DISCRETE SINE-GORDON EQUATION *

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The inverse scattering theory for the sine-Gordon equation discretized in space and both in space and time is considered.

1. Introduction

In the framework of integrable nonlinear evolution equations, Hirota recovered almost 25 years ago the Bianchi superposition formula for the solutions of the sine-Gordon equation, showing that it can be considered as an integrable doubly discrete (both in space and time) version of the sine-Gordon equation[1]. He found also its Lax pair, Bäcklund transformations and N-soliton solutions. In the following, this study was generalized to include the discrete (in space or time) case[2], soliton solutions were obtained by using the dressing method[3], the related inverse spectral transform was studied[4] and the nature of the numerical instabilities of the solutions was discussed[5,6].

Here we are interested in the spectral transform for the discrete and doubly discrete sine-Gordon equation with solutions decaying to 0 (modπ) at space infinity .

From one side, we re-examine the scattering theory of the spectral problem introduced by Orfanidis in[2] which depends on a discretized space variable n.

From the other side, we show that the equations of the matrix Orfanidis spectral problem can be decoupled and reduced to a scalar problem, giving

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as principal spectral problem the "exact discretization" of the Schrödinger operator introduced by Shabat[7]

$$\varphi_{n+2} = g_n \varphi_{n+1} - (1+\lambda)\varphi_n \tag{1}$$

and obtained iterating the Darboux transformations. We do not consider a Lax pair but a Lax triplet, i.e. we derive the discrete sine-Gordon equations as a compatibility condition of (1) with a pair of auxiliary spectral problems.

The spectral theory for the operator (1) was exhaustively studied in[9] and applied to a discrete version of the KdV. However, one must be advised that in this case the potential g_n is complex and the problem of characterizing the spectral data is left open.

By using this theory we introduce the spectral data, we find their time evolution and, therefore, according to the usual inverse scattering scheme, the Cauchy initial value problem is linearized. In particular, we show that the time evolution is discontinuous at the initial time $t = 0$.

The interested reader can find more details in[11] together with the full theory for the doubly discrete sine-Gordon.

2. Orfanidis Lax pair

Let us consider the Lax pair proposed by Orfanidis[2]

$$\chi_{n+1,m} = \begin{pmatrix} e^{-i(\theta_{n+1,m}-\theta_{n,m})/2} & ik \\ ik & e^{i(\theta_{n+1,m}-\theta_{n,m})/2} \end{pmatrix} \chi_{n,m} \tag{2}$$

$$\chi_{n,m+1} = \begin{pmatrix} 1 & \frac{\gamma}{ik} e^{-i(\theta_{n,m+1}+\theta_{n,m})/2} \\ \frac{\gamma}{ik} e^{i(\theta_{n,m+1}+\theta_{n,m})/2} & 1 \end{pmatrix} \chi_{n,m} \tag{3}$$

with $\theta_{n,m}$ real, k the spectral parameter and γ a real constant. The compatibility condition gives the Bianchi-Hirota equation

$$\sin\left(\frac{\theta_{n+1,m+1} - \theta_{n+1,m} - \theta_{n,m+1} + \theta_{n,m}}{4}\right) = \\ = \gamma \sin\left(\frac{\theta_{n+1,m+1} + \theta_{n+1,m} + \theta_{n,m+1} + \theta_{n,m}}{4}\right). \tag{4}$$

If we consider the compatibility condition of the Lax pair

$$\chi_{n+1} = \begin{pmatrix} e^{-i(\theta_{n+1}-\theta_n)/2} & ik \\ ik & e^{i(\theta_{n+1}-\theta_n)/2} \end{pmatrix} \chi_n \tag{5}$$

$$\partial_t \chi_n = \begin{pmatrix} 0 & \frac{\gamma}{ik} e^{-i\theta_n} \\ \frac{\gamma}{ik} e^{i\theta_n} & 0 \end{pmatrix} \chi_n \tag{6}$$

we get the semi-discrete sine-Gordon equation in the light cone coordinates

$$\partial_t \theta_{n+1} - \partial_t \theta_n = 4\gamma \sin\frac{1}{2}(\theta_{n+1} + \theta_n). \tag{7}$$

Notice also that in both cases, doubly discrete and space discrete cases, the principal spectral problem (2) is the same.

3. Direct scattering problem

Let us rewrite the spectral problem (2) as

$$\xi_{n+1}(1+ik\sigma_3) = (1+ik\sigma_1+Q_n)\xi_n \tag{8}$$

where the Pauli σ_i matrices are defined as usual and

$$\xi_n = \chi_n(1+ik\sigma_3)^{-n}, \qquad Q_n = e^{-\frac{i}{2}\sigma_3(\theta_{n+1}-\theta_n)} - 1. \tag{9}$$

In order to define the matrix Jost solutions μ_n, ν_n and to study their analytical properties with respect to the spectral parameter k it is convenient to consider separately the two columns of these matrices

$$\mu_n = (\mu_n^-, \mu_n^+), \quad \nu_n = (\nu_n^+, \nu_n^-). \tag{10}$$

The Jost solutions $\mu_n^\pm(k)$ are defined by the summation equations

$$\mu_n^+(k) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sum_{j=-\infty}^{n-1} \frac{(1+ik\sigma_1)^{n-j-1}}{(1-ik)^{n-j}} Q_j \mu_j^+(k) \tag{11}$$

$$\mu_n^-(k) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{j=-\infty}^{n-1} \frac{(1+ik\sigma_1)^{n-j-1}}{(1+ik)^{n-j}} Q_j \mu_j^-(k) \tag{12}$$

and $\nu^\pm$ by

$$\nu_n^+(k) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{1+ik} \sum_{j=n}^{+\infty} \frac{(1-ik\sigma_1)^{j-n+1}}{(1-ik)^{j-n+1}} Q_j \nu_j^+(k) \tag{13}$$

$$\nu_n^-(k) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{1}{1-ik} \sum_{j=n}^{+\infty} \frac{(1-ik\sigma_1)^{j-n+1}}{(1+ik)^{j-n+1}} Q_j \nu_j^-(k). \tag{14}$$

By using the standard method of the Neumann series one can prove that $\mu_n^\pm(k)$ are analytic, correspondingly, in the upper half plane and in the lower half plane of the spectral parameter k and continuous for $k_\Im \geq 0$ and $k_\Im \leq 0$ provided the potential θ_n is such that

$$\sum_{j=-\infty}^{+\infty} \left| e^{\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1 \right| < +\infty. \tag{15}$$

On the contrary, the summation equations defining the Jost solutions $\nu_n^\pm$ show explicitly that their kernel is singular at $k = \pm i$, i.e. in both the lower and upper half planes of k. This dissymmetry of the summation equations for the Jost solutions $\nu_n^\pm$ with respect to $\mu_n^\pm$ is peculiar of the discrete case, since it disappears in the continuous limit. In fact, in order to study the analytic properties of the Jost solutions one needs to define the same solutions $\nu_n^\pm(k)$ by introducing the alternative summation equations

$$\nu_n^+(k) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{j=n+1}^{+\infty} \frac{(1-ik\sigma_1)^{j-n-1}}{(1-ik)^{j-n}} \sigma_1 Q_{j-1} \sigma_1 \nu_j^+(k) \tag{16}$$

$$\nu_n^-(k) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sum_{j=n+1}^{+\infty} \frac{(1-ik\sigma_1)^{j-n-1}}{(1+ik)^{j-n}} \sigma_1 Q_{j-1} \sigma_1 \nu_j^-(k). \tag{17}$$

which can be obtained by exploiting the special structure of the potential Q_n and the symmetry property of the spectral problem (8) for $n \to -n$.

Following a procedure analogous to the one used for $\mu_n^\pm(k)$, one can prove that $\nu_n^\pm(k)$ are also analytic, correspondingly, in the upper half plane and in the lower half plane of the spectral parameter k and continuous for $k_\Im \geq 0$ and $k_\Im \leq 0$ provided the potential satisfies the condition (15).

Let us now define the Wronskian of any two vectors v and w as

$$W(v, w) = \det(v, w). \tag{18}$$

Two eigenvectors v_n and w_n of (5) are linearly independent if $W(v_n, w_n) \neq 0$ for all n. One can check that the eigenfunctions $\mu_n^+(1+ik\sigma_3)^n$ and $\mu_n^-(1+ik\sigma_3)^n$, as well as $\nu_n^+(1+ik\sigma_3)^n$, are independent and $\nu_n^-(1+ik\sigma_3)^n$ and, therefore, we can write

$$\frac{\mu_n^\pm(k)}{a^\pm(k)} = \frac{b^\pm(k)}{a^\pm(k)} \left(\frac{1 \pm ik}{1 \mp ik}\right)^n \nu_n^\pm(k) + \nu_n^\mp(k) \tag{19}$$

where we introduced the spectral data $a^\pm(k)$ and $b^\pm(k)$.

The scattering coefficients can also be given as explicit sum of the Jost solutions as follows

$$a^+(k) = 1 + \frac{1}{2(1-ik)} \sum_{j=-\infty}^{+\infty} \left[-\left(e^{-\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) \mu_j^{+,(1)}(k) + \left(e^{\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) \mu_j^{+,(2)}(k) \right] \tag{20}$$

$$b^+(k) = \frac{1}{2(1-ik)} \sum_{j=-\infty}^{+\infty} \left(\frac{1+ik}{1-ik}\right)^{-j-1} \left[\left(e^{-\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) \mu_j^{+,(1)}(k) + \left(e^{\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) \mu_j^{+,(2)}(k) \right]. \tag{21}$$

We have

$$a^\pm(k) = \mp\frac{1}{2} W\left(\mu_n^\pm(k), \nu_n^\pm(k)\right) \tag{22}$$

which proves that $a^+(k)$ can be analytically continued in the upper half plane (respectively $a^-(k)$ can be analytically continued in the lower half plane).

The discrete scattering problem (8) can possess discrete eigenvalues (bound states). These occur whenever $a^+(k_j^+) = 0$ for some k_j^+ in the upper half plane or $a^-(k_\ell^-) = 0$ for some k_ℓ^- in the lower half plane. Indeed, for such values of the spectral parameter from (22) it follows that, since $W\left(\mu_n^+(k_j^+), \nu_n^+(k_j^+)\right) = 0$ and $W\left(\mu_n^-(k_\ell^-), \nu_n^-(k_\ell^-)\right) = 0$, the eigenfunctions are linearly dependent and, therefore, have a decaying behaviour at both $n = \pm\infty$.

4. Inverse Scattering problem

Taking into account the analytic properties of the Jost solutions and of the scattering coefficients $a^\pm(k)$, as well as their asymptotic behavior at large k, we can use the Cauchy-Green formula to obtain from the "jump" conditions (19)

$$\nu_n^-(k) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sum_{j=1}^{J^+} \left(\frac{1+ik_j^+}{1-ik_j^+}\right)^n \frac{C_j^+}{k-k_j^+} \nu_n^+(k_j^+) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho^+(\zeta)}{\zeta - k + i0}\left(\frac{1+i\zeta}{1-i\zeta}\right)^n \nu_n^+(\zeta) d\zeta \tag{23}$$

$$\nu_n^+(k) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{j=1}^{J^-} \left(\frac{1-ik_j^-}{1+ik_j^-}\right)^n \frac{C_j^-}{k-k_j^-} \nu_n^-(k_j^-) - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho^-(\zeta)}{\zeta - k - i0}\left(\frac{1-i\zeta}{1+i\zeta}\right)^n \nu_n^-(\zeta) d\zeta. \tag{24}$$

where the $C_j^\pm$ are the so called normalization coefficients.

Equations (23) and (24) define a Riemann-Hilbert problem on the line which, in principle, allows one to solve the inverse problem, i.e. to reconstruct the Jost solutions from the scattering data.

In order to complete the inverse problem, we need to reconstruct the potential from the scattering data. We have

$$Q_n = \frac{1}{2}\left[\nu_{n+1}^{(-1)}\sigma_3 - \sigma_1 \nu_n^{(-1)}\right](1 + i\sigma_2) \tag{25}$$

where $\nu_n^{(-1)}$ is obtained from the asymptotic expansion at large k of $\nu_n = (\nu_n^+, \nu_n^-)$

$$\nu_n(k) = \nu_n^{(0)} + \frac{\nu_n^{(-1)}}{ik} + \ldots \tag{26}$$

The time evolution of the Jost solutions and, consequently, of the scattering data is fixed by (6).

If $\theta_n \to r\pi$ $(r \in \mathbb{N})$ we get $(k = k_{\mathbb{R}})$

$$b^\pm(k,t) = b^\pm(k,0)e^{\mp 2i\frac{\gamma}{k}t}, \qquad a^\pm(k,t) = a^\pm(k,0). \tag{27}$$

For the discrete spectral data one gets that the locations of poles at $k = k_j^\pm$ are fixed while

$$C_j^\pm(t) = C_j^\pm(0)e^{\mp 2i\frac{\gamma}{k_j^\pm}t}. \tag{28}$$

Notice that the switching on of the time evolution introduces a singularity at $k = 0$. One can prove that this does not spoil the good properties of the spectral transform.

5. Lax triplet

We can write the spectral problem (8) in component-form and decouple the two equations to get

$$\chi_{n+2}^{(1)} = g_n \chi_{n+1}^{(1)} - (1+k^2)\chi_n^{(1)} \tag{29}$$

$$\chi_n^{(2)} = \frac{1}{ik}\left[\chi_{n+1}^{(1)} - e^{-\frac{i}{2}(\theta_{n+1}-\theta_n)}\chi_n^{(1)}\right] \tag{30}$$

with

$$g_n = e^{\frac{i}{2}(\theta_{n+1}-\theta_n)} + e^{-\frac{i}{2}(\theta_{n+2}-\theta_{n+1})} \tag{31}$$

i.e. we recover the discrete Schrödinger equation whose scattering theory was given in[9]. In this case the time evolution is fixed by the pair of spectral problems

$$k^2\partial_t\chi_n^{(1)} = -\gamma e^{-i\theta_n}\chi_{n+1}^{(1)} + \gamma e^{-\frac{i}{2}(\theta_{n+1}+\theta_n)}\chi_n^{(1)} \tag{32}$$

$$\partial_t\chi_{n+1}^{(1)} = e^{-\frac{i}{2}(\theta_{n+1}-\theta_n)}\partial_t\chi_n^{(1)} + \gamma e^{-i\theta_{n+1}}\chi_n^{(1)}. \tag{33}$$

The compatibility requirement for the three spectral problems (29), (32) and (33) furnishes, after one integration, the semi-discrete sine-Gordon equation (7).

Note also that for θ_n vanishing at large n (modπ) we have from (31) that $g_n - 2 \to 0$ and the spectral theory developed in [9] is applicable. However, the condition required to be satisfied by $g_n - 2$ in [9] is stricter than (15).

We conclude that the Orfanidis matrix spectral problem can be reduced to a scalar Schrödinger spectral problem if the potential $\theta_n \to 0$ (modπ) sufficiently rapidly.

The evolution of the spectral data can be determined by considering the first and the third Lax operators, i.e. the spectral problem (29) and (32).

We introduce the spectral data $a^{+}(k)$ and $b^{+}(k)$ as defined in [9] and we get the following time evolution

$$a(t;k) \qquad = a(0;k) \tag{34}$$

$$b(t;k) = b(0;k)\exp\left[\frac{2i\eta\gamma}{k}t\right]. \tag{35}$$

References

1. R. Hirota, *J. Phys. Soc. of Japan* **43**, 2079 (1977)
2. S. J. Orfanidis, *Phys. Review D* **18**, 3822 and 3828 (1978)
3. D. Levi, O. Ragnisco and M. Bruschi, *Nuovo Cimento* **58A**, 56 (1980)
4. L. Pilloni, D. Levi, *Phys. Lett.* **92A**, 5 (1982)
5. M. J. Ablowitz, B. M. Herbst and C. Schober, *J. of Comput. Phys.* **126**, 299 (1996)
6. M. J. Ablowitz, B. M. Herbst and C. Schober, *J. of Comput. Phys.* **131**, 354 (1997)
7. A. Shabat, in *Nonlinearity, Integrability and All That. Twenty Years After NEEDS '79*, M. Boiti, L. Martina, F. Pempinelli, B. Prinari and G. Soliani eds, pp.331-342, World Scientific Pu. Co., Singapore (2000)
8. M. Boiti, J. Léon and F. Pempinelli, *Inverse Problems* **3**, 37 (1987)
9. M. Boiti, F. Pempinelli, B. Prinari and A. Spire, *Inverse Problems* **17**, 515 (2001)
10. H. S. Wall, *"Analytic Theory of Continued Fractions"* (New York: Chelsea, 1967)
11. M.Boiti, F. Pempinelli, P. Prinari and A. Spire, *Inverse Problems* **18** (2002) 1309

INTEGRABLE AND NON-INTEGRABLE EQUATIONS WITH PEAKONS

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We consider a one-parameter family of non-evolutionary partial differential equations which includes the integrable Camassa-Holm equation and a new integrable equation first isolated by Degasperis and Procesi. A Lagrangian and Hamiltonian formulation is presented for the whole family of equations, and we discuss how this fits into a bi-Hamiltonian framework in the integrable cases. The Hamiltonian dynamics of peakons and some other special finite-dimensional reductions are also described.

1. Introduction

In this note we consider the following one-parameter family of partial differential equations (PDEs):

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx} \tag{1.1}$$

(the parameter b is constant). In the particular case $b = 2$ this becomes the dispersionless version of the integrable Camassa-Holm equation, which is

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \tag{1.2}$$