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Inverse scattering transform for the perturbed 1-soliton potential of the heat equation ☆

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Abstract

Inverse scattering theory of the heat equation with potential being a perturbation of the one-soliton potential by means of a decaying two-dimensional function is presented. Modification of the Jost solutions and scattering data are given. © 2001 Elsevier Science B.V. All rights reserved.

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The operator

$$L=-\partial_{x_2}+\partial_{x_1}^2-u(x),\quad x=(x_1,x_2), \tag{1}$$

which defines the well-known equation

$$L\Phi(x,k)=0,\quad k\in\mathbb{C}, \tag{2}$$

of heat conduction, or heat equation for short, is known [1,2] to be associated to the Kadomtsev–Petviashvili equation

$$(u_t-6uu_{x_1}+u_{x_1x_1x_1})_{x_1}=-3u_{x_2x_2} \tag{3}$$

(more exactly, KPII equation). The inverse scattering transform for the heat equation with a real potential $u(x)$ was developed in [3–5], but only the case of potentials rapidly decaying at large distance on the x-plane was considered. On the other side, it is well known that KPII is a $(2+1)$-dimensional generalization of the famous KdV equation. Indeed, if the function $u_1(t,x_1)$ obeys KdV then, for an arbitrary constant $\mu\in\mathbb{R}$,

$$u(t,x_1,x_2)=u_1\bigl(t,x_1+\mu x_2-3\mu^2t\bigr) \tag{4}$$

solves KPII. Thus, it is natural to consider solutions of KPII that are not decaying in all directions at space infinity but have 1-dimensional rays with behaviour of type (4). A complete inverse scattering theory of the heat equation for such potentials is absent in the literature. Some preliminary results towards the formulation of this theory were obtained in [6,7], where special potentials generated by Bäcklund transformations were considered, and in [8], where, by using the so-called resolvent approach, the way was opened to the study of more generic potentials.

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Here, we consider the case of only one ray and we choose $\mu = 0$, since the generic direction of the ray can be reconstructed by the Galileo invariance of KPII. Thus in what follows $u_1(x) \equiv u_1(x_1)$ and, moreover, we consider for simplicity the case where u_1 is the 1-dimensional soliton potential

$$u_1(x) = \frac{-2a^2}{\cosh^2[a(x_1 - x_0)]}, \tag{5}$$

with $a > 0$ and x_0 real constants.

In this Letter we present results for the inverse scattering theory of the heat equation with a real potential $u(x)$ that is a perturbation of the 1-dimensional soliton potential $u_1(x)$ by means of a smooth potential $u_2(x)$ rapidly decaying in all directions,

$$u(x) = u_1(x) + u_2(x). \tag{6}$$

We introduce and describe the properties of the Jost solutions and scattering data and formulate the inverse problem relevant to this case. In what follows the special case of operator (1) with the 1-dimensional potential $u_1(x)$ is denoted as

$$L_1 = -\partial_{x_2} + \partial_{x_1}^2 - u_1(x). \tag{7}$$

First, let us consider the case where the perturbation $u_2 \equiv 0$ and let us re-formulate in the 2-dimensional space well known facts about Jost solutions of the 1-soliton potential (5). The Jost solutions of the differential equations $L_1\Phi_1(x,k) = 0$, $L_1^{\mathrm{d}}\Psi_1(x,k) = 0$, where L_1^{d} is the dual operator, are given explicitly by

$$\Phi_1(x;k) = \frac{k - ia\tanh[a(x_1 - x_0)]}{k - ia}\, e^{-i\ell(k)x}, \tag{8}$$

$$\Psi_1(x;k) = \frac{k + ia\tanh[a(x_1 - x_0)]}{k + ia}\, e^{i\ell(k)x}, \tag{9}$$

where $k \in \mathbb{C}$ and we introduced the two-component vector

$$\ell(k) = \bigl(k, -ik^2\bigr) \tag{10}$$

for simplicity. They obey the conjugation properties

$$\overline{\Phi_1(x,k)} = \Phi_1\bigl(x, -\bar{k}\bigr), \qquad \overline{\Psi_1(x,k)} = \Psi_1\bigl(x, -\bar{k}\bigr), \tag{11}$$

that are equivalent to the reality condition for the potential, and are normalized at k-infinity as follows:

$$\lim_{k\to\infty} e^{i\ell(k)x}\Phi_1(x;k) = 1,$$
$$\lim_{k\to\infty} e^{-i\ell(k)x}\Psi_1(x;k) = 1. \tag{12}$$

They are meromorphic functions in the complex domain of the spectral parameter k with poles at $k = ia$ and $k = -ia$, correspondingly. Thus, they obey the d-bar equations

$$\frac{\partial\Phi_1(x,k)}{\partial\bar{k}} = i\pi\Phi_{1,a}(x)\delta(k - ia),$$
$$\frac{\partial\Psi_1(x,k)}{\partial\bar{k}} = i\pi\Psi_{1,-a}(x)\delta(k + ia), \tag{13}$$

where we introduced the standard notations

$$\Phi_{1,a}(x) = -i \operatorname*{res}_{k=ia} \Phi_1(x,k),$$
$$\Psi_{1,-a}(x) = -i \operatorname*{res}_{k=-ia} \Psi_1(x,k). \tag{14}$$

Explicitly we have

$$\Phi_{1,a}(x) = \frac{ae^{ax_0 + a^2x_2}}{\cosh[a(x_1 - x_0)]},$$
$$\Psi_{1,-a}(x) = -\frac{ae^{ax_0 - a^2x_2}}{\cosh[a(x_1 - x_0)]}. \tag{15}$$

The formulation of the d-bar problem (13) is closed by means of the relations

$$\Phi_{1,a}(x) = c\Phi_{1,-a}(x), \qquad \Psi_{1,-a}(x) = -c\Psi_{1,a}(x), \tag{16}$$

where $c = 2ae^{2ax_0} > 0$ and $\Phi_{1,-a}$ and $\Psi_{1,a}(x)$ are values of the Jost solutions in the conjugated points,

$$\Phi_{1,-a}(x) = \Phi_1(x, -ia), \qquad \Psi_{1,a}(x) = \Psi_1(x, ia). \tag{17}$$

These formulae show that the embedding in two dimensions of the Jost solutions of the 1-soliton potential is trivial. On the contrary, the corresponding 2-dimensional Green's function G_1 of the operator L_1 is not a trivial extension of the 1-dimensional Green's function of the operator $k^2 + \partial_{x_1}^2 - u_1(x_1)$ and, in particular, it has more complicated singularities in k.

This Green's function G_1 can be obtained as a special case from the generic construction presented in [8] and can be expressed in terms of the standard objects introduced above as follows:

$$G_1(x,x',k) = \frac{1}{2\pi} \int\limits_{k'_{\Im} = k_{\Im}} dk' \bigl[\theta\bigl(|k_{\Re}| - |k'_{\Re}|\bigr) - \theta(x_2 - x'_2)\bigr]$$

$$\times \Phi_1(x,k')\Psi_1(x',k') + c\theta\big(a - |k_\Im|\big) \times \theta(x_2' - x_2)\Phi_{1,-a}(x)\Psi_{1,a}(x'). \tag{18}$$

One can check directly by means of the explicit formulae (8) and (9) that it obeys the differential equation $L_1 G_1(x,x',k) = \delta(x-x')$, and satisfies the following properties:

$$\overline{G_1(x,x',k)} \equiv G_1\big(x,x',-\bar{k}\big) \equiv G_1(x,x',k), \tag{19}$$

$$\lim_{k\to\infty}(-2ik)\frac{\partial}{\partial x_1}e^{i\ell(k)(x-x')}G_1(x,x',k) = \delta(x-x'), \tag{20}$$

$$\frac{\partial G_1(x,x',k)}{\partial \bar{k}} = \frac{\operatorname{sgn} k_\Re}{2\pi}\,\Phi_1\big(x,-\bar{k}\big)\Psi_1\big(x',-\bar{k}\big), \tag{21}$$

which are analogous to those satisfied by the Green's function of a 2-dimensional rapidly decaying potential. However, at the points $k = \pm ia$ (and only at these points) this Green's function is discontinuous. Precisely, we have from (18) in the neighborhood of these points:

$$G_1(x,x',k) = G_{1,\mathrm{reg}}(x,x') - \frac{c}{\pi}\left\{\operatorname{arccot}\frac{a-|k_\Im|}{|k_\Re|}\right\}\Phi_{1,-a}(x)\Psi_{1,a}(x'), \quad k \simeq \pm ia. \tag{22}$$

Since the second term in the right-hand side is annihilated by L_1 and its dual, $G_{1,\mathrm{reg}}(x,x')$ is also a Green's function. Notice that it is k independent and the same for both points. This property will be important in the following.

Now we are ready to construct the Jost solution of (2). By means of (6) and (7) we write L in (1) as $L = L_1 - u_2(x)$ and define the Jost solution $\Phi(x,k)$ (in fact, its generalization) as solution of the integral equation

$$\Phi(x,k) = \Phi_1(x,k) + \int dx'\,G_1(x,x',k)u_2(x')\Phi(x',k), \tag{23}$$

where G_1 is given in (18). In other words, we use the so called scattering theory on a non-trivial background [9]. In what follows we assume unique solvability of this equation for all $k \neq \pm ia$. There is an evidence that it can be proved by standard techniques under some small norm assumption on the perturbation u_2 [4,5]. Under this assumption, the properties of the Jost solution are prescribed by the properties of the Green's function G_1. The function $\Phi(x,k)$ is no more meromorphic, but its behaviour at k-infinity and conjugation property are the standard ones,

$$\lim_{k\to\infty} e^{i\ell(k)x}\Phi(x,k) = 1, \tag{24}$$

$$\lim_{k\to\infty} 2ik\frac{\partial}{\partial x_1}e^{i\ell(k)x}\Phi(x,k) = -u(x), \tag{25}$$

$$\overline{\Phi(x,k)} = \Phi\big(x,-\bar{k}\big). \tag{26}$$

Thanks to (23) they follow from the analogous properties of Φ_1 and from (6), (19) and (20).

Since $G_1(x,x',k)$ is discontinuous at the points $k = \pm ia$, the behaviour of the Jost solution in a neighborhood of these points must be studied in detail. To describe it we introduce the k-independent solution $\Phi_{\mathrm{reg}}(x)$ of (2) determined by the integral equation

$$\Phi_{\mathrm{reg}}(x) = \Phi_{1,-a}(x) + \int dx'\,G_{1,\mathrm{reg}}(x,x')u_2(x')\Phi_{\mathrm{reg}}(x'), \tag{27}$$

where we used the fact that $G_{1,\mathrm{reg}}$ is also a Green's function of the operator L_1. Let us now introduce the function

$$A(k) = \pi + \lambda\operatorname{arccot}\frac{a-|k_\Im|}{|k_\Re|}, \tag{28}$$

where λ denotes the constant

$$\lambda = c\int dx\,\Psi_{1,a}(x)u_2(x)\Phi_{\mathrm{reg}}(x). \tag{29}$$

Under our assumption of rapid decay of u_2 this constant exists (the integral is convergent), is real and obeys the inequality

$$1 + \lambda > 0, \tag{30}$$

so that the function $A(k)$ is real and positive. Then we can prove that

$$\Phi(x,k) = \frac{i\pi c\Phi_{\mathrm{reg}}(x)}{(k-ia)A(k)} + O(1), \quad k \simeq ia, \tag{31}$$

$$\Phi(x,k) = \frac{\pi\Phi_{\mathrm{reg}}(x)}{A(k)} + o(1), \quad k \simeq -ia, \tag{32}$$

that is, since for $\lambda \neq 0$ the function $A(k)$ is discontinuous at the points $k = \pm ia$ (and only at these points), the Jost solution $\Phi(x,k)$ results to have a more singular behaviour than the Jost solution $\Phi_1(x,k)$ of the unperturbed one soliton solution. Correspondingly, the properties of the d-bar derivative of the Jost solution as well as the properties of the scattering data are affected by this singular behaviour.

Let first $k \neq \pm ia$. Then as in the standard situation we get

$$\frac{\partial \Phi(x,k)}{\partial \bar{k}} = \Phi\bigl(x,-\bar{k}\bigr)r(k), \tag{33}$$

where

$$r(k) = \frac{\operatorname{sgn} k_{\Re}}{2\pi} \int dx\, \Psi_1\bigl(x,-\bar{k}\bigr)u_2(x)\Phi(x,k). \tag{34}$$

Thanks to (11) and (26) this scattering datum obeys the standard conjugation property

$$\overline{r(k)} = -r(k), \tag{35}$$

and by (31) and (32) it behaves at points $k \simeq \pm ia$ as

$$r(k) = \frac{i\lambda \operatorname{sgn} k_{\Re}}{2(k_{\Re} + i|k_{\Im}| - ia)A(k)} + O(1), \tag{36}$$

i.e., in both points it has a pole singularity multiplied by the discontinuous function $A(k)$. Taking into account that the singular behaviour of $\Phi(x,-\bar{k})$ is given by the denominator $\overline{(k-ia)}A(k)$ at point $k = ia$ and by $A(k)$ at point $k = -ia$ we see that the r.h.s. of (33) is integrable at the latter point but it has a singularity of the form $\operatorname{sgn} k_{\Re}|k - ia|^{-2}A(k)^{-2}$ at point $k = ia$, which is not integrable. Thus we have to give a meaning as a distribution to the r.h.s. of (33). Let $f(k)$ be a test function. Then using the fact that $\Phi(x,k)$ is integrable for any k we define its $\bar{\partial}$ derivative in the sense of distributions in the usual way

$$\int d^2k\, \frac{\partial \Phi(x,k)}{\partial \bar{k}} f(k) = -\int d^2k\, \Phi(x,k) \frac{\partial f(k)}{\partial \bar{k}}.$$

Since the distribution $\partial \Phi(x,k)/\partial \bar{k}$ is singular only at the point $k = ia$ we can write

$$\int d^2k\, \frac{\partial \Phi(x,k)}{\partial \bar{k}} f(k) = -\lim_{\varepsilon \to 0} \int_{|k-ia|>\varepsilon} d^2k\, \Phi(x,k) \frac{\partial f(k)}{\partial \bar{k}},$$

and then use the integral in the r.h.s. for deriving an explicit expression for the distribution $\partial \Phi(x,k)/\partial \bar{k}$. We have

$$\frac{\partial \Phi(x,k)}{\partial \bar{k}} = \Phi\bigl(x,-\bar{k}\bigr)r(k) + i\pi \Phi_a(x)\delta(k - ia), \tag{37}$$

where

$$\Phi_a(x) = \frac{-1}{2\pi} \lim_{\varepsilon \to 0} \oint_{|k-ia|=\varepsilon} dk\, \Phi(x,k), \tag{38}$$

and $\Phi(x,-\bar{k})r(k)$ is now a distribution in k defined by the following functional:

$$\text{p.v.} \int d^2k\, \Phi\bigl(x,-\bar{k}\bigr)r(k)f(k) = \lim_{\varepsilon \to 0} \int_{|k-ia|>\varepsilon} d^2k\, \Phi\bigl(x,-\bar{k}\bigr)r(k)f(k), \tag{39}$$

where we used the notation p.v. since the indicated regularization generalizes in a natural way the Cauchy principal value regularization procedure to the case of a singularity in the plane. Note that $i\Phi_a(x)$ in (38) can be considered as an extension of the definition of residuum to the case in which the pole singularity is multiplied by a discontinuous function at the same point. Following (38) we can define the "value" of $\Phi(x,k)$ at the point $k = -ia$ as

$$\Phi_{-a}(x) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \oint_{|k+ia|=\varepsilon} \frac{dk}{k+ia} \Phi(x,k). \tag{40}$$

From (31) and (32) we deduce that

$$\Phi_a(x) = c\frac{\log(1+\lambda)}{\lambda} \Phi_{\text{reg}}(x), \tag{41}$$

$$\Phi_{-a}(x) = \frac{\log(1+\lambda)}{\lambda} \Phi_{\text{reg}}(x), \tag{42}$$

and then

$$\Phi_a(x) = c\Phi_{-a}(x). \tag{43}$$

This last equality close the formulation of the inverse problem for (37) and demonstrates that the parameter c is not modified by the perturbation. Note also that in contrast with the 1-dimensional case continuous and discrete scattering data cannot be given independently since the singularity of the continuous scattering data, which in the generic case is necessarily present, depends on the parameters of the discrete scattering data.

Finally, taking into account the asymptotic behaviour (24), the definition of the principal value in (39) and definition (40) we can write the following system of integral equations:

$$\begin{aligned}\Phi(x,k) &= e^{-i\ell(k)x}\\ &\quad+\frac{1}{\pi}\,\text{p.v.}\int\frac{d^2k'}{k-k'}\,e^{i(\ell(k')-\ell(k))x}\Phi\big(x,-\bar{k}'\big)r(k')\\ &\quad+i\frac{e^{i(\ell(ia)-\ell(k))x}}{k-ia}\,\Phi_a(x),\end{aligned}\tag{44}$$

$$\begin{aligned}\frac{1}{c}\Phi_a(x) &= e^{-i\ell(-ia)x}\\ &\quad-\frac{1}{\pi}\,\text{p.v.}\int d^2k\,\frac{\Phi(x,-\bar{k})r(k)}{k+ia}e^{i(\ell(k)-\ell(-ia))x}\\ &\quad-\frac{e^{i(\ell(ia)-\ell(-ia))x}}{2a}\,\Phi_a(x),\end{aligned}\tag{45}$$

which solves the inverse problem for (37) and (43). The integrands in the r.h.s. of these two equations are not locally integrable, respectively, the first at $k=ia$ and the second at $k=\pm ia$. Correspondingly, their integrals are regularized by means of the principal value prescription, as defined in (39), at $k=ia$ and at $k=\pm ia$.

Under assumption of unique solvability one can derive from these equations and property (35) the conjugation property (26) of the Jost solution and one can also check that this system of equations is compatible with the singularities described in (31), (32), and (36). Notice also that only apparently the second equation can be obtained evaluating the first one at $k=-ia$ and, then, using (43). In fact, the first equation, due to (36), is singular at $k=-ia$ and one needs (40) in order to define coherently the "value" of $\Phi(x,k)$ at $k=-ia$.

The potential is reconstructed by means of (25) that thanks to (44) gives us

$$\begin{aligned}u(x) &= -\frac{2i}{\pi}\,\text{p.v.}\int d^2k\,\partial_{x_1}\big(e^{i\ell(k)x}\Phi\big(x,-\bar{k}\big)r(k)\big)\\ &\qquad+2\partial_{x_1}\big(e^{i\ell(ia)x}\Phi_a(x)\big).\end{aligned}\tag{46}$$

Properties (26) and (35) guarantee that this potential is real. Its properties under KPII time evolution must be specially investigated. Here we mention only that the singular behaviour (36) is preserved under such evolution since all three parameters a, c, and λ are constants of motion.

Conclusion

In this Letter we gave a short presentation of an extension of the inverse scattering theory for the heat operator to the case where the potential is a smooth (decaying at infinity) perturbation of the 1-dimensional one soliton potential. This is the first (to our knowledge) example of a inverse scattering theory including potentials with 1-dimensional rays. We showed that under such a perturbation the Jost solutions get singularities more complicated than poles. The study of these singularities enabled us to introduce the modified scattering data and describe their properties. Detailed proofs of the results presented here are based on the so-called resolvent approach (see [6,7,9,10] and, in particular, the case of one soliton on a background in [8]) and will be given in a forthcoming publication.

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