SOME NONDECAYING POTENTIALS FOR THE HEAT CONDUCTION EQUATION *

M. BOITI, F. PEMPINELLI, B. PRINARI

Dipartimento di Fisica dell'Università Sezione INFN, 73100 Lecce, ITALY
e-mail: boiti@le.infn.it, pempi@le.infn.it, prinari@le.infn.it

A. K. POGREBKOV

Steklov Mathematical Institute, Moscow, 117966 GSP-1 Russia
e-mail:pogreb@mi.ras.ru

Potentials of the heat conduction operator constructed by means of 2 binary Bäcklund transformations are studied in detail. Corresponding Darboux transformations of the Jost solutions are introduced. We show that these solutions obey modified integral equations and present their analyticity properties.

1 Introduction

In this article we investigate into the spectral theory of the heat conduction operator

$$L = -\partial_{x_2} + \partial_{x_1}^2 - u(x_1, x_2), \tag{1}$$

in the case in which u is a real function with "ray" type behavior. More exactly, u is supposed to be rapidly decaying in all directions on the x-plane with the exception of some finite number of directions. Spectral theory of operator L with potential of this class is interesting *per se* and because it is associated to the Kadomtsev–Petviashvili equation in its version called KPII[1]

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = -3u_{x_2x_2}. \tag{2}$$

In studying KPII equation, it was found [2] that there are significant differences with respect to the case of KPI. Specifically, the inverse problem cannot be formulated as a Riemann–Hilbert boundary value problem[3]. The role of the Riemann–Hilbert problem is now played by what is usually called a $\bar{\partial}$-problem. Anyway, it is always assumed that the potential is rapidly decaying for $x_1^2+x_2^2 \to \infty$, which excludes in particular the presence of line solitons, and a rigorous investigation of the IST and of the properties of solutions of KPII equation with nontrivial (for instance, one-dimensional) asymptotic behavior is still missing. Such extension of the spectral theory of the heat operator would provide the possibility to extend correspondingly the class of solutions of the KPII equation, including potentials with asymptotic one-dimensional behavior.

We propose an extension of the integral equations defining Jost solutions which allows also to include potentials with ray type behavior and we study in details the potentials obtained by superimposing, via binary Bäcklund transformations, one and two solitons to a generic decaying background.

*WORK SUPPORTED IN PART BY PRIN 97 "SINTESI" AND BY RFBR #99-01-00151

2 On Jost solutions

In the standard case one defines the Jost solution $\Phi(x,k)$ and dual Jost solution $\Psi(x,k)$ as the solutions of the heat conduction equation and its dual, respectively, and such that

$$\chi(x,k) = e^{ikx_1+k^2x_2}\Phi(x,k) \qquad \xi(x,k) = e^{-ikx_1-k^2x_2}\Psi(x,k) \tag{3}$$

satisfy the integral equations

$$\chi(x,k) = 1 + \int dx' G_0(x-x',k)u(x')\chi(x',k), \tag{4}$$

$$\xi(x,k) = 1 + \int dx' G_0(x'-x,k)u(x')\xi(x',k) \tag{5}$$

where $G_0(x,k)$ is the Green's function of Eq. (1) with zero potential, that is

$$G_0(x,k) = -\frac{\operatorname{sign} x_2}{2\pi}\int d\alpha\theta(\alpha(\alpha+2k_{\Re})x_2)e^{-i\alpha x_1-\alpha(\alpha+2k)x_2} \tag{6}$$

Functions χ and ξ are bounded for all x_1, x_2 (see, for instance, [4]) and

$$\lim_{k\to\infty}\chi(x,k) = \lim_{k\to\infty}\xi(x,k) = 1. \tag{7}$$

It follows from (6) that G_0 and, consequently, χ and ξ are nowhere analytic in the complex k-plane. One can check from (6) that

$$G_0(x, ik_{\Im}) = -\frac{\theta(x_2)}{2\sqrt{\pi x_2}}e^{-\frac{(x_1+2k_{\Im}x_2)^2}{4x_2}}$$

which is real and, up to the exponential factor $e^{-k_{\Im}x_1-k_{\Im}^2x_2}$, is the well-known Gauss-Weierstrass kernel (or heat kernel). Then χ and ξ can be computed at purely imaginary values of the spectral parameter k and they are real as well.

Solvability of integral equations under some small norm assumptions was proved in [5] and thanks to (4)–(6) it is easy to show that $\chi(x,k)$ and $\xi(x,k)$ tend to 1 at large x_1 independently on the direction. Moreover, one can prove [6] the following

Proposition 1 *Let u be a real, regular and rapidly decaying potential. The modified Jost solutions χ and ξ obey for any $n \in \mathbb{N}$ the following asymptotics*

$$\lim_{x_2 \to \pm\infty} x_2^{-n} \left(\chi(x, i\kappa) - 1\right) = 0, \tag{8}$$

$$\lim_{x_2 \to \pm\infty} x_2^{-n} \left(\xi(x, i\kappa) - 1\right) = 0. \tag{9}$$

Proposition 2 *Let u be a real potential in the Schwartz class. The modified Jost solutions χ and ξ for the heat conduction equation with potential u are such that $\chi(x, ik_\Im) \neq 0$, $\xi(x, ik_\Im) \neq 0$ $\forall x \in \mathbb{R}^2$, or equivalently, taking into account asymptotics (8)-(9)*

$$\chi(x, ik_\Im) > 0 \qquad \xi(x, ik_\Im) > 0, \qquad \forall x \in \mathbb{R}^2. \tag{10}$$

The main problem when dealing with a potential $u(x)$ not vanishing in all directions at large distances is that the integral equations (4) and (5) cannot be applied as the Green's function is slow decaying at space infinity. We suggest the following modification of these integral equations (see also [7])

$$\chi(x, k) = 1 + \int_{-k_\Im\infty}^{x_1} dy_1 \int dx' \partial_{y_1} G_0(y_1 - x_1', x_2 - x_2', k) u(x') \chi(x', k), \tag{11}$$

$$\xi(x, k) = 1 + \int_{k_\Im\infty}^{x_1} dy_1 \int dx' \partial_{y_1} G_0(x_1' - y_1, x_2' - x_2, k) u(x') \xi(x', k), \tag{12}$$

where the order of operations is explicitly prescribed. If the solution of this equation exists and is bounded, then

$$\lim_{x_1 \to -k_\Im\infty} \chi(x, k) = 1, \qquad \lim_{x_1 \to k_\Im\infty} \xi(x, k) = 1, \qquad k_\Im \neq 0. \tag{13}$$

We will study here the special but rather wide subclass of potentials of ray type that is obtained by applying recursively the so-called binary Bäcklund transformations [8] with complex spectral parameter to a decaying potential. As we are interested in the spectral characteristics of potentials u having nontrivial limits, we study also the corresponding Darboux transformations furnishing the Jost solutions of the transformed potentials and their analytical properties.

3 Binary Bäcklund transformations

We have at our disposal in [8] a rather simple and transparent method for performing binary Bäcklund transformations of the potential u and corresponding Darboux transformations of solutions of (1) and its dual. Let u_0 be a smooth, real and rapidly decaying potential and L_0 the corresponding heat operator

$$L_0 = -\partial_{x_2} + \partial_{x_1}^2 - u_0(x). \tag{14}$$

A new potential $u_0'(x)$ can be generated through an elementary BT

$$L_0' B_0 = B_0 L_0 \tag{15}$$

using a gauge operator $B_0 = \partial_{x_1} - (\partial_{x_1} \log \varphi_0(x))$. If we impose (15) we find that

$$u_0'(x) = u_0(x) - 2\partial_{x_1}^2 \log \varphi_0(x) \tag{16}$$

and that φ_0 has to be a solution of the original spectral problem.

Now let us consider an inverse elementary BT

$$L_0' B_0' = B_0' L_1 \tag{17}$$

through the gauge $B_0' = \partial_{x_1} + (\partial_{x_1} \log \psi_0'(x))$. One can easily check that in this case

$$u_1(x) = u_0'(x) - 2\partial_{x_1}^2 \log \psi_0' \tag{18}$$

where ψ_0' is an arbitrary solution of

$$\left(\partial_{x_2} + \partial_{x_1}^2 - u_0'(x)\right) \psi_0'(x) = 0. \tag{19}$$

Now we perform direct and inverse elementary Bäcklund transformations (15) and (17) and study the properties of operator L_1, First of all, from (16) and (18) we see that

$$u_1(x) = u_0(x) - 2\partial_{x_1}^2 \log \Delta_1(x) \tag{20}$$

where we introduced

$$\Delta_1(x) = \varphi_0(x) \psi_0'(x). \tag{21}$$

It is clear that a sufficient condition for u_1 in (20) to be real is to choose Δ_1 real, but this means that φ_0 cannot be an arbitrary solution of the original spectral problem. In particular, one can choose a real solution such as the Jost solution computed for purely imaginary value of spectral parameter, i.e.

$$\varphi_0(x) = \Phi_0(x, i\kappa_1). \tag{22}$$

To obtain $\psi_0'(x)$, and consequently Δ_1, we consider the Darboux transform of a solution $\psi_0(x)$ of the spectral problem $L_0^d \psi_0 = 0$. Then we get

$$\partial_{x_1} \left(\psi_0'(x) \varphi_0(x)\right) = \varphi_0(x) \psi_0(x). \tag{23}$$

and we choose for $\psi_0(x)$ the Jost solution of the dual heat equation computed for $k = i\alpha_1$, that is

$$\psi_0(x) = \Psi_0(x, i\alpha_1). \tag{24}$$

With this choice, equation (23) can be integrated with respect to x_1 giving

$$\psi_0'(x) = \frac{1}{\varphi_0(x)}\left[C(x_2) + \int_{-(\kappa_1-\alpha_1)\infty}^{x_1} dx_1' \Psi_0(x_1', x_2, i\alpha_1)\Phi_0(x_1', x_2, i\kappa_1)\right] \tag{25}$$

and the integral is convergent due to (22) and (24) and to the asymptotics (7). "Constant" of integration C depends, in general, on x_2 as well, but one can check that if c_1 does not depend on x_2 then ψ_0' is indeed a solution of (19). In this case we are considering

$$\Delta_1(x) = c_1 + \int_{-(\kappa_1-\alpha_1)\infty}^{x_1} dx_1' \Phi_0(x_1', x_2, i\kappa_1)\Psi_0(x_1', x_2, i\alpha_1), \tag{26}$$

where c_1 and α_1, κ_1 are real constants with $\alpha_1 \neq \kappa_1$.

Taking into account (26), Prop. 2 proves that it is always possible to formulate a small norm condition which ensures regularity of the dressed potential (20). Moreover Prop. 1 says that $\Phi_0(x, ik_\Re)$ and $\Psi_0(x, ik_\Re)$ are real and strictly positive for all $x \in \mathbb{R}^2$ (see (10)); then the integral in (26) is a monotonic function of x_1 and if we choose c_1 such that

$$(\kappa_1 - \alpha_1)c_1 > 0, \tag{27}$$

equation (26) gives a function Δ_1 which is regular, has no zeros in the x-plane and has the same sign as $\kappa_1 - \alpha_1$. Consequently, condition (27) ensures regularity of potential (20). Finally, one cane easily show that Δ_1 indeed superimposes one soliton on the generic (smooth and rapidly decaying) background u_0 along the direction $x_1 + (\kappa_1 + \alpha_1)x_2 = const.$

We need now to express the Jost solutions Φ_1 and Ψ_1 of the spectral equation and of its dual in terms of the Jost solutions Φ_0 and Ψ_0 of the original spectral problem, that is we have to construct the Darboux version of the fundamental Bäcklund transformation. From (17) we see that if ϕ_0' satisfies $L_0'\phi_0' = 0$ and we define ϕ_1 through

$$B_0'\phi_1 = \phi_0', \tag{28}$$

then ϕ_1 is such that $B_0'L_1\phi_1 = 0$. So, taking into account definition of B_0' in (3), from (28) we get

$$\partial_{x_1}(\phi_1(x,k)\psi_0'(x)) = \phi_0'(x,k)\psi_0'(x), \tag{29}$$

which allows to construct the solutions of heat conduction equation with potential u_1

$$\phi_1(x,k) = \Phi_0(x,k) - \frac{\Phi_0(x,i\kappa_1)}{\Delta_1(x)}\left[C_1(k) + \right.$$
$$\left. + \int_{-(k_\Re-\alpha_1)\infty}^{x_1} dx_1' \Phi_0(x_1', x_2, k)\Psi_0(x_1', x_2, i\alpha_1)\right] \tag{30}$$

and

$$F_1(x,k) = \Phi_0(x,k) - \frac{\Phi_0(x,i\kappa_1)}{\Delta_1(x)}\int_{-(k_\Re-\alpha_1)\infty}^{x_1} dx_1' \Phi_0(x_1', x_2, k)\Psi_0(x_1', x_2, i\alpha_1). \tag{31}$$

Eq. (31) shows that F_1 has the same analytical properties of Φ_0 and an additional pole for $k = i\alpha_1$.

$$F_1(x,k) = \frac{1}{ik+\alpha_1}\frac{\Phi_0(x,i\kappa_1)}{\Delta_1(x)} + reg \tag{32}$$

where reg denotes terms which are regular in the limit $k \to i\alpha_1$. Moreover one can check that there exist limits

$$\lim_{x_1\to\pm\infty} e^{ikx_1+k^2x_2}F_1(x,k) = A_1(\pm,k) \tag{33}$$

and they are given by

$$A_1(\pm,k) = 1 + \frac{(\kappa_1-\alpha_1)}{(ik+\alpha_1)}\theta(\pm(\kappa_1-\alpha_1)). \tag{34}$$

The Jost solution for potential u_1 is obtained as

$$\Phi_1(x,k) = \frac{F_1(x,k)}{A_1(-\operatorname{sign} k_\Re, k)}, \tag{35}$$

with F_1 given by (31), since one can easily check that χ_1 defined as $\chi_1(x,k) = e^{ikx_1+k^2x_2}\Phi_1(x,k)$ satisfies the modified integral equation (11). Taking into account (32) and (35), we see that if $\kappa_1\alpha_1 < 0$ Φ_1 has no poles in the complex k-plane.

As far as the dual solution is concerned, in the same way we get

$$\psi_1(x,k) = \Psi_0(x,k) - \frac{\Psi_0(x,i\alpha_1)}{\Delta_1(x)}\left[D_1(k) + \right.$$
$$\left. + \int_{(k_\Re-\kappa_1)\infty}^{x_1} dx_1' \Psi_0(x_1', x_2, k)\Phi_0(x_1', x_2, i\kappa_1)\right] \tag{36}$$

$$\Upsilon_1(x,k) = \Psi_0(x,k) - \frac{\Psi_0(x,i\alpha_1)}{\Delta_1(x)} \int_{(k_\Im-\kappa_1)\infty}^{x_1} dx_1' \Psi_0(x_1',x_2,k)\Phi_0(x_1',x_2,i\kappa_1). \tag{37}$$

Note that Υ_1, as a function of k, has the same analytical properties of Ψ_0 and an additional pole for $k = i\kappa_1$. Indeed, we have

$$\Upsilon_1(x,k) = \frac{1}{ik+\kappa_1}\frac{\Psi_0(x,i\alpha_1)}{\Delta_1(x)} + \text{reg.} \tag{38}$$

If we compute its asymptotic behavior, we see that

$$\lim_{x_1\to\pm\infty} e^{-ikx_1-k^2x_2}\Upsilon_1(x,k) = B_1(\pm,k) \tag{39}$$

with

$$B_1(\pm,k) = 1 + \frac{(\alpha_1-\kappa_1)}{(ik+\kappa_1)}\theta(\pm(\kappa_1-\alpha_1)) \tag{40}$$

and then, to obtain the dual Jost solution, satisfying the modified integral equation (12) we have to consider

$$\Psi_1(x,k) = \frac{\Upsilon_1(x,k)}{B_1(\text{sign}\, k_\Im, k)}. \tag{41}$$

Using (40) and (38) we can prove that Ψ_1 has no poles iff $\alpha_1\kappa_1 < 0$.

4 Two solitons on a generic background

Iterating once the procedure illustrated above we get

$$u_2 = u_0 - 2\partial_{x_1}^2 \log\det A_2(x) \tag{42}$$

where

$$A_2(x) = B_2(x) + C_2 \tag{43}$$

C_2 is a constant 2×2 matrix and

$$(B_2)_{jl}(x) = \int_{-(\kappa_l-\alpha_j)\infty}^{x_1} dx_1' \phi_0(x_1',x_2,i\kappa_l)\psi_0(x_1',x_2,i\alpha_j). \tag{44}$$

When κ_2 and α_2 obey

$$(\kappa_2-\alpha_2)(\kappa_1-\alpha_1) < 0, \tag{45}$$

and are such that the intervals κ_2, α_2 and κ_1, α_1 are contained one into another, the following conditions

$$c_{ij}(\kappa_j - \alpha_i) > 0, \qquad i,j = 1,2, \tag{46}$$

together with (45), ensure that $\Delta_1\Delta_2$ has no zeros in the x-plane and, consequently, that the dressed potential (42) is regular. Then the parameters will be ordered either according to

$$\kappa_1 < \alpha_2 < \kappa_2 < \alpha_1 \qquad \alpha_1 < \kappa_2 < \alpha_2 < \kappa_1 \tag{47}$$

or to

$$\kappa_2 < \alpha_1 < \kappa_1 < \bar\alpha_2 \qquad \alpha_2 < \kappa_1 < \alpha_1 < \kappa_2. \tag{48}$$

Now let us discuss asymptotic behavior of potential u_2. Along the direction of the first soliton, that is for $d \equiv d_1 = x_1 + (\kappa_1+\alpha_1)x_2$, if matrix C_2 is diagonal we get the expected soliton-like correction to the background (shifted $\cosh^{-2}$). But in the generic case the the off-diagonal terms "cancel" the soliton along direction d_1. Along any other direction the contribution of $\Delta 1\Delta_2$ to the potential u_2 is exponentially decaying, in contrast with KPI case where there are algebraically decaying corrections [7]. Note that if only one of them is different from zero one may obtain a single "tail" of the soliton along this direction. The same can be seen obviously along direction d_2, that is for $h = \kappa_2 + \alpha_2$. When $\tilde h = \kappa_1 + \kappa_2$ and $\tilde d = x_1 + \tilde h x_2$, if $c_{12} = c_{21} = 0$ we will have no contributions to the potential along this directions, while for $c_{12} \neq 0$ and $c_{21} \neq 0$ they will give the soliton contribution we would have expected for $h = \kappa_1 + \alpha_1$ and $h = \kappa_2 + \alpha_2$. If only one of the off-diagonal entries of matrix C_2 is different from zero, it will give a single tail of the soliton along this direction. The same can be proved for $h = \alpha_1 + \alpha_2$. Note that these features are absent in the case of KPI equation. In fact, for KPI the soliton directions do not depend on the choice of such matrix [9-10].

By using the Darboux procedure we get the Jost solutions for potential u_2

$$\Phi_2(x,k) = \frac{F_2(x,k)}{A_2(-k_\Im,k)} \qquad \Psi_2(x,k) = \frac{\Upsilon_2(x,k)}{B_2(k_\Im,k)}, \tag{49}$$

where

$$F_2(x,k) = \frac{1}{\det A_2(x)}\begin{vmatrix} A_{11}(x) & A_{12}(x) & \beta_1(x,k) \\ A_{21}(x) & A_{22}(x) & \beta_2(x,k) \\ \Phi_0(x,i\kappa_1) & \Phi_0(x,i\kappa_2) & \Phi_0(x,k) \end{vmatrix} \tag{50}$$

$$\Upsilon_2(x,k) = \frac{1}{\det A_2(x)}\begin{vmatrix} A_{11}(x) & A_{12}(x) & \beta_1^d(x,k) \\ A_{21}(x) & A_{22}(x) & \beta_2^d(x,k) \\ \Psi_0(x,i\alpha_1) & \Psi_0(x,i\alpha_2) & \Psi_0(x,k) \end{vmatrix}. \tag{51}$$

Matrix A_2 is given by (43) and

$$\beta_j(x,k) = \int_{-(k_\Im-\alpha_j)\infty}^{x_1} dx_1' \phi_0(x_1',x_2,k)\psi_0(x_1',x_2,i\alpha_j)$$

$$\beta_j^d(x,k) = \int_{(k_\Im-\kappa_j)\infty}^{x_1} dx_1' \Psi_0(x_1',x_2,k)\Phi_0(x_1',x_2,i\kappa_j).$$

$A_2(\pm,k)$ and $B_2(\pm,k)$ are given by the following recursion relations

$$A_2(\pm,k) = A_1(\pm,k)\left[1+\frac{\kappa_2-\alpha_2}{ik+\alpha_2}\theta(\pm(\kappa_2-\alpha_2))\right] \qquad (52)$$

$$B_2(\pm,k) = B_1(\pm,k)\left[1+\frac{\alpha_2-\kappa_2}{ik+\kappa_2}\theta(\pm(\kappa_2-\alpha_2))\right]. \qquad (53)$$

Proofs and more details will be given in [6].

References

1. B.B. Kadomtsev and V.I. Petviashvili, *Sov. Phys. Doklady* **192**, 539 (1970); V.S. Dryuma, *Sov. Phys. J. Exp. Theor. Phys. Lett.* **19**, 381 (1974); V.E. Zakharov and A.B. Shabat, *Funct. Anal. Appl.* **8**, 226 (1974).
2. M. Ablowitz, D. Bar Yacoov, A.S. Fokas, *Stud. Appl. Math.* **69**, 135 (1983).
3. V.E. Zakharov, S.V. Manakov, *Sov. Sci. Rev. Phys. Rev.* **1**, 133 (1979); S.V. Manakov, *Physica* **D 3**, 420 (1981); A.S. Fokas and M.J. Ablowitz, *Stud. Appl. Math.* **69**, 211 (1983).
4. M.J. Ablowitz and P.A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Lecture Notes Series **49**, University of Cambridge, Cambridge (1991).
5. M.V. Wickerhauser, *Commun. Math. Phys.* **108**, 67–89 (1987).
6. B. Prinari, On some non decaying potentials and related Jost solutions for the heat conduction equation, preprint, Lecce (December 1999).
7. M. Boiti, F. Pempinelli, A. Pogrebkov and B. Prinari, *Theor. Math. Phys.* **116**, 741 (1998).
8. V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin 1991).
9. M. Boiti, F. Pempinelli, A.K. Pogrebkov, B. Prinari, *Proceedings of the Steklov Institute of Mathematics* **126**, 42-62 (1999).
10. M. Boiti, F. Pempinelli, A.K. Pogrebkov, B. Prinari, Some nondecaying potentials for the nonstationary Schrödinger equation, in this volume.

A GEOMETRICAL APPROACH TO THE EMDEN-FOWLER EQUATION

L. A. BORDAG

BTU Cottbus, Postfach 101344, D-03013 Cottbus, Germany

E-mail: ljudmila@bordag.com

This paper is devoted to the investigation of Emden–Fowler equations of the type

$$y''(x) + e(x)\, y^p(x) = 0, \quad x \in \mathbb{R}.$$

The motivation to study such equations systematically came from astro-, atomic physics and Riemannian geometry. We will study the properties of Emden–Fowler equation, which are invariant under the general point transformations. The underlying geometrical theory of second order ordinary equations was developed by E. Cartan. He introduced the concept of the space of normal projective connection. The Emden-Fowler equation can then be considered as the equation of the geodesics in such space. In case of Emden-Fowler equation the corresponding space of normal projective connection can be immersed in the flat projective space $\mathbb{R}P^3$. The description of related surfaces will be given.

1 Introduction

The main object of our investigations is the general Emden-Fowler equation

$$y''(x) + e(x)\, y^p(x) = 0, \quad x \in \mathbb{R} \qquad (1)$$

In his celebrated work Emden [6] was led to the following equation, describing the density of stars

$$\Delta u(x) + a(r)u^p(x) = 0,\ x \in \mathbb{R}^3,\ r = ||x||. \qquad (2)$$

In the classical Emden–Fowler equation we have $a(r) = 1$, a more general model with $a(r) = r^s$ was proposed by Hénon (1973) [7] and Matukuma (1930) [8] considered $a(r) = \frac{1}{1+r^2}$. In radial case all these equations can be reduced to

$$y''(x) + a\left(x^{-1}\right) x^{-4} y^p = 0, \quad x \in \mathbb{R}, \qquad (3)$$

i.e. to the equation (1) with different positive functions $e(x)$.

The radial solutions of the problem (2) with negative $a(r)$ play an important role in diffusion processes and have attracted a lot of attention in the past decades. The corresponding Emden-Fowler equation is like (3) with a minus sign instead of a plus sign.

Similar equations come up in Riemannian geometry [3]. If we look for the scalar curvature of an N-dimensional ($N = 3$) Riemannian metric which is