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SOME NONDECAYING POTENTIALS FOR THE NONSTATIONARY SCHRÖDINGER EQUATION *

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We consider the special but rather wide class of nondecaying potentials obtained by superposimposing, via binary Bäcklund transformations, N solitons to an arbitrary decaying background. We introduce an exact recursion procedure for an arbitrary number of binary Bäcklund transformations and corresponding Darboux transformations for Jost solutions and solutions of the discrete spectrum. We show that Jost solutions obey modified integral equations and present their analytical properties. We formulate conditions of reality and regularity of the potentials constructed by these means and derive spectral data of the transformed Jost solutions. Finally, the recursion procedure is solved and the properties of potential are presented.

1 Introduction

Spectral theory of the nonstationary Schrödinger operator

$$L = i\partial_{x_2} + \partial_{x_1}^2 - u(x_1, x_2) \tag{1}$$

is interesting *per se* and because it is associated to the Kadomtsev-Petviashvili equation in its version called KPI [1]

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = 3u_{x_2x_2}. \tag{2}$$

The generalization of direct and inverse scattering transform for the nonstationary Schrödinger operator to the case of nondecaying potentials is still an open problem, twenty years after the pioneer papers [2]. We are interested in potentials with "ray" type behavior, that is rapidly decaying on the plane with the exception of some finite number of directions.

In the standard case of rapidly decaying potentials, the Jost solution

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$\Phi(x,k)$ can be given in terms of the solution of the integral equation

$$\chi(x,k) = 1 + \int dx' G_0(x-x',k)u(x')\chi(x',k), \tag{3}$$

where $G_0(x,k)$ is the Green's function of spectral problem (1) and

$$\chi(x,k) = e^{ikx_1+ik^2x_2}\Phi(x,k). \tag{4}$$

The Jost solution is an analytic function of k in the complex plane, $k_{\Im} \neq 0$ with finite limits on the real axis

$$\Phi^{\pm}(x,k) = \lim_{k_{\Im}\to\pm 0} \Phi(x,k). \tag{5}$$

The following definition of spectral data

$$\mathcal{F}(k,p) = \frac{1}{2\pi}\int dx_1' \overline{\Phi^+(x_1',x_2,k)}\Phi^+(x_1',x_2,p) - \delta(p-k) \tag{6}$$

allows to formulate the inverse problem as the nonlocal Riemann–Hilbert problem of construction of the function $\Phi(x,k)$ analytic in upper and bottom half planes with proper normalization and discontinuity at the real axis given by

$$\Phi^+(x,k) = \Phi^-(x,k) + \int dp\Phi^-(x,p)\mathcal{F}(p,k). \tag{7}$$

In the case of "ray" type potentials spectral theory is essentially more involved than the standard case of rapidly decaying potentials. Indeed, the standard integral equation cannot be applied in the case of a potential $u(x)$ not vanishing in all directions at large distances as the Green's function is slowly decaying at space infinity.

We suggest [3,4] the following modification of this integral equation

$$\chi(x,k) = 1 + \int_{-k_{\Im}\infty}^{x_1} dy_1 \int dx' \partial_{y_1} G_0(y_1 - x_1', x_2 - x_2', k)u(x')\chi(x',k) \tag{8}$$

where the order of operations is explicitly prescribed.

The full description of the solutions of the modified integral equation with ray potentials is absent till now. Up to now only multiple pure soliton solutions were constructed [5].

Potentials of "ray" type present new interesting features. For instance, it was shown [3] that the modified integral equation may have solutions with completely unexpected spectral properties, namely an additional cut in the plane of complex spectral parameter. For this reason, we are studying here

the special but rather wide subclass of ray potentials that is obtained by applying recursively the so-called binary Bäcklund transformations [6], with complex parameter, to a decaying potential. Since we are interested in the spectral properties of potential u, we study also the corresponding Darboux transformations furnishing the Jost solutions of the transformed potentials and their analytical properties, as well as the transformation of spectral data.

Binary Bäcklund Transformations

We have at our disposal in [6] a rather simple and transparent method for performing binary Bäcklund transformations of the potential u and corresponding Darboux transformations of solutions of (1). Let $\phi(x,k)$ be a solution of the nonstationary Schrödinger equation with potential $u(x)$. Then the transformed potential is equal to

$$\tilde{u}(x) = u(x) - 2\partial_{x_1}^2 \log\Delta(x), \tag{9}$$

where

$$\Delta(x) = \int^{x_1} dx_1' |\phi(x_1',x_2,\lambda)|^2. \tag{10}$$

The Darboux transform of $\phi(x,k)$,

$$\tilde{\phi}(x,k) = \phi(x,k) - \frac{\phi(x,\lambda)}{\Delta(x)}\int^{x_1} dx_1' \overline{\phi(x_1',x_2,\lambda)}\phi(x_1',x_2,k), \tag{11}$$

solves the equation

$$(i\partial_{x_2} + \partial_{x_1}^2 - \tilde{u}(x))\tilde{\phi}(x,k) = 0 \tag{12}$$

with transformed potential. It is natural to expect that this transformation for complex parameter λ would supply an example of a potential of ray type. Check of the fact that $\tilde{\phi}$ obeys Eq. (12) is based on the following identity for a pair of arbitrary solutions $f(x)$ and $g(x)$ of the Eq. (12):

$$i\partial_{x_2}(\overline{f(x)}g(x)) = -\partial_{x_1} W(\overline{f(x)}, g(x)), \tag{13}$$

where Wronskian

$$W(\overline{f(x)}, g(x)) = \overline{f(x)}\partial_{x_1} g(x) - g(x)\partial_{x_1}\overline{f(x)} \tag{14}$$

was introduced.

In order to make equations (9)–(11) determined it is necessary to substitute indefinite integrals by definite ones and to choose constants of integration in a way that the potential $\tilde{u}$ is real and regular. Thus in order to get transformations parametrized by constants and not by functions of x_2 obeying some differential equations it is natural to choose integrals in (10)–(11) to be from infinity to x_1. On the other side, infinite limits of these integrals require exact control of their convergency in the recursive procedure.

Recursion Procedure

Applying recursively the previous procedure we can compose an arbitrary number of binary Bäcklund transformations. Indeed, if $\phi_n(x,k)$ solves

$$i\partial_{x_2}\phi_n(x,k) + \partial^2_{x_1}\phi_n(x,k) = u_n(x)\phi_n(x,k), \tag{15}$$

we specify the equation for ϕ_{n+1} in the following way:

$$\phi_{n+1}(x,k) = \phi_n(x,k) - \frac{\phi_n(x,\lambda_{n+1})}{\Delta_{n+1}(x)} \times \\ \times \left[B'_{n+1}(k) + \int_{-(k_\Im+\lambda_{n+1\Im})\infty}^{x_1} dx'_1\, \overline{\phi_n(x'_1,x_2,\lambda_{n+1})}\phi_n(x'_1,x_2,k)\right]$$

where

$$\Delta_{n+1}(x) = c_{n+1} + \int_{-\lambda_{n+1\Im}\infty}^{x_1} dx'_1\, |\phi_n(x'_1,x_2,\lambda_{n+1})|^2 \tag{16}$$

and c_{n+1} and $B'_{n+1}(k)$ are some x-independent constant and function of k, respectively. This ϕ_{n+1} has to be a solution of the spectral problem with potential

$$u_{n+1}(x) = u_n(x) - 2\partial^2_{x_1}\log\Delta_{n+1}(x). \tag{17}$$

It is convenient to write all ϕ_n as sums of two solutions $\phi_n(x,k) = F_n(x,k) + f_n(x,k)$ that are given by the recursion relations

$$F_{n+1}(x,k) = F_n(x,k) - \frac{\phi_{n+1}(x,\lambda_{n+1})}{\Delta_{n+1}(x)} \times \\ \int_{-(k_\Im+\lambda_{n+1\Im})\infty}^{x_1} dx'_1\, \overline{\phi_n(x'_1,x_2,\lambda_{n+1})}F_n(x'_1,x_2,k) \tag{18}$$

$$f_{n+1}(x,k) = f_n(x,k) - \frac{\phi_{n+1}(x,\lambda_{n+1})}{\Delta_{n+1}(x)} \times$$

$$\left[B_{n+1}(k) + \int_{-\lambda_{n+1\Im}\infty}^{x_1} dx'_1\, \overline{\phi_n(x'_1,x_2,\lambda_{n+1})}f_n(x'_1,x_2,k)\right]. \tag{19}$$

We also put $u_0(x) = u(x)$, $F_0(x,k) = \Phi(x,k)$, $f_0(x,k) = 0$, so that we start with the standard, real, regular, and rapidly decaying at space infinity potential u. Properties of all these objects are given in the following

Theorem Let the potential $u(x)$ be real and rapidly decaying at space infinity and let $\Phi(x,k)$ be the corresponding Jost solution. Let also the sets of complex constants λ_1, λ_2, … and real nonzero constants c_1, c_2, … obey the following conditions:

$$\lambda_{n\Im} \neq 0, \qquad \lambda_{n\Im}c_n > 0 \qquad n = 1,2,\ldots, \tag{20}$$

$$|\lambda_{1\Im}| > |\lambda_{2\Im}| > \ldots, \tag{21}$$

and let be given some functions $B_1(k)$, $B_2(k)$, … of the complex parameter k. Then one has the following results:

1. Integrals in (16), (19), and (19) converge and define regular functions of x and k, for $k_\Im \neq 0$, $k_\Im + \lambda_{j\Im} \neq 0$ $(j = 1,2,\ldots,n)$. Functions $\Delta_{n+1}(x)$ have no zeroes.
2. Functions $F_n(x,k)$, $f_n(x,k)$, and $\phi_n(x,k)$ are solutions of the non-stationary Schrödinger equation with potential

$$u_n(x) = u(x) - 2\partial^2_{x_1}\log\prod_{j=1}^{n}\Delta_j(x). \tag{22}$$

3. There exist limits

$$\lim_{x_1\to\pm k_\Im\infty} e^{ikx_1+ik^2x_2}F_n(x,k) = A_n(\pm,k). \tag{23}$$

They are independent on x_2 and obey the recursion relation

$$A_0(\pm,k) = 1, \\ A_{n+1}(\pm,k) = A_n(\pm,k)\times\left[1 + \theta(\pm k_\Im\lambda_{n+1\Im})\frac{2i\lambda_{n+1\Im}}{\overline{\lambda_{n+1}} - k}\right] \tag{24}$$

4. Functions $\Phi_n(x,k)$ defined as

$$\Phi_n(x,k) = \frac{F_n(x,k)}{A_n(-,k)}, \qquad n = 1,2,\ldots, \tag{25}$$

are the Jost solutions of the spectral problem with potential u_n, i.e. functions

$$\chi_n(x,k) = e^{ikx_1 + ik^2 x_2}\Phi_n(x,k) \tag{26}$$

obey the modified integral equation with potential $u_n(x)$.

Corollary $F_n(x,k)$ is a sectionally meromorphic function of k with discontinuity at the real axis and poles at points $k = \overline{\lambda}_j$, $j = 1, \ldots, n$.

Corollary $A_n(-,k)$ is a meromorphic function of k discontinuous at $k_{\Im} = 0$ and with simple poles at $k = \overline{\lambda}_j$, $j = 1, \ldots, n$.

Corollary $\Phi_n(x,k)$ is an analytic function in the complex plane of k with possible discontinuity at the real axis. For limiting values of the Jost solution at the real axis we have relation

$$\frac{\Phi_n^+(x,k)}{a_n^+(k)} = \Phi_n^-(x,k) + \int dp\, \Phi_n^-(x,p)\mathcal{F}_n(p,k) \tag{27}$$

where $a_n(k) = \frac{A_n(+,k)}{A_n(-,k)}$ and continuous part of the spectral data is given by

$$\mathcal{F}_n(k,p) = \mathcal{F}(k,p)\prod_{j=1}^{n}\left(\frac{(k-\lambda_j)(p-\overline{\lambda}_j)}{(k-\overline{\lambda}_j)(p-\lambda_j)}\right)^{\theta(\lambda_{j\Im})} \quad . \tag{28}$$

3 Resolution of the recursion relations

The recursion procedure formulated above is solved in terms of the original potential $u(x)$ and its Jost solution $\Phi(x,k)$. In this way we get a solution depending on $n(n+1)$ complex parameters describing n solitons superimposed to a generic background. Let us introduce for $l, m = 1, 2, \ldots, n$

$$B_{l,m}(x) = \int_{-(\lambda_{l\Im}+\lambda_{m\Im})\infty}^{x_1} dx_1' \,\overline{\Phi(x_1', x_2, \lambda_l)}\,\Phi(x_1', x_2, \lambda_m), \tag{29}$$

$$\beta_l(x,k) = \int_{-(k_{\Im}+\lambda_{l\Im})\infty}^{x_1} dx_1' \,\overline{\Phi(x_1', x_2, \lambda_l)}\,\Phi(x_1', x_2, k). \tag{30}$$

Let us introduce the $n \times n$ matrix $B_n(x)$

$$B_n(x) = \|B_{l,m}(x)\|_{l,m=1,\ldots,n} \tag{31}$$

and rows and column

$$\Phi(x) = (\Phi(x,\lambda_1), \ldots, \Phi(x,\lambda_n)), \tag{32}$$

$$\beta(x,k) = (\beta_1(x,k), \ldots, \beta_n(x,k))^{\mathrm{T}}, \tag{33}$$

$$\gamma(x,k) = (\gamma_1(k), \ldots, \gamma_n(k))^{\mathrm{T}}, \tag{34}$$

where superscript T means transposition and $\gamma_n(k)$ are some given functions such that matrix

$$C_n = \|c_{l,m}\|_{l,m=1,\ldots,n}, \qquad c_{l,m} = \gamma_l(\lambda_m), \tag{35}$$

is Hermitian. Let A_n denote

$$A_n(x) = C_n + B_n(x). \tag{36}$$

In order to formulate conditions of regularity of the potential $u_n(x)$ we introduce also matrices

$$C_n(\pm) = \|\{c_{l,m}| \ \pm\lambda_{l\Im} > 0,\ \pm\lambda_{m\Im} > 0\}\|_{l,m=1,\ldots,n} \,. \tag{37}$$

These matrices are Hermitian also and if all rows and columns are removed, then by definition we put $\det C_n(\pm) = 1$. We proved the following

Theorem Let be given $\lambda_1, \ldots, \lambda_n$ with $\lambda_{j\Im} \neq 0$ for $j = 1, \ldots, n$ and $|\lambda_{1\Im}| > |\lambda_{2\Im}| > \ldots$. Let us also assume that constants $c_{l,m}$ are such that matrices $C_n(\pm)$ satisfy the following condition

$$\pm C_n(\pm) > 0. \tag{38}$$

Then for any x and $n = 1, 2, \ldots$

$$\det A_0(x) = 1, \qquad \det A_n(x)\prod_{l=1}^{n}\lambda_{l\Im} > 0, \tag{39}$$

and the solutions of the recursion equations are given by means of the following relations:

$$F_n(x,k) = \frac{1}{\det A_n(x)}\begin{vmatrix} A_n(x) & \beta(x,k) \\ \Phi(x) & \Phi(x,k) \end{vmatrix} \tag{40}$$

$$f_n(x,k) = \frac{1}{\det A_n(x)}\begin{vmatrix} A_n(x) & \gamma(k) \\ \Phi(x) & 0 \end{vmatrix} \tag{41}$$

$$u_n(x) = u(x) - 2\partial_{x_1}^2 \log\det A_n(x). \tag{42}$$

Corollary

$$\Phi_n(x,\lambda_m)=\sum_{l=1}^{n} d_{l,m}\Phi_n(x,\overline{\lambda}_l), \tag{43}$$

where we introduced the constants

$$d_{l,m} = 2c_{l,m}\lambda_{l\Im}\prod_{j=1,j\neq l}^{n}\left(\frac{\overline{\lambda}_l-\lambda_j}{\overline{\lambda}_l-\overline{\lambda}_j}\right)^{\theta(\lambda_{l\Im}\lambda_{j\Im})}\times\prod_{j=1}^{n}\left(\frac{\lambda_m-\overline{\lambda}_j}{\lambda_m-\lambda_j}\right)^{\theta(-\lambda_{m\Im}\lambda_{j\Im})}. \tag{44}$$

Corollary Condition $|\lambda_{1\Im}|>|\lambda_{2\Im}|>\dots$ can be omitted. Indeed, this condition was relevant only for the proof that these formulas obey the recursion procedure, while final formulas for potential and solutions f_n and F_n are invariant under permutation of λ_j's.

4 Properties of potentials

We proved that potentials obtained above are real and regular and these properties are equivalent to Hermiticity of the matrix C_n and conditions (38). In order to demonstrate that they are "ray" potentials we have to study their asymptotic behavior when x_1 is replaced with $x_1-2\mu x_2$ and $x_2\to\infty$ (with fixed x_1). Real parameter μ determines the direction of asymptotics on the plane. It is easy to prove that the leading term of the potential is decaying for all $\mu\neq\lambda_{1\Re},\dots,\lambda_{n\Re}$. Taking into account that the original potential is equally decaying we have from (42) for this limit

$$\lim_{x_2\to\pm\infty} u_n(x_1-2\lambda_{j\Re}x_2,x_2)=-\frac{2\lambda_{j\Im}^2}{\cosh^2(\lambda_{j\Im}x_1+\epsilon_{j,\pm})}. \tag{45}$$

The two rays belonging to each direction are mutually shifted and the shift has been explicitly computed[4].

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