ON THE IST FOR DISCRETE NONLINEAR SCHRÖDINGER SYSTEMS AND POLARIZATION SHIFT FOR DISCRETE VECTOR SOLITONS

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An integrable discretization of the vector nonlinear Schrödinger equation is considered within the framework of the inverse scattering method. The problem of multisoliton collisions is investigated. An analytic formula for the polarization shift of discrete vector solitons as a consequence of their interaction is obtained.

1. Introduction

The vector nonlinear Schrödinger equation (VNLS)

$$i\mathbf{q}_t = \mathbf{q}_{xx} + 2\left\|\mathbf{q}\right\|^2 \mathbf{q} \tag{1}$$

$\mathbf{q}$ being an N component vector and $\|\cdot\|$ the Euclidean norm, arises, physically, under conditions similar to those described by the celebrated NLS equation when there are multiple wavetrains moving with nearly the same group velocity [1,2]. Also, VNLS models systems where the field has more than one component (for example, in optical fibers and waveguides, the propagating electric field has two components transverse to the direction of propagation). This system which asymptotically models the propagation of the electric field in a waveguide was first examined by Manakov [3]. Subsequently, it was derived as a key model for light-wave propagation in optical fibers [4–6].

The VNLS has a natural matrix generalization in the system

$$i\mathbf{Q}_t = \mathbf{Q}_{xx} - 2\mathbf{QRQ}, \qquad -i\mathbf{R}_t = \mathbf{R}_{xx} - 2\mathbf{RQR} \tag{2}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are $N \times M$ and $M \times N$ matrices, respectively. When $\mathbf{R} = -\mathbf{Q}^H$ (the superscript ^{H} denotes the Hermitian), the system (2) reduces to the single matrix equation

$$i\mathbf{Q}_t = \mathbf{Q}_{xx} + 2\mathbf{QQ}^H\mathbf{Q} \tag{3}$$

which we refer to as matrix NLS (MNLS). VNLS corresponds to the special case when $\mathbf{Q}$ is an N-component row vector and $\mathbf{R}$ is an N-component column vector or vice-versa.

Both NLS and VNLS admit integrable discretizations which, besides being used as the basis for constructing numerical schemes for the continuous counterparts, also have physical applications as discrete systems (see ref.[11]).

A natural discretization of NLS, referred here as integrable discrete NLS (or IDNLS)

$$i\frac{d}{dt}q_n = \frac{1}{h^2}\left(q_{n+1} - 2q_n + q_{n-1}\right) + |q_n|^2\left(q_{n+1} + q_{n-1}\right) \tag{4}$$

was shown to be integrable via the IST [7] and a matrix analog of IDNLS is given by

$$i\frac{d\mathbf{Q}_n}{d\tau} = \mathbf{Q}_{n+1} - 2\mathbf{Q}_n + \mathbf{AQ}_n + \mathbf{Q}_n\mathbf{B} + \mathbf{Q}_{n-1} - \mathbf{Q}_{n+1}\mathbf{R}_n\mathbf{Q}_n - \mathbf{Q}_n\mathbf{R}_n\mathbf{Q}_{n-1} \tag{5}$$

$$-i\frac{d\mathbf{R}_n}{d\tau} = \mathbf{R}_{n+1} - 2\mathbf{R}_n + \mathbf{BR}_n + \mathbf{R}_n\mathbf{A} + \mathbf{R}_{n-1} - \mathbf{R}_{n+1}\mathbf{Q}_n\mathbf{R}_n - \mathbf{R}_n\mathbf{Q}_n\mathbf{R}_{n-1} \tag{6}$$

where $\mathbf{Q}_n$ and $\mathbf{R}_n$ are, respectively, $N \times M$ and $M \times N$ matrices, $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ and $M \times M$ diagonal matrices. This system (IDMNLS) results from the compatibility condition of the following linear equations [8–10]

$$\mathbf{v}_{n+1} = \begin{pmatrix} z\mathbf{I}_N & \mathbf{Q}_n \\ \mathbf{R}_n & z^{-1}\mathbf{I}_M \end{pmatrix}\mathbf{v}_n \tag{7}$$

$$-i\frac{d\mathbf{v}_n}{d\tau} = \begin{pmatrix} \mathbf{Q}_n\mathbf{R}_{n-1} - \frac{1}{2}(z - z^{-1})^2\mathbf{I}_N + \mathbf{A} & -z\mathbf{Q}_n + z^{-1}\mathbf{Q}_{n-1} \\ z^{-1}\mathbf{R}_n - z\mathbf{R}_{n-1} & -\mathbf{R}_n\mathbf{Q}_{n-1} + \frac{1}{2}(z - z^{-1})^2\mathbf{I}_M + \mathbf{B} \end{pmatrix}\mathbf{v}_n \tag{8}$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix and $\mathbf{I}_M$ is the $M \times M$ identity matrix. The matrices $\mathbf{A}$ and $\mathbf{B}$ can be absorbed by the gauge transformation

$$\hat{\mathbf{Q}}_n = e^{i\tau\mathbf{A}}\mathbf{Q}_n e^{i\tau\mathbf{B}}, \quad \hat{\mathbf{R}}_n = e^{-i\tau\mathbf{B}}\mathbf{R}_n e^{-i\tau\mathbf{A}}. \tag{9}$$

This corresponds to solving eqs. (5)–(6) in the gauge $\mathbf{A} = \mathbf{B} = \mathbf{0}$. It should be noted that the system (5)-(6) does not, in general, admit the reduction $\mathbf{R}_n = \mp\mathbf{Q}_n^H$ and the asymmetry depends on the non-commutativity of matrix multiplication. However, if one takes $M = N$ and restricts $\mathbf{R}_n$ and $\mathbf{Q}_n$ to be such that

$$\mathbf{R}_n\mathbf{Q}_n = \mathbf{Q}_n\mathbf{R}_n = \alpha_n\mathbf{I}_N \tag{10}$$

then

$$\mathbf{R}_n = \mp\mathbf{Q}_n^H, \tag{11}$$

becomes a consistent reduction. For instance, in the case $N = 2$, the condition (10) is accomplished choosing

$$\mathbf{R}_n = (-1)^{n+1}\sigma_2\mathbf{Q}_n^T\sigma_2 \tag{12}$$

σ_2 being the Pauli matrix, i.e.

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\mathbf{Q}_n = \begin{pmatrix} Q_n^{(1)} & Q_n^{(2)} \\ (-1)^n R_n^{(2)} & (-1)^{n+1}R_n^{(1)} \end{pmatrix}, \quad \mathbf{R}_n = \begin{pmatrix} R_n^{(1)} & (-1)^n Q_n^{(2)} \\ R_n^{(2)} & (-1)^{n+1}Q_n^{(1)} \end{pmatrix}. \tag{13}$$

In this case $\alpha_n = (R_n^{(1)}Q_n^{(1)} + R_n^{(2)}Q_n^{(2)})$ and $\mathbf{q}_n = h^{-1}(Q_n^{(1)}, Q_n^{(2)})^T$, $\mathbf{r}_n = h^{-1}(R_n^{(1)}, R_n^{(2)})^T$ satisfy the discrete VNLS system

$$i\frac{d}{dt}\mathbf{q}_n = \frac{1}{h^2}\left(\mathbf{q}_{n+1} - 2\mathbf{q}_n + \mathbf{q}_{n-1}\right) - \mathbf{r}_n \cdot \mathbf{q}_n\left(\mathbf{q}_{n+1} + \mathbf{q}_{n-1}\right) \tag{14}$$

$$-i\frac{d}{dt}\mathbf{r}_n = \frac{1}{h^2}\left(\mathbf{r}_{n+1} - 2\mathbf{r}_n + \mathbf{r}_{n-1}\right) - \mathbf{r}_n \cdot \mathbf{q}_n\left(\mathbf{r}_{n+1} + \mathbf{r}_{n-1}\right) \tag{15}$$

where $t = h^2\tau$ and $\cdot$ denotes the scalar product. Then with (11) and $\mathbf{r}_n = \mp\mathbf{q}_n^*$ we have

$$i\frac{d}{dt}\mathbf{q}_n = \frac{1}{h^2}\left(\mathbf{q}_{n+1} + \mathbf{q}_{n-1} - 2\mathbf{q}_n\right) \pm ||\mathbf{q}_n||^2\left(\mathbf{q}_{n+1} + \mathbf{q}_{n-1}\right) \tag{16}$$

which here is referred to as integrable discrete vector NLS (or IDVNLS) equation.

The symmetry (10) between $\mathbf{Q}_n$ and $\mathbf{R}_n$ imposes restrictions on the choice of the gauge matrices $\mathbf{A}$ and $\mathbf{B}$. Indeed, if the symmetry condition (12) is imposed at the initial time, it is preserved by the time evolution iff

$$\mathbf{B} = 2\mathbf{I} - \sigma_2\mathbf{B}^T\sigma_2, \qquad \mathbf{A} = 2\mathbf{I} - \sigma_2\mathbf{A}^T\sigma_2 \tag{17}$$

and this condition is accomplished, for instance, when $\mathbf{A} = \mathbf{B} = \mathbf{I}$. In this case, in order to obtain solutions of the original system (5)–(6), one first determines the evolution of the potentials $\hat{\mathbf{Q}}_n$ and $\hat{\mathbf{R}}_n$ such that $\hat{\mathbf{Q}}_n(0) = \mathbf{Q}_n(0)$, $\hat{\mathbf{R}}_n(0) = \mathbf{R}_n(0)$. Then from (9)

$$\mathbf{Q}_n(\tau) = e^{2i\tau}\hat{\mathbf{Q}}_n(\tau), \qquad \mathbf{R}_n(\tau) = e^{-2i\tau}\hat{\mathbf{R}}_n(\tau)$$

yields the solution to (5)–(6) in the gauge $\mathbf{A} = \mathbf{B} = \mathbf{0}$. To get equations (14)-(15) from (5)-(6), we need the gauge $\mathbf{A} = \mathbf{B} = \mathbf{0}$. However, in this gauge, equations (17) are not satisfied and the symmetry (11) will be broken by evolution. Thus, in order to determine the solution of (14)-(15) for all time we have to solve for the evolution of the "hat" variables and then transform back.

2. Inverse Scattering Transform for IDMNLS

In this section a very brief description of the IST for the scattering problem (7) is given. Details and proofs can be found in [11]. The scattering problem (7) for decaying potentials admits the asymptotic solutions

$$\phi_n(z) \sim z^n \begin{pmatrix} \mathbf{I}_N \\ 0 \end{pmatrix}, \qquad \bar{\phi}_n(z) \sim z^{-n} \begin{pmatrix} 0 \\ \mathbf{I}_M \end{pmatrix} \qquad n \to -\infty$$

and

$$\psi_n(z) \sim z^{-n} \begin{pmatrix} 0 \\ \mathbf{I}_M \end{pmatrix}, \qquad \bar{\psi}_n(z) \sim z^n \begin{pmatrix} \mathbf{I}_N \\ 0 \end{pmatrix} \qquad n \to +\infty$$

matrix-valued functions of $(N+M) \times N$ and $(N+M) \times M$ dimensions.

The eigenfunctions with constant boundary conditions as $n \to \pm\infty$

$$\mathbf{M}_n(z) = z^{-n}\phi_n(z), \qquad \bar{\mathbf{M}}_n(z) = z^n\bar{\phi}_n(z) \tag{18}$$

$$\mathbf{N}_n(z) = z^n\psi_n(z), \qquad \bar{\mathbf{N}}_n(z) = z^{-n}\bar{\psi}_n(z) \tag{19}$$

$$\mathbf{M}_n = \begin{pmatrix} \mathbf{I}_N \\ 0 \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \mathbf{G}^{\ell}_{n-k}\tilde{\mathbf{Q}}_k\mathbf{M}_k, \quad \mathbf{N}_n = \begin{pmatrix} 0 \\ \mathbf{I}_M \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \mathbf{G}^{r}_{n-k}\tilde{\mathbf{Q}}_k\mathbf{N}_k \tag{20}$$

$$\bar{\mathbf{N}}_n = \begin{pmatrix} \mathbf{I}_N \\ 0 \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \bar{\mathbf{G}}^{r}_{n-k}\tilde{\mathbf{Q}}_k\bar{\mathbf{N}}_k, \quad \bar{\mathbf{M}}_n = \begin{pmatrix} 0 \\ \mathbf{I}_M \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \bar{\mathbf{G}}^{\ell}_{n-k}\tilde{\mathbf{Q}}_k\bar{\mathbf{M}}_k \tag{21}$$

where

$$\tilde{\mathbf{Q}}_n = \begin{pmatrix} 0 & \mathbf{Q}_n \\ \mathbf{R}_n & 0 \end{pmatrix}$$

and the Green's functions are given by

$$\mathbf{G}^{\ell}_n(z) = z^{-1}\theta(n-1)\begin{pmatrix} \mathbf{I}_N & 0 \\ 0 & z^{-2(n-1)}\mathbf{I}_M \end{pmatrix} \qquad \mathbf{G}^{r}_n(z) = -z\theta(-n)\begin{pmatrix} z^{2(n-1)}\mathbf{I}_N & 0 \\ 0 & \mathbf{I}_M \end{pmatrix}$$

$$\bar{\mathbf{G}}^{r}_n(z) = -z^{-1}\theta(-n)\begin{pmatrix} \mathbf{I}_N & 0 \\ 0 & z^{-2(n-1)}\mathbf{I}_M \end{pmatrix} \qquad \bar{\mathbf{G}}^{\ell}_n(z) = z\theta(n-1)\begin{pmatrix} z^{2(n-1)}\mathbf{I}_N & 0 \\ 0 & \mathbf{I}_M \end{pmatrix}$$

$\theta(n)$ being the discrete version of the Heaviside function.

The existence of the eigenfunctions can be proved by iteration (i.e. Neumann series). Precisely, one can prove that if $\|\mathbf{Q}\|_1, \|\mathbf{R}\|_1 < \infty$ ($\|\mathbf{Q}\|_1 = \sum_{n=-\infty}^{+\infty}\|\mathbf{Q}_n\|_a$, where $\|\mathbf{Q}_n\|_a$ is any matrix norm), then $\mathbf{M}_n(z), \mathbf{N}_n(z)$ defined by (20) are analytic in z for $|z| > 1$, and continuous for $|z| \geq 1$ and $\bar{\mathbf{N}}_n(z), \bar{\mathbf{M}}_n(z)$ defined by (21) are analytic in z for $|z| < 1$ and continuous for $|z| \leq 1$. Moreover, these eigenfunctions are unique in the space of continuous functions. All proofs can be found in ref. [11].

The eigenfunctions $\bar{\psi}_n(z)$ and $\psi_n(z)$ together constitute $N + M$ independent solutions of the scattering problem as well as $\phi_n(z)$ and $\bar{\phi}_n(z)$. Therefore the two sets are linearly dependent on each other, i.e.

$$\phi_n(z) = \psi_n(z)\mathbf{b}(z) + \bar{\psi}_n(z)\mathbf{a}(z), \quad \bar{\phi}_n(z) = \psi_n(z)\bar{\mathbf{a}}(z) + \bar{\psi}_n(z)\bar{\mathbf{b}}(z) \tag{22}$$

or the counterpart

$$\psi_n(z) = \phi_n(z)\mathbf{d}(z) + \bar{\phi}_n(z)\mathbf{c}(z), \quad \bar{\psi}_n(z) = \phi_n(z)\bar{\mathbf{c}}(z) + \bar{\phi}_n(z)\bar{\mathbf{d}}(z) \tag{23}$$

are valid for any z such that all the scattering coefficients are well defined, in particular they hold for $|z| = 1$.

The scattering coefficients can be expressed as explicit sums of the eigenfunctions and the potentials, for instance

$$\mathbf{a}(z) = \mathbf{I}_N + \sum_{k=-\infty}^{+\infty} z^{-1}\mathbf{Q}_k\mathbf{M}^{(\mathrm{dn})}_k(z), \quad \bar{\mathbf{a}}(z) = \mathbf{I}_M + \sum_{k=-\infty}^{+\infty} z\mathbf{R}_k\bar{\mathbf{M}}^{(\mathrm{up})}_k(z) \tag{24}$$

the superscripts $^{\mathrm{up}}$ and $^{\mathrm{dn}}$ indicating, respectively, the first N rows and the last M rows of a matrix $\mathbf{A}$ (i.e. $\mathbf{A}^{(\mathrm{up})}$ is the $N \times J$ upper block and $\mathbf{A}^{(\mathrm{dn})}$ is the lower $M \times J$ block of matrix $\mathbf{A}$).

The expressions (24) imply that $\mathbf{a}(z)$ is analytic in the same region as $\mathbf{M}_n(z)$, i.e. for $|z| > 1$, and $\bar{\mathbf{a}}(z)$ is analytic in the same region as $\bar{\mathbf{M}}_n(z)$, i.e. for $|z| < 1$. Moreover, one can show that $\mathbf{a}(z), \bar{\mathbf{a}}(z)$ are even functions of the spectral parameter z.

In terms of the functions

$$\mu_n(z) = \mathbf{M}_n(z)\mathbf{a}^{-1}(z), \qquad \bar{\mu}_n(z) = \bar{\mathbf{M}}_n(z)\bar{\mathbf{a}}^{-1}(z), \tag{25}$$

the relations (22) can be written as

$$\mu_n(z) - \bar{\mathbf{N}}_n(z) = z^{-2n}\mathbf{N}_n(z)\rho(z), \quad \bar{\mu}_n(z) - \mathbf{N}_n(z) = z^{2n}\bar{\mathbf{N}}_n(z)\bar{\rho}(z) \quad (26)$$

with $\rho(z) = \mathbf{b}(z)\mathbf{a}(z)^{-1}$, $\bar{\rho}(z) = \bar{\mathbf{b}}(z)\bar{\mathbf{a}}(z)^{-1}$.

The system (26) is the starting point for the inverse problem. Indeed, if $\rho(z)$ and $\bar{\rho}(z)$ are known functions on the unit circle $|z| = 1$, then (26) is the boundary condition of a Riemann-Hilbert problem with poles. Such poles, corresponding to the discrete eigenvalues of the scattering problem, are the points where $\det \mathbf{a}(z) = 0$, $\det \bar{\mathbf{a}}(z) = 0$ and since $\mathbf{a}(z)$ and $\bar{\mathbf{a}}(z)$ are even functions of z, they appear in pairs $\{\pm z_j : |z_j| > 1\}_{j=1}^{J}$ and $\{\pm \bar{z}_\ell : |\bar{z}_\ell| < 1\}_{\ell=1}^{J}$.

The functions $\mu_n(z)$ and $\bar{\mu}_n(z)$ have poles precisely at these points and assuming simple poles the residues are given by

$$\operatorname{Res}(\mu_n; z_j) = z_j^{-2n}\mathbf{N}_n(z_j)\mathbf{C}_j \qquad \operatorname{Res}(\bar{\mu}_n; \bar{z}_\ell) = \bar{z}_\ell^{2n}\bar{\mathbf{N}}_n(\bar{z}_\ell)\bar{\mathbf{C}}_\ell$$

where the matrices

$$\mathbf{C}_j = \lim_{z \to z_j} (z - z_j)\rho(z) = \frac{1}{a'(z_j)}\mathbf{b}(z_j)\alpha(z_j) \quad (27)$$

$$\bar{\mathbf{C}}_\ell = \lim_{z \to \bar{z}_\ell} (z - \bar{z}_\ell)\bar{\rho}(z) = \frac{1}{\bar{a}'(\bar{z}_\ell)}\bar{\mathbf{b}}(\bar{z}_\ell)\bar{\alpha}(\bar{z}_\ell) \quad (28)$$

are referred to as "norming constants". In the previous equations $a(z) = \det \mathbf{a}(z)$, $\alpha(z)$ is the cofactor matrix of $\mathbf{a}(z)$ and $'$ denotes the derivative with respect to z (analogously for $\bar{\mathbf{a}}(z)$). Note that the norming constant associated with $-z_j$ (respectively, $-\bar{z}_\ell$) is equal to the norming constant associated with $+z_j$ (respectively, $+\bar{z}_\ell$).

In order to reduce the IDMNLS to the discrete VNLS, we require the two symmetries (10) and (11), each of them inducing a symmetry in the scattering data. For simplicity, we restrict to the two-component case. If $\mathbf{R}_n$ is chosen as in (12), the reflection coefficients satisfy the symmetry

$$\bar{\rho}(i/z) = i\sigma_2\, \rho^T(z)\, \sigma_2, \quad (29)$$

$\hat{z}_j = i/z_j$ is an eigenvalue such that $|\hat{z}_j| < 1$ if and only if z_j is an eigenvalue such that $|z_j| > 1$ and the associated norming constants are such that

$$\hat{\mathbf{C}}_j = z_j^{-2}\sigma_2\, \mathbf{C}_j^T\sigma_2. \quad (30)$$

If, in addition, the potentials satisfy the symmetry (11), then the reflection coefficients are such that

$$\bar{\rho}(z) = \mp\rho^H(1/z^*), \quad (31)$$

$\bar{z}_j = 1/z_j^*$ is an eigenvalue such that $|\bar{z}_j| < 1$ if, and only if, z_j is an eigenvalue such that $|z_j| > 1$ and, consequently, the number of eigenvalues inside the unit circle is equal to the number of eigenvalues outside and the eigenvalues appear in sets of eight $\{\pm z_j,\ \pm i/z_j, \pm 1/z_j^*, \pm i z_j^*\}_{j=1}^{J}$; the norming constants associated with these paired eigenvalues satisfy the symmetry

$$\bar{\mathbf{C}}_j = \pm(z_j^*)^{-2}\mathbf{C}_j^H. \quad (32)$$

In order to solve the inverse problem, one has to take into account that the boundary conditions depend on $\mathbf{Q}_n$ and $\mathbf{R}_n$ which are unknowns in the inverse problem. Therefore, we introduce the following modified functions

$$\mathbf{N}'_n = \begin{pmatrix} \mathbf{I}_N & 0 \\ 0 & \Delta_n \end{pmatrix} \mathbf{N}_n = \begin{pmatrix} -z^{-1}\mathbf{Q}_n\Delta_n^{-1} + O(z^{-3}) \\ \mathbf{I}_M + O(z^{-2}) \end{pmatrix} \quad (33)$$

$$\mu'_n = \begin{pmatrix} \mathbf{I}_N & 0 \\ 0 & \Delta_n \end{pmatrix} \mu_n = \begin{pmatrix} \mathbf{I}_N + O(z^{-2}) \\ z^{-1}\Delta_n\mathbf{R}_{n-1} + O(z^{-3}) \end{pmatrix} \quad (34)$$

$$\bar{\mathbf{N}}'_n = \begin{pmatrix} \mathbf{I}_N & 0 \\ 0 & \Delta_n \end{pmatrix} \bar{\mathbf{N}}_n = \begin{pmatrix} \Omega_n^{-1} + O(z^2) \\ -z\Delta_n\mathbf{R}_n\Omega_n^{-1} + O(z^3) \end{pmatrix} \quad (35)$$

$$\bar{\mu}'_n = \begin{pmatrix} \mathbf{I}_N & 0 \\ 0 & \Delta_n \end{pmatrix} \bar{\mu}_n = \begin{pmatrix} z\mathbf{Q}_{n-1} + O(z^3) \\ \Delta_n + O(z^2) \end{pmatrix} \quad (36)$$

where

$$\Delta_n = \prod_{k=n,\ \text{left}}^{+\infty} (\mathbf{I}_N - \mathbf{R}_k\mathbf{Q}_k), \qquad \Omega_n = \prod_{k=n,\ \text{left}}^{+\infty} (\mathbf{I}_N - \mathbf{Q}_k\mathbf{R}_k) \quad .$$

and "left" indicates that the matrix with index k occurs to the left of the matrix with index $k-1$.

These modified functions satisfy the same jump conditions (26) and the constant boundary conditions $\mathbf{N}'_n \sim (0, \mathbf{I}_M)^T$, $\mu'_n \sim (\mathbf{I}_N, 0)^T$ as $z \to \infty$.

The jump conditions (26) and these boundary conditions, along with the poles and their associated norming constants, constitute a Riemann-Hilbert problem with poles. This Riemann-Hilbert problem determines the sectionally meromorphic functions $\mu'_n(z)$, $\bar{\mu}'_n(z)$ and the sectionally analytic functions $\mathbf{N}'_n(z)$, $\bar{\mathbf{N}}'_n(z)$ in terms of the reflection coefficients $\rho(z)$, $\bar{\rho}(z)$, the eigenvalues (poles) and their respective norming constants. From the Riemann-Hilbert problem, one can obtain the linear algebraic-integral system

$$\begin{aligned} \bar{\mathbf{N}}_n^{\prime(\text{up})}(z) &= \mathbf{I}_N + 2\sum_{j=1}^{J} \frac{z_j^{-2n+1}}{z^2 - z_j^2}\mathbf{N}_n^{\prime(\text{up})}(z_j)\mathbf{C}_j - \\ &\lim_{\zeta \to z,\ |\zeta|<1} \frac{1}{2\pi i}\oint_{|w|=1} \frac{w^{-2n}}{w - \zeta}\mathbf{N}_n^{\prime(\text{up})}(w)\rho(w)\,dw \end{aligned} \quad (37)$$

$$\mathbf{N}_n^{\prime(\mathrm{up})}(z) = 2\sum_{j=1}^{J} \frac{\bar{z}_j^{2n} z}{z^3 - \bar{z}_j^2} \bar{\mathbf{N}}_n^{\prime(\mathrm{up})}(\bar{z}_j)\bar{\mathbf{C}}_j + \lim_{\zeta \to z,\ |\zeta|>1} \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{2n}}{w-\zeta} \bar{\mathbf{N}}_n^{\prime(\mathrm{up})}(w)\bar{\rho}(w)dw. \qquad (38)$$

To close the system (37)-(38), we obtain $\mathbf{N}^{\prime(\mathrm{up})}(z_j)$ by evaluating (37) at each of the eigenvalues $\{z_j : |z_j| > 1,\ \mathrm{Re} z_j \geq 0\}_{j=1}^{J}$ and, similarly, we obtain $\bar{\mathbf{N}}^{\prime(\mathrm{up})}(\bar{z}_j)$ by evaluating (38) at each of the poles $\{\bar{z}_j : |\bar{z}_j| < 1,\ \mathrm{Re}\bar{z}_j \geq 0\}_{j=1}^{J}$. The lower components satisfy a similar system.

The potential $\mathbf{Q}_n$ is recovered from the $O(z^{-1})$ term in the Laurent expansion of $\mathbf{N}^{\prime(\mathrm{up})}(z)$ around $z = \infty$. By the Laurent expansion of the rhs of (37), we obtain

$$\mathbf{Q}_{n-1} = -2\sum_{j=1}^{J} \bar{z}_j^{2(n-1)} \bar{\mathbf{N}}_n^{\prime(\mathrm{up})}(\bar{z}_j)\bar{\mathbf{C}}_j + \frac{1}{2\pi i}\oint_{|w|=1} w^{2(n-1)}\bar{\mathbf{N}}_n^{\prime(\mathrm{up})}(w)\bar{\rho}(w)dw.$$

Analogously, $\mathbf{R}_n$ can be obtained by the power-series expansion of the equations for the lower components.

Recall that, if the potentials satisfy (12), then the eigenvalues appear in octets and the norming constants and reflection coefficients satisfy (29)-(30). Therefore, in the case where (12) is known to hold, one must account for the symmetries of the reflection coefficients, norming constants and eigenvalues in the solution of (37)-(38).

Next we explicitly calculate the potential in the case where there is one octet of eigenvalues/norming constants and no continuous spectrum. If $\rho(z) = \bar{\rho}(z) = 0$ on $|z| = 1$, then the system for $\mathbf{N}_n^{\prime(\mathrm{up})}(z_j)$, $\bar{\mathbf{N}}_n^{\prime(\mathrm{up})}(\bar{z}_j)$ reduces to a system of linear algebraic equations. In the two-component case, we define

$$\mathbf{C}_1 = \begin{pmatrix} \gamma_1^{(1)} & \delta_1^{(2)} \\ \gamma_1^{(2)} & -\delta_1^{(1)} \end{pmatrix}, \qquad \bar{\mathbf{C}}_1 = \bar{z}_1^2 \begin{pmatrix} \bar{\gamma}_1^{(1)} & \bar{\gamma}_1^{(2)} \\ \bar{\delta}_1^{(2)} & -\bar{\delta}_1^{(1)} \end{pmatrix}$$

and

$$\gamma_1 = \begin{pmatrix} \gamma_1^{(1)} \\ \gamma_1^{(2)} \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} \delta_1^{(1)} \\ \delta_1^{(2)} \end{pmatrix}, \quad \bar{\gamma}_1 = \begin{pmatrix} \bar{\gamma}_1^{(1)} \\ \bar{\gamma}_1^{(2)} \end{pmatrix}, \quad \bar{\delta}_1 = \begin{pmatrix} \bar{\delta}_1^{(1)} \\ \bar{\delta}_1^{(2)} \end{pmatrix}.$$

When the symmetries (10), (11) hold, $\bar{z}_1 = 1/z_1^*$, $\bar{\gamma}_1 = \gamma_1^*$ and $\bar{\delta}_1 = \delta_1^*$. Imposing the additional condition $\delta_1 = 0$, we obtain the potential

$$\begin{pmatrix} Q_n^{(1)} \\ Q_n^{(2)} \end{pmatrix} = -\frac{\gamma_1^*}{\|\gamma_1\|} e^{2ib_1(n+1)} \sinh(2a_1)\ \mathrm{sech}\,[2a_1(n+1)+d] \qquad (39)$$

where $z_1 = e^{a_1+ib_1}$ and $d = \log[\sinh(2a_1)] - \log\|\gamma_1\|$.

In order to obtain the complex-modulated sech potential, we imposed an additional condition on the norming constants that is not required by the symmetry, namely $\delta_1 = 0$ with $\gamma_1 \neq 0$. Equivalently, one can obtain a complex-modulated sech potential by the alternative condition $\gamma_1 = 0$, $\delta_1 \neq 0$. These potentials correspond to solutions of IDVNLS that we refer to as the *fundamental* solitons. For a fixed eigenvalue octet, the two potentials have the same amplitude but different frequencies of complex modulation.

There is, however, a more general potential corresponding to a single octet of eigenvalues in the case where both $\gamma_1 \neq 0$ and $\delta_1 \neq 0$. These potentials correspond to solutions of IDVNLS that we refer to as *composite* solitons. A single composite soliton can be regarded as "two fundamental soliton solution". A composite soliton has two complex carrier modulations and, in general, two peaks. In the special case where both $\gamma_1 \neq 0$ and $\delta_1 \neq 0$ but $\gamma_1 \cdot \delta_1 = 0$, one obtains a potential with a single peak, but two complex carrier frequencies. In a sense, a composite soliton is composed of two fundamental solitons - each corresponding to the same octet of eigenvalues, but with a distinct carrier frequency - both travelling at the same speed.

The time-dependence operator (8) determines the evolution of the eigenfunctions. From this we deduce the time evolution of the scattering data and, consequently, the evolution of a solution of IDMNLS.

For decaying potentials and in the gauge $\mathbf{A} = \mathbf{B} = \mathbf{I}$, the time evolution of the scattering data is given by

$$\mathbf{b}(z,\tau) = e^{i(z^2+z^{-2})\tau}\mathbf{b}(z,0) \qquad \mathbf{a}(z,\tau) = \mathbf{a}(z,0) \qquad (40)$$

$$\bar{\mathbf{a}}(z,\tau) = \bar{\mathbf{a}}(z,0) \qquad \bar{\mathbf{b}}(z,\tau) = e^{-i(z^2-z^{-2})\tau}\bar{\mathbf{b}}(z,0). \qquad (41)$$

hence from (27)-(28)

$$\mathbf{C}_j(\tau) = e^{i(z_j^2-z_j^{-2})\tau}\mathbf{C}_j(0), \qquad \bar{\mathbf{C}}_j(\tau) = e^{-i(\bar{z}_j^2-\bar{z}_j^{-2})\tau}\bar{\mathbf{C}}_j(0). \qquad (42)$$

Taking into account the time dependence, we obtain from (39)

$$\begin{pmatrix} Q_n^{(1)}(\tau) \\ Q_n^{(2)}(\tau) \end{pmatrix} = \frac{\gamma_1^*(0)}{\|\gamma_1(0)\|}\sinh(2a_1)e^{i\pi+2i(n+1)b_1-2i\omega\tau} \times \mathrm{sech}\,[2(n+1)a_1 - v\tau + d(0)] \qquad (43)$$

where $v = -\sinh(2a_1)\sin(2b_1)$, $\omega = \cosh(2a_1)\cos(2b_1) - 1$ and $d(0) = \log\sinh 2a_1 - \log\|\gamma_1(0)\|$.

The potential (43) is a discrete, vector version of the familiar sech profile with complex modulation that is characteristic of NLS solitons. Although a single soliton is, in effect, governed by the scalar IDNLS, the vector nature of the solitons (i.e. the associated polarization vector) manifests itself in the interaction of vector solitons with one another. In particular, when vector solitons interact, they shift one another's polarization vectors. We examine this issue in the following section.

3. Vector soliton interactions

Vector NLS solitons (both continuous and discrete) are distinguished from scalar NLS by the fact that, to fully characterize a vector soliton, one must specify a (complex) polarization vector. The polarization vector, like the location of the peak and unlike the amplitude and complex-modulation frequency, is a "phase" variable that is shifted in interaction with other solitons. We investigate the effects of multisoliton collision on the polarization by examining the individual solitons in the asymptotic limit in which they are well-separated.

For a generic multi-soliton solution (in which the solitons travel with different velocities), the solitons are well-separated in the long-time limits $\tau \to \pm\infty$. In each of these long-time limits, we can define a polarization for each soliton. Each of these polarizations is specified independently of the other parameters that characterize any of the solitons. In the generic case, when $\tau \to \pm\infty$, $\mathbf{q}_n^{\pm} \sim \sum_{j=1}^{J} \mathbf{q}_n^{(j),\pm}$ where $\mathbf{q}_n^{(j),\pm} = q_n^{(j),\pm}\mathbf{p}_j^{\pm}$ and $q_n^{(j),\pm}$ is a one soliton solution of IDNLS. We then identify $\mathbf{p}_j^{\pm}$ as the polarization of the j-th soliton before $(-)$ and after $(+)$ the soliton interaction. For the matrix system, we will write $\mathbf{Q}_n^{\pm} \sim \sum_{j=1}^{J} \mathbf{Q}_n^{(j),\pm}$ and for simplicity we restrict ourselves to the case of fundamental solitons when all the norming constants $\mathbf{C}_j$ have the form

$$\mathbf{C}_j(0) = \begin{pmatrix} \gamma_j^{(1)}(0) & 0 \\ \gamma_j^{(2)}(0) & 0 \end{pmatrix} \qquad j = 1, \ldots, J.$$

Let us fix the values of the soliton parameters for $\tau \to -\infty$, i.e. for each eigenvalue z_j we assign a matrix $\mathbf{S}_j^-$ that completely determines $\mathbf{Q}_n^{(j),-}$. For $\tau \to +\infty$ we denote the corresponding matrices by $\mathbf{S}_j^+$. Assume that the velocities of the individual solitons satisfy the condition $v_1 < v_2 < \ldots < v_J$. Then, as $\tau \to -\infty$ the solitons are distributed along the n-axis in the order corresponding to $v_J, v_{J-1}, \ldots, v_1$; the order of the soliton sequence is reversed as $\tau \to +\infty$. If we denote the soliton coordinates (i.e. the center of the soliton) at the instant of time τ by $n_j(\tau)$, when $\tau \to -\infty$ then $n_J \ll n_{J-1} \ll \ldots \ll n_1$.

The matrix-valued function $\phi_n(z_j)$ has the form

$$\phi_n(z_j) \sim z_j^n \begin{pmatrix} \mathbf{I}_N \\ 0 \end{pmatrix} \qquad n \ll n_J.$$

After passing through the J-th soliton it will be of the form

$$\phi_n(z_j) \sim z_j^n \begin{pmatrix} \mathbf{I}_N \\ 0 \end{pmatrix} \mathbf{a}_J(z_j, \mathbf{S}_J^-)$$

where $\mathbf{a}_J(z, \mathbf{S}_J^-)$ is the transmission coefficient relative to the J-th soliton and the additional argument is to take into account the dependence on the relative norming constant.

Repeating the argument we find that

$$\phi_n(z_j) \sim z_j^n \begin{pmatrix} \mathbf{I}_N \\ 0 \end{pmatrix} \mathbf{a}_{j+1}(z_j, \mathbf{S}_{j+1}^-)\mathbf{a}_{j+2}(z_j, \mathbf{S}_{j+2}^-) \ldots \mathbf{a}_J(z_j, \mathbf{S}_J^-)$$

$$\equiv z_j^n \begin{pmatrix} \mathbf{I}_N \\ 0 \end{pmatrix} \prod_{l=j+1,\ \mathrm{right}}^{J} \mathbf{a}_l(z_j, \mathbf{S}_l^-) \qquad n_{j+1} \ll n \ll n_j.$$

Since the j-th soliton corresponds to a bound state, passing through the j-th soliton yields

$$\phi_n(z_j) \sim z_j^{-n} \begin{pmatrix} 0 \\ \mathbf{I}_M \end{pmatrix} \mathbf{S}_j^- \prod_{l=j+1,\ \mathrm{right}}^{J} \mathbf{a}_l(z_j, \mathbf{S}_l^-) \qquad n_j \ll n \ll n_{j-1}.$$

On the other hand, starting from $n \gg n_1$ and proceeding in a similar way we find for the eigenfunction ψ_n the following asymptotic behavior

$$\psi_n(z_j) \sim z_j^{-n} \begin{pmatrix} 0 \\ \mathbf{I}_M \end{pmatrix} \mathbf{c}_{j-1}(z_j, \mathbf{S}_{j-1}^-)\mathbf{c}_{j-2}(z_j, \mathbf{S}_{j-2}^-) \ldots \mathbf{c}_1(z_j, \mathbf{S}_1^-)$$

$$\equiv z_j^{-n} \begin{pmatrix} 0 \\ \mathbf{I}_M \end{pmatrix} \prod_{l=1,\ \mathrm{left}}^{j-1} \mathbf{c}_l(z_j, \mathbf{S}_l^-) \qquad n_j \ll n \ll n_{j-1}$$

At an eigenvalue $\phi_n(z_j) = \psi_n(z_j)\mathbf{B}_j$ and therefore from the comparison we get

$$\mathbf{B}_j \sim \prod_{l=1,\ \mathrm{right}}^{j-1} \left(\mathbf{c}_l(z_j, \mathbf{S}_l^-)\right)^{-1} \mathbf{S}_j^- \prod_{l=j+1,\ \mathrm{right}}^{J} \mathbf{a}_l(z_j, \mathbf{S}_l^-) \qquad \tau \to -\infty.$$

Proceeding in a similar fashion as $\tau \to +\infty$ and taking into account that the order of solitons is reversed, we get

$$\mathbf{B}_j \sim \prod_{l=j+1,\ \mathrm{left}}^{J} \left(\mathbf{c}_l(z_j, \mathbf{S}_l^+)\right)^{-1} \mathbf{S}_j^+ \prod_{l=1,\ \mathrm{left}}^{j-1} \mathbf{a}_l(z_j, \mathbf{S}_l^+) \qquad \tau \to +\infty$$

and the comparison between the two representations for $\mathbf{B}_j$ yields for $j = J$

$$\mathbf{S}_J^+ = \prod_{l=1,\ \mathrm{right}}^{J-1} (\mathbf{c}_l(z_J, \mathbf{S}_l^-))^{-1}\ \mathbf{S}_J^- \prod_{l=1,\ \mathrm{right}}^{J-1} (\mathbf{a}_l(z_J, \mathbf{S}_l^+))^{-1} \tag{44}$$

and for $j = 1, \dots, J-1$

$$\mathbf{S}_j^+ = \prod_{l=j+1,\ \mathrm{right}}^{J} \mathbf{c}_l(z_j, \mathbf{S}_l^+) \prod_{l=1,\ \mathrm{right}}^{j-1} (\mathbf{c}_l(z_j, \mathbf{S}_l^-))^{-1}\ \mathbf{S}_j^- \times$$
$$\times \prod_{l=j+1,\ \mathrm{right}}^{J} \mathbf{a}_l(z_j, \mathbf{S}_l^-) \prod_{l=1,\ \mathrm{right}}^{j-1} (\mathbf{a}_l(z_j, \mathbf{S}_l^+))^{-1}. \tag{45}$$

Note that in the pure soliton case we can show (see [11])

$$\mathbf{a}_j(z) = \begin{pmatrix} \frac{z^2 - z_j^2}{z^2 - \bar{z}_j^2} & 0 \\ 0 & \frac{z^2 + \bar{z}_j^{-2}}{z^2 + z_j^{-2}} \end{pmatrix} \tag{46}$$

$$\mathbf{c}_j(z, \mathbf{p}_j) = \frac{\bar{z}_j^2 + z^{-2}}{z_j^2 + z^{-2}} \left[\mathbf{I}_M + \frac{(\bar{z}_j^2 - z_j^2)(\bar{z}_j^2 + z_j^{-2})}{(z^2 - \bar{z}_j^2)(\bar{z}_j^2 + z^{-2})} \mathbf{p}_j^* \mathbf{p}_j^T \right] \tag{47}$$

and since $\mathbf{a}_j$ actually do not depend on $\mathbf{S}_j$ formulae (44)-(45) are effective to determine $\mathbf{S}_j^+$ in terms of the $\mathbf{S}_l^-$.

Let us consider in detail the interaction of two solitons. In this case

$$\mathbf{S}_2^+ = \mathbf{c}_1(z_2, \mathbf{S}_1^-)^{-1}\ \mathbf{S}_2^-\ \mathbf{a}_1(z_2)^{-1} \tag{48}$$
$$\mathbf{S}_1^+ = \mathbf{c}_2(z_1, \mathbf{S}_2^+)\ \mathbf{S}_1^-\ \mathbf{a}_2(z_1) \tag{49}$$

and using the explicit expressions (46)-(47), one can solve (48) for $\mathbf{S}_2^+$ and then substitute it into the right-hand side of (49) in order to get $\mathbf{S}_1^+$. If we denote by $\mathbf{s}_j^\pm$ the vectors composing the first column of the matrices $\mathbf{S}_j^\pm$, the unit polarization vectors before and after the interaction are written, respectively, as

$$\mathbf{p}_j{}^\pm = \frac{(\mathbf{s}_j^\pm)^*}{\|\mathbf{s}_j^\pm\|} \tag{50}$$

$$\mathbf{s}_2^+ = \frac{z_2^2 - \bar{z}_1^2}{z_2^2 - z_1^2} \mathbf{c}_1(z_2, \mathbf{p}_1^-)^{-1} \mathbf{s}_2^-, \qquad \mathbf{s}_1^+ = \frac{z_1^2 - z_2^2}{z_1^2 - \bar{z}_2^2} \mathbf{c}_2(z_1, \mathbf{p}_2^+)\ \mathbf{s}_1^-. \tag{51}$$

Substituting (47) into (51), one can explicitly express $\mathbf{p}_j^+$ in (50) in terms of the parameters of the colliding solitons, i.e. the discrete eigenvalues and the incoming polarizations $\mathbf{p}_1^-, \mathbf{p}_2^-$, namely

$$\mathbf{p}_2^+ = \frac{1}{\chi} \frac{(z_1^2 - \bar{z}_2^2)(\bar{z}_1^2 + \bar{z}_2^{-2})}{(\bar{z}_1^2 - \bar{z}_2^2)(z_1^2 + \bar{z}_2^{-2})} \left[\mathbf{p}_2^- + \frac{(z_1^2 - \bar{z}_1^2)(\bar{z}_1^2 + z_1^{-2})}{(\bar{z}_1^2 - \bar{z}_2^2)(\bar{z}_1^2 + \bar{z}_2^{-2})} (\mathbf{p}_1^{-\,*} \cdot \mathbf{p}_2^-) \mathbf{p}_1^- \right] \tag{52}$$

$$\mathbf{p}_1^+ = \frac{1}{\chi} \frac{(z_1^2 - \bar{z}_2^2)(z_2^2 + z_1^{-2})}{(z_1^2 - z_2^2)(\bar{z}_2^2 + z_1^{-2})} \left[\mathbf{p}_1^- + \frac{(\bar{z}_2^2 - z_2^2)(z_2^2 + \bar{z}_2^{-2})}{(z_2^2 - z_1^2)(z_2^2 + z_1^{-2})} (\mathbf{p}_2^{-\,*} \cdot \mathbf{p}_1^-) \mathbf{p}_2^- \right] \tag{53}$$

where

$$\chi^2 = \left| \frac{(z_1^2 - \bar{z}_2^2)(\bar{z}_1^2 + \bar{z}_2^{-2})}{(\bar{z}_1^2 - \bar{z}_2^2)(z_1^2 + \bar{z}_2^{-2})} \right|^2 \tilde{\chi}^2 \tag{54}$$

$$\tilde{\chi}^2 = 1 + \frac{(z_1^2 - \bar{z}_1^2)(z_2^2 - \bar{z}_2^2)(\bar{z}_1^2 + z_1^{-2})(\bar{z}_2^{-2} + z_2^2)}{(z_2^2 - z_1^2)(\bar{z}_1^2 - \bar{z}_2^2)(z_2^2 + z_1^{-2})(\bar{z}_1^2 + \bar{z}_2^{-2})} |\mathbf{p}_1^{-\,*} \cdot \mathbf{p}_2^-|^2. \tag{55}$$

Without loss of generality, one can parameterize the incoming polarization vectors of the solitons as follows

$$\mathbf{p}_1^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{p}_2^- = \begin{pmatrix} \rho e^{i\varphi} \\ \sqrt{1 - \rho^2} \end{pmatrix} \tag{56}$$

with $0 \le \rho \le 1$ and $0 \le \varphi \le 2\pi$. Hence from (52)–(55) it follows that the polarization vectors after the interaction are such that

$$\left| p_1^{+\,(1)} \right|^2 = \frac{1}{\chi^2} \left| \frac{(z_1^2 - \bar{z}_2^2)(z_2^2 + z_1^{-2})}{(z_1^2 - z_2^2)(\bar{z}_2^2 + z_1^{-2})} \right|^2 \left| 1 + \frac{(\bar{z}_2^2 - z_2^2)(z_2^2 + \bar{z}_2^{-2})}{(z_2^2 - z_1^2)(z_2^2 + z_1^{-2})} \rho^2 \right|^2$$

$$\left| p_2^{+\,(1)} \right|^2 = \frac{1}{\chi^2} \bar{z}_1^4 z_1^{-4} \left| \frac{z_1^2 - \bar{z}_2^2}{\bar{z}_1^2 - \bar{z}_2^2} \right|^4 \rho^2$$

with

$$\chi^2 = \left| \frac{(z_1^2 - \bar{z}_2^2)(\bar{z}_1^2 + \bar{z}_2^{-2})}{(\bar{z}_1^2 - \bar{z}_2^2)(z_1^2 + \bar{z}_2^{-2})} \right|^2 \left[1 + \frac{(z_1^2 - \bar{z}_1^2)(z_2^2 - \bar{z}_2^2)(\bar{z}_1^2 + z_1^{-2})(\bar{z}_2^{-2} + z_2^2)}{(z_2^2 - z_1^2)(\bar{z}_1^2 - \bar{z}_2^2)(z_2^2 + z_1^{-2})(\bar{z}_1^2 + \bar{z}_2^{-2})} \rho^2 \right].$$

In Fig. 1 we plot both $|p_1^{+\,(1)}|^2$ and $|p_2^{+\,(1)}|^2$ as functions of ρ^2 for a specific choice of the eigenvalues. In the figure, the continuous curve represents the continuous limit, i.e. the limit with $h \to 0$, where $z_j = e^{-ik_j h}$ and $\bar{z}_j \equiv 1/z_j^* = e^{-ik_j^* h}$. In the continuous limit, the formulae (52) and (53) coincide with Manakov's formulae for the polarization shift in VNLS[3]. Note that, just as in the continuous case, the magnitudes of the components of the soliton polarizations are invariant only in the cases that the colliding solitons are either orthogonal or parallel.

Finally, we note that, from (44)-(45), one can show that the interaction of J solitons is equivalent to the net result of the $J(J-1)/2$ pairwise interactions of the solitons and is independent of the order in which these pairwise interactions take place [11].

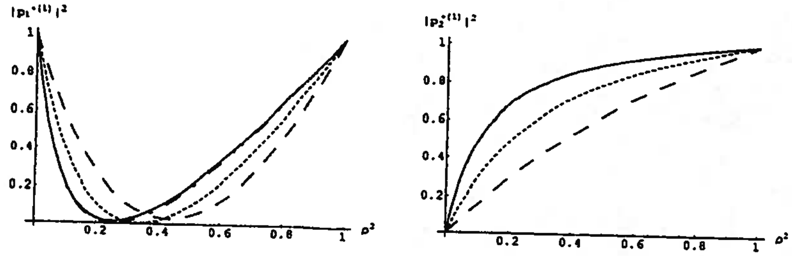

Figure 1. Intensity shift induced by two-soliton interaction of 2-component IDMNLS. The soliton parameters are: $z_j = \exp(a_j + ib_j)h$ with $a_1 = 1$, $a_2 = 2$, $b_1 = -b_2 = 0.1$ and $h = 1$ (dashed), $h = 0.5$ (dotted), $h = 0.1$ (dot-dashed). The solid curves represent the continuous limit, i.e. $h \to 0$. The initial polarizations are given by (56).

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SOLITON SOLUTIONS OF COUPLED NONLINEAR KLEIN-GORDON EQUATIONS

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The coupled nonlinear Klein-Gordon equations are analyzed for their integrability properties in a systematic manner through Painlevé(P-) test. From the P-test, one-soliton solutions are identified. The connection between the system equations and sine-Gordon equation has been pointed out from the bilinear form. Then, the results are generalized to the two-coupled Klein-Gordon equations.

1. Introduction

In recent decades, the study of integrability and soliton solutions of nonlinear ordinary and partial differential equations (ODEs, PDEs) and differential-difference equations (DDEs) has been the major field of research in dynamical systems and the soliton communities. Different techniques have been developed so far to determine whether or not PDEs belong to privileged class of completely integrable equations[1]. One of the systematic,powerful and widely used techniques is the Painlevé test, named after the famous French Mathematician Paul Painlevé (1863-1993). In essence, the Painlevé test verifies whether or not solutions of differential equations in the complex plane are single-valued in the neighborhood of all their movable singularities.

2. Coupled Nonlinear Klein-Gordon Equations

The nonlinear Klein-Gordon equation is one of the most important soliton systems which is attributed to so-called classical Φ^4 field theory in the physics of elementary particles and fields. Scott[6] analyzed a nonlinear Klein-Gordon system to determine the nonlinear pulse and periodic wave forms. Fordy and Gibbons[7] have shown integrable systems of nonlinear Klein-Gordon equations of which the usual sine and sinh-Gordon equations