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Analysing quality with generalized kinetic methods☆

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Abstract

A mathematical structure is developed with the aim of analysing the time evolution of the quality of a composite system such as a medical service inside an hospital. The approach belongs to the so-called Generalized Kinetic Theory, and consists of a set of balanced statistical equations on the probability distribution functions of the system populations over a state variable that represents the perceived quality. Internal and external actions are taken into account by means of direct interactions and ensemble terms. The mathematical framework is developed for a general setting. As a particular case, a model is suggested with reference to the quality of a specific medical service.

Keywords: Generalized kinetic model; Integro-differential equations; Evolution equations; Quality analysis; Social behaviours; Population dynamics

1. Introduction

The interest in mathematical models that describe and analyse living elements is increasing. In particular, the field known in the literature as Generalized Kinetic Theory is proving to be sufficiently general and conveniently adaptable to perform the difficult task of treating advanced and complex systems such as those that refer to human individuals.

Simulations are promptly processed and easily presented, and common opinion positively reacts to mathematical anticipations. Hence, models may be formulated and tested even when they refer to partially hidden and highly nonlinear underlying systems. On the other hand, mathematical theories well-known and tested in some research areas may highly contribute to the creation of new methods and perspectives even in fields remote to those that motivated their introduction.

It is the authors' opinion that the so-called *Generalized Kinetic Models* represent a fruitful predictive and descriptive tool in the area of the Social Sciences and of high level Human Structures. As it is well-known these models transfer the methodology developed for systems of a great number of interacting particles (typically in the

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field of kinetic theory of classical particles, [1]) to various other fields of research, such as traffic dynamics (see, among others, [2–5]), cellular dynamics (see, among others, [6–9]), aspects of social dynamics [10–13], population dynamics [14], and biological systems in general. Further research towards the generalization of classical models of the mathematical kinetic theory have been the object of systematic studies developed by various authors; among others [15–17]. Interesting reviews of the various aspects of the theory may be found in [18,6,19].

In what follows, and on the lines of the analysis started in [20,21], a statistical picture that only loosely may be addressed as a "Boltzmann model" is developed to describe the time evolution of a macroscopic variable related to the *quality* of a fully developed system such as a medical service. This is done by means of a microscopic state variable, which in the literature is denoted by *activity* of the actors, driven by actions both of internal and of external nature.

On account of the natural fuzziness of the observations that concern a property such as the estimated quality of a structure, not only due to the absence of repeatable measurements but also due to the intrinsic difficulty of defining what is meant as *quality* of a system, the activity is assumed to be a *random variable*. The object of the present study is to propose a convenient set of evolution equations that may describe the dynamics of the *probability density functions* about this variable when the latter is individually related to each of the actors in the system. The analysis is developed keeping in mind the idea of singling out and characterizing the physical variables that control the long times (or average) dynamics, and to predict the effect that possible readjustments of the structure may produce on it.

The content of the paper is organized into six sections. Section 2, that follows this introduction, contains a brief discussion on the definition of the *actors* of the play. Section 3 introduces the mathematical framework in the case of a continuous status variable. Section 4 treats the analogous, although not equivalent, framework developed for a discrete status variable. In Section 5 a model referred to a specific Medical Service is anticipated. Section 6 is reserved for a brief discussion.

2. Structures with weakly defined quality levels

This section is devoted to the depiction of the main characteristics of the structure that we intend to describe using the methods mentioned in the introduction and developed in the following sections. In particular, since data are available on the specific case of the Neuropsychiatric Ward: *Dipartimento Salute Mentale ASL – BR* which is under the direct responsibility of one of the authors, special reference will be made to a medical service inside an hospital. However it is easy to realize that the structure under consideration, hereafter called *service*, has characteristics that are widely general and common to many other services, from a store to a firm. In fact, by conveniently renaming some of the terms that will be used, the whole construction may be adapted even to inert actors, such as the items of a production process. They are briefly listed below.

1. The service has living objects (humans, in particular) as targets of its efforts. For instance: patients, customers, and users, respectively for a medical service, a store, and a general service. They constitute the first population of elements, or *actors*, that play a role in the picture.
2. The service is run by living objects (humans, in particular) that provide a utility to the former actors. For instance: medical staff, clerks, agents, respectively. They constitute a second population of actors.
3. The service takes advantage of a set of resources (hardware, in particular) that help the agents to perform the specific tasks that are supplied to the users. They constitute a third population of actors.
4. The service is subject to a dynamics that runs on different time scales, often related to the different populations of actors. For instance: first aid to patients on a fast time scale, turnover shifts on a median time scale, staff reorganizations or plant renovations on a long time scale.
5. The service is not provided with a definite quality variable, although it is very clear to all the actors that the dynamics of the service, and the mutual relations among the actors, strictly depend on the quality that is offered by the service. Only subjective effects are considered, that are produced on each of the actors by the performance of the others. These effects may be (and have been) measured on a statistical basis.
6. Actors are identified (by their populations and) by a unique state variable: the *activity*, which is meant to be related to the *quality* of the service. No kinematic variables such as the spatial position and/or velocity of the actors have relevance to the service dynamics, and consequently they are not taken into account.

Although, as stated above, actors are addressed by only one (scalar) state variable, nevertheless it is clear that if referred to actors of different populations the activity has different meanings. For instance, in the case of a patient the quality is related to the *satisfaction* he feels about the service he receives, for an agent it is related to the *stress* he is subject to when performing his duty, for a piece of hardware it simply depends on its *efficiency* and suitability.

On the other hand, per each of these different quality variables, a global judgment of the service quality is convenient. Therefore we shall assume that a meaningful normalization of the various indices is possible, and in particular some appropriate coding of their values to corresponding "state spaces", that allow computations and comparisons one with the others.

On the contrary, as already noted, unification of the time scales is not acceptable. The dynamics of the service are indeed strongly dependent on the shift variations as well as on a patient's charge and discharge, and these may not be measured on the same time scale of the various interactions among the actors. Indeed the last ones happen much more frequently than the former, and have as well an influence on the system evolution.

A picture that may certainly be of interest is the one with a *discrete time* variable, though it needs a deeper analysis on the relations among the various time scales. In what follows here, the evolution is assumed to develop on a sequence of time intervals of equal length, for instance $\tau = 24$ h, and time is a continuous real variable defined on

$$\begin{aligned} &[0, T] = [0 =: t_0, t_1) \cup [t_2, t_3) \cup \cdots \cup [t_{M-2}, t_{M-1}) \cup [t_{M-1}, t_M = T], \\ &t_m := m\tau, \quad m = 1, 2, \ldots, M \in \mathbb{N}. \end{aligned} \tag{2.1}$$

The system is assumed to be specified by its values *at each of the instants* $t_m, m = 0, 1, \ldots, M - 1$, and discontinuous changes in the composition of populations are allowed at each of these instants.

Concerning the activity variable, not only the recourse to the statistical interpretation of the random variable is almost necessary, but also separate pictures, the one with a continuous state space and the other with a discrete one, are of relevance. Indeed, and in particular when subjective evaluations are requested or filed, the recourse to a proper codification induces output tables that may have as well an intrinsic continuous structure or a discrete one. In both cases, the activity is assumed to be a scalar random variable over some (hidden) measure space of elementary events, and realized by three distinct variables, one per each of the system populations. In the continuous case, activity means the variables

$$u_1, u_2, u_3 : \quad u_i \in \mathbb{I}_i := [a_i, b_i] \subset \mathbb{R}, \quad i = 1, 2, 3; \tag{2.2}$$

for a convenient six-tuple of real values: $(a_1, \ldots, b_3)$, whereas in the discrete case it means the variables u_1, u_2, u_3 whose values $u_{i,h}$ are specified by the sets

$$\mathbb{D}_i := \left\{u_{i,1}, u_{i,2}, \ldots, u_{i,h}, \ldots, u_{i,H_i}\right\} \subset \mathbb{R}, \quad i = 1, 2, 3. \tag{2.3}$$

Remark 2.1. It may happen, when computations are concerned, that recourse to more convenient variables is made, such as the normalized ones:

$$u_1, u_2, u_3 : \quad u_i \in \mathbb{I}_i \equiv [0, 1] \subset \mathbb{R}, \quad i = 1, 2, 3; \tag{2.4}$$

in the continuous case and, in the discrete case, variables u_1, u_2, u_3 whose values $u_{i,h}$ are in the sets

$$\mathbb{D}_i := \{1, 2, \ldots, h, \ldots, (H_i - 1), H_i\} \subset \mathbb{R}, \quad i = 1, 2, 3; \tag{2.5}$$

however the mathematical framework is unaffected by these particular choices, and is developed here for the general case.

To be more precise, what in fact is of interest are not the mentioned variables (whichever their character be, continuous or discrete), rather the random processes represented by the family of such state variables over the interval $[0, T]$. That is:

$$u_1, u_2, u_3 : \quad u_i : t \in [0, T] \mapsto u_i(t) \in \mathbb{I}_i \subset \mathbb{R}, \quad i = 1, 2, 3; \tag{2.6}$$

in the continuous case and, in the discrete case, the set of functions

$$u_{i,j} : t \in [0, T] \mapsto u_{i,j}(t) \in \mathbb{D}_i \tag{2.7}$$

whose values $u_{i,j}(t)$ are randomly distributed in the sets

$$\mathbb{D}_i := \left\{u_{i,1}, u_{i,2}, \dots, u_{i,h}, \dots, u_{i,H_i}\right\} \subset \mathbb{R}, \quad i = 1, 2, 3. \tag{2.8}$$

The corresponding probability density functions

$$f_i : (t, x) \in [0, T] \times \mathbb{I}_i \subset \mathbb{R}^2 \mapsto f_i(t, x) \in [0, \infty) \tag{2.9}$$

in the continuous case and, respectively, the set of functions

$$f_{i,j}(t) : [0, T] \subset \mathbb{R} \mapsto f_{i,j}(t) \in [0, \infty), \quad i = 1, 2, 3,\ j = 0, 1, \dots, H_i \tag{2.10}$$

are the objects of our study.

By means of functions (2.9) and (2.10), by use of the consequent expectations of appropriate state functions, and possibly making reference to convenient time averages, the statistical description of the system is meant to be analysed and the quality of the entire institution is described.

3. The mathematical framework

This section is devoted to the construction of a possible framework for the mathematical representation and description of the variables cited in Section 2, and to deduce the corresponding evolution equations when the system state variables are assumed to be continuous, see Eqs. (2.6) and (2.9). As cited above, the approach is statistical on a (hidden) probability space of elementary events.

Notation. Unless explicitly stated, throughout the paper *lowercase letters* denote real numbers, real variables, and real valued functions of real variables. Moreover, for simplicity reasons and unless clarity is harmed, it may happen that functions are recalled by the (incorrect but more expressive) notation of denoting their values at given arguments: $f(x)$ rather than $f : x \mapsto f(x)$. Therefore small letters also denote the values taken by real valued random variables. *Capital letters* denote expectations with respect to probability density functions; in particular zeroth or first-order moments. For shortness reasons, in writing expectation values the dependence on the density functions is not explicitly indicated. Conversely, the implicit dependence on the densities is recalled (within square brackets) when functions of the expectations are introduced. *Blackboard fonts* are reserved to denote sets. *Calligraphic fonts* are used to denote explicit functionals of the densities. *Bold face* fonts are used to denote *n*-tuples of elements.

Assumption 3.1. The system under consideration is composed of three populations $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$, of actors. Population $\mathbb{P}_1$ is composed of $N_1 = N_1(t)$ human individuals, the *users* of the service; population $\mathbb{P}_2$ of $N_2 = N_2(t)$ human individuals, the service *agents*; population $\mathbb{P}_3$ is composed of $N_3 = N_3(t)$ pieces of hardware, the service *resources*. Mass functions $N_i : t \in [0, T] \mapsto N_i(t), i = 1, 2, 3$, are assumed to be stepwise constant, integer valued, right continuous, and *known as data of the problem*. Discontinuities may be found only at the endpoints $t_m := m\tau$, $m = 0, 1, \dots, M \in \mathbb{N}$, of the evolution intervals.

Assumption 3.2. Individuals of the same population are identical. Individuals are not personally identified, but only addressed by the state (random) variable denoting their *activity*. Individuals of the same population may share the same activity value with other individuals of the same population. The system description is obtained by the set of density functions over the individual states.

What is under examination is whether the probability that the activity u_i of an element e_i randomly selected out of population $\mathbb{P}_i$ is found in (any) real interval $[c_1, c_2] \subset \mathbb{I}_i$. Or:

Assumption 3.3. The system variables are defined by a set of stochastic processes over the interval $[0, T]$:

$$u_1, u_2, u_3 : \qquad u_i : t \in [0, T] \mapsto u_i(t) \in \mathbb{I}_i \subset \mathbb{R}, \quad i = 1, 2, 3; \tag{3.1}$$

that associate with each instant of time $t \in [0, T]$ the values of the corresponding (random) state variables u_i. Under examination are the *probability density functions*

$$f_i : (t, \cdot) \in \mathbb{I}_i \subset \mathbb{R} \mapsto f_i(t, \cdot) \in [0, \infty), \quad i = 1, 2, 3, \tag{3.2}$$

such that the event: $u_i \in [c_i, c_2] \subset \mathbb{I}_i$, which refers to the outcome of the cited (random, uncertain) activity measurement at time $t \in [0, T] \subset [0, \infty)$, has probability given by

$$P(t; c_1 \le u_i \le c_2) = \int_{c_1}^{c_2} f_i(t, x)\mathrm{d}x, \quad i \in \{1, 2, 3\}. \tag{3.3}$$

Functions $f_i : (t, \cdot) \in \mathbb{I}_i \subset \mathbb{R} \to f_i(t, \cdot) \in [0, \infty)$ are sufficiently regular; in particular they verify: $f_i(t, \cdot) \in \mathcal{C}^2(\mathbb{I}_i)$, and are such that

$$\int_{\mathbb{I}_i} f_i(t, x)\mathrm{d}x = 1, \quad \forall t \in [0, T], i \in \{1, 2, 3\}. \tag{3.4}$$

Functions $f_i : (\cdot, x) \in [0, T] \subset \mathbb{R} \to f_i(\cdot, x) \in [0, \infty)$ verify

$$f_i(\cdot, x) \in \bigcap_{m=0}^{M} \left(\mathcal{C}^1([m\tau, (m+1)\tau])\right). \tag{3.5}$$

In addition to the cited assumptions, all the density functions are assumed to be sufficiently regular such that the calculations and integrations hereafter introduced may be meaningful. In particular, the following particular macroscopic variables are assumed to exist.

Mean activity of population $\mathbb{P}_i$

$$U_i(t) := N_i(t) \int_{\mathbb{I}_i} x f_i(t, x)\mathrm{d}x, \quad \text{for } t \in [0, T], i \in \{1, 2, 3\}. \tag{3.6}$$

On these accounts, we assume that real normalizing constants $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ may be properly defined such that

$$U(t) := \sum_{i=1}^{3} \alpha_i U_i(t) \tag{3.7}$$

correctly refers to the (global) *Expected Quality of the Service* at time t.

Concerning the system evolution, as it is already mentioned in Section 2 and is by now commonly referred to in the literature of Generalized Kinetic modelling, evolution equations on functions $\boldsymbol{f} := (f_1, f_2, f_3)$ are in fact *mass balance* equations in the state space. By this we mean that:

Assumption 3.4. The total variation rate of each density function f_i, namely the sum of the direct variation with respect to time plus a flow term due to external actions and (possibly) to internal actions of global character, is equal to the balance between a "gain" term and a "loss" term that refers to the specified density and is due to internal "interactions". In formulas:

$$\partial_t f_i + \partial_x \Phi_i[\boldsymbol{f}] = \mathcal{G}_i[\boldsymbol{f}] - \mathcal{L}_i[\boldsymbol{f}], \quad i = 1, 2, 3. \tag{3.8}$$

Unlike other Generalized Kinetic models, here no change in populations is (obviously) allowed. Moreover, no input or output of actors are taken into account except those that happen at the cited instants $t_m = m\tau, m = 0, 1, 2, \ldots, M$, when the evolution undergoes its global periodic variations.

The terms that appear in the balance equation (3.8) may be identified, as it is explained in the sequel, by means of a set of appropriate functions that refer to how these "mesoscopic" statistical terms are constructed starting from elementary microscopic actions. Specifically:

Assumption 3.5. Each individual of the system is subject to actions of external and of internal nature. Actions of internal nature, or interactions, are further subdivided into actions of social character and actions of direct character; the former ones due to means computed over specific actors' ensembles, the latter to direct relations among single actors.

Actions of external nature act by means of a term with the structure of a field.

Interactions among the actors are identified, as usual in kinetic theories, by means of convenient encounter frequencies and change of state probabilities, as specified later on.

Interaction rules *parametrically* depend on appropriate expectations and averages performed over the system densities. Only instantaneous (i.e. with no time delays), interactions are considered.

Evolution equations are meant to refer to the dynamics of the probability density functions f_i referring to the status of actors of population $\mathbb{P}_i$ at time t. More precisely, $N_i(t) f_i(t,x)$ denotes the expected fraction of the total number $N_i(t)$ of actors of population $\mathbb{P}_i$ that are in status x. The equation on f_i relates its time variations with the various actions that produce these variations. When, among them, an interaction term is included, the probability density function is about a *test* actor, and refers to is status immediately after the interaction with other *field* actors. By this we mean that the random measurement on the test actor status is performed immediately after the event considered by the interaction, and has value distributed as referred to $f_i(t,x)$.

As the flux terms are concerned, we borrow our reasoning from the theory of a (hypothetical) flow, and state the following

Assumption 3.6. The convective term on p.d.f. $f_i(t,x)$ has the structure of a (local) net flow: $\partial_x \Phi_i$, $i = 1,2,3$. The flow Φ_i may be identified by

$$\Phi_i[\boldsymbol{f}](t,x) = \mathcal{K}_i[\boldsymbol{f}](t,x) f_i(t,x) + c_i(t,x)\partial_x f_i(t,x), \quad x \in \mathbb{I}_i, i = 1,2,3. \tag{3.9}$$

In the first term of Eq. (3.9) the coefficient $\mathcal{K}_i[\boldsymbol{f}]$, to be thought of as a drift velocity, represents the internally induced speed of change due to actions of global character that may be ascribed to ensembles of actors of the various populations, namely: as if they were other single actors. The drift $\mathcal{K}_i$ is written as

$$\mathcal{K}_i[\boldsymbol{f}](t,x) := \sum_{j=1}^{3} \mathcal{F}_{i,j}[\boldsymbol{f}](t,x), \quad i = 1,2,3, \tag{3.10}$$

$$\mathcal{F}_{i,j}[\boldsymbol{f}](t,x) := \int_{\mathbb{L}_j \subset \mathbb{I}_j} \mathcal{P}_{i,j}(t,x,y) f_j(t,y)\mathrm{d}y, \quad y \in \mathbb{I}_j, x \in \mathbb{I}_i, \tag{3.11}$$

where the kernels $\mathcal{P}_{i,j}$ need to be properly modelled on account of the specific ensembles to be considered.

The second contribution in Eq. (3.9) is thought to be due to global dynamics induced on the actors (and on their states) by actions of external origins: social or psychological reasons. It may be represented by a flow proportional to the gradient of the probability density function through "diffusion" coefficients $c_i : [0,T] \times \mathbb{I}_i \to \mathbb{R}$ that need to be properly modelled as well.

For instance, $\mathbb{L}_j \equiv \mathbb{I}_j$ and

$$\mathcal{P}_{i,j}(t,x,y) = \chi(x)\alpha_j y N_j(t)$$

furnish a drift $\mathcal{K}_i$ that is proportional to the macroscopic variable U. The regularizing term $\chi : \mathbb{I}_i \to [0,\infty)$ is conveniently considered in order that the flow through the end points of the intervals $\mathbb{I}_i$ may be null at each instant of time.

As it concerns the balance on the right-hand side of the evolution equation (3.8), it refers to the relations, or "interactions", that directly involve the actors. The structure here is quite similar to those well-known in the Generalized Kinetic Theory, see in particular [18]. In what follows, single-actor interactions are also considered, and triple interactions, but not those of greater multiplicity, that are assumed to be rare. Minor relevance is attributed to triple interactions which involve two actors of the same population as well; they are assumed to be rare with respect to the occurrence of a pairwise interaction between the two "equal" actors followed by a further pairwise interaction between their outcome and the third element of the triplet. In this sense: "similar elements interact at first". Triplet of actors of the same population are rare as well.

Assumption 3.7. Internal direct events modify the state of a test actor, not its population, with a rate specified by convenient functions: η_i, $\eta_{i,j}$, $\eta_{i,j,\ell}$, that refer to the frequencies of his relations with field actors. State changes are stochastic events, specified by convenient *transition (conditional) probability density functions* ψ_i, $\psi_{i,j}$, $\psi_{i,j,\ell}$.

Specifically, the following sets of (regular) functions are assumed to exist. In all of them, $t \in [0,T]$, $i, j, \ell \in \{1,2,3\}$. Each of them may depend on time not only as an external parameter, but also because their dependence on expectations over the densities $\boldsymbol{f}$. This kind of implicit dependence on the densities $\boldsymbol{f}$ is recalled by the square bracket notation.

- $\eta_i(t,\cdot) : x \in \mathbb{I}_i \mapsto \eta_i(t,x) \in [0,\infty)$ rate of events wherein an individual of population $\mathbb{P}_i$ autonomously reflects on modifying his state x.
- $\eta_{i,j}(t,\cdot) : (x,y) \in \mathbb{I}_i \times \mathbb{I}_j \mapsto \eta_{i,j}(t,x,y) \in [0,\infty)$ rate of events wherein an individual of population $\mathbb{P}_i$ in the state $x \in \mathbb{I}_i$ "encounters" an individual of population $\mathbb{P}_j$ in the state $y \in \mathbb{I}_j$.
- $\eta_{i,j,\ell}(t,\cdot) : (x,y,z) \in \mathbb{I}_i \times \mathbb{I}_j \times \mathbb{I}_\ell \mapsto \eta_{i,j,\ell}(t,x,y,z) \in [0,\infty)$ rate of events wherein an individual of population $\mathbb{P}_i$ in the state $x \in \mathbb{I}_i$ "encounters" both an individual of population $\mathbb{P}_j$ in the state $y \in \mathbb{I}_j$ and an individual of population $\mathbb{P}_\ell$ in the state $z \in \mathbb{I}_\ell$.
- $\psi_i[\boldsymbol{f}](t,x;x') \in [0,\infty)$ probability density function on the outgoing state $x' \in \mathbb{I}_i$ of a test individual of population $\mathbb{P}_i$ in the state $x \in \mathbb{I}_i$ after an event wherein he autonomously reflects on the possibility of modifying his state x.
- $\psi_{i,j}[\boldsymbol{f}](t,x,y;x') \in [0,\infty)$ probability density function on the outgoing state $x' \in \mathbb{I}_i$ of a test individual of population $\mathbb{P}_i$ in the state $x \in \mathbb{I}_i$ after an event wherein he encounters an individual of population $\mathbb{P}_j$ in the state $y \in \mathbb{I}_j$.
- $\psi_{i,j,\ell}[\boldsymbol{f}](t,x,y,z;x') \in [0,\infty)$ probability density function on the outgoing state $x' \in \mathbb{I}_i$ of a test individual of population $\mathbb{P}_i$ in the state $x \in \mathbb{I}_i$ after an event wherein he simultaneously encounters two individuals: one of population $\mathbb{P}_j$ in the state $y \in \mathbb{I}_j$, the other of population $\mathbb{P}_\ell$ in the state $z \in \mathbb{I}_\ell$.

Transition probability density functions ψ_i, $\psi_{i,j}$, $\psi_{i,j,\ell}$, that describe the test actor state changes according to the above scheme, are probability density functions with respect to the outgoing test actor state.

Assumption 3.8. Convenient boundary conditions may be stated and specified in order that each of the total population sizes $N_i(t)$ remain constant along each of the intervals $[m\tau,(m+1)\tau]$, $m = 0,1,2,\dots,M$. The flow of individuals through both the boundaries of $\mathbb{I}_i$ are null at each instant $t \in [0,T]$. Population sizes may only change at instants $t_m := m\tau$, $m = 0,1,2,\dots,M-1$, to refer to inlet and outlet of actors of each population. Functions $f_i(\cdot,x)$ satisfy the regularity condition specified by Eqs. (3.4), (3.5) and are such that, for $m = 1,2,\dots,M$

$$f_i(m\tau,x) = \frac{1}{N_i(m\tau)}\left[N_i^{+,m}\, g_i^{(m)}(x) + \left(\lim_{\varepsilon\to 0} N_i(m\tau-\varepsilon) - N_i^{-,m}\right) f_i(m\tau-\varepsilon,x)\right], \tag{3.12}$$

for $x \in \mathbb{I}_i$, and $i = 1,2,3$; where $N_i^{+,m}$ and $N_i^{-,m}$ are the number of actors of population $\mathbb{P}_i$ that respectively enter and leave the system at time $t_m = m\tau$, and where

$$g_i^{(m)} : x \in \mathbb{I}_i \mapsto g_i^{(m)}(x) \in [0,\infty) \quad \text{are such that} \int_{\mathbb{I}_i} g_i^{(m)}(x)\mathrm{d}x = 1 \tag{3.13}$$

and refer about the inlet distributions of actors at time t_m. As such, the set of numbers $N_i^{+,m}$ and $N_i^{-,m}$ and the set of functions $\{g_i^{(m)},\ m = 1,\dots,M\}$ are to be considered as part of the data of the problem.

The following assumption concludes the axiomatic description of the mathematical framework of the problem. It consists of sufficient, and far from necessary, conditions; in fact, several other similar hypotheses may be proposed and only a specific analysis may properly suggest one of them. On the other hand, the total freedom that has been left to the above analysis and to the various parameters, points towards the next one.

Assumption 3.9. Making recourse to the functions listed above, the statistical balances that appear on the right-hand sides of Eq. (3.8) verify the scheme that follows

$$\mathcal{G}_i - \mathcal{L}_i := \mathcal{G}_i^{(1)} - \mathcal{L}_i^{(1)} + \mathcal{G}_i^{(2)} - \mathcal{L}_i^{(2)} + \mathcal{G}_i^{(3)} - \mathcal{L}_i^{(3)}, \quad i = 1,2,3, \tag{3.14}$$

where, let δ denote the well-known *Dirac's distribution*, and $x' \in \mathbb{I}_i$, $t \in [0,T]$,

$$\left(\mathcal{G}_i^{(1)} - \mathcal{L}_i^{(1)}\right)[\boldsymbol{f}](t,x') := \int_{\mathbb{I}_i} \eta_i(t,x)\left(\psi_i[\boldsymbol{f}](t,x;x') - \delta(x-x')\right) f_i(t,x)\mathrm{d}x \tag{3.15}$$

$$\begin{aligned}\left(\mathcal{G}_i^{(2)}-\mathcal{L}_i^{(2)}\right)[\boldsymbol{f}](t,x') := & \sum_{j=1}^{3}\int_{\mathbb{I}_j}\int_{\mathbb{I}_i}\eta_{i,j}(t,x,y)(\psi_{i,j}[\boldsymbol{f}](t,x,y;x') \\ & -\delta(x-x'))f_i(t,x)f_j(t,y)\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{3.16}$$

$$\begin{aligned}\left(\mathcal{G}_i^{(3)}-\mathcal{L}_i^{(3)}\right)[\boldsymbol{f}](t,x') := & \sum_{\ell=1}^{3}\int_{\mathbb{I}_\ell}\sum_{j=1}^{3}\int_{\mathbb{I}_j}\int_{\mathbb{I}_i}\eta_{i,j,\ell}(t,x,y,z)(\psi_{i,j,\ell}[\boldsymbol{f}](t,x,y,z;x') \\ & -\delta(x-x'))f_i(t,x)f_j(t,y)f_\ell(t,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z.\end{aligned} \tag{3.17}$$

Easy consequence of the preceding equations is that, for each $i=1,2,3$,

$$\int_{\mathbb{I}_i}\left(\mathcal{G}_i^{(h)}-\mathcal{L}_i^{(h)}\right)[\boldsymbol{f}](t,x')\mathrm{d}x'=0,\quad h=1,2,3. \tag{3.18}$$

In conclusion, the complete evolution equations for the system densities f_i, $i=1,2,3$, for $t\in[0,T]$ and $u\in\mathbb{I}_i$, are written here.

$$\frac{\partial f_i}{\partial t}(t,u)+\frac{\partial}{\partial u}\left(\mathcal{K}_i[\boldsymbol{f}]f_i+c_i\partial_u f_i\right)(t,u)=\mathcal{G}_i[\boldsymbol{f}](t,u)-\mathcal{L}_i[\boldsymbol{f}](t,u), \tag{3.19}$$

where

$$\begin{aligned}\mathcal{G}_i[\boldsymbol{f}](t,u) = & \int_{\mathbb{I}_i}\eta_i(t,x)\psi_i[\boldsymbol{f}](t,x;u)f_i(t,x)\mathrm{d}x \\ & +\sum_{j=1}^{3}\int_{\mathbb{I}_j}\int_{\mathbb{I}_i}\eta_{i,j}(t,x,y)\psi_{i,j}[\boldsymbol{f}](t,x,y;u)f_i(t,x)f_j(t,y)\mathrm{d}x\mathrm{d}y \\ & +\sum_{\ell=1}^{3}\int_{\mathbb{I}_\ell}\sum_{j=1}^{3}\int_{\mathbb{I}_j}\int_{\mathbb{I}_i}\eta_{i,j,\ell}(t,x,y,z)\psi_{i,j,\ell}[\boldsymbol{f}](t,x,y,z;u)f_i(t,x)f_j(t,y)f_\ell(t,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z\end{aligned} \tag{3.20}$$

$$\begin{aligned}\mathcal{L}_i[\boldsymbol{f}](t,u) = & f_i(t,u)\left(\eta_i(t,u)+\sum_{j=1}^{3}\int_{\mathbb{I}_j}\eta_{i,j}(t,u,y)f_j(t,y)\mathrm{d}y\right. \\ & \left.+\sum_{\ell=1}^{3}\int_{\mathbb{I}_\ell}\sum_{j=1}^{3}\int_{\mathbb{I}_j}\eta_{i,j,\ell}(t,u,y,z)f_j(t,y)f_\ell(t,z)\mathrm{d}y\mathrm{d}z\right)\end{aligned} \tag{3.21}$$

and

$$\mathcal{K}_i[\boldsymbol{f}](t,u)=\sum_{j=1}^{3}\int_{\mathbb{L}_j\subset\mathbb{I}_j}\mathcal{P}_{i,j}(t,u,y)f_j(t,y)\mathrm{d}y. \tag{3.22}$$

Moreover, the following conditions are assumed to be verified for all $t\in[0,T]$, $x\in\mathbb{I}_i$, $y\in\mathbb{I}_j$, $z\in\mathbb{I}_\ell$, and each $i=1,2,3$.

$$\int_{\mathbb{I}_i}f_i(t,u)\mathrm{d}u=1, \tag{3.23}$$

$$\int_{\mathbb{I}_i}\psi_i[\boldsymbol{f}](t,x;u)\mathrm{d}u=1, \tag{3.24}$$

$$\int_{\mathbb{I}_i}\psi_{i,j}[\boldsymbol{f}](t,x,y;u)\mathrm{d}u=1, \tag{3.25}$$

$$\int_{\mathbb{I}_i}\psi_{i,j,\ell}[\boldsymbol{f}](t,x,y,z;u)\mathrm{d}u=1, \tag{3.26}$$

$$\mathcal{K}_i[\boldsymbol{f}](t,a_i)=\mathcal{K}_i[\boldsymbol{f}](t,b_i)=0,\quad [a_i,b_i]:=\mathbb{I}_i, \tag{3.27}$$

$$c_i(t, a_i) = c_i(t, b_i) = \partial_u c_i(t, a_i) = \partial_u c_i(t, b_i) = 0. \tag{3.28}$$

4. Mathematical framework (the discrete case)

On the basis of the mathematical framework discussed in the former section, in the present one an analogous, though not equivalent, picture is developed with the assumption that the actors' state variables, rather than being continuous real valued random variables, are instead discrete real valued ones. This approach, although seemingly "less precise" than the former one, is under many respects even better and more suitable to describe the specific case taken here into account in the next Section 6. Indeed the specific activity variable: *quality* has not been (even if it could be) measured as a real variable. On the contrary its values have been attributed, in fact according to a phenomenological measure scale, to a finite set of values

$$\mathbb{D}_i := \left\{u_{i,1}, u_{i,2}, \ldots, u_{i,h}, \ldots, u_{i,H_i}\right\} \subset \mathbb{R}, \quad i = 1, 2, 3 \tag{4.1}$$

which may even be, and more conveniently, considered as a *code space*. Namely, the particular activity values may be totally disregarded, and simply represented by integers:

$$\mathbb{D}_i := \{1, 2, \ldots, h, \ldots, (H_i - 1), H_i\} \subset \mathbb{R}, \quad i = 1, 2, 3. \tag{4.2}$$

Correlating these sets with a global quality index is in fact a subsequent problem to be separately analysed when a particular model is developed. This transformation, on the other hand, may require techniques and reasonings that go far beyond the present exposure and that belong to the engineering rather than to the mathematical field of interest, see for instance [23,24].

In what follows, we shall restrict ourselves to the real valued discrete case that is a consequence of the preceding discussion. By this we mean that Assumptions 3.1 and 3.2 may be restated here, and set identically as Assumptions 4.3 and 4.4. On the contrary, the subsequent ones need to be rephrased according to the fact that now each of the continuous probability density function $f_i(t, \cdot)$ gives rise to as many values $f_{i,h}(t)$ as many points there are in the ith set $\mathbb{D}_i$ which, unless otherwise stated, may be thought of as either the ones mentioned in Eq. (4.1) or (4.2).

Assumption 4.3. The system variables are defined by a triplet of discrete stochastic processes over the interval $[0, T]$:

$$u_1, u_2, u_3 : \quad u_i : t \in [0, T] \mapsto u_i(t) \in \mathbb{D}_i \subset \mathbb{R}, \quad i = 1, 2, 3; \tag{4.3}$$

that associate with each instant of time $t \in [0, T]$ a triplet of discrete random variables $u_1(t), u_2(t), u_3(t)$, with values in the finite real sets $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$ respectively. The related *probability distributions*

$$\left\{f_{i,1}(t), f_{i,2}(t), \ldots, f_{i,h}(t), \ldots, f_{i,H_i}(t)\right\} \subset \bigtimes_{h=1}^{H_i} [0, \infty), \quad i = 1, 2, 3 \tag{4.4}$$

of the event: $u_i(t) = u_{i,h} \in \mathbb{D}_i$ which refers to the outcome of the (random, uncertain) activity measurement at time $t \in [0, T] \subset [0, \infty)$, are complete

$$\sum_{h=1}^{H_i} f_{i,h}(t) = 1, \quad i \in \{1, 2, 3\}. \tag{4.5}$$

Functions $f_{i,h} : t \in [0, T] \mapsto f_{i,h}(t) \in [0, \infty)$ are such that

$$f_{i,h} \in \bigcap_{m=0}^{M-1} \left(\mathcal{C}^1([m\tau, (m+1)\tau])\right), \quad h = 1, \ldots, H_i, i = 1, 2, 3. \tag{4.6}$$

In other words, probabilities $f_{i,h}$ refer to the expectation that a random measurement of population $\mathbb{P}_i$'s activity ends up with the value $u_{i,h}$ that represents (all the quality levels that are felt by) the actors that belong to the "cell" identified by number h.

This point of view becomes more reasonable if one takes into account the fact that the quality is already an estimate, and not an exact variable as particle position x or velocity v. In particular, no attempt on any kind of convergence from the discrete picture to the continuous one will be tried. Rather, the aim here is to develop a model different from the continuous one, that may contain proper coefficients and evolutions, and that may only be compatible with the other.

On the other hand, the idea is to maintain as strict as possible a connection with the continuous picture, in order to have a guideline to attribute a meaning (and the consequent behaviour) to the terms that compose the evolution equations on *masses* $f_{i,h}$.

Indeed, borrowing from hydrodynamic fluid motion is again convenient in the present case as well as it has been in the former one. Specifically, the evolution equations on functions $\boldsymbol{f} := \{f_{i,h}; h = 1, \dots, H_i, i = 1, 2, 3\}$ are again mass balance equations on a system of particles. They are conjectured here to be flowing through aligned cells, numbered from 1 to H_i. The product $N_i(t) f_{i,h}(t)$ gives the number of particles that at time t is expected to be found inside cell number h.

Along these lines, we state the following.

Assumption 4.4. The variation rate of each function $f_{i,h}$ is the sum of a term with the character of a local flow, plus a balance between a gain term and a loss term due to internal interactions among the system actors. In formula

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{i,h} + \Phi_{i,h} = \mathcal{G}_{i,h}[\boldsymbol{f}] - \mathcal{L}_{i,h}[\boldsymbol{f}], \quad h = 1, \dots, H_i, i = 1, 2, 3. \tag{4.7}$$

The term $\Phi_{i,h}$ on the left-hand side of Eq. (4.7) refers to the expected mass variations of the set of actors identified by (i, h) due to actions of external and of ensemble (social) character; the right-hand side refers to those due to direct interactions with the other actors of the system.

Once again making reference to a hydrodynamic picture, to represent the effects on the total dynamics due to external and ensemble actions, we assume that functions

$$\mathcal{K}_{i,h}[\boldsymbol{f}] : [0, T] \to \mathbb{R}, \quad h = 2, \dots, H_i,$$

may be defined with the meaning of convective speeds (wind drifts) between cell $h - 1$ and cell h of population $\mathbb{P}_i$, for each $i = 1, 2, 3$. On using this approach, a possible setting for the terms $\Phi_{i,h}$ is the following.

Let the cells be (contiguously) aligned according to the natural order, so that $\mathcal{K}_{i,h} > 0$ means that the wind blows from cell $h - 1$ to cell h. Let $\Phi^-_{i,h}$ represent the (algebraic) flow from cell number $h - 1$ to cell number h; loosely speaking, $\Phi^-_{i,h} > 0$ is the mass (per unit time) that enters cell number h coming from cell number $(h - 1)$. Hence, $\Phi^+_{i,h} := \Phi^-_{i,h+1}[\boldsymbol{f}]$ is the one leaving cell h towards cell $h + 1$, for $h = 2, \dots, H_i - 1$. For completeness, also denote by $\Phi^-_{i,1}$ the external inflow to cell number 1 and by $\Phi^+_{i,H_i} =: \Phi^-_{i,H_i+1}$ the external outflow from cell number H_i; both of them are assumed to be null at each instant of time. With this in mind, we state the next assumption.

Assumption 4.5. The terms $\Phi_{i,h}[\boldsymbol{f}] : [0, T] \to \mathbb{R}$ on the left-hand side of Eq. (4.7) may be represented by the following (restricted) balances, to be verified at each instant $t \in [0, T]$.

$$\begin{aligned} &\Phi_{i,h}[\boldsymbol{f}] = \Phi^+_{i,h}[\boldsymbol{f}] - \Phi^-_{i,h}[\boldsymbol{f}] \equiv \Phi^-_{i,h+1}[\boldsymbol{f}] - \Phi^-_{i,h}[\boldsymbol{f}], \\ &\Phi^-_{i,H_i+1} := \Phi^-_{i,1} := 0, \quad h = 1, \dots, H_i, i = 1, 2, 3. \end{aligned} \tag{4.8}$$

Functions $\mathcal{K}_{i,h}[\boldsymbol{f}] : [0, T] \to \mathbb{R}$, $h = 2, \dots, H_i$, may be identified such that:

$$\Phi^-_{i,h}[\boldsymbol{f}] = \begin{cases} \mathcal{K}_{i,h}[\boldsymbol{f}] f_{i,h-1} & \text{if } \mathcal{K}_{i,h} > 0 \\ \mathcal{K}_{i,h}[\boldsymbol{f}] f_{i,h} & \text{otherwise} \end{cases} \quad h = 2, \dots, H_i, i = 1, 2, 3. \tag{4.9}$$

Remark 4.1. The terms $\Phi_{i,h}$, meant as due to external and ensemble actions, and the effects that they consequently produce on the dynamics, are not only conceptually distinct, but also actually (and technically) different from those that may arise from a general picture of direct interactions which furnishes the balances on the right-hand side of Eq. (4.7).

Assumption 4.6. Direct interactions are events that, on account of the possible singleton, double, or triple, sets of actors: test and field ones, refer to the possible changes of the test actor state (not of his population). They are identified by a set of probabilities on the state changes, and by a set of rates on their occurrence.

Interaction probabilities and rates are hereafter specified; they consist of the following (regular) functions of time, where the time dependence is not only direct but also through expectations over the densities $\boldsymbol{f}$; parametrical dependence on the activity values themselves may also happen.

- $\eta_{i,h}[\boldsymbol{f}] : [0, T] \to [0, \infty)$ rate of the events wherein an individual of population $\mathbb{P}_i$ in the state $u_{i,h} \in \mathbb{D}_i$ autonomously reflects on modifying his state.
- $\eta_{i,h,j,k}[\boldsymbol{f}] : [0, T] \to [0, \infty)$ rate of the double events: a test individual of population $\mathbb{P}_i$ in the state $u_{i,h} \in \mathbb{D}_i$ and a field individual of population $\mathbb{P}_j$ in the state $u_{j,k} \in \mathbb{D}_j$.
- $\eta_{i,h,j,k,\ell,n}[\boldsymbol{f}] : [0, T] \to [0, \infty)$ rate of the triple events: a test individual of population $\mathbb{P}_i$ in the state $u_{i,h} \in \mathbb{D}_i$, a field individual of population $\mathbb{P}_j$ in the state $u_{j,k} \in \mathbb{D}_j$ and a field individual of population $\mathbb{P}_\ell$ in the state $u_{\ell,n} \in \mathbb{D}_\ell$.
- $\psi_{i,h;h'}[\boldsymbol{f}] : [0, T] \to [0, 1]$ probability that the state of an individual of a population $\mathbb{P}_i$ changes from $u_{i,h} \in \mathbb{D}_i$ to $u_{i,h'} \in \mathbb{D}_i$ because of an event wherein he autonomously reflects on modifying his initial state.
- $\psi_{i,h,j,k;h'}[\boldsymbol{f}] : [0, T] \to [0, 1]$ probability that the state of an individual of a population $\mathbb{P}_i$ changes from $u_{i,h} \in \mathbb{D}_i$ to $u_{i,h'} \in \mathbb{D}_i$ because of a double event wherein he is the test individual with a field individual of population $\mathbb{P}_j$ in the state $u_{j,k} \in \mathbb{D}_j$.
- $\psi_{i,h,j,k,\ell,n;h'}[\boldsymbol{f}] : [0, T] \to [0, 1]$ probability that the state of an individual of a population $\mathbb{P}_i$ changes from $u_{i,h} \in \mathbb{D}_i$ to $u_{i,h'} \in \mathbb{D}_i$ because of a triple event wherein he is test individual with two field individuals: one of population $\mathbb{P}_j$ in the state $u_{j,k} \in \mathbb{D}_j$ and one of population $\mathbb{P}_\ell$ in the state $u_{\ell,n} \in \mathbb{D}_\ell$.

Remark 4.2. We realize that the requirement that the above functions' dependence on $\boldsymbol{f}$ occurs at most through statistical expectations, such as the variables n-order moments, is actually an implicit assumption that the multiple events are considered here on an independence basis: the product structure underlines this hypothesis. In fact, this might be considered as an unacceptable condition, especially when the total numbers of the actors is not so relevant. On the other hand, one may consider that rates η's may already contain, in themselves, the correlation coefficients related to the various kinds of multiple encounters. In fact, maintaining the polynomial structure of the r.h.s. of Eq. (4.7), and the limited size of the variables, suggest interpreting them as expansions with significative coefficients.

The next, final assumption concludes the framework.

Assumption 4.7. The balances on the right-hand side of Eq. (4.7) are identified, for $h = 1, \ldots, H_i$, $i = 1, 2, 3$, and all $t \in [0, T]$, by

$$\mathcal{G}_{i,h}[\boldsymbol{f}] - \mathcal{L}_{i,h}[\boldsymbol{f}] = \mathcal{G}^{(1)}_{i,h}[\boldsymbol{f}] - \mathcal{L}^{(1)}_{i,h}[\boldsymbol{f}] + \mathcal{G}^{(2)}_{i,h}[\boldsymbol{f}] - \mathcal{L}^{(2)}_{i,h}[\boldsymbol{f}] + \mathcal{G}^{(3)}_{i,h}[\boldsymbol{f}] - \mathcal{L}^{(3)}_{i,h}[\boldsymbol{f}] \tag{4.10}$$

where (on using the well-known Kronecker symbol δ)

$$\begin{aligned}
\mathcal{G}^{(1)}_{i,h}[\boldsymbol{f}] - \mathcal{L}^{(1)}_{i,h}[\boldsymbol{f}] &= \sum_{l=1}^{H_i} \eta_{i,l} \left[\psi_{i,l;h} - \delta_{l,h}\right] f_{i,l} \\
\mathcal{G}^{(2)}_{i,h}[\boldsymbol{f}] - \mathcal{L}^{(2)}_{i,h}[\boldsymbol{f}] &= \sum_{j=1}^{3} \sum_{k=1}^{H_j} \sum_{l=1}^{H_i} \eta_{i,l,j,k} \left[\psi_{i,l,j,k;h} - \delta_{l,h}\right] f_{i,l} f_{j,k} \\
\mathcal{G}^{(3)}_{i,h}[\boldsymbol{f}] - \mathcal{L}^{(3)}_{i,h}[\boldsymbol{f}] &= \sum_{\ell=1}^{3} \sum_{n=1}^{H_\ell} \sum_{j=1}^{3} \sum_{k=1}^{H_j} \sum_{l=1}^{H_i} \eta_{i,l,j,k,\ell,n} \left[\psi_{i,l,j,k,\ell,n;h} - \delta_{l,h}\right] f_{i,l} f_{j,k} f_{\ell,n}.
\end{aligned} \tag{4.11}$$

In conclusion, the evolution equations that rule the system when the action variable is assumed to be discrete consist of a set of ordinary differential equations, for $h = 1, \ldots, H_i$, $i = 1, 2, 3$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f_{i,h} + \Phi_{i,h}[\boldsymbol{f}] &= \sum_{l=1}^{H_i} \eta_{i,l} \left[\psi_{i,l;h} - \delta_{l,h}\right] f_{i,l} + \sum_{j=1}^{3} \sum_{k=1}^{H_j} \sum_{l=1}^{H_i} \eta_{i,l,j,k} \left[\psi_{i,l,j,k;h} - \delta_{l,h}\right] f_{i,l} f_{j,k} \\
&\quad + \sum_{\ell=1}^{3} \sum_{n=1}^{H_\ell} \sum_{j=1}^{3} \sum_{k=1}^{H_j} \sum_{l=1}^{H_i} \eta_{i,l,j,k,\ell,n} \left[\psi_{i,l,j,k,\ell,n;h} - \delta_{l,h}\right] f_{i,l} f_{j,k} f_{\ell,n}
\end{aligned} \tag{4.12}$$

together with the following conditions (for $i, j, \ell \in \{1, 2, 3\}$; $l = 1, \dots, H_i$; $k = 1, \dots, H_j$; $n = 1, \dots, H_\ell$; $H_i, H_j, H_\ell, M \in \mathbb{N}$; $\tau \in \mathbb{R}$)

$$f_{i,h}(t_m) = f_{i,h}^{(m)} \in \mathbb{R}, \quad t_m = m\tau, m = 0, 1, \dots, M; \tag{4.13}$$

$$\sum_{h=1}^{H_i} f_{i,h}^{(m)} = 1, \qquad \sum_{h=1}^{H_i} f_{i,h}(t) = 1, \quad \forall i, m, t \in [0, T]; \tag{4.14}$$

$$\sum_{h=1}^{H_i} \psi_{i,l;h}(t) = 1, \quad \forall i, l, t \in [0, T]; \tag{4.15}$$

$$\sum_{h=1}^{H_i} \psi_{i,l,j,k;h}(t) = 1, \quad \forall i, l, j, k, t \in [0, T]; \tag{4.16}$$

$$\sum_{h=1}^{H_i} \psi_{i,l,j,k,\ell,n;h}(t) = 1, \quad \forall i, l, j, k, \ell, n, t \in [0, T]; \tag{4.17}$$

and where the functions $\Phi_{i,h}$ are specified by Eqs. (4.8).

5. A specific model

In this section the overall picture is sketched of a model that, within the preceding mathematical framework, aims to describe the time development of the performance of a medical service inside an hospital.

This is done by considering a specific index of quality as a *macroscopic variable* of the system, and by computing the corresponding expectation variable in a framework such as the one described in Section 4. The idea is to achieve the target introducing the lowest possible number of controlling parameters.

In a forthcoming paper [22], a detailed version of the model is presented together with a set of simulations, the discussion about the various terms further extended, and, on the basis of the available data, validation of the whole picture analysed.

At first we formally introduce the problem that the model is meant to solve. In the discrete case it consists of an initial value problem of the form:

Let $3(M-1)$ arbitrary sets of probabilities $\{f_{i,h}^{(m)}, h = 1, \dots, H_i\}$, $i = 1, 2, 3$, $m = 0, 1, \dots, (M-1) \in \mathbb{N}$ be assigned. Analyse the dynamics,and its dependence on the controlling parameters, of the sets of functions $\{f_{i,h} : [0, T] \to [0, 1], h = 1, \dots, H_i\}$, $i = 1, 2, 3$, determined by Eq. (4.12) under the constraints (4.13)–(4.17), with initial conditions, at each of the initial instants $\{t_m := m\tau, \quad m = 0, 1, \dots, M-1\}$ of the evolution intervals $[t_m, t_{m+1})$, represented by the values $\{f_{i,h}^{(m)}, h = 1, \dots, H_i\}$.

In particular, probabilities $\{f_{i,h}^{(m)}, h = 1, \dots, H_i\}$, $i = 1, 2, 3$, $m = 0, 1, \dots, M-1$, are written in the form

$$f_{i,h}^{(m)} = \frac{1}{N_i^{(m)}} \left[(N_i^{(m-1)} - N_i^{-,m}) \lim_{\varepsilon \to 0} f_{i,h}(t_m - \varepsilon) + N_i^{+,m} g_{i,h}^{(m)} \right] \tag{5.1}$$

where $\{N_i^{(0)}, N_i^{+,m}, N_i^{-,m}, \ m = 1, \dots, M-1\}$ is an assigned set of non negative integer numbers such that $N_i^{(m)} = N_i^{(m-1)} + N_i^{+,m} - N_i^{-,m}$, $N_i^{-,m} \leq N_i^{(m-1)}$, and the distributions

$$g_{i,h}^{(m)} \quad \text{such that} \sum_{h=1}^{H_i} g_{i,h}^{(m)} = 1, \quad i = 1, 2, 3, m = 0, 1, \dots, M-1, \tag{5.2}$$

are (conveniently assumed) to represent the input/output changes of actors in the service at every τ period of time.

On the basis of this statement, a set of axioms is now presented that depict the main strategy and tools that are going to be used in the specific case under examination. Again it must be recalled that they partially depend on what is already available about the system, in that they have been elaborated on the basis of actual data, and that under this respect they are meant to be used to validate the model in its *descriptive* role of the system. As the *predictive* part is concerned, a simulation on a statistical basis is hopefully of use, with characteristics inherited by the observed ones. In particular, as already mentioned in Section 2, identification of a quality index is not an easy task. On the other hand, in the specific case of interest, the problem has been approached using an already developed (although subjective) structure.

The following set of assumptions is directly suggested by the available data. Here they are specified for the continuous case, but are meant to hold for the discrete case as well. They concern three macroscopic variables, which prove to be of main interest both under the descriptive and the predictive point of view. Although referred to the specific service under examination: a medical (in particular, a neuropsychiatric) service, they are stated in such a way that they can easily be adapted to more general systems.

Assumption 5.1. The statistical expectations (first-order moments of the variables u_i)

$$U_i(t) = N_i(t) \sum_{h=1}^{H_i} u_{i,h} f_{i,h}(t), \quad i = 1, 2, 3 \tag{5.3}$$

refer to the quality levels perceived, on the mean, by the actors of population $\mathbb{P}_i$ at each instant of time $t \in [0, T]$. On their basis, and the sets of values $\mathbb{D}_i$:

$$\mathbb{D}_i := \left\{u_{i,1}, u_{i,2}, \ldots, u_{i,h}, \ldots, u_{i,H_i}\right\} \subset \mathbb{R}, \quad i = 1, 2, 3,$$

see Eq. (4.1), nonnegative constants $\alpha_1, \alpha_2, \alpha_3$ may be identified such that the *Quality of the service* is delivered by

$$U(t) = \sum_{i=1}^{3} \alpha_i U_i(t), \quad i = 1, 2, 3. \tag{5.4}$$

Assumption 5.2. The system is indirectly driven by a set of events that happen, on the temporal frame, nonuniformly and unexpectedly. They are of two kinds: events of external nature, or *Extra events*, and of internal origins, or *Critical events*. Their action on the system is *indirect* in the sense that they act on the *controlling functions*, already introduced in Assumptions 3.6 and 3.7, and directly neither on the state variables nor on their probability density functions.

We leave to a future paper the detailed description of both these kind of events. Here, just for an example, we mention that good extra events may be the arrival of a new piece of hardware, or the regular dismissal of a patient, whereas a bad extra event may be the extra work due to the arrival of a great number of new patients and the dismissal of an equal number of old ones: a negligible net flow yet a heavy change-of-turn duty. On the other side, critical events may be the recourse to strain-jacket or to constriction bed.

Assumption 5.3. Each extra event is identified by a real value e_q selected in a finite set

$$\mathbb{E} := \{e_1, e_2, \ldots, e_E\}, \qquad e_q \in \mathbb{R}, \quad q = 1, 2, \ldots, E \in \mathbb{N}, \tag{5.5}$$

that collects all the possible chances, both for “good” events and “bad” events.

The effects of an extra event start at the beginning of an evolution subinterval: $[t_m, t_{m+1})$ (see Assumptions 3.1 and 3.8) and last for that interval only.

The final action of all extra events upon the system is of collective nature, and is identified by a stepwise constant, real valued, macroscopic function

$$E(t) := \sum_{q=1}^{E} e_q \delta_q^{(m)}, \quad t \in [t_m, t_{m+1}), m = 0, 1, \ldots, M-1, \tag{5.6}$$

where $(\delta_1^{(m)}, \delta_2^{(m)}, \ldots, \delta_E^{(m)})$ is an E-tuple of values $\delta_q^{(m)} \in \{0, 1\}$, $q \in \{1, \ldots, E\}$, that testify the occurrence of qth event, in that

$$\delta_q^{(m)} = 1 \quad \text{if the } q\text{th event happened at time } t_m; \qquad \delta_q^{(m)} = 0 \quad \text{otherwise.}$$

Assumption 5.4. Critical events are all bad events. Their occurrence is controlled by the values of suitable internal macroscopic variables

$$\underline{\text{Critical control variables}} \quad C_i : t \in [0, T] \mapsto C_i(t) \in [0, 1], \quad i = 1, 2, 3, \tag{5.7}$$

and by a triplet of real numbers, the *critical thresholds* $C_1^c, C_2^c, C_3^c \in [0, 1]$.

A critical event happens when an instant $t_c \in (0, T)$ exists such that

$$\begin{cases} C_i(t) < C_i^c & \text{for } t < t_c \text{ and } i = 1, 2, 3, \\ C_i(t) \geq C_i^c & \text{for } t \geq t_c \text{ and at least one } i \in \{1, 2, 3\}. \end{cases} \tag{5.8}$$

The effects of a critical event start at time t_c and last up to a certain instant t_c' that equals one of the instants t_m mentioned above (see Assumptions 3.1 and 3.8) for some $m \in \{1, 2, \ldots, M\}$; the length $t_c' - t_c$ depends on the set of values

$$\{C_i(t) | t > t_c, \text{ and } C_i(t) > C_i^c\}. \tag{5.9}$$

In the specific case of interest, the critical functions C_i are meant to be defined by

$$C_i(t) := \int_{u_i^c}^{b_i} f_i(t, x) \mathrm{d}x, \quad i = 1, 2, 3, \tag{5.10}$$

where b_i is the highest possible value for the state variable u_i of population $\mathbb{P}_i$, and where $0 < u_i^c \leq b_i$ are proper values. In fact we note that what is of relevance is not so much how big the value $C_i(t)$ is, rather the fact that $C_i(t)$ is positive when u_i^c is conveniently high.

Assumptions are here stated in a restricted, qualitative form; description of the specific functions is postponed to a future paper. In particular, in order that the number of the undetermined coefficients may be kept as low as possible, which especially in the discrete case is uncomfortably relevant, the following assumptions are stated for the continuous case discussed in Section 3. This approach will allow one to attribute a meaning, or at least a set of correlations, to entire subsets of the coefficients stated in Assumptions 4.5 and 4.6 and that which appeared in Eq. (4.12). Therefore, in the last part of this section reference is made only to the functions stated in Assumptions 3.6 and 3.7. Moreover, to ease up the exposition, all the three state variables are meant to have the same "direction", meaning

u_1 to address the psychotic behaviour of the patients;

u_2 to address the stress of the agents;

u_3 to address the inefficiency of the hardware;

the three values (b_1, b_2, b_3) represent therefore the worst possible quality values felt by the actors of the service, and (a_1, a_2, a_3) the corresponding highest ones.

Let $\mathbb{1}_{[x_1,x_2]} \subset \mathcal{C}_0^2$ denote the set of the compactly supported, twice differentiable real valued functions on $\mathbb{R}$ such that there exist reals $\varepsilon_1, \varepsilon_2 > 0$:

$$\chi \in \mathbb{1}_{[x_1,x_2]} \quad \text{when} \quad \begin{cases} \chi(x) = 0 & \text{if } x \leq x_1 \text{ or } x \geq x_2 \\ \chi(x) = 1 & \text{if } x_1 + \varepsilon_1 \leq x \leq x_2 - \varepsilon_2 \\ \chi(x) & \text{monotone on } [x_1, x_1 + \varepsilon_1] \text{ and on } [x_2 - \varepsilon_2, x_2] \\ \chi(x_1) = \chi'(x_1) = \chi(x_2) = \chi'(x_2) = 0 \end{cases} \tag{5.11}$$

and denote by $1_{[a,b]}(x)$ the characteristic function of the set $[a, b]$, namely:

$$1_{[a,b]}(x) = 1 \quad \text{if } x \in [a, b], \qquad 1_{[a,b]}(x) = 0 \quad \text{otherwise}. \tag{5.12}$$

Assumption 5.5. Coefficients $\mathcal{F}_{i,j}$ of Eq. (3.11) are specialized by

$$\mathcal{F}_{i,j}[\boldsymbol{f}](t, x) = \chi_i(x) \left[\int_{\mathbb{I}_j} \left(\mathcal{P}_{i,j}^{(1)}(t, y) + 1_{[u_j^c, b_j]}(y) \mathcal{P}_{i,j}^{(2)}(t, y) \right) f_j(t, y) \mathrm{d}y + \mathcal{P}_{i,j}^{(3)}(t) \right], \tag{5.13}$$

where the functions χ_i and $\chi_{[t_c,t'_c]}$ are suitably selected in the set $\mathbb{1}_{\mathbb{I}_i}$ and $\mathbb{1}_{[t_c,t'_c]}$ respectively, and reals $\{p^{(1)}_{i,j}, p^{(2)}_{i,j}, p^{(3)}_{i,j}, p^{(4)}_{i,j}\}$, $i, j = 1, 2, 3$, exist such that

$$\begin{aligned} \mathcal{P}^{(1)}_{i,j}(t, y) &= p^{(1)}_{i,j}(\alpha_j y - 1)N_j(t) \\ \mathcal{P}^{(2)}_{i,j}(t, y) &= p^{(2)}_{i,j}\chi_{[t_c,t'_c]}(t) \\ \mathcal{P}^{(3)}_{i,j}(t, y) &= p^{(3)}_{i,j}\chi_{[t_c,t'_c]}(t) + p^{(4)}_{i,j}E(t). \end{aligned} \tag{5.14}$$

Coefficients $c_i(t, x)$ of Eq. (3.9) have the form

$$c_i(t, x) = \chi_i(x)c_i(t); \qquad c_i(t) := c_i + c^c_i\chi_{[t_c,t'_c]}(t), \quad c_i, c^c_i \in \mathbb{R}, i = 1, 2, 3. \tag{5.15}$$

On account of the preceding assumption, the left-hand side terms of the evolution equation (3.8) are given by

$$\frac{\partial}{\partial t}f_i(t, u) + c_i(t)\frac{\partial}{\partial u}\left(\chi_i(u)\frac{\partial}{\partial u}f_i(t, u)\right) + \phi_i[\boldsymbol{f}](t)\frac{\partial}{\partial u}(\chi_i(u)f_i(t, u)) \quad i = 1, 2, 3,$$

where $c_i(t) := c_i + c^c_i\chi_{[t_c,t'_c]}(t)$, and

$$\phi_i[\boldsymbol{f}](t) := \sum_{j=1}^{3} p^{(1)}_j\alpha_jU_j(t) + p^{(2)}_{i,j}C_j(t) + p^{(3)}_{i,j}\chi_{[t_c,t'_c]}(t) + p^{(4)}_{i,j}E(t).$$

Finally, let 1, 2, 3 respectively denote the patients, the medical staff, and the hardware, of the service. Then denote by $\mathbb{T}(1, 2, 3)$ the set of (odd and even) permutations without replacements of the three numbers 1, 2, 3.

Assumption 5.6. Functions η stated in Assumption 3.7 do not depend on time. Functions $\eta_{i,j,\ell}$ are such that

$$\begin{aligned} &\eta_{i,j,\ell} = \eta_{i',j',\ell'} \quad \forall(i', j', \ell') \in \mathbb{T}(1, 2, 3), \\ &\eta_{i,j,\ell} = 0 \quad \forall(i, j, \ell) \notin \mathbb{T}(1, 2, 3), \end{aligned}$$

and, as a first approximation, the following hold true

$$\eta_{3,3} = 0; \qquad \eta_{3,2} = \eta_{2,3} = 0; \quad \eta_3, \eta_{3,1}, \eta_{1,3}(\varepsilon).$$

In the former assumption, ε is meant to be a number much smaller than the significative values of $\eta_1, \eta_2, \eta_{1,1}, \eta_{1,2}, \eta_{2,2}, \eta_{1,2,3}$ still to be specified.

The next assumption concludes the section.

Assumption 5.7. Change of state probabilities depend not only on the state variables of the denoted actors, but also on the macroscopic variables $U_i, C_i, E, i = 1, 2, 3$. Moreover, they may also directly depend on the time, but only through the functions $1_{[t_c,t'_c]}(t)$ defined in Eq. (5.8). Probabilities are computed as expectations of suitable probability density functions ψ, stated in Assumption 3.7, that may be selected among those that are a priori identified by their most probable value μ and their uncertainty σ.

For instance, it has already proved to be convenient a normal distribution truncated on a set $\mathbb{I} \subset \mathbb{R}$, with mean (on $\mathbb{R}$) $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$:

$$\langle n\rangle(u; \mu, \sigma) := \left(\exp -\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2\right) \Big/ \left(\int_{\mathbb{I}} \exp -\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2 \mathrm{d}u\right). \tag{5.16}$$

However, concerning the various values μ's and σ's relative to the probability density functions $\psi_i, \psi_{i,j}, \psi_{i,j,\ell}$ cited in Assumption 3.7, their specific behaviour strictly depends on the model to be described and hence a more detailed, future, analysis is needed.

6. Discussion and research perspectives

We conclude the paper with a brief discussion on the picture outlined above, and some consequent research perspectives.

In the former sections we proposed a possible method, based on a statistical approach, to analyse and predict the quality of a structure.

Although suggested by a specific case, the frameworks developed in this paper are of quite general use. Small changes may be made to the variables defined here and, on the basis of these frameworks, helpful models may be developed to describe and (or) predict the evolution of the quality of a system that involves users, agents, and hardware. As well, models of the same kind may be thought as structures that are identified by goods, providers, and resources.

Clearly, the problem of coherent measurableness of the quality of (or felt by) the actors of the system is a completely separate matter, and here it is considered as already solved. What we propose is a possible tool to analyse, or even predict, the quality *evolution* with respect to time when the external events are known.

The tool we propose is of statistical nature, as it is the whole picture of the quality measurement, and takes advantage of a methodology that, although inherited from the physics of particles, is gaining a growing relevance also in remote fields.

The balanced evolution of statistical variables based on a microscopic "easy" point of view is the core of our tool.

We realize that, in this way, a number of controlling parameters is necessarily introduced which is much greater than that suitable for a totally macroscopic phenomenological model. However, owing to the fact that the microscopic description is more plain than the macroscopic one, the parameters needed to build the microscopic picture may be more clearly identified than the others, and better related to the actual processes involved.

On these lines more research work is needed, and in particular to develop a detailed analysis of the rules that control the relations at the microscopic scale. Ultimately, what we sustain is that the macroscopic picture may indeed be constructed staring from the microscopic one, and that this may be true also for complex structures.

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