int dk\,\Phi_{\pm}(k)\otimes\Psi_{\mp}(k), \tag{2.56}$$

where we introduced the functions $\Phi_{\pm}(x,k)$ and $\Psi_{\pm}(x,k)$, which, in analogy with (2.50) and using the same notation, are defined by

$$\Phi_{\pm}(k)=G_{\pm}\overleftarrow{\mathcal{L}}_0\Phi_0(k),\quad \Psi_{\pm}(k)=\Psi_0(k)\vec{\mathcal{L}}_0G_{\pm}. \tag{2.57}$$

Next we consider in detail the properties of all the objects introduced so far, $G(\mathbf{k})$, $\Phi(\mathbf{k})$, $\Psi(\mathbf{k})$, $G_{\pm}(\mathbf{k})$, $\Phi_{\pm}(k)$, and $\Psi_{\pm}(k)$.

3. Properties of the Green's function

Thanks to (2.17) it is clear that $G(\mathbf{k})$ defined in (2.49) is a Green's function of the operator $\mathcal{L}$ depending on the complex parameter $\mathbf{k}$,

$$\vec{\mathcal{L}}G(\mathbf{k})=G(\mathbf{k})\overleftarrow{\mathcal{L}}=I. \tag{2.58}$$

Properties of this Green's function follow from the properties of $M(s)$. In particular, thanks to the definition (2.4) and to (2.38) (i.e., reality of the potential u) we get

$$\overline{G(x,x',\mathbf{k})}=G(x',x,\overline{\mathbf{k}}). \tag{2.59}$$

Applying the reduction (2.49) to equalities (2.37) we get that this function obeys the integral equations

$$G(\mathbf{k})=G_0(\mathbf{k})+G_0(\mathbf{k})UG(\mathbf{k}),\quad G(\mathbf{k})=G_0(\mathbf{k})+G(\mathbf{k})UG_0(\mathbf{k}), \tag{2.60}$$

where the Green's function $G_0(\mathbf{k})$ of the operator $\mathcal{L}_0$ is defined by the general formula (2.51) in terms of M_0 and thanks to (2.27) it equals

$$G_0(x,x',\mathbf{k})=\frac{1}{2\pi i}\int dk'\,[\,\theta(x_2-x_2')-\theta(\mathbf{k}_\Im k')\,]\,\Phi_0(x,\mathbf{k}'+k)\Psi_0(x',\mathbf{k}+k'). \tag{2.61}$$

By (2.60) it is easy to check that the function

$$g(x,x',\mathbf{k})=e^{i\mathbf{k}(x_1-x_1')+i\mathbf{k}^2(x_2-x_2')}G(x,x',\mathbf{k}) \tag{2.62}$$

is a bounded function of its arguments and that

$$\lim_{\mathbf{k}\to\infty}g(x,x',\mathbf{k})=0 \tag{2.63}$$

if the potential $u(x)$ decays rapidly enough. The function $G(x,x',\mathbf{k})$ is a continuously differentiable function of $\mathbf{k}$ in the whole complex plane $\mathbb{C}$ with exception of the real axis $\mathbf{k}_\Im=0$, as follows from (2.49). By (2.45) and (2.46) for $\mathbf{k}_\Im\neq 0$ we have

$$\frac{\partial G(\mathbf{k})}{\partial \mathbf{k}_\Re}=\frac{\operatorname{sgn}\mathbf{k}_\Im}{2\pi i}\Phi(\mathbf{k})\otimes\Psi(\mathbf{k}),\quad \frac{\partial G(\mathbf{k})}{\partial \mathbf{k}_\Im}=\frac{\operatorname{sgn}\mathbf{k}_\Im}{2\pi}\Phi(\mathbf{k})\otimes\Psi(\mathbf{k}), \tag{2.64}$$

so that in the complex domain this Green's function is analytic,

$$\frac{\partial G(\mathbf{k})}{\partial\overline{\mathbf{k}}}=0,\quad \mathbf{k}_\Im\neq 0, \tag{2.65}$$

and discontinuous at the real axis.

The Hilbert identity (2.41) allows us also to find relations among the Green's function at different values of the parameter $\mathbf{k}$. For this sake we choose $\mathbf{s}_1'=\mathbf{s}_1=s_1\in\mathbb{R}$ and let $\mathbf{s}_2',\mathbf{s}_2\in\mathbb{C}$. By (2.23) $\mathcal{L}(s_1,\mathbf{s}_2')-\mathcal{L}(s_1,\mathbf{s}_2)=i(\mathbf{s}_2'-\mathbf{s}_2)I$. Inserting this equality in (2.41) we can divide both parts by $\mathbf{s}_2'-\mathbf{s}_2$ and thanks to condition (2.8) we get

$$M(s_1,\mathbf{s}_2')M(s_1,\mathbf{s}_2)=i\frac{M(s_1,\mathbf{s}_2')-M(s_1,\mathbf{s}_2)}{\mathbf{s}_2'-\mathbf{s}_2}. \tag{2.66}$$

In terms of the hat-kernel of the resolvent by (2.18) we then have

$$\int dx'\, e^{(\mathbf{s}_2'-\mathbf{s}_2)x_2'}\hat{M}(x,x';s_1,\mathbf{s}_{2\mathfrak{R}}')\hat{M}(x',x'';s_1,\mathbf{s}_{2\mathfrak{R}})$$

$$=i\,\frac{e^{(\mathbf{s}_2'-\mathbf{s}_2)x_2''}\hat{M}(x,x'';s_1,\mathbf{s}_{2\mathfrak{R}}')-e^{(\mathbf{s}_2'-\mathbf{s}_2)x_2}\hat{M}(x,x'';s_1,\mathbf{s}_{2\mathfrak{R}})}{\mathbf{s}_2'-\mathbf{s}_2}.$$

Next we perform the Fourier transform with respect to the difference $\mathbf{s}_{2\mathfrak{I}}'-\mathbf{s}_{2\mathfrak{I}}$ and denote $\mathbf{s}_{2\mathfrak{R}}'=s_2'$, $\mathbf{s}_{2\mathfrak{R}}=s_2$. We get for $s_1'=s_1$

$$\int dx_1'\,\hat{M}(x,x';s')\hat{M}(x',x'';s)=i\hat{M}(x,x'';s')[\theta(x_2''-x_2')-\theta(s_2-s_2')]$$

$$-i\hat{M}(x,x'';s)[\theta(x_2-x_2')-\theta(s_2-s_2')].$$

Finally, we introduce $\mathbf{k}_{\mathfrak{I}}=s_1'=s_1$, $\mathbf{k}_{\mathfrak{R}}=s_2/(2s_1)$, $k'=(s_2'-s_2)/(2s_1)$. Then $s'=(\mathbf{k}_{\mathfrak{I}},2(\mathbf{k}_{\mathfrak{R}}+k')\mathbf{k}_{\mathfrak{I}})$ and $s=(\mathbf{k}_{\mathfrak{I}},2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}})$ and using definition (2.49) we can rewrite the above equality in terms of the Green's function as

$$\int dx_1'\,G(x,x',\mathbf{k}+k')G(x',x'',\mathbf{k})$$

$$=iG(x,x'',\mathbf{k}+k')[\theta(x_2''-x_2')-\theta(-\mathbf{k}_{\mathfrak{I}}k')]-iG(x,x'',\mathbf{k})[\theta(x_2-x_2')-\theta(-\mathbf{k}_{\mathfrak{I}}k')],$$

$$\mathbf{k}\in\mathbb{C},\quad k'\in\mathbb{R}. \tag{2.67}$$

Properties of the functions $G_\pm(x,x')$ as well follow from the properties of the resolvent. Both of them are also Green's functions of the operator (1.1) and its dual,

$$\vec{\mathcal{L}}G_\pm=G_\pm\overleftarrow{\mathcal{L}}=I, \tag{2.68}$$

obey the conjugation property

$$\overline{G_\pm(x,x')}=G_\mp(x',x), \tag{2.69}$$

and the integral equations

$$G_\pm=G_{0,\pm}+G_{0,\pm}UG_\pm,\quad G_\pm=G_{0,\pm}+G_\pm UG_{0,\pm}, \tag{2.70}$$

where $G_{0,\pm}$ is given in (2.54).

The limiting values of the Green's function $G(\mathbf{k})$ on the real axis in a sense must be close to the Green's functions $G_\pm$. Indeed, introducing notations

$$G^\pm(x,x',k)=G(x,x',k\pm i0),\quad k\in\mathbb{R}, \tag{2.71}$$

we see by (2.49) that they are given by the following limits of the resolvent

$$G^\pm(x,x',k)=\lim_{\varepsilon\to+0}\hat{M}(x,x';\pm\varepsilon,\pm2k\varepsilon), \tag{2.72}$$

so like in (2.52) they correspond to $\hat{M}(s)$ at $s=0$. But the resolvent is discontinuous at this point, so in the generic situation these limits are different. In order to find relations between them we again start from (2.42) choosing both s' and s to be real. Performing for s' the limiting procedure (2.72) and for s the one in (2.52) we get

$$G^\sigma(k)-G_\pm=(G_\pm\overleftarrow{\mathcal{L}}_0)(G_0^\sigma(k)-G_{0,\pm})(\vec{\mathcal{L}}_0G^\sigma(k)), \tag{2.73}$$

where $k\in\mathbb{R}$ and $\sigma=+,-$. Then, thanks to (2.54) and definitions (2.57) we get from (2.73)

$$G^{\sigma}(k)-G_{\pm}=\frac{\mp 1}{2\pi i}\int dk'\ \theta(\mp\sigma(k-k'))\Phi_{\pm}(k')\otimes(\Psi_0(k')\vec{\mathcal{L}}_0G^{\sigma}(k)), \tag{2.74}$$

where for the "covector" $\Psi_0(k')\vec{\mathcal{L}}_0G^{\sigma}(k)$ we used a shorthand notation analogous to that in (2.50). Finally, let us mention that the relation between G_+ and G_- was given in (2.56).

4. Jost and advanced/retarded solutions. Bilinear representation for the resolvent

We have shown that the derivatives of the resolvent and its discontinuities are given in terms of the functions $\Phi(\mathbf{k})$, $\Psi(\mathbf{k})$, $\Phi_{\pm}(k)$, and $\Psi_{\pm}(k)$ which in their turn are obtained as special reductions of the resolvent. Here we describe the properties of these functions. Thanks to (2.33) and (2.58) it follows directly from definitions (2.50) that the functions $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$ obey the nonstationary Schrödinger equation with potential $u(x)$ and its dual, that is,

$$\vec{\mathcal{L}}\Phi(\mathbf{k})=0,\quad \Psi(\mathbf{k})\overleftarrow{\mathcal{L}}=0, \tag{2.75}$$

where we use the same notation as in (2.33). Thanks to (2.59) we get the conjugation property

$$\overline{\Phi(x,\mathbf{k})}=\Psi(x,\overline{\mathbf{k}}) \tag{2.76}$$

and the integral equations for these functions,

$$\Phi(\mathbf{k})=\Phi_0(\mathbf{k})+G_0(\mathbf{k})U\Phi(\mathbf{k}),\quad \Psi(\mathbf{k})=\Psi_0(\mathbf{k})+\Psi(\mathbf{k})UG_0(\mathbf{k}), \tag{2.77}$$

follow if we apply $\overleftarrow{\mathcal{L}}_0\Phi_0$ from the right to the first equation in (2.60) and $\Psi_0\vec{\mathcal{L}}_0$ from the left to the second one. Thanks to (2.61) they are just the standard[12] equations for the Jost solution of the nonstationary Schrödinger equation and its dual. Thus, in what follows we refer to $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$ as Jost solutions and to $G(x,x',\mathbf{k})$ as to the Green's function of the Jost solutions. By (2.65) these solutions are analytic functions of the spectral parameter $\mathbf{k}$ for $\mathbf{k}_{\mathfrak{I}}\neq 0$ and are discontinuous at the real axis. Inserting (2.51) into (2.50) we get by (2.18) and (2.28) the standard representation

$$\Phi(x,\mathbf{k})=e^{-i\mathbf{k}x_1-i\mathbf{k}^2x_2}\chi(x,\mathbf{k}),\quad \Psi(x,\mathbf{k})=e^{i\mathbf{k}x_1+i\mathbf{k}^2x_2}\xi(x,\mathbf{k}), \tag{2.78}$$

where

$$\chi(x,\mathbf{k})=\int dx'\ (ML_0)(x,x';\mathbf{s}),\quad \xi(x',\mathbf{k})=\int dx\ (L_0M)(x,x';\mathbf{s}), \tag{2.79}$$

for $\mathbf{s}=(-i\mathbf{k},-i\mathbf{k}^2)$. From (2.77) we get that χ and ξ are normalized at infinity,

$$\lim_{\mathbf{k}\to\infty}\chi(x,\mathbf{k})=\lim_{\mathbf{k}\to\infty}\xi(x,\mathbf{k})=1, \tag{2.80}$$

while the potential $u(x)$ is reconstructed by their means as

$$u(x)=-2i\lim_{\mathbf{k}\to\infty}\mathbf{k}\partial_{x_1}\chi(x,\mathbf{k})=2i\lim_{\mathbf{k}\to\infty}\mathbf{k}\partial_{x_1}\xi(x,\mathbf{k}). \tag{2.81}$$

Solutions $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$ obey orthonormality relation, i.e., their "scalar product" equals

$$\frac{1}{2\pi}\int dx_1\ \Psi(x,\mathbf{k}+k')\Phi(x,\mathbf{k})=\delta(k'),\quad \mathbf{k}\in\mathbb{C},\quad k'\in\mathbb{R}. \tag{2.82}$$

In order to prove it we apply $\tilde{\mathcal{L}}_0\Phi_0(\mathbf{k})$ to (2.67) from the right. Thanks to (2.28) and (2.62) the integrand in the first term on the r.h.s. of (2.67) is rapidly decaying when $x_2''\to\infty$ and we can integrate by parts, that gives zero due to (2.33). Thus by recalling (2.50) we obtain

$$\int dx_1'\, G(x,x',\mathbf{k}+k')\Phi(x',\mathbf{k})=-i[\theta(x_2-x_2')-\theta(-\mathbf{k}_{\mathfrak{I}}k')]\Phi(x,\mathbf{k}) \tag{2.83}$$

and by taking the complex conjugate and using the conjugation properties (2.59) and (2.76)

$$\int dx_1'\Psi(x',\mathbf{k}+k')G(x',x,\mathbf{k})=i[\theta(x_2-x_2')-\theta(-\mathbf{k}_{\mathfrak{I}}k')]\Psi(x,\mathbf{k}+k'). \tag{2.84}$$

Next we apply $\Psi_0(\mathbf{k}+k')\vec{\mathcal{L}}_0$ to (2.83) from the left and recalling (2.50) and (2.28) we have

$$\begin{aligned}\int dx_1'\,\Psi(x',\mathbf{k}+k')\Phi(x',\mathbf{k})=\int dx\,\Psi_0(x,\mathbf{k}+k')\,\delta(x_2-x_2')\Phi(x,\mathbf{k})-i\int dx\,\Psi_0(x,\mathbf{k}+k')\\ \times[\theta(x_2-x_2')-\theta(-\mathbf{k}_{\mathfrak{I}}k')]u(x)\Phi(x,\mathbf{k}).\end{aligned} \tag{2.85}$$

Now (2.82) follows if we consider (2.84) for G_0 and Ψ_0, insert it in the second term on the r.h.s. of (2.85) and use the first equation in (2.77).

The properties of the functions $\Phi_\pm(x,k)$ and $\Psi_\pm(x,k)$ are derived analogously from their definition (2.57). By (2.68) they obey

$$\vec{\mathcal{L}}\Phi_\pm(k)=0,\quad \Psi_\pm(k)\tilde{\mathcal{L}}=0. \tag{2.86}$$

Their conjugation property

$$\overline{\Phi_\pm(x,k)}=\Psi_\mp(x,k) \tag{2.87}$$

and the integral equations

$$\Phi_\pm(k)=\Phi_0(k)+G_{0,\pm}U\Phi_\pm(k),\quad \Psi_\pm(k)=\Psi_0(k)+\Psi_\pm(k)UG_{0,\pm}, \tag{2.88}$$

follow from their definition and Eqs. (2.70), where the Green's function $G_{0,\pm}$ is defined in (2.54). Thus $\Phi_\pm(x,k)$ and $\Psi_\pm(x,k)$ are just the standard[12] advanced/retarded solutions of the nonstationary Schrödinger equation and its dual. Correspondingly, in what follows we refer to $G_\pm(x,x')$ as to the Green's function of the advanced/retarded solutions. In analogy with derivation of (2.82) we can get the orthonormality relation

$$\frac{1}{2\pi}\int dx_1\,\Psi_\pm(x,k+k')\Phi_\mp(x,k)=\delta(k'),\quad k,k'\in\mathbb{R}. \tag{2.89}$$

The properties of the Jost solutions enable us to reconstruct $M(s)$ from (2.45) in the form

$$\hat{M}(x,x';s)=\frac{1}{2\pi i}\int_{\mathbf{k}_{\mathfrak{I}}=s_1}d\mathbf{k}_{\mathfrak{R}}[\theta(x_2-x_2')-\theta(2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}}-s_2)]\,\Phi(x,\mathbf{k})\Psi(x',\mathbf{k}) \tag{2.90}$$

that generalizes (2.27) to the case of nonzero potentials. Indeed, thanks to (2.78) and to the boundedness of χ and ξ the integral, on the interval fixed by the θ-functions, is convergent, the kernel $M(x,x';s)=e^{-s(x-x')}\hat{M}(x,x';s)$ has zero limit for $s_2\to\infty$ in accordance with (2.39), and expression (2.45) for the derivative of $M(s)$ with respect to s_1 follows from (2.90) at $s_1\neq 0$ thanks to the analyticity of the Jost solutions at $\mathbf{k}_{\mathfrak{I}}\neq 0$. At the same time we know that the resolvent must be continuous at $s_1=0$ when $s_2\neq 0$. In order to formulate this property, we introduce in analogy with (2.71) a specific notation for the limiting values of the Jost solutions at the real axis, i.e.,

$$\Phi^{\pm}(x,k)=\Phi(x,k\pm i0), \quad \Psi^{\pm}(x,k)=\Psi(x,k\pm i0). \tag{2.91}$$

Then the above mentioned condition of continuity reads as

$$\int dk\,\Phi^{+}(k)\otimes\Psi^{+}(k)=\int dk\,\Phi^{-}(k)\otimes\Psi^{-}(k). \tag{2.92}$$

Representation (2.90) plays a crucial role in the resolvent approach since it enables us to express all objects of the spectral theory in terms of the Jost solutions. In particular, thanks to (2.51), for the Green's function of the Jost solutions we get

$$G(x,x',\mathbf{k})=\frac{1}{2\pi i}\int dk'\,[\theta(x_2-x_2')-\theta(\mathbf{k}_{\mathfrak{I}}k')]\,\Phi(x,\mathbf{k}+k')\Psi(x',\mathbf{k}+k'), \tag{2.93}$$

that in its turn generalizes (2.61). Moreover, from (2.90) one can get an expression of the advanced/retarded Green's functions in terms of the limiting values of the Jost solutions. Applying the limiting procedure (2.52) to (2.90) we get

$$G_{\pm}(x,x')=\frac{\pm\theta(\pm(x_2-x_2'))}{2\pi i}\int dk\,\Phi^{\sigma}(x,k)\Psi^{\sigma}(x',k), \tag{2.94}$$

where $\sigma=+,-$ and we used notation (2.91) for the limiting values of the Jost solutions at the real axis. Condition (2.92) guarantees that the $G_{\pm}$'s are independent of the sign σ as it must be.

From the behavior at large s_2 of the bilinear representation (2.90), comparing with (2.39) we obtain also the following important property of the Jost solutions:

$$\frac{1}{2\pi}\int_{x_2'=x_2} d\mathbf{k}_{\mathfrak{R}}\,\Psi(x',\mathbf{k})\Phi(x,\mathbf{k})=\delta(x_1-x_1'), \tag{2.95}$$

which can be considered a completeness relation.

5. Relations among Jost and advanced/retarded solutions. Spectral data

In the previous section, in the framework of the extended resolvent approach, we obtained in (2.93) a bilinear representation in terms of the Jost solutions of the Green's function for the Jost solutions and in (2.94) of the Green's function for the advanced/retarded solutions. We derived also Eq. (2.74) relating these Green's functions. We can now exploit these results for deriving relations among the advanced/retarded and Jost solutions on the real axis and use them to introduce the spectral data. In fact, applying $\overleftarrow{\mathcal{L}}_0\Phi_0(k)$ to (2.74) from the right, recalling definitions (2.50) and (2.57), we get (notice that $k\in\mathbb{R}$)

$$\Phi^{\sigma}(k)=\int dk'\,\Phi_{\pm}(k')r_{\pm}^{\sigma}(k',k), \quad \sigma=+,-, \tag{2.96}$$

while the relations among $\Psi^{\sigma}(k)$ and $\Psi_{\pm}(k)$ can be derived analogously or follow by conjugation properties (2.76) and (2.87). Here we introduced the spectral data

$$r_{\pm}^{\sigma}(k',k)=\delta(k'-k)\mp\theta(\pm\sigma(k'-k))r^{\sigma}(k',k), \quad k',k\in\mathbb{R}, \tag{2.97}$$

$$r^{\sigma}(k',k)=\frac{\Psi_0(k')\vec{\mathcal{L}}_0G^{\sigma}(k)\overleftarrow{\mathcal{L}}_0\Phi_0(k)}{2\pi i}\equiv\frac{\Psi_0(k')\vec{\mathcal{L}}_0\Phi^{\sigma}(k)}{2\pi i}. \tag{2.98}$$

Recalling "vector" and "covector" notations [cf. (2.47), (2.48), and (2.50)], the "expectation values" at the numerator have the following explicit expressions:

$$\Psi_0(k')\vec{\mathcal{L}}_0 G^\sigma(k)\vec{\mathcal{L}}_0\Phi_0(k)=\int dx\int dx'\,\Psi_0(x,k')(\mathcal{L}_0(x,\partial_x)\mathcal{L}_0^{\rm d}(x',\partial_{x'})G^\sigma(x,x',k))\Phi_0(x',k), \tag{2.99}$$

$$\Psi_0(k')\vec{\mathcal{L}}_0\Phi^\sigma(k)=\int dx\,\Psi_0(x,k')\mathcal{L}_0(x,\partial_x)\Phi^\sigma(x,k), \tag{2.100}$$

showing that they are functions of k' and k only.

In order to get the advanced/retarded solutions in terms of the boundary values of the Jost ones we use (2.94) and the limiting values (2.71) of (2.93) on the real axis. Then

$$G^\sigma(k)-G_\pm=\frac{\mp 1}{2\pi i}\int dk'\;\theta(\pm\sigma(k'-k))\Phi^\sigma(k')\otimes\Psi^\sigma(k'),\quad \sigma=+,-, \tag{2.101}$$

and in the same way as above we readily derive

$$\Phi_\pm(k)=\int dk'\;\Phi^\sigma(k')\overline{r_\pm^{-\sigma}(k,k')}. \tag{2.102}$$

Now inserting these expressions into (2.96) and taking into account the orthonormality properties (2.82) and (2.89) of the Jost and advanced/retarded solutions we derive that the spectral data obey characterization equations,[17] which we discuss below in Sec. IV E 2 in a more general situation.

The alternative spectral data defined as

$$F^\sigma(k,k')=\int dk''\,\overline{r_\pm^{-\sigma}(k'',k)}r_\pm^{-\sigma}(k'',k'), \tag{2.103}$$

where the r.h.s. is independent of the choice of $\pm$ thanks to the above mentioned characterization equations, allow expressing the discontinuity of the Jost solutions across the real axis as

$$\Phi^\sigma(k)=\int dk'\;\Phi^{-\sigma}(k')F^{-\sigma}(k',k). \tag{2.104}$$

The inverse problem can be formulated in the standard way by using the fact that the Jost solution $\Phi(x,\mathbf{k})$ is an analytic function of $\mathbf{k}$ for $\mathbf{k}_\Im\neq 0$, has the discontinuity (2.104) at the real axis and obeys the normalization condition given by (2.78) and (2.80). In Ref. 5 we also demonstrated that the inverse problem can be formulated in terms of the resolvent itself. Here we skip this for brevity, as well as many other results that follow from the extended resolvent approach, like the introduction and properties of the dressing operators,[5] derivation of the time evolution of the spectral data corresponding to the KPI equation and algorithmic construction of time evolutions compatible with the given linear problem.[4]

III. CASE OF ONE-DIMENSIONAL POTENTIAL

A. Main definitions

In this section we consider the spectral theory of the differential operator (1.1) when the perturbation $u_2(x)$ in (1.4) is identically zero, i.e., we consider the extended differential operator

$$L_1(s)=L_0(s)-U_1,\quad U_1(x,x';s)=u_1(x_1)\,\delta(x-x'), \tag{3.1}$$

where $L_0(s)$ is defined in (2.22). Here and in the following we use the subscript 1 for all objects related to the case of a one-dimensional potential. In addition we choose u_1 to be the one-soliton solution of the KdV equation at the initial time $t=0$, that is,

$$u_1(x_1)=\frac{-2\kappa^2}{\cosh^2\kappa(x_1-x_0)}, \tag{3.2}$$

where x_0 and $\kappa>0$ are arbitrary constants. This choice is sufficient for showing the main peculiar aspects of the imbedding of a one-dimensional potential into two dimensions and is particularly convenient, since all objects relevant to the extended resolvent approach to inverse scattering can be explicitly given.

The Jost and dual Jost solutions [cf. (2.78)] are given by

$$\Phi_1(x,\mathbf{k})=e^{-i\mathbf{k}x_1-i\mathbf{k}^2x_2}\chi_1(x,\mathbf{k}),\quad \Psi_1(x,\mathbf{k})=e^{i\mathbf{k}x_1+i\mathbf{k}^2x_2}\xi_1(x,\mathbf{k}), \tag{3.3}$$

where as always we use boldface font for the complex spectral parameter $\mathbf{k}$, and

$$\chi_1(x_1,\mathbf{k})=\frac{\mathbf{k}-i\kappa\tanh\kappa(x_1-x_0)}{\mathbf{k}-i\kappa},\quad \xi_1(x_1,\mathbf{k})=\frac{\mathbf{k}+i\kappa\tanh\kappa(x_1-x_0)}{\mathbf{k}+i\kappa}. \tag{3.4}$$

They satisfy the differential equations

$$\vec{\mathcal{L}}_1\Phi_1(\mathbf{k})=0,\quad \Psi_1(\mathbf{k})\overleftarrow{\mathcal{L}}_1=0, \tag{3.5}$$

and are analytic in the complex $\mathbf{k}$-plane with the exception of a pole at $\mathbf{k}=i\kappa$ and at $\mathbf{k}=-i\kappa$, correspondingly. Strictly speaking, in order to deal with Jost solutions defined in the standard way one would have to consider, for instance, $\Phi_1(x,\mathbf{k})$ multiplied by $(\mathbf{k}-i\kappa)/(\mathbf{k}+i\kappa\,\mathrm{sgn}\,\mathbf{k}_\Im)$, but this would introduce unnecessary complications. So, these Jost solutions have no discontinuity at the real axis and when we are considering their value at the real axis we can omit the superscript $\pm$. They obey, however, the conjugation property (2.76).

It is convenient to introduce the residua of functions $\Phi_1(x,\mathbf{k})$ and $\Psi_1(x,\mathbf{k})$ at the poles and their values at the conjugate points. We write

$$\operatorname*{res}_{\mathbf{k}=i\kappa}\Phi_1(x,\mathbf{k})=i\sqrt{\frac{\kappa}{\pi}}e^{\kappa x_0}\varphi_1(x),\quad \Phi_1(x,-i\kappa)=\frac{e^{-\kappa x_0}}{2\sqrt{\kappa\pi}}\varphi_1(x), \tag{3.6}$$

$$\operatorname*{res}_{\mathbf{k}=-i\kappa}\Psi_1(x,\mathbf{k})=-i\sqrt{\frac{\kappa}{\pi}}e^{\kappa x_0}\psi_1(x),\quad \Psi_1(x,i\kappa)=\frac{e^{-\kappa x_0}}{2\sqrt{\kappa\pi}}\psi_1(x), \tag{3.7}$$

where

$$\varphi_1(x)=\frac{\sqrt{\kappa\pi}\,e^{i\kappa^2x_2}}{\cosh\kappa(x_1-x_0)},\quad \psi_1(x)=\overline{\varphi_1(x)}. \tag{3.8}$$

Functions φ_1 and ψ_1 satisfy the differential equations

$$\vec{\mathcal{L}}_1\varphi_1=0,\quad \psi_1\overleftarrow{\mathcal{L}}_1=0. \tag{3.9}$$

The equation

$$\operatorname*{res}_{\mathbf{k}=i\kappa}\Phi_1(x,\mathbf{k})=2i\kappa e^{2\kappa x_0}\Phi_1(x,-i\kappa), \tag{3.10}$$

or an analogous equation for Ψ_1, closes the formulation of the inverse problem for this Jost solution.

The orthonormality relation for the Jost solutions is not modified with respect to (2.82),

$$\frac{1}{2\pi}\int dx_1\,\Psi_1(x,\mathbf{k}+k')\Phi_1(x,\mathbf{k})=\delta(k'),\quad k'\in\mathbb{R},\quad \mathbf{k}\in\mathbb{C},\tag{3.11}$$

but now it is accompanied by the scalar products

$$\int dx_1\,\psi_1(x)\Phi_1(x,\mathbf{k})=\int dx_1\,\Psi_1(x,\mathbf{k})\varphi_1(x)=0,\quad |\mathbf{k}_\Im|<\kappa,\tag{3.12}$$

$$\frac{1}{2\pi}\int dx_1\psi_1(x)\varphi_1(x)\equiv\frac{1}{2\pi}\int dx_1|\psi_1(x)|^2\equiv\frac{1}{2\pi}\int dx_1|\varphi_1(x)|^2=1,\tag{3.13}$$

where condition $|\mathbf{k}_\Im|<\kappa$ in (3.12) is necessary for the convergence of the integral. On the other side, the completeness relation is essentially modified with respect to the case of decaying potential (2.95) considered above, since we have

$$\int_{x_2'=x_2} d\mathbf{k}_\Re\,\Phi_1(x,\mathbf{k})\Psi_1(x',\mathbf{k})+\theta(\kappa-|\mathbf{k}_\Im|)\varphi_1(x)\psi_1(x')\bigg|_{x_2'=x_2}=2\pi\delta(x_1-x_1').\tag{3.14}$$

B. Resolvent

The resolvent $M_1(s)$ according to our general definition is the inverse, in the sense of (2.26), of the extended operator (3.1) and its hat-kernel obeys the equations

$$\vec{\mathcal{L}}_1\hat{M}_1(s)=\hat{M}_1(s)\overleftarrow{\mathcal{L}}_1=I.\tag{3.15}$$

Taking into account that the completeness relation (3.14) is modified with respect to (2.95) and that this relation is the first order term of the asymptotic expansion of the resolvent as $s_2\to\infty$, it is natural to expect that the bilinear representation (2.90) is also modified by an additional term proportional to the product $\varphi_1(x)\psi_1(x')$. In fact we get

$$\begin{aligned}\hat{M}_1(x,x';s)=\frac{1}{2\pi i}\bigg(&\int_{\mathbf{k}_\Im=s_1} d\mathbf{k}_\Re[\theta(x_2-x_2')-\theta(2\mathbf{k}_\Re\mathbf{k}_\Im-s_2)]\,\Phi_1(x,\mathbf{k})\Psi_1(x',\mathbf{k})\\&+\theta(\kappa-|s_1|)[\theta(x_2-x_2')-\theta(-s_2)]\varphi_1(x)\psi_1(x')\bigg).\end{aligned}\tag{3.16}$$

Performing the transformation (2.13) and taking into account the properties of the Jost solutions and of the functions $\varphi_1(x)$ and $\psi_1(x')$ one can check that $M_1(x,x';s)\in\mathcal{S}'$ and obeys integral equations of the kind (2.37) with $U=U_1$. The second term in (3.16) is different from zero in the interval $|s_1|<\kappa$, it compensates the discontinuities of the first term at $s_1=\pm\kappa$, which are consequence of the pole singularities of $\Phi_1(x,\mathbf{k})$ and $\Psi_1(x,\mathbf{k})$. On the other side, this term introduces a discontinuity in $M_1(s)$ at $s_2=0$ for all $|s_1|<\kappa$ and not only at $s_1=0$ as it was for the resolvent of the decaying case. By direct calculations one can prove the regularity with respect to s-variables of

$$\hat{M}_{1,\mathrm{reg}}(s)=\hat{M}_1(s)-\Gamma(s)\,\varphi_1(x)\psi_1(x'),\tag{3.17}$$

where

$$\Gamma(s)=\frac{\operatorname{sgn}s_1}{(2\pi)^2}\log\frac{s_2+2is_1(s_1-\kappa)}{s_2+2is_1(s_1+\kappa)},\tag{3.18}$$

the logarithm being the principal part with the cut along the negative real axis. Thus, we see that the extended resolvent in the case of one-dimensional potentials gets logarithmic singularities at the points $s=(\pm\kappa,0)$. The singular part of the resolvent has the structure of a direct product.

C. Properties of the resolvent and Green's functions

For the discontinuity of the resolvent at $s_2=0$ we get from (3.16)

$$\lim_{s_2\to+0}\hat{M}_1(s)-\lim_{s_2\to-0}\hat{M}_1(s)=\frac{\theta(\kappa-|s_1|)}{2\pi i}\varphi_1\otimes\psi_1, \tag{3.19}$$

which also follows from (3.17). For all other values of s the resolvent (3.16) has derivatives with respect to s of the form of (2.34), (2.35):

$$\frac{\partial\hat{M}_1(s)}{\partial s_1}=\frac{i}{\pi}\int_{\mathbf{k}_{\mathfrak{I}}=s_1}d\mathbf{k}_{\mathfrak{R}}\,\overline{\mathbf{k}}\,\delta(2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}}-s_2)\,\Phi_1(\mathbf{k})\otimes\Psi_1(\mathbf{k}), \tag{3.20}$$

$$\frac{\partial\hat{M}_1(s)}{\partial s_2}=\frac{1}{2\pi i}\int_{\mathbf{k}_{\mathfrak{I}}=s_1}d\mathbf{k}_{\mathfrak{R}}\,\delta(2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}}-s_2)\,\Phi_1(\mathbf{k})\otimes\Psi_1(\mathbf{k}), \tag{3.21}$$

which at the vicinity of the point $s_1=0$ have to be considered in the distributional sense. Thus, we can introduce the Green's function $G_1(x,x',\mathbf{k})$ of the Jost solutions by using the same definition as in (2.51). Then, from (3.16) we get the bilinear representation

$$\begin{aligned}G_1(x,x',\mathbf{k})=\frac{1}{2\pi i}\int dk'\,[\theta(x_2-x_2')-\theta(\mathbf{k}_{\mathfrak{I}}k')]\,\Phi_1(x,k'+\mathbf{k})\Psi_1(x',k'+\mathbf{k})\\+\frac{\theta(\kappa-|\mathbf{k}_{\mathfrak{I}}|)}{2\pi i}[\theta(x_2-x_2')-\theta(-\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}})]\varphi_1(x)\psi_1(x'),\end{aligned} \tag{3.22}$$

which was already derived in Ref. 16, but starting from another point of view. It is easy to check that this Green's function obeys the conjugation property (2.59) and that the function $g_1(x,x',\mathbf{k})$ defined like in (2.62) has property (2.63). Taking into account that the resolvent obeys (2.37) with $U=U_1$ we get that the Green's function obeys integral equations of the type (2.60),

$$G_1(\mathbf{k})=G_0(\mathbf{k})+G_0(\mathbf{k})U_1G_1(\mathbf{k}),\quad G_1(\mathbf{k})=G_0(\mathbf{k})+G_1(\mathbf{k})U_1G_0(\mathbf{k}), \tag{3.23}$$

it is analytic when $\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}}\neq0$, continuous up to the borders of these quadrants, and in this region in analogy with (2.64) we obtain

$$\frac{\partial G_1(\mathbf{k})}{\partial\mathbf{k}_{\mathfrak{I}}}=\frac{\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}{2\pi}\Phi_1(\mathbf{k})\otimes\Psi_1(\mathbf{k}). \tag{3.24}$$

In addition to the standard discontinuity at the real axis, the Green's function has also a discontinuity

$$G_1(+0+i\mathbf{k}_{\mathfrak{I}})-G_1(-0+i\mathbf{k}_{\mathfrak{I}})=\frac{\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}{2\pi i}\theta(\kappa-|\mathbf{k}_{\mathfrak{I}}|)\varphi_1\otimes\psi_1 \tag{3.25}$$

at the imaginary axis when $|\mathbf{k}_{\mathfrak{I}}|<\kappa$. Using (3.17) one can prove that the function $G_1(\mathbf{k})-\gamma(\mathbf{k})\,\varphi_1\otimes\psi_1$, where

$$\gamma(\mathbf{k})=\Gamma(s)|_{s=(\mathbf{k}_{\mathfrak{I}},2\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}})}\equiv\frac{\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}{(2\pi)^2}\log\frac{\mathbf{k}-i\kappa}{\mathbf{k}+i\kappa}, \tag{3.26}$$

is a bounded continuous function of $\mathbf{k}$ in vicinity of the cut $\mathbf{k}_{\mathfrak{R}}=0$, $|\mathbf{k}_{\mathfrak{I}}|<\kappa$ with exception of the point $\mathbf{k}=0$. As a consequence the regularized function

$$G_{1,\mathrm{reg}}(\mathbf{k})=G_1(\mathbf{k})+2\pi i\,\gamma(\mathbf{k})(\mathbf{k}\,\mathrm{sgn}\,\mathbf{k}_{\mathfrak{I}}-i\kappa)\Phi_1(\mathbf{k})\otimes\Psi_1(\mathbf{k}) \tag{3.27}$$

has finite limits at the points $\mathbf{k}=\pm i\kappa$, while it is discontinuous at $\mathbf{k}_{\mathfrak{R}}=0$ and $\mathbf{k}_{\mathfrak{I}}=0$. In (3.27) the multiplier $(\mathbf{k}\,\mathrm{sgn}\,\mathbf{k}_{\mathfrak{I}}-i\kappa)$ compensates the poles of $\Phi_1(\mathbf{k})$ and $\Psi_1(\mathbf{k})$. Note that the regularized function $g_{1,\mathrm{reg}}(\mathbf{k})$ constructed from $G_{1,\mathrm{reg}}(x,x',\mathbf{k})$ like in (2.62) is bounded with respect to x and has finite limits when $x\to\infty$.

Let us mention that in the case of the Jost solutions Φ_1 and Ψ_1 the integrals in definitions (2.47) and (2.48) are divergent and they must be conveniently regularized in the following way:

$$\Phi_1(x,\mathbf{k})=\lim_{\varepsilon\to+0}\int dx'\,(G_1(\mathbf{k})\overleftarrow{\mathcal{L}}_0)(x,x')\Phi_0(x',\mathbf{k})e^{i\varepsilon\mathbf{k}_{\mathfrak{I}}x_2'}, \tag{3.28}$$

$$\Psi_1(x,\mathbf{k})=\lim_{\varepsilon\to+0}\int dx'\,e^{-i\varepsilon\mathbf{k}_{\mathfrak{I}}x_2'}\Psi_0(x',\mathbf{k})(\overrightarrow{\mathcal{L}}_0G_1(\mathbf{k}))(x',x), \tag{3.29}$$

where $\mathbf{k}_{\mathfrak{I}}\neq 0$. As the result of this regularization we get (see Refs. 2 and 4) a modification of the integral equations for the Jost solutions with respect to the standard ones[10,12–15] that are not applicable in the case of one-dimensional potentials. In particular, in spite of the cut that $G_1(x,x',\mathbf{k})$ has on the imaginary axis, we have

$$\Phi_1(+0+i\mathbf{k}_{\mathfrak{I}})-\Phi_1(-0+i\mathbf{k}_{\mathfrak{I}})=2i\sqrt{\kappa\pi}e^{\kappa x_0}\delta(\mathbf{k}_{\mathfrak{I}}-\kappa)\varphi_1 \tag{3.30}$$

in agreement with (3.6).

When $s_1=0$ the kernel $\hat{M}_1(s)$ is discontinuous at $s_2=0$. This means that we have to introduce the advanced/retarded Green's functions like in (2.52). By (3.16) we get the following bilinear representation for these Green's functions in terms of the Jost solutions on the real axis:

$$G_{1,\pm}(x,x')=\frac{\pm\theta(\pm(x_2-x_2'))}{2\pi i}\left(\int dk\,\Phi_1(x,k)\Psi_1(x',k)+\varphi_1(x)\psi_1(x')\right). \tag{3.31}$$

The singular part of these Green's functions is equal to zero, as follows from (3.18). These Green's functions obey the conjugation property (2.69).

Now turning back to the discontinuity of the resolvent with respect to s_2 when $s_1\neq 0$ we get from (2.51) for these limits

$$\lim_{s_2\to\pm 0\mathbf{k}_{\mathfrak{I}}}(\hat{M}_1(s)|_{s_1=\mathbf{k}_{\mathfrak{I}}})=G_1(\pm 0+i\mathbf{k}_{\mathfrak{I}}). \tag{3.32}$$

Relations among Green's functions follow from the bilinear representation (3.16) for the resolvent. For the boundary values of the Green's function of the Jost solutions at the real axis we use notations (2.72). Then by (3.22) they are finite for all finite k but discontinuous at $k=0$ and by (3.31)

$$G_1^{\sigma}(k)-G_{1,\pm}=\frac{\mp 1}{2\pi i}\int dk'\;\theta(\mp\sigma(k-k'))\Phi_1(k')\otimes\Psi_1(k')\mp\frac{\theta(\mp\sigma k)}{2\pi i}\varphi_1\otimes\psi_1,\quad \sigma=+,-. \tag{3.33}$$

Other relations of such kind, say, among advanced and retarded Green's functions, can be derived analogously.

IV. INVERSE SCATTERING TRANSFORM ON NONTRIVIAL BACKGROUND: TWO-DIMENSIONAL PERTURBATION OF THE SOLITON SOLUTION

A. Resolvent

We investigate, now, the operator $\mathcal{L}$ in (1.1) with a potential of the form (1.4), where $u_1(x_1)$ is the one-soliton solution (3.2) and $u_2(x)$ is a generic real, smooth and rapidly decaying function on the plane (x_1,x_2), "small" in some sense, so that it can be considered a two-dimensional perturbation of the one-soliton solution. According to our general approach we introduce the extension $L(s)$ of $\mathcal{L}$ and the inverse of this extension, i.e., its extended resolvent $M(s)$. In the previous sections we considered the special case $u_2(x)\equiv 0$ and studied in detail the properties of the corresponding resolvent $M_1(s)$. Therefore, it is convenient to exploit these results and to use for constructing the resolvent $M(s)$, instead of the integral equations (2.37), the following ones,

$$M(s)=M_1(s)+M_1(s)U_2M(s), \quad M(s)=M_1(s)+M(s)U_2M_1(s), \tag{4.1}$$

where $U_2(x,x';s)=u_2(x)\delta(x-x')$. We assume that, thanks to the "smallness" requirement on $u_2(x)$, both equations admit a unique solution $M(x,x';s)$ in $\mathcal{S}'(\mathbb{R}^6)$. This requirement, in addition, guarantees that the solution $M(x,x';s)$ inherits the properties of $M_1(x,x';s)$, in particular that $M(x,x';s)$ is a distribution with respect to $x-x'$, a smooth function with respect to $x+x'$ and a continuous function of s, when $s_1\neq 0$ and $s_2\neq 0$. Therefore, the Hilbert identities (2.40) and (2.42) are valid also in our case, since the composition of operators appearing in them exist and are associative. The Hilbert identity (2.40) can be used for proving that equations (4.1) have the same solution. In fact, if we choose M to be a solution of the first equation in (4.1) [and then solution of the second equation in (2.26)] and M' to obey the second equation in (4.1) [the first in (2.26), correspondingly], then we obtain $M'=M$. Notice that, in addition, this proves that $M(s)$ is Hermitian, i.e., it obeys (2.38) like in the decaying and one-dimensional cases. Instead, the Hilbert identity (2.42) can be used for studying the properties of the resolvent in the s variable. However, as we did for the integral equations (4.1) defining $M(s)$, it is more convenient to rewrite it in terms of the one-dimensional operator $L_1(s)$ and resolvent $M_1(s)$ as follows:

$$M(\mathbf{s}')-M(\mathbf{s})=M(\mathbf{s}')L_1(\mathbf{s}')(M_1(\mathbf{s}')-M_1(\mathbf{s}))L_1(\mathbf{s})M(\mathbf{s}). \tag{4.2}$$

Then, in analogy with (2.44) we get for the derivatives of the hat-kernel of the resolvent

$$\frac{\partial \hat{M}(s)}{\partial s_j}=\hat{M}(s)\overleftarrow{\mathcal{L}}_1\frac{\partial \hat{M}_1(s)}{\partial s_j}\overrightarrow{\mathcal{L}}_1\hat{M}(s), \quad j=1,2, \quad s_2\neq 0.$$

Inserting here (3.20) and (3.21) we get for the derivatives of $\hat{M}(s)$ the same equations (2.45) and (2.46) of the decaying case, where the Jost solutions are defined now as [cf. (2.50)]

$$\Phi(\mathbf{k})=G(\mathbf{k})\overleftarrow{\mathcal{L}}_1\Phi_1(\mathbf{k}), \quad \Psi(\mathbf{k})=\Psi_1(\mathbf{k})\overrightarrow{\mathcal{L}}_1G(\mathbf{k}), \tag{4.3}$$

with the Green's function $G(x,x',\mathbf{k})$ defined by (2.51) as in the decaying case. We excluded here the value $s_2=0$, where $M_1(s)$ is discontinuous.

For the study of the discontinuity at $s_2=0$ of $M(s)$ inherited from $M_1(s)$, following the results of Sec. III, we have to consider separately the cases $s_1=0$ and $s_1\neq 0$. The boundary values of the resolvent in the first case, $\lim_{s_2\to\pm 0}\hat{M}(s)|_{s_1=0}$, define the advanced/retarded Green's functions like in (2.52). In the case $s_1\neq 0$ for the boundary values of the resolvent thanks to (2.51) we have in analogy with (3.32) that

$$\lim_{s_2\to\pm 0\mathbf{k}_\Im}(\hat{M}(s)|_{s_1=\mathbf{k}_\Im})=G(\pm 0+i\mathbf{k}_\Im). \tag{4.4}$$

Now we consider the properties of these Green's functions in detail.

B. The Green's function of the Jost solutions

1. Properties of $G(\mathbf{k})$

We proved above that the definition (2.51) of the Green's function $G(\mathbf{k})$ of the Jost solutions is applicable in the case we are considering now. Then directly by (4.1) we get that $G(\mathbf{k})$ satisfies the integral equations

$$G(\mathbf{k})=G_1(\mathbf{k})+G_1(\mathbf{k})U_2G(\mathbf{k}), \quad G(\mathbf{k})=G_1(\mathbf{k})+G(\mathbf{k})U_2G_1(\mathbf{k}). \tag{4.5}$$

Taking into account that $\mathcal{L}=\mathcal{L}_1-U_2$ it is easy to check that $G(\mathbf{k})$ obeys the differential equations (2.58) and the conjugation property (2.59). According to the remark we already made above for the resolvent $M(s)$, $G(\mathbf{k})$ has the same domains of analyticity as $G_1(\mathbf{k})$. Precisely, it is analytic in the region $\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}}\neq 0$, where it obeys (2.64) and (2.65), and continuous up to the borders, it has a cut at $\mathbf{k}_{\mathfrak{I}}=0$ as in the decaying case and an additional cut at $\mathbf{k}_{\mathfrak{R}}=0$ when $|\mathbf{k}_{\mathfrak{I}}|<\kappa$. Using notation (2.71) for the boundary values of the Green's function at the real axis, we get

$$\lim_{\sigma\mathbf{k}_{\mathfrak{I}}\to+0} G(\pm 0+i\mathbf{k}_{\mathfrak{I}})=\lim_{\pm k\to+0} G^{\sigma}(k)\equiv G^{\sigma}(\pm 0), \quad \sigma=+,-. \tag{4.6}$$

Notice that by inserting (4.5) into the definitions (4.3) of the Jost solutions we get the integral equations

$$\Phi(\mathbf{k})=\Phi_1(\mathbf{k})+G_1(\mathbf{k})U_2\Phi(\mathbf{k}), \quad \Psi(\mathbf{k})=\Psi_1(\mathbf{k})+\Psi(\mathbf{k})U_2G_1(\mathbf{k}), \tag{4.7}$$

which can be also considered to define the Jost solutions.

2. Discontinuity of $G(\mathbf{k})$ across the imaginary axis

As always in order to study the properties of the resolvent at a discontinuity we use the Hilbert identity. Choosing in (4.2) both $\mathbf{s}$ and $\mathbf{s}'$ to be real and $s_1=s_1'$, we have

$$\lim_{s_2\to+0}\hat{M}(s)-\lim_{s_2\to-0}\hat{M}(s)=(\lim_{s_2\to\pm 0}\hat{M}(s)\overleftarrow{\mathcal{L}}_1)(\lim_{s_2\to+0}\hat{M}_1(s)-\lim_{s_2\to-0}\hat{M}_1(s))(\vec{\mathcal{L}}_1\lim_{s_2\to\mp 0}\hat{M}(s)),$$

which, by (4.4), is the discontinuity of the Green's function across the imaginary axis. By using (3.22) for the discontinuity of $G_1(\mathbf{k})$ we obtain

$$G(+0+i\mathbf{k}_{\mathfrak{I}})-G(-0+i\mathbf{k}_{\mathfrak{I}})=\frac{\theta(\kappa-|\mathbf{k}_{\mathfrak{I}}|)}{2\pi i\,\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}(G(\pm 0+i\mathbf{k}_{\mathfrak{I}})\overleftarrow{\mathcal{L}}_1\varphi_1)\otimes(\psi_1\vec{\mathcal{L}}_1G(\mp 0+i\mathbf{k}_{\mathfrak{I}})), \tag{4.8}$$

where in brackets "vector" and "covector" notation like in (2.50) is used and the direct product is defined as in (2.36). Applying operations $\psi_1\vec{\mathcal{L}}_1$ and $\overleftarrow{\mathcal{L}}_1\varphi_1$ to this equality from the left and from the right, correspondingly, we get in the region $\kappa>|\mathbf{k}_{\mathfrak{I}}|>0$

$$\left[1-\frac{\psi_1\vec{\mathcal{L}}_1G(-0+i\mathbf{k}_{\mathfrak{I}})\overleftarrow{\mathcal{L}}_1\varphi_1}{2\pi i\,\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}\right]\left[1+\frac{\psi_1\vec{\mathcal{L}}_1G(+0+i\mathbf{k}_{\mathfrak{I}})\overleftarrow{\mathcal{L}}_1\varphi_1}{2\pi i\,\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}\right]=1, \tag{4.9}$$

where the "expectation values" in the numerator have explicit expressions analogous to those in (2.99). Under our assumptions on the potential both multipliers on the l.h.s. are regular in this region so they have no zeroes. Denoting the right multiplier as $A(\mathbf{k}_{\mathfrak{I}})$ we get

$$A(\mathbf{k}_{\mathfrak{I}})=\left(1\pm\frac{\psi_1\vec{\mathcal{L}}_1G(\pm 0+i\mathbf{k}_{\mathfrak{I}})\overleftarrow{\mathcal{L}}_1\varphi_1}{2\pi i\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}\right)^{\pm 1}, \tag{4.10}$$

which has no zeroes for any $|\mathbf{k}_{\mathfrak{I}}|<\kappa$,

$$A(\mathbf{k}_{\mathfrak{I}})\neq 0, \tag{4.11}$$

and by (2.59) and the second equality in (3.8) obeys the conjugation property

$$\overline{A(\mathbf{k}_{\mathfrak{I}})}=A(-\mathbf{k}_{\mathfrak{I}}). \tag{4.12}$$

The function $A(\mathbf{k}_{\mathfrak{I}})$ is defined inside the interval $|\mathbf{k}_{\mathfrak{I}}|<\kappa$, it is continuous for all $\mathbf{k}_{\mathfrak{I}}\neq 0$ and using for the limiting values at $\mathbf{k}_{\mathfrak{I}}=\pm 0$ notation of the type (2.71), $A^{\pm}=\lim_{\mathbf{k}_{\mathfrak{I}}\to\pm 0}A(\mathbf{k}_{\mathfrak{I}})$, we get by (4.6) and (4.10)

$$A^{\pm}=1\pm\frac{\psi_1\vec{\mathcal{L}}_1 G^{\pm}(+0)\overleftarrow{\mathcal{L}}_1\varphi_1}{2\pi i}. \tag{4.13}$$

Thanks to (4.11) both these constants $A^{\pm}$ are different from zero and by (4.12) they are mutually conjugate,

$$\overline{A^{-}}=A^{+}. \tag{4.14}$$

If we introduce the functions

$$\varphi(\mathbf{k}_{\mathfrak{I}})=G(+0+i\mathbf{k}_{\mathfrak{I}})\overleftarrow{\mathcal{L}}_1\varphi_1,\quad \psi(\mathbf{k}_{\mathfrak{I}})=A(\mathbf{k}_{\mathfrak{I}})(\psi_1\vec{\mathcal{L}}_1 G(-0+i\mathbf{k}_{\mathfrak{I}})), \tag{4.15}$$

where the multiplier $A(\mathbf{k}_{\mathfrak{I}})$ in the last equality is introduced for convenience, the discontinuity of the Green's function at the imaginary axis (4.8) can be written in their terms as

$$G(+0+i\mathbf{k}_{\mathfrak{I}})-G(-0+i\mathbf{k}_{\mathfrak{I}})=\theta(\kappa-|\mathbf{k}_{\mathfrak{I}}|)\frac{\varphi(\mathbf{k}_{\mathfrak{I}})\otimes\psi(\mathbf{k}_{\mathfrak{I}})}{2\pi i A(\mathbf{k}_{\mathfrak{I}})\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}. \tag{4.16}$$

Thanks to (4.10) this equality gives the following relations symmetric to (4.15),

$$\varphi(\mathbf{k}_{\mathfrak{I}})=A(\mathbf{k}_{\mathfrak{I}})(G(-0+i\mathbf{k}_{\mathfrak{I}})\overleftarrow{\mathcal{L}}_1\varphi_1),\quad \psi(\mathbf{k}_{\mathfrak{I}})=\psi_1\vec{\mathcal{L}}_1 G(+0+i\mathbf{k}_{\mathfrak{I}}). \tag{4.17}$$

Notice that using the first equality in (4.15) and the second equality in (4.17) we get by (4.5) the integral equations for these functions

$$\varphi(\mathbf{k}_{\mathfrak{I}})=\varphi_1+G_1(+0+i\mathbf{k}_{\mathfrak{I}})U_2\varphi(\mathbf{k}_{\mathfrak{I}}),\quad \psi(\mathbf{k}_{\mathfrak{I}})=\psi_1+\psi(\mathbf{k}_{\mathfrak{I}})U_2G_1(+0+i\mathbf{k}_{\mathfrak{I}}), \tag{4.18}$$

$|\mathbf{k}_{\mathfrak{I}}|<\kappa$, which can also be considered to define them.

The values of the Green's function on the two sides of the cut along the imaginary axis can be obtained by using the derivatives of the Green's function (2.64), whose validity can be extended to the case we are now considering if the Jost solutions are defined as in (4.3). Performing the limits $\mathbf{k}_{\mathfrak{R}}\to\pm 0$ we get in the strip $|\mathbf{k}_{\mathfrak{I}}|<\kappa$

$$G(\pm 0+i\mathbf{k}_{\mathfrak{I}})=G^{\sigma}(\pm 0)+\frac{\sigma}{2\pi}\int_0^{\mathbf{k}_{\mathfrak{I}}}d\alpha\,\Phi(\pm 0+i\alpha)\otimes\Psi(\pm 0+i\alpha),\quad \sigma=\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}, \tag{4.19}$$

where we used the second equality in (4.6).

3. Behavior of $G(\mathbf{k})$ at the end points of the cut

In order to study the behavior of the Green's function in the vicinities of the points $\mathbf{k}=\pm i\kappa$ [where $G_1(\mathbf{k})$ is singular] we use (3.27) and introduce the regularized Green's function $G_{\mathrm{reg}}(\mathbf{k})$ as solution of the integral equation

$$G_{\mathrm{reg}}(\mathbf{k})=G_{1,\mathrm{reg}}(\mathbf{k})+G_{1,\mathrm{reg}}(\mathbf{k})U_2G_{\mathrm{reg}}(\mathbf{k}), \tag{4.20}$$

which thanks to properties of $G_{1,\mathrm{reg}}$ has finite limits at points $\mathbf{k}=\pm i\kappa$. Subtracting (4.20) from (4.5) and using (3.27) we get

$$G(\mathbf{k})-G_{\mathrm{reg}}(\mathbf{k})=-2\pi i\gamma(\mathbf{k})(\mathbf{k}\,\mathrm{sgn}\,\mathbf{k}_{\mathfrak{J}}-i\kappa)\Phi_1(\mathbf{k})\otimes\Psi_1(\mathbf{k})[I+U_2G(\mathbf{k})]$$
$$+G_{1,\mathrm{reg}}(\mathbf{k})U_2[G(\mathbf{k})-G_{\mathrm{reg}}(\mathbf{k})].$$

Under the assumption of unique solvability of (4.20) and thanks to the identity $I+U_2G(\mathbf{k})=\vec{\mathcal{L}}_1G(\mathbf{k})$ we derive from this equality for the Green's function in vicinity of these points the following representation:

$$G(\mathbf{k})=G_{\mathrm{reg}}(\mathbf{k})-\frac{2\pi i\gamma(\mathbf{k})(\mathbf{k}\,\mathrm{sgn}\,\mathbf{k}_{\mathfrak{J}}-i\kappa)(G_{\mathrm{reg}}(\mathbf{k})\overleftarrow{\mathcal{L}}_1\Phi_1(\mathbf{k}))\otimes(\Psi_1(\mathbf{k})\vec{\mathcal{L}}_1G_{\mathrm{reg}}(\mathbf{k}))}{1+2\pi i\gamma(\mathbf{k})(\mathbf{k}\,\mathrm{sgn}\,\mathbf{k}_{\mathfrak{J}}-i\kappa)(\Psi_1(\mathbf{k})\vec{\mathcal{L}}_1G_{\mathrm{reg}}(\mathbf{k})\overleftarrow{\mathcal{L}}_1\Phi_1(\mathbf{k}))}. \tag{4.21}$$

In analogy with (4.15), (4.17) we introduce the regular parts of the functions $\varphi(x,\mathbf{k}_{\mathfrak{J}})$, $\psi(x,\mathbf{k}_{\mathfrak{J}})$ at $\mathbf{k}_{\mathfrak{J}}=\pm\kappa$:

$$\varphi_{\mathrm{reg},\pm}=G_{\mathrm{reg}}(\pm i\kappa)\overleftarrow{\mathcal{L}}_1\varphi_1,\quad \psi_{\mathrm{reg},\pm}=\psi_1\vec{\mathcal{L}}_1G_{\mathrm{reg}}(\pm i\kappa) \tag{4.22}$$

and the "expectation values"

$$B_\pm=\psi_1\vec{\mathcal{L}}_1G_{\mathrm{reg}}(\pm i\kappa)\overleftarrow{\mathcal{L}}_1\varphi_1. \tag{4.23}$$

Then by (3.6), (3.7) we have that

$$2\pi\lim_{\mathbf{k}\to\pm i\kappa}(\mathbf{k}\,\mathrm{sgn}\,\mathbf{k}_{\mathfrak{J}}-i\kappa)(\Psi_1(\mathbf{k})\vec{\mathcal{L}}_1G_{\mathrm{reg}}(\mathbf{k})\overleftarrow{\mathcal{L}}_1\Phi_1(\mathbf{k}))=iB_\pm,$$

and thus using the above notations we get the asymptotic relation

$$G(\mathbf{k})=G_{\mathrm{reg}}(\pm i\kappa)+\frac{\gamma(\mathbf{k})\varphi_{\mathrm{reg}}(\pm\kappa)\otimes\psi_{\mathrm{reg}}(\pm\kappa)}{1-\gamma(\mathbf{k})B_\pm}+o(1),\quad \mathbf{k}\to\pm i\kappa. \tag{4.24}$$

Summarizing, the Green's function $G(\mathbf{k})$, like $G_1(\mathbf{k})$, has a discontinuity on the segment $|\mathbf{k}_{\mathfrak{J}}|<\kappa$ of the imaginary axis, but in general it is not necessarily singular at the end points of this segment. Indeed, if $B_+=0$, then at $\mathbf{k}\to i\kappa$ the logarithmic behavior of $\gamma(\mathbf{k})$ in the numerator of (4.24) is not compensated, so the Green's function $G(\mathbf{k})$ has a logarithmic singularity at the point $\mathbf{k}=i\kappa$, like in the special case considered in Sec. II. However, if

$$B_+\neq 0, \tag{4.25}$$

then the logarithmic behaviors of numerator and denominator in (4.24) compensate and we get a finite result. Taking into account the conjugation property $B_-=\overline{B_+}$, this is also valid for the point $\mathbf{k}=-i\kappa$, so in this case at the end points of the cut we have

$$G(\pm i\kappa)=G_{\mathrm{reg}}(\pm i\kappa)-\frac{\varphi_{\mathrm{reg},\pm}\otimes\psi_{\mathrm{reg},\pm}}{B_\pm}. \tag{4.26}$$

Applying to (4.24) the operations $\psi_1\vec{\mathcal{L}}_1$ from the left and $\overleftarrow{\mathcal{L}}_1\varphi_1$ from the right we get for $A(\mathbf{k}_{\mathfrak{J}})$ by (4.10) that

$$A(\pm\kappa)=1 \tag{4.27}$$

independently of the value of B_+.

C. Jost solutions

1. Properties of Jost solutions

From the properties of the Green's function $G(\mathbf{k})$ studied in Sec. II it follows that the Jost solutions $\Phi(\mathbf{k})$ and $\Psi(\mathbf{k})$ introduced in (4.3) are analytic in the region $\mathbf{k}_{\mathfrak{R}}\mathbf{k}_{\mathfrak{I}}\neq 0$, and continuous up to the borders, have a discontinuity at $\mathbf{k}_{\mathfrak{I}}=0$ as in the decaying case and an additional discontinuity at $\mathbf{k}_{\mathfrak{R}}=0$ when $|\mathbf{k}_{\mathfrak{I}}|<\kappa$. By (2.59) the Jost solutions obey the conjugation property (2.76). Moreover, the functions $\chi(x,\mathbf{k})$ and $\xi(x,\mathbf{k})$ defined as in (2.78) are bounded both in the x-variables and obey condition (2.80).

The discontinuity of the Jost solution $\Phi(\mathbf{k})$ at $\mathbf{k}_{\mathfrak{R}}\to\pm 0$ follows from (4.16) by means of (4.3), i.e., by applying $\overleftarrow{\mathcal{L}}_1\Phi_1(\pm 0+i\mathbf{k}_{\mathfrak{I}})$ to (4.16) from the right. By (3.30) the discontinuity of $\Phi_1(\mathbf{k})$ at the imaginary axis is proportional to φ_1. Under condition (4.25) we have representation (4.26) thanks to which $G(i\kappa)\overleftarrow{\mathcal{L}}_1\varphi_1=0$ by (4.22), (4.23). So the discontinuity of $\Phi_1(\mathbf{k})$ in the first equality in (4.3) does not contribute to the discontinuity of $\Phi(\mathbf{k})$ and we get by (4.16)

$$\Phi(x,+0+i\mathbf{k}_{\mathfrak{I}})-\Phi(x,-0+i\mathbf{k}_{\mathfrak{I}})=\theta(\kappa-|\mathbf{k}_{\mathfrak{I}}|)\varphi(x,\mathbf{k}_{\mathfrak{I}})w(\mathbf{k}_{\mathfrak{I}}), \tag{4.28}$$

where

$$w(\mathbf{k}_{\mathfrak{I}})=\frac{\psi(\mathbf{k}_{\mathfrak{I}})\overleftarrow{\mathcal{L}}_1\Phi_1(i\mathbf{k}_{\mathfrak{I}})}{2\pi iA(\mathbf{k}_{\mathfrak{I}})\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}},\quad |\mathbf{k}_{\mathfrak{I}}|<\kappa, \tag{4.29}$$

and again thanks to (4.25) we can omit specification of the limiting procedure for $\Phi_1(\pm 0 +i\mathbf{k}_{\mathfrak{I}})$ in the r.h.s. Below in (4.38) and (4.42) we give the behavior of $\varphi(\mathbf{k}_{\mathfrak{I}})$ and $w(\mathbf{k}_{\mathfrak{I}})$ that enables to prove that in the limit $B(\kappa)\to 0$ one gets $\delta(\mathbf{k}_{\mathfrak{I}}-\kappa)$ in the r.h.s. of (4.28).

If we use for the numerator of (4.29) the explicit expression

$$\psi(\mathbf{k}_{\mathfrak{I}})\overleftarrow{\mathcal{L}}_1\Phi_1(i\mathbf{k}_{\mathfrak{I}})=\int dx(\mathcal{L}_1^{\mathrm{d}}(x,\partial_x)\psi(x,\mathbf{k}_{\mathfrak{I}}))\Phi_1(x,i\mathbf{k}_{\mathfrak{I}}),$$

we recover the spectral data $w(\mathbf{k}_{\mathfrak{I}})$ describing the discontinuity of the Jost solutions introduced in Ref. 16. Using (4.3) and (4.15)–(4.17) $w(\mathbf{k}_{\mathfrak{I}})$ can be rewritten as

$$w(\mathbf{k}_{\mathfrak{I}})=\frac{\psi_1\vec{\mathcal{L}}_1G(-0+i\mathbf{k}_{\mathfrak{I}})\overleftarrow{\mathcal{L}}_1\Phi_1(i\mathbf{k}_{\mathfrak{I}})}{2\pi i\ \operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}\equiv\frac{\psi_1\vec{\mathcal{L}}_1\Phi(-0+i\mathbf{k}_{\mathfrak{I}})}{2\pi i\ \operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}, \tag{4.30}$$

or

$$w(\mathbf{k}_{\mathfrak{I}})=\frac{\psi_1\vec{\mathcal{L}}_1G(+0+i\mathbf{k}_{\mathfrak{I}})\overleftarrow{\mathcal{L}}_1\Phi_1(i\mathbf{k}_{\mathfrak{I}})}{2\pi iA(\mathbf{k}_{\mathfrak{I}})\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}\equiv\frac{\psi_1\vec{\mathcal{L}}_1\Phi(+0+i\mathbf{k}_{\mathfrak{I}})}{2\pi iA(\mathbf{k}_{\mathfrak{I}})\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}}. \tag{4.31}$$

From the first equality here and the conjugation properties of all the objects involved we have

$$\overline{w(-\mathbf{k}_{\mathfrak{I}})}=\frac{\Psi_1(i\mathbf{k}_{\mathfrak{I}})\vec{\mathcal{L}}_1\varphi(\mathbf{k}_{\mathfrak{I}})}{2\pi iA(\mathbf{k}_{\mathfrak{I}})\operatorname{sgn}\mathbf{k}_{\mathfrak{I}}},\quad |\mathbf{k}_{\mathfrak{I}}|<\kappa. \tag{4.32}$$

Notice that by (2.59), (4.15), and (4.17) we get the conjugation property

$$\overline{\varphi(x,\mathbf{k}_{\mathfrak{I}})}=\psi(x,-\mathbf{k}_{\mathfrak{I}}), \tag{4.33}$$

which motivates the introduction of the multiplier $A(\mathbf{k}_{\mathfrak{I}})$ in (4.15). Then, by conjugation of (4.28) we get for the dual Jost solution

$$\Psi(x,+0+i\mathbf{k}_{\mathfrak{I}})-\Psi(x,-0+i\mathbf{k}_{\mathfrak{I}})=\theta(\kappa-|\mathbf{k}_{\mathfrak{I}}|)\psi(x,\mathbf{k}_{\mathfrak{I}})\overline{w(-\mathbf{k}_{\mathfrak{I}})}. \tag{4.34}$$

Thus we see that the discontinuity of the Jost solutions across the imaginary axis is given not in their terms, but in terms of the functions $\varphi(x,\mathbf{k}_\Im)$, $\psi(x,\mathbf{k}_\Im)$ introduced in (4.15) [or (4.17)]. From their definition it follows that they obey equations (2.75): $\vec{\mathcal{L}}\varphi(\mathbf{k}_\Im)=0$ and $\psi(\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}=0$, and can be considered a generalization of the functions $\varphi_1(x)$, $\psi_1(x)$ to the case u_2 different from zero. However, they acquire a nontrivial dependence on $\mathbf{k}_\Im$ and cannot be obtained as specific values of the Jost solutions. Due to the properties of the Green's function these solutions are discontinuous at $\mathbf{k}_\Im=0$. In the following we refer to $\varphi(x,\mathbf{k}_\Im)$ and $\psi(x,\mathbf{k}_\Im)$ as auxiliary Jost solutions.

2. Behavior of Jost solutions and spectral data at the end points of the cut

The asymptotic behavior of the Jost solutions easily results from the asymptotic property of the Green's function given in (4.24). Applying $\overleftarrow{\mathcal{L}}_1\varphi_1$ from the right and using definitions (4.22) and (4.23) we get

$$G(\mathbf{k})\overleftarrow{\mathcal{L}}_1\varphi_1=\frac{\varphi_{\mathrm{reg},\pm}}{1-\gamma(\mathbf{k})B_\pm}+o(1),\quad \mathbf{k}\to\pm i\kappa. \tag{4.35}$$

Then, from the definitions (4.3) of the Jost solution and properties (3.6), (3.7) of the Jost solutions of the one-soliton case, we derive the asymptotic behavior

$$\Phi(\mathbf{k})=\sqrt{\frac{\kappa}{\pi}}\frac{ie^{\kappa x_0}\varphi_{\mathrm{reg},+}}{(\mathbf{k}-i\kappa)(1-\gamma(\mathbf{k})B_+)}+O(1),\quad \mathbf{k}\to+i\kappa, \tag{4.36}$$

$$\Phi(\mathbf{k})=\frac{e^{-\kappa x_0}\varphi_{\mathrm{reg},-}}{2\sqrt{\kappa\pi}(1-\gamma(\mathbf{k})B_-)}+o(1),\quad \mathbf{k}\to-i\kappa, \tag{4.37}$$

while the behavior of $\Psi(\mathbf{k})$ follows from conjugation property (2.76) and relations $\overline{\varphi_{\mathrm{reg},\pm}}=\psi_{\mathrm{reg},\mp}$ that follow from (4.22). As regards the auxiliary Jost solution $\varphi(\mathbf{k}_\Im)$, by its definition (4.15) and (4.35) we get in the same way

$$\varphi(\mathbf{k}_\Im)=\frac{\varphi_{\mathrm{reg},\pm}}{1-\gamma(+0+i\mathbf{k}_\Im)B_\pm}+o(1),\quad \mathbf{k}_\Im\to\pm\kappa, \tag{4.38}$$

and asymptotics for $\psi(\mathbf{k}_\Im)$ follow again by the conjugation property (4.33). We see that the behavior of the Jost and auxiliary Jost solutions at the points $\mathbf{k}_\Im=\pm\kappa$ is determined by the value of B_+. If $B_+=0$ the behavior of the Jost solutions is like in the case $u_2\equiv0$ [cf. (3.6) and (3.7)] and the auxiliary Jost solutions are finite and nonzero at these points. If $B_+\neq0$, the behavior of both types of Jost solutions is modified at these points by the multiplier $(\log\lvert\lvert\mathbf{k}_\Im\rvert-\kappa\rvert)^{-1}$ as follows from (3.26). In particular, in this case we have

$$\varphi(\pm\kappa)=\psi(\pm\kappa)=0. \tag{4.39}$$

Now we apply $\overleftarrow{\mathcal{L}}_1\varphi_1$ from the right to (4.19) with the upper sign and cancel $G^\sigma(\pm0)\overleftarrow{\mathcal{L}}_1\varphi_1$ thanks to (4.39). Then by (4.31) and the conjugation properties (2.76) and (4.12) we get the following expression of $\varphi(x,\mathbf{k}_\Im)$ in terms of the Jost solution,

$$\varphi(x,\mathbf{k}_\Im)=i\int_{\kappa\,\mathrm{sgn}\,\mathbf{k}_\Im}^{\mathbf{k}_\Im}d\alpha\,A(\alpha)\overline{w(-\alpha)}\Phi(x,+0+i\alpha), \tag{4.40}$$

which, as we will see below, is necessary for closing the inverse problem.

For the spectral data $A(\mathbf{k}_\Im)$ in (4.10) we can use the first equality in (4.17) and in the case $B_+\neq0$ also (4.40), and, then, by (4.27) and (4.31) we obtain the relation

$$A(\mathbf{k}_\Im)=1+i\int_{\kappa\,\mathrm{sgn}\,\mathbf{k}_\Im}^{\mathbf{k}_\Im} d\alpha\, w(\alpha)\overline{w(-\alpha)}A^2(\alpha),$$

giving the explicit expression of $A(\mathbf{k}_\Im)$ in terms of the spectral data $w(\mathbf{k}_\Im)$,

$$A(\mathbf{k}_\Im)=\left(1-i\int_{\kappa\,\mathrm{sgn}\,\mathbf{k}_\Im}^{\mathbf{k}_\Im} d\alpha\, w(\alpha)\overline{w(-\alpha)}\right)^{-1}, \tag{4.41}$$

which also is necessary for closing the inverse problem.

For the spectral data $w(\mathbf{k}_\Im)$ we can use the second equality in (4.31) and taking into account that by (4.22) and (4.23) $\psi_1\vec{\mathcal{L}}_1\varphi_{\mathrm{reg},\pm}=B_\pm$ we get by (4.36) and (4.37) the asymptotic behaviors

$$w(\mathbf{k}_\Im)=\frac{i\sqrt{\kappa}e^{\kappa x_0}B_+}{2\pi^{3/2}(\kappa-\mathbf{k}_\Im)(1-\gamma(+0+i\mathbf{k}_\Im)B_+)}+O(1),\quad \mathbf{k}_\Im\to\kappa-0, \tag{4.42}$$

$$w(\mathbf{k}_\Im)=\frac{ie^{-\kappa x_0}B_-}{4\sqrt{\kappa}\pi^{3/2}(1-\gamma(+0+i\mathbf{k}_\Im)B_-)}+o(1),\quad \mathbf{k}_\Im\to-\kappa+0. \tag{4.43}$$

Notice that for $B_+=0$ both these asymptotics are equal to zero, while as we mentioned above, in the product on the r.h.s. of (4.28) one gets δ-function in this case.

Concluding this study of the properties of the Jost solutions and spectral data at the points $\mathbf{k}_\Im=\pm\kappa$, let us emphasize that, in order to get relations among different spectral data, which are crucial for solving the inverse problem, as for instance (4.32) used in getting (4.40), it is necessary to use different representations for the spectral data, like (4.30) and (4.31), in terms of different solutions of the nonstationary Schrödinger equation and its dual. These representations can be obtained in the framework of the resolvent approach, while they cannot be derived if one deals only with equations for the Jost solutions (cf. Ref. 16).

3. Bilinear representation for the resolvent and Green's functions

Now, we can derive a bilinear representation of the resolvent in terms of the Jost solutions. We proved that for $s_2\neq 0$ its derivative with respect to s_2 obeys (2.46), so using notations (4.4) for the limiting values at $s_2=0$ we get

$$\begin{aligned}\hat{M}(x,x';s)=\frac{1}{2\pi i}\int dk\,[\theta(x_2-x_2')-\theta(2ks_1-s_2)]\,\Phi(x,k+is_1)\Psi(x',k+is_1)\\ +\mathrm{sgn}\,s_1[\theta(x_2-x_2')-\theta(-s_2)][G(x,x';+0+is_1)-G(x,x';-0+is_1)].\end{aligned}$$

Then, thanks to (4.16), we get the following generalization of the bilinear representation (3.16) for the resolvent of the perturbed potential:

$$\begin{aligned}\hat{M}(x,x';s)=\frac{1}{2\pi i}\int dk\,[\theta(x_2-x_2')-\theta(2ks_1-s_2)]\,\Phi(x,k+is_1)\Psi(x',k+is_1)\\ +\frac{\theta(\kappa-|s_1|)}{2\pi iA(s_1)}[\theta(x_2-x_2')-\theta(-s_2)]\varphi(x,s_1)\psi(x',s_1).\end{aligned} \tag{4.44}$$

If we consider the kernel $M(x,x';s)$ defined by (2.13), then both terms on the r.h.s. give distributions belonging to $\mathcal{S}'$. For the first term this follows from the mentioned properties of the functions χ and ξ, defined as in the decaying case by (2.78), and for the second term from the fact that $e^{-s_1x_1}\varphi(x,s_1)$ and $e^{s_1x_1}\psi(x,s_1)$ are bounded at space infinity when $|s_1|<\kappa$. Equation (2.45) at $s_1 \neq 0$ follows from the analyticity of the Jost solutions and from the derivatives of (4.40) and of the analogous equation for $\psi(s_1)$. The absence of a discontinuity at $s_1 = 0$ in the case $s_2 \neq 0$ like in the derivation of (2.92) is equivalent to the condition

$$\int dk\, \Phi^+(k) \otimes \Psi^+(k) + \frac{\varphi^+ \otimes \psi^+}{A^+} = \int dk\, \Phi^-(k) \otimes \Psi^-(k) + \frac{\varphi^- \otimes \psi^-}{A^-}, \tag{4.45}$$

where we used notation (2.91) also for the limiting values of the auxiliary Jost solutions on the real axis

$$\varphi^{\pm}(x) = \lim_{\pm \mathbf{k}_{\mathfrak{I}} \to +0} \varphi(x, \mathbf{k}_{\mathfrak{I}}), \quad \psi^{\pm}(x) = \lim_{\pm \mathbf{k}_{\mathfrak{I}} \to +0} \psi(x, \mathbf{k}_{\mathfrak{I}}). \tag{4.46}$$

Thanks to (2.76) and (4.33) these limiting values obey the conjugation properties

$$\overline{\Phi^{\pm}(x,k)} = \Psi^{\mp}(x,k), \quad \overline{\varphi^{\pm}(x)} = \psi^{\mp}(x), \quad k \in \mathbb{R}. \tag{4.47}$$

By (2.49) the bilinear representation (4.44) for the resolvent leads to the following bilinear representation for the Green's function of the Jost solutions:

$$G(x,x',\mathbf{k}) = \frac{1}{2\pi i} \int dk' \, [\theta(x_2 - x_2') - \theta(\mathbf{k}_{\mathfrak{I}} k')] \, \Phi(x, k' + \mathbf{k}) \Psi(x', k' + \mathbf{k})$$
$$+ \frac{\theta(\kappa - |\mathbf{k}_{\mathfrak{I}}|)}{2\pi i A(\mathbf{k}_{\mathfrak{I}})} [\theta(x_2 - x_2') - \theta(-\mathbf{k}_{\mathfrak{R}} \mathbf{k}_{\mathfrak{I}})] \varphi(x, \mathbf{k}_{\mathfrak{I}}) \psi(x', \mathbf{k}_{\mathfrak{I}}), \tag{4.48}$$

which generalizes (3.22). Below we use this bilinear representation for deriving relations between the Jost and advanced/retarded solutions.

D. Discontinuity of the resolvent at $s = 0$

1. The advanced/retarded Green's functions and solutions

Above we investigated the behavior of the resolvent when at least one of the variables s_1 or s_2 is different from zero. As we have seen in the case of decaying potentials, it is just the study of the discontinuity of the resolvent at $s = 0$ that leads to relations between Jost and advanced/retarded solutions which can be used for defining the spectral data. In the present case of a perturbed one-soliton potential the method is pretty close and, therefore, we omit details and we mainly present the peculiarities of this case. First, we introduce the advanced/retarded Green's functions as specific limits of the resolvent in the same way as in (2.52). It is straightforward to prove that they obey equations (2.68) and (2.69) and integral equations (2.70). In order to find the difference between the advanced and the retarded Green's function we use the Hilbert identity in the form (4.2), which, according to our previous discussion, in the case of a perturbed one-soliton potential is more convenient than (2.42). So we choose in (4.2) both s and s' real, $s_1' = s_1 = 0$ and consider the limits $s_2' \to \pm 0$, $s_2 \to \mp 0$. We get

$$G_+ - G_- = G_{\pm} \tilde{\mathcal{L}}_1 (G_{1,+} - G_{1,-}) \vec{\mathcal{L}}_1 G_{\mp},$$

which by (3.31) gives for the discontinuity the representation

$$G_+ - G_- = \frac{1}{2\pi i} \left(\int dk\, \Phi_{\pm}(k) \otimes \Psi_{\mp}(k) + \varphi_{\pm} \otimes \psi_{\mp} \right), \tag{4.49}$$

where we introduced not only, in analogy with (2.57), the advanced/retarded solutions $\Phi_{\pm}(k)$, $\Psi_{\pm}(k)$ but also the auxiliary advanced/retarded solutions $\varphi_{\pm}$, $\psi_{\pm}$ defined by

$$\Phi_{\pm}(k)=G_{\pm}\tilde{\vec{\mathcal{L}}}_1\Phi_1(k), \quad \varphi_{\pm}=G_{\pm}\tilde{\vec{\mathcal{L}}}_1\varphi_1, \tag{4.50}$$

$$\Psi_{\pm}(k)=\Psi_1(k)\vec{\mathcal{L}}_1G_{\pm}, \quad \psi_{\pm}=\psi_1\vec{\mathcal{L}}_1G_{\pm}. \tag{4.51}$$

It is easy to check that these functions $\Phi_{\pm}(x,k)$, $\Psi_{\pm}(x,k)$ and $\varphi_{\pm}(x)$, $\psi_{\pm}(x)$ are solutions of the differential equations (2.86) and obey a conjugation property, i.e., (2.87) for $\Phi_{\pm}(x,k)$, $\Psi_{\pm}(x,k)$, and

$$\overline{\varphi_{\pm}}=\psi_{\mp} \tag{4.52}$$

for the auxiliary solutions.

The bilinear representation (4.44) for the resolvent, thanks to (2.52), gives the following representation for the advanced/retarded Green's functions in terms of the Jost and auxiliary Jost solutions on the real axis,

$$G_{\pm}(x,x')=\frac{\pm\theta(\pm(x_2-x_2'))}{2\pi i}\left(\int dk\,\Phi^{\sigma}(x,k)\Psi^{\sigma}(x',k)+\frac{\varphi^{\sigma}(x)\psi^{\sigma}(x')}{A^{\sigma}}\right), \tag{4.53}$$

where we used notations (2.91) for the limiting values of the Jost solutions at the real axis and where the l.h.s. is independent on the sign $\sigma=+,-$ due to condition (4.45).

2. Relations among Green's functions $G^{\pm}(k)$ and $G_{\pm}$

The difference among the limiting values of the Green's function $G(\mathbf{k})$ on the two sides of the real axis and the advanced/retarded Green's functions, like in the case of the decaying potential, can be presented in two forms. The first one follows from (4.48) in the limits $\mathbf{k}\rightarrow k\pm i0$ and (4.53):

$$G^{\sigma}(k)-G_{\pm}=\frac{\mp 1}{2\pi i}\left(\int dk'\;\theta(\pm\sigma(k'-k))\,\Phi^{\sigma}(k')\otimes\Psi^{\sigma}(k')+\theta(\mp\sigma k)\,\frac{\varphi^{\sigma}\otimes\psi^{\sigma}}{A^{\sigma}}\right). \tag{4.54}$$

A second type of relation, analogous to (2.74), can be derived from the Hilbert identity (4.2). Taking into account definitions (2.72) and (2.52) we get

$$G^{\sigma}(k)-G_{\pm}=G_{\pm}\tilde{\vec{\mathcal{L}}}_1(G_1^{\sigma}(k)-G_{1,\pm})\vec{\mathcal{L}}_1G^{\sigma}(k), \quad k\in\mathbb{R},$$

so that we can use (3.33) and definitions (4.50) and (4.51) to derive

$$\begin{aligned}G^{\sigma}(k)-G_{\pm}=&\frac{\mp 1}{2\pi i}\int dk'\;\theta(\mp\sigma(k-k'))\Phi_{\pm}(k')\otimes(\Psi_1(k')\vec{\mathcal{L}}_1G^{\sigma}(k))\\&\mp\frac{\theta(\mp\sigma k)}{2\pi i}\varphi_{\pm}\otimes(\psi_1\vec{\mathcal{L}}_1G^{\sigma}(k)), \quad \sigma=+,-.\end{aligned} \tag{4.55}$$

Now we can use relations (4.54) and (4.55) in order to get relations among Jost and advanced/retarded solutions, introduce spectral data and get characterization equations such that, in particular, condition (4.45) is accomplished.

E. Spectral data

1. Relations among Jost and advanced/retarded solutions

In order to derive these relations we again use definitions (4.3) and (4.15) for $\mathbf{k}=k\in\mathbb{R}$ and apply $\tilde{\vec{\mathcal{L}}}_1\Phi_1(k)$ and $\tilde{\vec{\mathcal{L}}}_1\varphi_1$ to (4.55) from the right (the latter one to the limit of (4.55) at $k\rightarrow+0$). Then by (2.91), (4.13), (4.46), and (4.50) we get

$$\Phi^{\sigma}(k)=\int dk'\;\Phi_{\pm}(k')r_{\pm}^{\sigma}(k',k)+\varphi_{\pm}r_{\pm}^{\sigma}(k), \quad \sigma=+,-, \tag{4.56}$$

$$\varphi^{\sigma}=\int dk'\,\Phi_{\pm}(k')\tau_{\pm}^{\sigma}(k')+\varphi_{\pm}(A^{\sigma})^{\theta(\mp\sigma)}, \tag{4.57}$$

generalizing (2.96), where we introduced spectral data [cf. (2.97) and (2.98)] by means of relations

$$r_{\pm}^{\sigma}(k,k')=\delta(k-k')\mp\theta(\pm\sigma(k-k'))r^{\sigma}(k,k'), \tag{4.58}$$

$$r^{\sigma}(k,k')=\frac{\Psi_1(k)\vec{\mathcal{L}}_1G^{\sigma}(k')\overleftarrow{\mathcal{L}}_1\Phi_1(k')}{2\pi i}\equiv\frac{\Psi_1(k)\vec{\mathcal{L}}_1\Phi^{\sigma}(k')}{2\pi i}, \tag{4.59}$$

$$r_{\pm}^{\sigma}(k)=\mp\theta(\mp\sigma k)r^{\sigma}(k), \tag{4.60}$$

$$r^{\sigma}(k)=\frac{\psi_1\vec{\mathcal{L}}_1G^{\sigma}(k)\overleftarrow{\mathcal{L}}_1\Phi_1(k)}{2\pi i}\equiv\frac{\psi_1\vec{\mathcal{L}}_1\Phi^{\sigma}(k)}{2\pi i}, \tag{4.61}$$

$$\tau_{\pm}^{\sigma}(k)=\mp\theta(\pm\sigma k)\tau^{\sigma}(k), \tag{4.62}$$

$$\tau^{\sigma}(k)=\frac{\Psi_1(k)\vec{\mathcal{L}}_1G^{\sigma}(+0)\overleftarrow{\mathcal{L}}_1\varphi_1}{2\pi i}\equiv\frac{\Psi_1(k)\vec{\mathcal{L}}_1\varphi^{\sigma}}{2\pi i}. \tag{4.63}$$

We write equations (4.56) and (4.57) in a more compact form as

$$(\Phi^{\sigma},\varphi^{\sigma})=(\Phi_{\pm},\varphi_{\pm})*\mathcal{R}_{\pm}^{\sigma}, \tag{4.64}$$

and omit equations for Ψ and ψ that can be obtained analogously, or by conjugation. Here we introduced the matrix operator

$$\mathcal{R}_{\pm}^{\sigma}(k,k')=\begin{pmatrix} r_{\pm}^{\sigma}(k,k') & \tau_{\pm}^{\sigma}(k) \\ r_{\pm}^{\sigma}(k') & (A^{\sigma})^{\theta(\mp\sigma)}\end{pmatrix}. \tag{4.65}$$

$\mathcal{R}^{\dagger}$ denotes the Hermitian conjugate operator. Solutions $\Phi(x,k)$ and $\varphi(x)$ [$\Psi(x,k)$ and $\psi(x)$] have been combined in a row (correspondingly, column), where only the first element depends on k. For two operators $\mathcal{R}$ and $\mathcal{R}'$ with elements $\mathcal{R}_{ij}$, $\mathcal{R}'_{ij}$ with dependence on the arguments k, k' like in (4.65) we introduce a composition * that gives a matrix of the same form with elements

$$\begin{aligned}
(\mathcal{R}*\mathcal{R}')_{11}(k,k')&=\int dk''\,\mathcal{R}_{11}(k,k'')\mathcal{R}'_{11}(k'',k')+\mathcal{R}_{12}(k)\mathcal{R}'_{21}(k'),\\
(\mathcal{R}*\mathcal{R}')_{12}(k,k')&=\int dk''\,\mathcal{R}_{11}(k,k'')\mathcal{R}'_{12}(k'')+\mathcal{R}_{12}(k)\mathcal{R}'_{22},\\
(\mathcal{R}*\mathcal{R}')_{21}(k,k')&=\int dk''\,\mathcal{R}_{21}(k'')\mathcal{R}'_{11}(k'',k')+\mathcal{R}_{22}\mathcal{R}'_{21}(k'),\\
(\mathcal{R}*\mathcal{R}')_{22}(k,k')&=\int dk''\,\mathcal{R}_{21}(k'')\mathcal{R}'_{12}(k'')+\mathcal{R}_{22}\mathcal{R}'_{22}.
\end{aligned} \tag{4.66}$$

The unity matrix in the set of such operators is given by

$$\mathcal{I}(k,k')=\begin{pmatrix}\delta(k-k') & 0\\ 0 & 1\end{pmatrix}. \tag{4.67}$$

In the same way we define the product of the matrix by the row (from the left) and by column (from the right) of the type used in (4.64). This composition is also denoted by an asterisk to emphasize that the integration is performed in the first (upper) element only.

Equation (4.64) gives the boundary values of the Jost solutions in terms of the advanced/retarded ones. In order to get the inverse relations we apply the same procedures as above to Eq. (4.54) and in analogy with (2.102) in terms of the introduced notations we get

$$(\Phi_{\pm}, \varphi_{\pm}) = (\Phi^{\sigma}, \varphi^{\sigma}) * (W^{\sigma})^{-1} (\mathcal{R}_{\pm}^{-\sigma})^{\dagger}, \tag{4.68}$$

where we introduced the constant matrix

$$W^{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & A^{\sigma} \end{pmatrix}. \tag{4.69}$$

Alternative spectral data that relate values of the Jost solutions computed on the two sides of the real axis appear if we insert $(\Phi_{\pm}, \varphi_{\pm})$ from (4.68) into (4.64) for opposite sign of σ. Thus we get

$$(\Phi^{\sigma}, \varphi^{\sigma}) = (\Phi^{-\sigma}, \varphi^{-\sigma}) * (W^{-\sigma})^{-1} \mathcal{F}^{-\sigma}, \tag{4.70}$$

where we denoted

$$\mathcal{F}^{\sigma} = (\mathcal{R}_{\pm}^{-\sigma})^{\dagger} * \mathcal{R}_{\pm}^{-\sigma}. \tag{4.71}$$

Finally let us note that Eqs. (4.58)–(4.63) together with Eqs. (4.31), (4.40), and (4.41) solve completely the so-called direct problem, furnishing all spectral data and the auxiliary function $\varphi(x, \mathbf{k}_{\Im})$ in terms of the Jost solution $\Phi(x, \mathbf{k})$ and the functions $\Phi_1(x, \mathbf{k})$ and $\varphi_1(x)$ relative to the one-soliton potential.

Below we discuss the properties of these spectral data and relations between them.

2. Characterization equations for spectral data

Inserting (4.68) in (4.64) we get relation $(\Phi^{\sigma}, \varphi^{\sigma}) = (\Phi^{\sigma}, \varphi^{\sigma})(W^{\sigma})^{-1}(\mathcal{R}_{\pm}^{-\sigma})^{\dagger} * \mathcal{R}_{\pm}^{\sigma}$, $\sigma = +, -$, so taking into account the asymptotic behavior of the Jost solutions with respect to the x-variables we derive the characterization equation

$$(\mathcal{R}_{\pm}^{-\sigma})^{\dagger} * \mathcal{R}_{\pm}^{\sigma} (W^{\sigma})^{-1} = \mathcal{I} \tag{4.72}$$

for the spectral data $\mathcal{R}_{\pm}^{\sigma}$. Another characterization equation follows by inserting (4.64) into (4.68):

$$\mathcal{R}_{\pm}^{\sigma} (W^{\sigma})^{-1} * (\mathcal{R}_{\pm}^{-\sigma})^{\dagger} = \mathcal{I}, \tag{4.73}$$

and the third one is the condition that the r.h.s. of (4.71) is independent of the choice of the sign $\pm$:

$$(\mathcal{R}_{+}^{\sigma})^{\dagger} * \mathcal{R}_{+}^{\sigma} = (\mathcal{R}_{-}^{\sigma})^{\dagger} * \mathcal{R}_{-}^{\sigma}. \tag{4.74}$$

These equations generalize the characterization equations of the case of decaying potential.[17] We see that (4.72) and (4.73) are just the condition that the "triangular operators" $\mathcal{R}_{\pm}^{\sigma}$ are invertible and that $(\mathcal{R}_{\pm}^{-\sigma})^{\dagger}$ is the inverse of $\mathcal{R}_{\pm}^{\sigma}(W^{\sigma})^{-1}$. They define, say, $\mathcal{R}_{\pm}^{-}$ if $\mathcal{R}_{\pm}^{+}$ is given. Thus essential restrictions on the spectral data are imposed by (4.74), that is a consequence of the reality condition on the potential $u(x)$. Thanks to (4.64) this equation proves (4.45). By (4.65) we write (4.74) as a system of three equations (for some sign σ):

$$\left(\int_{k'}^{+\infty}-\int_{-\infty}^{k}\right)dk''\,\overline{r^{\sigma}(k'',k)}\,r^{\sigma}(k'',k')=\operatorname{sgn}k\;\theta(kk')\,\overline{r^{\sigma}(k)}\,r^{\sigma}(k')+\sigma r^{\sigma}(k,k')+\sigma\overline{r^{\sigma}(k',k)},\tag{4.75}$$

$$[\theta(k)A^{\sigma}+\theta(-k)]\overline{r^{\sigma}(k)}=\sigma\left(\int_{0}^{+\infty}-\int_{-\infty}^{k}\right)dk'\,\overline{r^{\sigma}(k',k)}\,\tau^{\sigma}(k')-\tau^{\sigma}(k),\tag{4.76}$$

$$|A^{\sigma}|^{2}=\int dk\,\operatorname{sgn}k\,|\tau^{\sigma}(k)|^{2}+1.\tag{4.77}$$

Characterization equations for the spectral data (4.71) look simpler,

$$(\mathcal{F}^{\sigma})^{\dagger}=\mathcal{F}^{\sigma},\quad (\mathcal{F}^{\sigma})^{-1}=(W^{-\sigma})^{-1}\mathcal{F}^{-\sigma}(W^{\sigma})^{-1},\tag{4.78}$$

to which we have to add the requirement that $\mathcal{F}^{\sigma}$ can be decomposed in the product (4.71) of two sets of triangular operators (4.65).

3. Discontinuities of spectral data

The cuts of the Jost solutions intersect on the complex plane at $k=0$. Thanks to property (4.6) for the Green's function we get by (4.3) that the Jost solution obeys property

$$\lim_{\sigma\mathbf{k}_{\Im}\to+0}\Phi(\pm 0+i\mathbf{k}_{\Im})=\Phi^{\sigma}(\pm 0),\qquad \sigma=+,-,\tag{4.79}$$

which means that it is necessary to pay special attention to the properties of the spectral data at this point $k=0$. Thus taking the limits $\mathbf{k}_{\Im}\to\sigma 0$ in (4.28) we get for the boundary values of the Jost and auxiliary Jost solutions the relation

$$\Phi^{\sigma}(+0)-\Phi^{\sigma}(-0)=\varphi^{\sigma}w^{\sigma}.\tag{4.80}$$

The spectral data $r^{\sigma}(k,k')$ are defined in (4.59) in terms of the Jost solution $\Psi_1(k)$, which is a continuous function of k, and of $\Phi^{\sigma}(k')$ which is discontinuous at $k'=0$. Therefore $r^{\sigma}(k,k')$ is continuous with respect to k and k' with a possible discontinuity at $k'=0$. Precisely, by (4.59), (4.63), and (4.80) we get

$$r^{\sigma}(k,+0)-r^{\sigma}(k,-0)=\tau^{\sigma}(k)w^{\sigma}.\tag{4.81}$$

The spectral data $\tau^{\sigma}(k)$ are continuous at $k=0$ and from their definition (4.63) and the definitions (4.10) and (4.29) of A and w we derive

$$\tau^{\sigma}(0)=\sigma A^{\sigma}\overline{w^{-\sigma}}.\tag{4.82}$$

The spectral data $r^{\sigma}(k)$ are discontinuous at $k=0$ as follows from their definition (4.61). Taking into account (4.30) and (4.31), we get the relations

$$r^{\sigma}(+0)=\sigma A^{\sigma}w^{\sigma},\tag{4.83}$$

$$r^{\sigma}(-0)=\sigma w^{\sigma}.\tag{4.84}$$

These behaviors at $k=0$ are compatible with the characterization equations. In fact, choosing $k'=\pm 0$ in (4.75) we get (4.76) and analogously (4.76) at $k=\pm 0$ gives (4.77).

F. Inverse problem and time evolution

Let now be given the spectral data, i.e., function $w(\mathbf{k}_{\Im})$ and, say, $\mathcal{R}^{+}_{\pm}$. Function $w(\mathbf{k}_{\Im})$ is continuously differentiable on the interval $|\mathbf{k}_{\Im}|<\kappa$ except for the point $\mathbf{k}_{\Im}=0$, where it has bounded limits $w^{\pm}$. Moreover, $w(\mathbf{k}_\Im)$ has the asymptotic behaviors (4.42), (4.43) and $A(\mathbf{k}_\Im)$ defined by (4.41) is bounded on this interval and has finite limits $A^{\pm}$ at zero. The spectral data $\mathcal{R}_{\pm}^{+}$ obey the characterization equation (4.74), or, more explicitly the characterization equations (4.75)–(4.77) for its elements in (4.65) and relations (4.81)–(4.84). The matrix operator $\mathcal{R}_{\pm}^{+}$ also admits inverse, so that the spectral data $\mathcal{R}_{\pm}^{-}$ can be introduced by (4.72) or (4.73) with the triangularity property given in (4.65), which means that the spectral data $\mathcal{F}^{-}$ can be defined by (4.71). Taking into account that the Jost solution $\Phi(x,\mathbf{k})$ is analytic in the complex $\mathbf{k}$-plane with a discontinuity across the real axis, given by the first equation in (4.70) for $\sigma=+$, and with a discontinuity across the interval $|\mathbf{k}_\Im|<\kappa$ of the imaginary axis, given by (4.28), and has an asymptotic normalization at $\mathbf{k}\to\infty$ given by (2.78) and (2.80), the Cauchy–Green formula yields

$$\Phi(x,\mathbf{k})=e^{-i\mathbf{k}x_1-i\mathbf{k}^2x_2}+\frac{1}{2\pi i}\int dk'\frac{e^{i(k'-\mathbf{k})x_1+i(k'^2-\mathbf{k}^2)x_2}}{k'-\mathbf{k}}\left(\int dk''\Phi^{-}(x,k'')[\mathcal{F}_{11}^{-}(k'',k')\right.$$
$$\left.-\delta(k''-k')]+\frac{\varphi^{-}(x)}{A^{-}}\mathcal{F}_{21}^{-}(k')\right)-\frac{1}{2\pi i}\int_{-\kappa}^{\kappa}\frac{d\alpha\, w(\alpha)}{\alpha+i\mathbf{k}}e^{-(\alpha+i\mathbf{k})x_1-i(\alpha^2+\mathbf{k}^2)x_2}\varphi(x,\alpha). \tag{4.85}$$

Notice that the singular behavior (4.42) of $w(\alpha)$ at $\alpha=\kappa$ is smoothed by (4.38) and, consequently, the integral on the r.h.s. is well defined. It is necessary to emphasize that equalities (4.40) and (4.85) were derived under condition (4.25) that excludes the case $u_2(x)\equiv 0$. Indeed, in this case $G_{\mathrm{reg}}(i\kappa)=G_{1,\mathrm{reg}}(i\kappa)$ so that by (4.23) $B(\kappa)=0$.

The integral equation (4.85) together with (4.40) solves the inverse problem. In fact, considering (4.40) and the limits of (4.85) as $\mathbf{k}\to k-i0$, $k\in\mathbb{R}$, and as $\mathbf{k}\to+0+i\mathbf{k}_\Im$, we obtain a system of three linear integral equations for $\{\Phi^{-}(x,k),\Phi(x,+0+i\mathbf{k}_\Im),\varphi(x,\mathbf{k}_\Im)\}$, whose solution is uniquely determined in terms of the spectral data. When these boundary values are determined, the Jost solution for generic $\mathbf{k}\in\mathbb{C}$ is given by (4.85) and then the potential is reconstructed in the standard way by (2.78), (2.81) as

$$u(x)=\frac{1}{\pi}\frac{\partial}{\partial x_1}\left[\int dk\, e^{ikx_1+ik^2x_2}\left(\int dk'\Phi^{-}(x,k')[\mathcal{F}_{11}^{-}(k',k)-\delta(k'-k)]+\frac{\varphi^{-}(x)}{A^{-}}\mathcal{F}_{21}^{-}(k)\right)\right.$$
$$\left.-i\int_{-\kappa}^{\kappa}d\alpha\, w(\alpha)\, e^{-\alpha x_1-i\alpha^2x_2}\varphi^{-}(x,\alpha)\right]. \tag{4.86}$$

Since, in the resolvent approach, all spectral data can be given as special reductions of the extended resolvent, one can derive directly (see Ref. 4 for details) the time evolution of the spectral data related to a solution of (1.2). We omit here these calculations and give only the final result on the time evolution of the spectral data:[16]

$$\mathcal{F}^{-}(k,k',t)=\begin{pmatrix} e^{-4ik^3t} & 0\\ 0 & 1\end{pmatrix}\mathcal{F}^{-}(k,k')\begin{pmatrix} e^{4ik'^3t} & 0\\ 0 & 1\end{pmatrix},\quad k,k'\in\mathbb{R}, \tag{4.87}$$

$$w(\mathbf{k}_\Im,t)=e^{4\mathbf{k}_\Im^3t}w(\mathbf{k}_\Im), \tag{4.88}$$

$$A(\mathbf{k}_\Im,t)=A(\mathbf{k}_\Im). \tag{4.89}$$

Taking the time evolution (4.88) into account we get that (4.42) and (4.43) are preserved under the dynamics and like in the pure one-soliton case

$$\kappa(t)=\kappa,\quad x_0(t)=x_0+4\kappa^2t, \tag{4.90}$$

where $x_0(t)$ is the parameter of the one-soliton solution (3.2).

V. CONCLUSION

We presented the main aspects of what we call the extended resolvent approach to the spectral theory of differential operators and we tested it in the two-dimensional case of the nonstationary Schrödinger equation with decaying and nondecaying potentials at large distances in the plane. We showed that a specific advantage of this approach is that the extended resolvent $M(s)$ of the differential operator (1.1), which depends on a parameter $s \in \mathbb{R}^2$, can be used as a general tool for finding all mathematical objects, such as Green's functions, Jost and auxiliary Jost solutions, advanced/retarded solutions and spectral data, which result to be necessary for developing the spectral theory. In fact, the introduction of all these quantities follow naturally from the study of departure from analyticity of the extended resolvent and of its discontinuities, together with its reductions. In particular, the different Green's functions involved in the theory are values (in general limiting values) of the resolvent $M(s)$ at different specific points s, which are suggested by the behavior in s of $M(s)$ itself. Moreover, the study of the properties of all these objects is reduced to the study of the resolvent $M(s)$, which can be accomplished by using Hilbert-type identities and the bilinear representation (2.90). This representation gives the extended resolvent in terms of the Jost solutions and supplies a simple and straightforward way of deriving relations between different kinds of solutions of the linear problem, constructing spectral data and deriving relations between them, as well as characterization equations.

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