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Inverse scattering transform for 3-level coupled Maxwell–Bloch equations with inhomogeneous broadening

S. Chakravarty [a,*], B. Prinari [a,b], M.J. Ablowitz [c]

[a] Department of Mathematics, University of Colorado Colorado Springs, CO 80918, USA

[b] Dip di Matematica e Fisica "Ennio de Giorgi" and Sezione INFN, Univ del Salento, Italy

[c] Department of Applied Mathematics, University of Colorado, Boulder, CO 80309-0526, USA

HIGHLIGHTS

- Pulse propagation in an optical medium with 3-level atomic transitions is studied.
- The study is performed via inverse scattering transform on the associated Lax pair.
- Soliton interactions and polarization shifts for the optical pulses are investigated.
- Expression for the *n*-soliton scattering matrix is derived.

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ABSTRACT

In this paper we study the propagation of optical pulses in an optical medium with coherent three-level atomic transitions. The interaction between the pulses and the medium is described by the coupled Maxwell–Bloch equations, which we investigate by applying the method of inverse scattering transform. The details of the inverse scattering method and the non-trivial evolution of the associated scattering data are discussed. The one- and two-soliton solutions, polarization shifts due to two-soliton interactions, and the explicit form of the transmission matrix associated with pure soliton solutions are also derived.

1. Introduction

The nonlinear interaction between radiation and a multilevel optical medium has received considerable attention over the past decades. The phenomenon that describes the effect of a coherent medium response to an incident electric field, to which the medium is totally transparent and which undergoes lossless propagation, is known as self-induced transparency (SIT). SIT was first discovered by McCall and Hahn [1] in the case of resonant optical media undergoing a pure two-level atomic transition. A large variety of special solutions as well as an infinite number of conservation laws associated with the Maxwell–Bloch equations governing the SIT phenomenon in a two-level medium were found by Lamb [2,3] and others [4,5]. The general initial value problem for the propagation of a pulse through a resonant two-level optical medium for the SIT case was solved by applying the Inverse Scattering Transform (IST) in [6]. In [7–9], the IST was employed to solve the Maxwell–Bloch equations in a more general setting of two-level unstable optical media to study the superfluorescence phenomenon and related problems in laser optics. The complete integrability of the reduced Maxwell–Bloch equations in the presence of a permanent dipole was investigated in [10], and also in [11] where the effect of inhomogeneous broadening was taken into account. In these papers, the authors obtained the Lax pairs, a hierarchy of commuting flows, and Bäcklund transformations for the reduced Maxwell–Bloch equations.

It is also possible to formulate the propagation of optical pulses in a three-level optical medium in the framework of the IST. Optical pulse propagation in a three-level medium under two-photon or double one-photon resonance conditions has been studied extensively, both theoretically and experimentally, by various authors since the 1970s. Specifically, there have been investigations in connection with SIT and simultons [12–14], lasing without inversion [15,16], electromagnetically induced transparency [17–19], and other related topics [20–22]. In these problems, one typically

* Corresponding author.
E-mail address: chuck@math.uccs.edu (S. Chakravarty).

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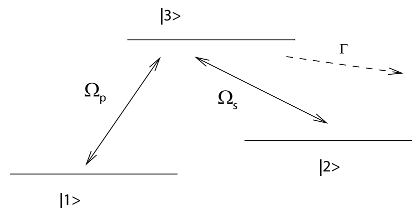

Fig. 1. Schematic of a three-level system, with excited state $|3\rangle$ and lower states $|1\rangle$ and $|2\rangle$. Ω_p and Ω_s are the Rabi frequencies of the control and secondary pulses E_1 and E_2 at $|3\rangle \leftrightarrow |1\rangle$ and $|3\rangle \leftrightarrow |2\rangle$ transitions. Γ is the decay rate of $|3\rangle$ to states external to the three-level system.

considers Λ-configuration media, i.e., three-level media with a degenerate ground level (see Fig. 1), other possible configurations being the V (inverted version of the Λ-configuration) and the Ladder (or Cascade) types of three-level systems. A large class of solutions for the physical problems mentioned above have been obtained by using the IST technique and other methods in earlier works [23,13,24], as well as in more recent papers [25,26]. A comprehensive review of the earlier studies (during the 1980s) on the application of IST to the theory of pulse propagation in multi-level optical media can be found in [27].

The basic physical problem considered in this paper is the propagation of two optical pulses in a medium of three level atoms in the Λ configuration as shown in Fig. 1. In this scheme, the excited state $|3\rangle$ decays at a rate Γ to states other than $|1\rangle$ and $|2\rangle$. The electric fields E_1 and E_2 corresponding to the individual optical pulses are resonantly coupled to the $|3\rangle \leftrightarrow |1\rangle$ and $|3\rangle \leftrightarrow |2\rangle$ atomic transitions, respectively. The material properties of the optical medium are described by the Bloch density matrix $\tilde{\boldsymbol{\rho}}$, which is a 3×3 Hermitian matrix. The diagonal elements of $\tilde{\boldsymbol{\rho}}$ are determined by the population densities of the atomic levels, and the off-diagonal elements describe the complex valued material polarizability envelopes of the optical medium. The equations governing the temporal evolution of the atomic levels in the optical medium and the propagation of the optical pulses through the medium can be derived from the Schrödinger and Maxwell's equations using a slowly varying envelope approximation. The resulting system of equations are known as the coupled Maxwell–Bloch (CMB) equations. In the lossless case ($\Gamma = 0$), and under the assumption that the propagation constants, which depend on the dipole moments and the atomic number density for the two optical pulses through the medium, are the same (simulton conditions), these equations can be written as

$$\boldsymbol{\Omega}_\tau = \frac{1}{4}\left[\tilde{\boldsymbol{\rho}}_{\rm av}, \mathbf{J}\right] \tag{1.1a}$$

$$\tilde{\boldsymbol{\rho}}_x = \left[(\boldsymbol{\Omega} + i\alpha\mathbf{J}), \tilde{\boldsymbol{\rho}}\right] \quad \alpha \in \mathbb{R}. \tag{1.1b}$$

In the above equations, $[\cdot,\cdot]$ denotes the matrix commutator, and

$$\mathbf{J} = \mathrm{diag}\,(-1,-1,1)\,, \qquad \boldsymbol{\Omega} = \frac{1}{2}\begin{pmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ -\Omega_1^* & -\Omega_2^* & 0 \end{pmatrix}, \tag{1.2}$$

$$\tilde{\boldsymbol{\rho}}_{\rm av} = \int_{-\infty}^{\infty} \tilde{\boldsymbol{\rho}}(x,\tau,\alpha) g(\alpha) d\alpha,$$

where $\Omega_1(x,\tau)$ and $\Omega_2(x,\tau)$ are the slowly varying complex envelopes of the electric fields $E_1(x,\tau)$ and $E_2(x,\tau)$. Note that besides the coordinates x and τ, the density matrix $\tilde{\boldsymbol{\rho}}$ depends also on a real parameter α representing the de-tuning from the exact quantum transition frequency due to the Doppler shift caused by the thermal motion of the atoms in the medium. The last integral term in (1.2) accounts for the inhomogeneous broadening effect by averaging over the range of detuning with the (atomic) velocity distribution function $g(\alpha)$, which satisfies the normalization condition $\int_{-\infty}^{\infty} g(\alpha)d\alpha = 1$. The coordinates (x,τ) adopted in this paper are the natural coordinates for the IST scheme (see Section 2). These are related to the physical coordinates as follows:

$$x = t - z/c, \qquad \tau = z,$$

where $t \in \mathbb{R}$ is the normalized time, $z \geq 0$ is the normalized propagation distance along the optical medium, and c is the speed of light. In particular, Eq. (1.1a) is written in the frame moving to the right with speed c.

It follows from Eq. (1.1b) that $\partial_x \left(\mathrm{Tr}\,\tilde{\boldsymbol{\rho}}^n(x,\tau,\alpha)\right) = 0$ for any positive integer n because $\tilde{\boldsymbol{\rho}}^n$ also satisfies (1.1b), and the trace of any commutator vanishes identically. In particular, for $n = 1$, the trace of the density matrix $\tilde{\boldsymbol{\rho}}$ is independent of x, which corresponds to the conservation of the net population density in the three atomic levels as x denotes a time-like variable. In fact, since the CMB equations are invariant with respect to a gauge transformation of the form $\tilde{\boldsymbol{\rho}} \to \tilde{\boldsymbol{\rho}} - \mu(\tau,\alpha)\mathbf{I}_3$ where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix, one can set $\mathrm{Tr}\,\tilde{\boldsymbol{\rho}} = 1$ by appropriately choosing the function $\mu(\tau,\alpha)$. Furthermore, in some physical applications such as electromagnetically induced transparency (see e.g., [18]) or matched pulse propagation through absorbing media [28], the Bloch matrix $\tilde{\boldsymbol{\rho}}$ is of rank 1, and its elements can be expressed as $\tilde{\rho}_{ij} = \gamma_i\gamma_j^*$ in terms of the probability amplitudes γ_j of the atomic levels. The optical pulse envelopes $\Omega_1(x,\tau)$, $\Omega_2(x,\tau)$ are localized in x for all values of τ since they represent temporal pulses. They satisfy the vanishing boundary conditions $\boldsymbol{\Omega} \to 0$ as $x \to \pm\infty$ for all τ. Note, however, that only one set of boundary conditions can be prescribed for the Bloch matrix $\tilde{\boldsymbol{\rho}}$ since (1.1b) can be regarded as an ordinary differential equation for $\tilde{\boldsymbol{\rho}}$ in x given $\boldsymbol{\Omega}(x,\tau)$ for any (fixed) value of τ. Thus, for instance, if $\tilde{\boldsymbol{\rho}}$ as $x \to -\infty$ is prescribed, then $\tilde{\boldsymbol{\rho}}$ as $x \to +\infty$ is determined uniquely by (1.1b). For later convenience we define these boundary values for $\tilde{\boldsymbol{\rho}}$ as $\tilde{\boldsymbol{\rho}}_- := \tilde{\boldsymbol{\rho}}(x \to -\infty)$, and $\tilde{\boldsymbol{\rho}}_+ := \tilde{\boldsymbol{\rho}}(x \to \infty)$.

In this paper we present an IST procedure to solve the general initial value problem for the CMB equations with inhomogeneous broadening, (1.1), describing the propagation of localized optical pulses decaying sufficiently rapidly as $x \to \pm\infty$, through a three-level medium. Furthermore, these solutions are determined for generic initial preparation of the medium, i.e., for a sufficiently broad class of specified boundary conditions $\tilde{\boldsymbol{\rho}}_-$. A comprehensive treatment of this problem via IST has not been previously carried out for generic choice of $\tilde{\boldsymbol{\rho}}_-$. Another key issue, not addressed elsewhere before, is to determine the final state of the medium given by the Bloch matrix after the interaction with the electromagnetic field, i.e. the asymptotic value $\tilde{\boldsymbol{\rho}}_+$ of $\tilde{\boldsymbol{\rho}}$ as $x \to \infty$. In this work, we demonstrate how to recover this asymptotic medium configuration explicitly in terms of the associated scattering data. Using the IST method we construct exact n-soliton solutions for the optical pulses and study the soliton interaction properties, including polarization shifts for the optical pulses. Moreover, as a by-product of our IST analysis we derive an explicit formula for the scattering matrix in the case of pure n-soliton solution.

The paper is organized as follows. In Section 2 we discuss the direct as well as the inverse scattering problem associated with the CMB equations by introducing the eigenfunctions and the scattering data. The inverse problem is posed as a Riemann–Hilbert problem on the real axis, with poles in the upper/lower half-plane of the scattering parameter. Then we derive the time evolution of the eigenfunctions and the scattering data, which is nontrivial and the key issue in this problem. We close Section 2 with a discussion of the so-called sharp-line limit, corresponding to the case when there is no inhomogeneous broadening and the broadening function $g(\alpha)$ is taken to be a Dirac delta function. Section 3 is devoted to reflection-less scattering and pure soliton solutions. In particular, we derive explicit formulas for the n-soliton electric field envelope $\boldsymbol{\Omega}(x,\tau)$ and the density matrix $\tilde{\boldsymbol{\rho}}(x,\tau,\alpha)$, and investigate the dynamics of one- and two-soliton solutions, including the collision-induced polarization shifts due to two-soliton interactions. In addition, the explicit form of the reflection-less scattering transmission matrix together with its analyticity properties is also derived in this section. Finally, we make some concluding remarks in Section 4. The sections in the main text are supplemented by Appendices A–C. In particular, Appendix B provides further properties of the scattering transmission matrix in the reflection-less case derived in Section 3.4, and Appendix C contains some preliminary results for the scattering matrix for a one-soliton solution on a small radiative background.

2. Scattering problem

It is well-known that the CMB system (1.1) for the matrices $\boldsymbol{\Omega}$ and $\tilde{\boldsymbol{\rho}}$ can be expressed as the compatibility condition of the following Lax pair:

$$v_x = (ik\mathbf{J} + \boldsymbol{\Omega})\, v \tag{2.1a}$$

$$v_\tau = \mathbf{T} v \tag{2.1b}$$

where $v = v(x,\tau,k)$ is a three component vector, k the complex spectral parameter, and $\mathbf{T} = i\langle\tilde{\boldsymbol{\rho}}\rangle$. For convenience, here and henceforth we employ the following notation:

$$\langle A\rangle(x,\tau,k) = \frac{1}{4}\fint_{-\infty}^{\infty}\frac{g(\alpha)}{\alpha - k}A(x,\tau,\alpha)\,d\alpha, \quad k\in\mathbb{R} \tag{2.2}$$

with $\fint_{-\infty}^{\infty}$ denoting the Cauchy principal value integral. When $k\notin\mathbb{R}$, the integral above is understood to be over the entire real line instead of principal value.

The compatibility condition between (2.1a) and (2.1b) yields $\boldsymbol{\Omega}_\tau = \mathbf{T}_x + [\mathbf{T}, ik\mathbf{J} + \boldsymbol{\Omega}]$, which can be rewritten as,

$$\boldsymbol{\Omega}_\tau = \frac{i}{4}\fint_{-\infty}^{\infty}\frac{g(\alpha)}{\alpha - k}\Big\{i(k-\alpha)[\tilde{\boldsymbol{\rho}}(\alpha),\mathbf{J}] + \tilde{\boldsymbol{\rho}}_x(\alpha) + [\tilde{\boldsymbol{\rho}}(\alpha), \boldsymbol{\Omega} + i\alpha\mathbf{J}]\Big\}d\alpha.$$

Equating the terms with the same k-dependence from the above expression, one indeed recovers the CMB equations (1.1).

The scattering problem (2.1a) is the same as that for the vector nonlinear Schrödinger (VNLS) equation in the focusing case, with $\boldsymbol{\Omega}$ playing the role of the potential. For $\boldsymbol{\Omega}$ decaying rapidly enough as $x\to\pm\infty$, the IST is well-established in this case (see e.g., [29]). Below we provide a brief review of the direct and inverse scattering for Eq. (2.1a).

2.1. Direct problem

One refers to the solutions of the scattering problem (2.1a) as eigenfunctions with respect to the spectral parameter k. When $\boldsymbol{\Omega}$ decays rapidly as $x\to\pm\infty$, the eigenfunctions are asymptotic to the solutions of the differential equation $v_x = ik\mathbf{J}v$ as $|x|\to\infty$. Hence, the scattering problem (2.1a) admits the eigenfunction bases

$$\boldsymbol{\Phi}(x,k)\sim e^{ik\mathbf{J}x}, \quad x\to-\infty \tag{2.3a}$$

$$\boldsymbol{\Psi}(x,k)\sim e^{ik\mathbf{J}x}, \quad x\to+\infty \tag{2.3b}$$

with $\mathbf{J} = \mathrm{diag}\,(-1,-1,1)$, as given in (1.2). These eigenfunctions with the assigned boundary conditions as $x\to\pm\infty$ for any $k\in\mathbb{R}$ can be written in terms of the following integral equations

$$\boldsymbol{\Phi}(x,k) = e^{ik\mathbf{J}x} + \int_{-\infty}^{x} e^{ik\mathbf{J}(x-x')}\boldsymbol{\Omega}(x')\boldsymbol{\Phi}\left(x',k\right)dx',$$

$$\boldsymbol{\Psi}(x,k) = e^{ik\mathbf{J}x} - \int_{x}^{+\infty} e^{ik\mathbf{J}(x-x')}\boldsymbol{\Omega}(x')\boldsymbol{\Psi}\left(x',k\right)dx'.$$

Neumann iteration on the integral equations allows one to prove that the first two columns of $\boldsymbol{\Phi}$, denoted by $\boldsymbol{\phi}$, can be analytically continued to the upper half of the complex k-plane, while the last column of $\boldsymbol{\Phi}$, denoted by $\bar{\boldsymbol{\phi}}$, can be analytically continued to the lower half-plane. Similarly, the first two columns of $\boldsymbol{\Psi}$, denoted by $\bar{\boldsymbol{\psi}}$, and the last column of $\boldsymbol{\Psi}$, denoted by $\boldsymbol{\psi}$, admit analytic continuation onto the lower and upper half-planes, respectively. Moreover, integration by parts on the above integral equations yields

$$\boldsymbol{\Phi}(x,k)e^{-ik\mathbf{J}x}\sim\mathbf{I}_3, \qquad \boldsymbol{\Psi}(x,k)e^{-ik\mathbf{J}x}\sim\mathbf{I}_3, \quad \text{as } k\to\infty$$

where the asymptotic behavior in each column is taken in the proper half-plane of analytic continuation.

It can be shown that both $\boldsymbol{\Phi} = \left(\boldsymbol{\phi}\;\bar{\boldsymbol{\phi}}\right)$ and $\boldsymbol{\Psi} = \left(\bar{\boldsymbol{\psi}}\;\boldsymbol{\psi}\right)$ are two complete sets of basis eigenfunctions for the scattering problem (2.1a). Then one can define the scattering data $\mathbf{S}(k)$ by the relation

$$\boldsymbol{\Psi} = \boldsymbol{\Phi}\mathbf{S}, \qquad \mathbf{S}(k) = \begin{pmatrix}\bar{\mathbf{a}} & \mathbf{b}\\ \bar{\mathbf{b}}^\dagger & a\end{pmatrix}, \quad k\in\mathbb{R}, \tag{2.4}$$

where $\bar{\mathbf{a}}(k)$ is a 2×2 matrix, $a(k)$ is a scalar, $\mathbf{b}(k)$, $\bar{\mathbf{b}}(k)$ are 2-component column vectors, and † denotes the Hermitian conjugate.

Properties of the scattering matrix: It follows from the analyticity properties of the basis eigenfunctions that the matrix $\bar{\mathbf{a}}(k)$ is analytic in the lower half k-plane, while the scalar function $a(k)$ is analytic in the upper half-plane. The vectors $\mathbf{b}$, $\bar{\mathbf{b}}$ in general cannot be continued off the real k-axis. Moreover, the scattering matrix $\mathbf{S}(k)\sim\mathbf{I}_3$ as $k\to\infty$ since $\boldsymbol{\Phi},\boldsymbol{\Psi}\sim e^{ik\mathbf{J}x}$ as $k\to\infty$. Then the analyticity properties of the various components of $\mathbf{S}(k)$ imply that

$$\bar{\mathbf{a}}(k)\sim\mathbf{I}_2 \quad \text{as } k\to\infty,\ \mathrm{Im}(k)\le 0,$$

$$a(k)\sim 1 \quad \text{as } k\to\infty,\ \mathrm{Im}(k)\ge 0,$$

while the off-diagonal scattering coefficients $\mathbf{b}(k),\bar{\mathbf{b}}(k)\to 0$ as $k\to\infty$ on the real axis.

From Eq. (2.1a) for $\boldsymbol{\Phi}$ together with its Hermitian conjugate equation, and the symmetry for the matrix potential $\boldsymbol{\Omega}^\dagger = -\boldsymbol{\Omega}$, one can show that for $k\in\mathbb{R}$, $\boldsymbol{\Phi}^\dagger(x,k)\boldsymbol{\Phi}(x,k)$ is independent of x. Applying the boundary condition (2.3a) as $x\to-\infty$ then gives $\boldsymbol{\Phi}^\dagger(k)\boldsymbol{\Phi}(k) = \mathbf{I}_3$. In a similar fashion, one can show that $\boldsymbol{\Psi}^\dagger(k)\boldsymbol{\Psi}(k) = \mathbf{I}_3$ for $k\in\mathbb{R}$. Then (2.4) implies that $\mathbf{S}$ is a unitary matrix for $k\in\mathbb{R}$, i.e.,

$$\mathbf{S}(k)\mathbf{S}^\dagger(k) = \mathbf{S}^\dagger(k)\mathbf{S}(k) = \mathbf{I}_3.$$

Furthermore, if $\mathbf{S}(k)$ can be analytically continued off the real k-axis, then similar arguments as above lead to the complex unitarity conditions $\boldsymbol{\Phi}^\dagger(k^*)\boldsymbol{\Phi}(k) = \boldsymbol{\Psi}^\dagger(k^*)\boldsymbol{\Psi}(k) = \mathbf{I}_3$ for the eigenfunctions, and the symmetry relation

$$\mathbf{S}(k)\mathbf{S}^\dagger(k^*) = \mathbf{S}^\dagger(k^*)\mathbf{S}(k) = \mathbf{I}_3,$$

or explicitly

$$\bar{\mathbf{a}}(k)\bar{\mathbf{a}}^\dagger(k^*) + \mathbf{b}(k)\mathbf{b}^\dagger(k^*) = \mathbf{I}_2,\qquad a^*(k^*)a(k) + \bar{\mathbf{b}}^\dagger(k)\bar{\mathbf{b}}(k^*) = 1, \tag{2.5a}$$

$$\bar{\mathbf{a}}(k)\bar{\mathbf{b}}(k^*) + a^*(k^*)\mathbf{b}(k) = 0, \tag{2.5b}$$

$$\bar{\mathbf{a}}^\dagger(k^*)\bar{\mathbf{a}}(k) + \bar{\mathbf{b}}(k^*)\bar{\mathbf{b}}^\dagger(k) = \mathbf{I}_2,\qquad a^*(k^*)a(k) + \mathbf{b}^\dagger(k^*)\mathbf{b}(k) = 1, \tag{2.5c}$$

$$\bar{\mathbf{a}}^\dagger(k^*)\mathbf{b}(k) + a(k)\bar{\mathbf{b}}(k^*) = 0, \tag{2.5d}$$

which are valid wherever all scattering coefficients are simultaneously defined.

From the scattering problem (2.1a) for $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$, it also follows that

$$(\ln\det\boldsymbol{\Phi})_x = (\ln\det\boldsymbol{\Psi})_x = \mathrm{Tr}(ik\mathbf{J} + \boldsymbol{\Omega}) = -ik,$$

since $\mathrm{Tr}(\boldsymbol{\Omega}) = 0$. Integrating these equations and applying the boundary conditions (2.3a) and (2.3b), one obtains $\det \boldsymbol{\Phi} = \det \boldsymbol{\Psi} = e^{-ikx}$. Consequently, $\det \mathbf{S}(k) = 1$ due to (2.4). From the symmetry relation above one can write $[\mathbf{S}^\dagger(k^*)]_{33} = [\mathbf{S}^{-1}(k)]_{33}$, which together with the unimodularity of the scattering matrix $\mathbf{S}(k)$ leads to the relation

$$\det \bar{\mathbf{a}}(k) = a^*(k^*), \tag{2.6}$$

for any k in the lower half-plane. The zeros of $\det \bar{\mathbf{a}}(k)$ in the lower half plane and those of $a(k)$ in the upper half plane play the role of discrete eigenvalues for the scattering problem (2.1a). If $a(k)$ has a finite number of simple zeros in the region $\mathrm{Im}\, k > 0$, say $k_1, \ldots, k_n$, then $\det \bar{\mathbf{a}}(k)$ has simple zeros at points $k_1^*, \ldots, k_n^*$ in the region $\mathrm{Im}\, k < 0$. We also introduce the *reflection coefficients* $\mathbf{r}(k)$, $\bar{\mathbf{r}}(k)$ defined on the real k-axis as

$$\mathbf{r}(k) = \mathbf{b}(k)/a(k), \qquad \bar{\mathbf{r}}(k) = (\bar{\mathbf{a}}^\dagger)^{-1}(k)\bar{\mathbf{b}}(k), \tag{2.7}$$

for which the symmetry relations (2.5d) yield $\bar{\mathbf{r}}(k) = -\mathbf{r}(k)$, $k \in \mathbb{R}$.

2.2. Inverse problem

Let us now formulate the inverse problem to solve for the eigenfunctions $\boldsymbol{\Phi}, \boldsymbol{\Psi}$ in terms of the scattering data. Recall that $\boldsymbol{\Phi} = (\boldsymbol{\phi}\ \bar{\boldsymbol{\phi}})$ and $\boldsymbol{\Psi} = (\bar{\boldsymbol{\psi}}\ \boldsymbol{\psi})$, where the matrices $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are analytic on $\mathrm{Im}\, k > 0$, and $\bar{\boldsymbol{\phi}}$, $\bar{\boldsymbol{\psi}}$ are analytic on $\mathrm{Im}\, k < 0$. From (2.4), one obtains the matrix relations

$$\bar{\boldsymbol{\psi}}\, \bar{\mathbf{a}}^{-1} = \bar{\boldsymbol{\phi}}\, \bar{\mathbf{r}}^\dagger + \boldsymbol{\phi}, \qquad \boldsymbol{\psi}/a = \boldsymbol{\phi}\, \mathbf{r} + \bar{\boldsymbol{\phi}},$$

for $k \in \mathbb{R}$, where $\mathbf{r}$, $\bar{\mathbf{r}}$ are defined in (2.7). Using the second relation above, one finds that for $k \in \mathbb{R}$,

$$\det(\boldsymbol{\phi}, \boldsymbol{\psi}) = \det\left(\boldsymbol{\phi}, \boldsymbol{\phi}\mathbf{b} + a\bar{\boldsymbol{\phi}}\right) = a \det \boldsymbol{\Phi} = a(k)e^{-ikx}.$$

Since $\det(\boldsymbol{\phi}, \boldsymbol{\psi})$ can be continued analytically onto the region $\mathrm{Im}\, k > 0$, it follows that $\det(\boldsymbol{\phi}, \boldsymbol{\psi}) = 0$ at each of the zeros (assumed to be simple) $k_1, \ldots, k_n$ of $a(k)$ in the upper-half plane. Consequently, at each k_j, the column vector $\boldsymbol{\psi}$ is spanned by the two column vectors constituting $\boldsymbol{\phi}$, i.e.,

$$\boldsymbol{\psi}(k_j) = \boldsymbol{\phi}(k_j)\boldsymbol{\eta}_j$$

where $\boldsymbol{\eta}_j$ is a 2-component column vector. Then the quantity $\boldsymbol{\psi}/a$ is meromorphic in the upper-half plane with simple poles at $k = k_j$, $j = 1, 2, \ldots, n$, and the residue at each pole is given by

$$\mathrm{Res}_{k=k_j}\left(\frac{\boldsymbol{\psi}}{a}\right) = \boldsymbol{\phi}(k_j)\boldsymbol{\beta}_j, \quad \boldsymbol{\beta}_j = \frac{\boldsymbol{\eta}_j}{a'(k_j)},$$

where a' denotes the derivative with respect to k, and $\boldsymbol{\beta}_j$ is the *norming constant* associated with k_j. In addition, from the large k behavior $\boldsymbol{\Psi} \sim e^{ik\mathbf{J}x}$ and $a(k) \sim 1$, it follows that $(\boldsymbol{\psi}/a)e^{-ikx} \sim \mathbf{e}_3$ as $k \to \infty$ where $\mathbf{e}_3 = (0, 0, 1)^T$. Thus we can formulate a vector Riemann–Hilbert problem for a piece-wise analytic function via the equation

$$\frac{\boldsymbol{\psi}}{a}e^{-ikx} - \left(\mathbf{e}_3 + \sum_{j=1}^n \frac{\boldsymbol{\phi}(k_j)\boldsymbol{\beta}_j}{k - k_j}e^{-ik_jx}\right) = \boldsymbol{\phi}\mathbf{r}e^{-ikx} + \left(\bar{\boldsymbol{\phi}}e^{-ikx} - \mathbf{e}_3 - \sum_{j=1}^n \frac{\boldsymbol{\phi}(k_j)\boldsymbol{\beta}_j}{k - k_j}e^{-ik_jx}\right).$$

The left hand side of the above equation is analytic in the upper-half k-plane and vanishes as $k \to \infty$, whereas the bracketed quantity in the right hand side is analytic in the lower-half k-plane and vanishes as $k \to \infty$, and $\boldsymbol{\phi}\mathbf{r}e^{-ikx}$ is the jump discontinuity on the real k-axis. Then the solution of this Riemann–Hilbert problem can be written as

$$\bar{\boldsymbol{\phi}}(k)e^{-ikx} = \mathbf{e}_3 + \sum_{j=1}^n \frac{\boldsymbol{\phi}(k_j)\boldsymbol{\beta}_j}{k - k_j}e^{-ik_jx} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\boldsymbol{\phi}(\xi)\mathbf{r}(\xi)e^{-i\xi x}}{\xi - (k - i0)}d\xi. \tag{2.8a}$$

An equation similar to (2.8a) can be derived for $\boldsymbol{\phi}$ starting from the relation $\bar{\boldsymbol{\psi}}\, \bar{\mathbf{a}}^{-1} = \bar{\boldsymbol{\phi}}\, \bar{\mathbf{r}}^\dagger + \boldsymbol{\phi}$ or, equivalently, $\bar{\boldsymbol{\psi}} = \bar{\boldsymbol{\phi}}\, \bar{\mathbf{b}}^\dagger + \boldsymbol{\phi}\bar{\mathbf{a}}$. Then for $k \in \mathbb{R}$,

$$\det\left(\bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}}\right) = \det \bar{\mathbf{a}} \det \boldsymbol{\Phi} = \det \bar{\mathbf{a}}(k)e^{-ikx}.$$

Since $\det\left(\bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}}\right)$ and $\det \bar{\mathbf{a}}(k)$ are analytic in the lower-half plane, and $\det \bar{\mathbf{a}}(k) = a^*(k^*)$ has simple zeros at $k = k_j^*$ for $j = 1, \ldots, n$, it follows that the columns of the matrix $\left(\bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}}\right)$ are linearly dependent. Then it can be deduced (see Appendix A) by using the relations $\boldsymbol{\Phi}^\dagger(k^*)\boldsymbol{\Phi}(k) = \boldsymbol{\Psi}^\dagger(k^*)\boldsymbol{\Psi}(k) = \mathbf{I}_3$ that

$$\left(\bar{\boldsymbol{\psi}}\bar{\mathbf{a}}^{-1} \det \bar{\mathbf{a}}\right)(k_j^*) = \left(\bar{\boldsymbol{\psi}}\bar{\mathbf{a}}^{-1}\right)(k_j^*)a^*(k_j) = -\bar{\boldsymbol{\phi}}(k_j^*)\boldsymbol{\eta}_j^\dagger,$$

where the 2-component vector $\boldsymbol{\eta}_j$ was introduced above. Hence, $\left(\bar{\boldsymbol{\psi}}\bar{\mathbf{a}}^{-1}\right)(k)$ is meromorphic in the lower-half k-plane with simple poles at $k = k_j^*$, $j = 1, 2, \ldots, n$, and

$$\mathrm{Res}_{k=k_j^*}\left(\bar{\boldsymbol{\psi}}\, \bar{\mathbf{a}}^{-1}\right) = -\bar{\boldsymbol{\phi}}(k_j^*)\boldsymbol{\beta}_j^\dagger.$$

Taking into account that both $\bar{\boldsymbol{\psi}}\, \bar{\mathbf{a}}^{-1}e^{ikx} \sim (\mathbf{e}_1\, \mathbf{e}_2)$ and $\boldsymbol{\phi}e^{ikx} \sim (\mathbf{e}_1\, \mathbf{e}_2)$ as $k \to \infty$, where $\mathbf{e}_1 = (1, 0, 0)^T$, $\mathbf{e}_2 = (0, 1, 0)^T$, one can formulate the following Riemann–Hilbert problem

$$\boldsymbol{\phi}e^{ikx} - \left((\mathbf{e}_1\, \mathbf{e}_2) - \sum_{j=1}^n \frac{\bar{\boldsymbol{\phi}}(k_j^*)\, \boldsymbol{\beta}_j^\dagger}{k - k_j^*}e^{ik_j^*x}\right) = \bar{\boldsymbol{\phi}}\mathbf{r}^\dagger e^{ikx} + \left(\bar{\boldsymbol{\psi}}\, \bar{\mathbf{a}}^{-1}e^{ikx} - (\mathbf{e}_1\, \mathbf{e}_2) + \sum_{j=1}^n \frac{\bar{\boldsymbol{\phi}}(k_j^*)\, \boldsymbol{\beta}_j^\dagger}{k - k_j^*}e^{ik_j^*x}\right),$$

where the left-hand side is analytic for $\mathrm{Im}\, k > 0$ and vanishes as $k \to \infty$, while the right-hand side consists of a bracketed term that is analytic for $\mathrm{Im}\, k < 0$, and vanishes as $k \to \infty$, plus the jump discontinuity $\bar{\boldsymbol{\phi}}\mathbf{r}^\dagger e^{ikx}$ on the real k-axis. The solution of this problem yields, for $\mathrm{Im}\, k > 0$,

$$\boldsymbol{\phi}e^{ikx} = (\mathbf{e}_1\, \mathbf{e}_2) - \sum_{j=1}^n \frac{\bar{\boldsymbol{\phi}}(k_j^*)\boldsymbol{\beta}_j^\dagger}{k - k_j^*}e^{ik_j^*x} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\bar{\boldsymbol{\phi}}(\xi)\mathbf{r}^\dagger(\xi)e^{i\xi x}}{\xi - (k + i0)}d\xi. \tag{2.8b}$$

Eqs. (2.8a) and (2.8b) form a closed system of integro-algebraic equations for the eigenfunctions $\boldsymbol{\phi}$ and $\bar{\boldsymbol{\phi}}$. The inverse problem amounts to solving this system for $\boldsymbol{\Phi} = (\boldsymbol{\phi}\bar{\boldsymbol{\phi}})$ in terms of the scattering data which consists of $\left\{k_j,\ \boldsymbol{\beta}_j\right\}_{j=1}^n$ together with the reflection coefficient $\mathbf{r}(k)$ for $k \in \mathbb{R}$. Once the eigenfunction basis $\boldsymbol{\Phi}$ is found, the potential $\boldsymbol{\Omega}$, the full scattering matrix $\mathbf{S}(k)$ and the Bloch matrix $\tilde{\boldsymbol{\rho}}(x, \tau, \alpha)$ can be obtained from $\boldsymbol{\Phi}$ in ways described below.

The electric fields Ω_1, Ω_2 are reconstructed via a large k expansion of the basis eigenfunctions. If one sets $\boldsymbol{\Phi} = \mu e^{ik\mathbf{J}x}$, then it follows from the integral equation for $\boldsymbol{\Phi}$ in Section 2.1 that

$$\mu(x, k) = \mathbf{I}_3 + \frac{\mu_1(x)}{k} + O(k^{-2}) \quad \text{as } k \to \infty,$$

with $\mu_1 = -\mu_1^\dagger$, and (2.1a) implies that $\mu(x, k)$ satisfies

$$\mu_x + ik[\mu, \mathbf{J}] = \boldsymbol{\Omega}\mu.$$

Substituting the large k expansion for μ in the equation above, and taking the limit as $k \to \infty$, gives

$$\boldsymbol{\Omega} = i[\mu_1, \mathbf{J}] \Rightarrow \Omega_1 = 4i[\mu_1]_{13} = -4i\,([\mu_1]_{31})^*\,, \quad \Omega_2 = 4i[\mu_1]_{23} = -4i\,([\mu_1]_{32})^*\,, \tag{2.9}$$

where μ_1 can be obtained as the k^{-1}-coefficient in the large k expansions of (2.8a) and (2.8b). In (2.9), $[\mu_1]_{ij}$ indicates the (ij)th entry of the matrix μ_1.

The scattering matrix can be recovered from (2.4) and the large x-asymptotics of the eigenfunctions

$$\mathbf{S}(k) = \lim_{x\to+\infty} \boldsymbol{\Phi}^{-1}(x,k)\boldsymbol{\Psi}(x,k) = \lim_{x\to+\infty} \boldsymbol{\Phi}^{\dagger}(x,k^*)e^{ik\mathbf{J}x} = \lim_{x\to+\infty} e^{-ik\mathbf{J}x}\mu^{\dagger}(x,k^*)e^{ik\mathbf{J}x}. \tag{2.10}$$

The Bloch matrix $\tilde{\boldsymbol{\rho}}$ can be reconstructed from $\boldsymbol{\Phi}$ via the "squared" eigenfunction $\mathbf{F} = \boldsymbol{\Phi}\mathbf{C}\boldsymbol{\Phi}^{-1}$ where $\mathbf{C} = \mathbf{C}(\tau, k)$ is an x-independent matrix. It is easy to verify using (2.1a) that $\mathbf{F}$ satisfies

$$\mathbf{F}_x + [\mathbf{F}, ik\mathbf{J} + \boldsymbol{\Omega}] = 0,$$

which is the same ODE (1.1b) for $\tilde{\boldsymbol{\rho}}(x,\tau,\alpha)$ when $k = \alpha \in \mathbb{R}$. Therefore, $\tilde{\boldsymbol{\rho}}$ is uniquely determined by

$$\mathbf{F}(x,\tau,k=\alpha) = \boldsymbol{\Phi}(x,\tau,\alpha)\mathbf{C}(\tau,\alpha)\,\boldsymbol{\Phi}^{-1}\,(x,\tau,\alpha) = \tilde{\boldsymbol{\rho}}(x,\tau,\alpha), \tag{2.11}$$

provided that the matrix $\mathbf{C}(\tau,\alpha)$ is specified by the boundary condition $\tilde{\boldsymbol{\rho}}_-$ as $x \to -\infty$. That is,

$$\tilde{\boldsymbol{\rho}}(x,\tau,\alpha) \sim e^{i\alpha\mathbf{J}x}\mathbf{C}e^{-i\alpha\mathbf{J}x} = \tilde{\boldsymbol{\rho}}_- \quad \text{as } x \to -\infty. \tag{2.12}$$

Since the Bloch matrix $\tilde{\boldsymbol{\rho}}$ is Hermitian and $\boldsymbol{\Phi}^{\dagger}(\alpha) = \boldsymbol{\Phi}^{-1}(\alpha)$ for $\alpha \in \mathbb{R}$, it follows from (2.11) that $\mathbf{C}(\tau,\alpha)$ is a Hermitian matrix. Once $\mathbf{C}$ is fixed from the assigned boundary condition $\tilde{\boldsymbol{\rho}}_-$ as $x \to -\infty$ corresponding to the initial preparation of the 3-level atomic states in the optical medium, the Bloch matrix $\tilde{\boldsymbol{\rho}}$ is then determined by (2.11) for all later $x \in \mathbb{R}$. In particular, the final state of the 3-level medium after interaction with the electric field is then given in terms of the scattering matrix as follows:

$$\tilde{\boldsymbol{\rho}} = \boldsymbol{\Psi}\mathbf{S}^{-1}\mathbf{C}\mathbf{S}\boldsymbol{\Psi}^{-1} \sim e^{i\alpha\mathbf{J}x}(\mathbf{S}^{-1}\mathbf{C}\mathbf{S})e^{-i\alpha\mathbf{J}x} = \tilde{\boldsymbol{\rho}}_+ \quad \text{as } x \to \infty, \tag{2.13}$$

where we have used Eqs. (2.11) and (2.4).

Let us summarize the method of solving the CMB system (1.1) for the matrices $\boldsymbol{\Omega}$ and $\tilde{\boldsymbol{\rho}}$ via the Lax pair (2.1a)–(2.1b). We assume that $\boldsymbol{\Omega}$ is assigned at $\tau = 0$ for all $x \in \mathbb{R}$ and a boundary condition $\tilde{\boldsymbol{\rho}}_-$ for $\tilde{\boldsymbol{\rho}}$ is assigned as $x \to -\infty$. The solution of the problem via IST then amounts to

(i) first determining the basis eigenfunctions $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ (solutions of (2.1a)) specified by their asymptotic behavior as $x \to \pm\infty$ as in (2.3a) and (2.3b), and the corresponding scattering matrix $\mathbf{S}(k)$ at $\tau = 0$ in terms of $\boldsymbol{\Omega}(x, 0)$;
(ii) using (2.1b) to determine the τ-evolutions of the eigenfunctions and the scattering data (see Section 2.3);
(iii) reconstructing the potential $\boldsymbol{\Omega}(x,\tau)$ from (2.9), and $\tilde{\boldsymbol{\rho}}(x,\tau,\alpha)$ using the boundary condition $\tilde{\boldsymbol{\rho}}_-$ and (2.11) via the inverse problem for $\tau > 0$.

As mentioned earlier in the introduction, since Ω_1, Ω_2 correspond to localized optical pulse envelopes we assume here that $\boldsymbol{\Omega} \to 0$ as $x \to \pm\infty$ for all $\tau > 0$. A necessary condition for this to happen is

$$\partial_\tau \boldsymbol{\Omega} \to 0 \quad \text{as } x \to \pm\infty \text{ for all } \tau.$$

(Note that this condition alone would still be meaningful in the case of potentials $\boldsymbol{\Omega}$ going to a constant, or to $e^{i\kappa x}$ as $x \to \pm\infty$.) Then Eq. (1.1a) implies that the following condition

$$\lim_{x\to\pm\infty} \left[\tilde{\boldsymbol{\rho}}_{\text{av}}, \mathbf{J}\right] = \int_{-\infty}^{\infty} [\tilde{\boldsymbol{\rho}}_{\pm}, \mathbf{J}](\alpha)g(\alpha)d\alpha = 0$$

must be satisfied for all τ. From Eqs. (2.12) and (2.13), one finds that the non-vanishing matrix elements of the commutator $[\tilde{\boldsymbol{\rho}}_{\pm}, \mathbf{J}]$ are proportional to $e^{\pm 2i\alpha x}$. If we assume that the coefficients of $e^{\pm 2i\alpha x}$ in the commutator are in $L_1(\mathbb{R})$, then the integral over the commutator vanishes as $x \to \pm\infty$ due to the Riemann–Lebesgue lemma.

Reality condition: The CMB equations (1.1) admit real solutions

$$\boldsymbol{\Omega}^* = \boldsymbol{\Omega}, \qquad \tilde{\boldsymbol{\rho}}^* = \tilde{\boldsymbol{\rho}}$$

when the parameter $\alpha = 0$ in (1.1b). The reality of the solutions induces additional symmetries on the eigenfunctions as well as the scattering coefficients. If $\boldsymbol{\Phi}(x,k)$, $\boldsymbol{\Psi}(x,k)$ are eigenfunction bases for (2.1a) when $\boldsymbol{\Omega}^* = \boldsymbol{\Omega}$, then so are the functions $\boldsymbol{\Phi}^*(x,-k^*)$, $\boldsymbol{\Psi}^*(x,-k^*)$, and they satisfy the same boundary conditions as $\boldsymbol{\Phi}(x,k)$, $\boldsymbol{\Psi}(x,k)$ when $x \to \pm\infty$. Therefore, one has additional symmetries of the eigenfunctions,

$$\boldsymbol{\Phi}(x,k) = \boldsymbol{\Phi}^*(x,-k^*), \qquad \boldsymbol{\Psi}(x,k) = \boldsymbol{\Psi}^*(x,-k^*).$$

Then $\det(\boldsymbol{\phi},\boldsymbol{\psi})(k) = a(k)e^{-ikx}$ implies the symmetry $a(k) = a^*(-k^*)$. Consequently, one has

$$a(k_j) = 0 \Rightarrow a(-k_j^*) = 0 \quad \forall j, \qquad \text{Im}(k_j) = \text{Im}(-k_j^*) > 0.$$

Note that the zeros of $\det \bar{\mathbf{a}}(k) = a^*(k^*)$ then occur at $k = (k_j^*, -k_j)$ in the lower-half plane.

Thus, the zeros of $a(k)$ occur either in distinct pairs $\{k_j, -k_j^*\}$, or are pure imaginary when $k_j = -k_j^* = i\eta_j$. In this case, if one imposes the symmetry $\boldsymbol{\Phi}(x,k) = \boldsymbol{\Phi}^*(x,-k^*)$ in Eqs. (2.8), then equating the residues at the simple poles $k \in \{k_j, -k_j^*, i\eta_j\}$, and changing the variable $\xi \to -\xi$ in the integral term for $\boldsymbol{\Phi}^*(x,-k^*)$, one obtains the following symmetries for the norming constants and reflection coefficient

$$\boldsymbol{\beta}|_{k=k_j^*} = -(\boldsymbol{\beta}|_{k=k_j})^* = -\boldsymbol{\beta}_j^*, \qquad \boldsymbol{\beta}|_{k=i\eta_j} = i\mathbf{b}_j, \quad \mathbf{b}_j \in \mathbb{R}^2,$$
$$\mathbf{r}^*(-\xi) = \mathbf{r}(\xi).$$

Therefore, the scattering data corresponding to real solutions for $\boldsymbol{\Omega}$, $\tilde{\boldsymbol{\rho}}$ consists of

$$\left\{(k_j,\boldsymbol{\beta}_j), (-k_j^*, -\boldsymbol{\beta}_j^*)\right\}_{j=1}^p, \qquad \left\{i\eta_j, i\mathbf{b}_j,\ \eta_j \in \mathbb{R},\ \mathbf{b}_j \in \mathbb{R}^2\right\}_{j=1}^q,$$

and the reflection coefficient $\mathbf{r}(k)$, $k \in \mathbb{R}$. In Section 3, we will use these symmetries for the norming constants to construct real n-soliton solutions.

Thus far our discussion of the scattering problem for the CMB system (1.1) has been primarily based on the first equation of the Lax pair i.e., Eq. (2.1a), which is the same as for the VNLS equation. In the following subsection we consider the second equation (2.1b) of the Lax pair in order to obtain the τ-evolution of the scattering matrix $\mathbf{S}(k)$. It is this feature of the scattering problem that is substantially different from that of the VNLS equation. In contrast to the VNLS equation, which is purely an initial value problem, the CMB system constitutes an initial–boundary value problem, where, in addition to the initial data $\boldsymbol{\Omega}(x, 0)$, the boundary values $\tilde{\boldsymbol{\rho}}_{\pm}$ of the Bloch matrix significantly influence the solution method. More specifically, if $\tilde{\boldsymbol{\rho}}_- \neq \tilde{\boldsymbol{\rho}}_+$, then in (2.1b) the forms of the asymptotic τ-evolution matrices $\mathbf{T}_{\pm}$ as $x \to \pm\infty$, are also distinct. A consequence of this is that the τ-evolution of the scattering matrix $\mathbf{S}(k)$ associated with the CMB system becomes highly nontrivial compared to the VNLS case, as illustrated next.

2.3. Evolution of the scattering data

First, we consider the τ-evolution of the eigenfunctions $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$. Since the boundary conditions (2.3a) and (2.3b) hold for all τ, one must have $\boldsymbol{\Phi}_\tau \to 0$ as $x \to -\infty$, and $\boldsymbol{\Psi}_\tau \to 0$ as

$x \to \infty$. Therefore, it is necessary to modify Eq. (2.1b) for $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ as follows:

$$\boldsymbol{\Phi}_\tau = \mathbf{T}\boldsymbol{\Phi} - \boldsymbol{\Phi}\mathbf{T}_-, \qquad \boldsymbol{\Psi}_\tau = \mathbf{T}\boldsymbol{\Psi} - \boldsymbol{\Psi}\mathbf{T}_+,$$

where $\mathbf{T}_\pm$ depend only on τ, k, and are given by

$$\mathbf{T}_\pm(k,\tau) = \lim_{x\to\pm\infty}\left(e^{-ik\mathbf{J}x}\mathbf{T}(x,\tau,k)e^{ik\mathbf{J}x}\right) = i\lim_{x\to\pm\infty}\langle e^{-ik\mathbf{J}x}\tilde{\boldsymbol{\rho}}e^{ik\mathbf{J}x}\rangle$$

with $\langle\cdot\rangle$ defined as in (2.2). Moreover, using the asymptotic values $\tilde{\boldsymbol{\rho}}_\pm$ in (2.12) and (2.13)

$$\mathbf{T}_-(k,\tau) = i\lim_{x\to-\infty}\langle e^{-i(k-\alpha)\mathbf{J}x}\mathbf{C}e^{i(k-\alpha)\mathbf{J}x}\rangle,$$

$$\mathbf{T}_+(k,\tau) = i\lim_{x\to\infty}\langle e^{-i(k-\alpha)\mathbf{J}x}\mathbf{S}^{-1}\mathbf{C}\mathbf{S}e^{i(k-\alpha)\mathbf{J}x}\rangle.$$

2.3.1. Evolution of $\mathbf{S}(k)$

The evolution of the scattering matrix $\mathbf{S}$ is determined from the evolution equations for $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ above, and Eq. (2.4). It is straightforward to verify that $\mathbf{S}(k,\tau)$ satisfies the ordinary differential equation

$$\mathbf{S}_\tau = \mathbf{T}_-\mathbf{S} - \mathbf{S}\mathbf{T}_+, \tag{2.14}$$

with k as a parameter. In order to derive the evolution equations for the various components of the matrix $\mathbf{S}(k,\tau)$, it is necessary to calculate the quantities $\mathbf{T}_\pm(k,\tau)$ explicitly, for $k \in \mathbb{R}$. For that purpose, we first express the matrices $\mathbf{C}$ and $\mathbf{S}^{-1}\mathbf{C}\mathbf{S}$ in block form as

$$\mathbf{C} = \begin{pmatrix} \mathbf{H} & \mathbf{m} \\ \mathbf{m}^\dagger & h \end{pmatrix}, \qquad \mathbf{S}^{-1}\mathbf{C}\mathbf{S} = \begin{pmatrix} \tilde{\mathbf{H}} & \tilde{\mathbf{m}} \\ \tilde{\mathbf{m}}^\dagger & \tilde{h} \end{pmatrix}, \tag{2.15}$$

where the 2×2 matrices $\mathbf{H}, \tilde{\mathbf{H}}$ satisfy $\mathbf{H}^\dagger = \mathbf{H}$ and $\tilde{\mathbf{H}}^\dagger = \tilde{\mathbf{H}}$, the scalars $h, \tilde{h} \in \mathbb{R}$, and $\mathbf{m}, \tilde{\mathbf{m}}$ are 2-component complex vectors. Then, using the well-known formula

$$\lim_{x\to\pm\infty} ⨍_{-\infty}^{\infty}\frac{f(\alpha,\tau)}{\alpha-k}e^{2i(k-\alpha)x}d\alpha = \mp i\pi f(k,\tau) \quad k\in\mathbb{R}, \tag{2.16}$$

we obtain

$$\begin{aligned}\mathbf{T}_-(k,\tau) &= \frac{i}{4}\lim_{x\to-\infty}⨍_{-\infty}^{\infty}\frac{g(\alpha)}{\alpha-k}\begin{pmatrix}\mathbf{H} & e^{2i(k-\alpha)x}\mathbf{m}\\ e^{-2i(k-\alpha)x}\mathbf{m}^\dagger & h\end{pmatrix}d\alpha\\ &= \begin{pmatrix} i\langle\mathbf{H}\rangle & -\frac{\pi}{4}g\mathbf{m}\\ \frac{\pi}{4}g\mathbf{m}^\dagger & i\langle h\rangle\end{pmatrix},\\ \mathbf{T}_+(k,\tau) &= \frac{i}{4}\lim_{x\to\infty}⨍_{-\infty}^{\infty}\frac{g(\alpha)}{\alpha-k}\begin{pmatrix}\tilde{\mathbf{H}} & e^{2i(k-\alpha)x}\tilde{\mathbf{m}}\\ e^{-2i(k-\alpha)x}\tilde{\mathbf{m}}^\dagger & \tilde{h}\end{pmatrix}d\alpha\\ &= \begin{pmatrix} i\langle\tilde{\mathbf{H}}\rangle & \frac{\pi}{4}g\tilde{\mathbf{m}}\\ -\frac{\pi}{4}g\tilde{\mathbf{m}}^\dagger & i\langle\tilde{h}\rangle\end{pmatrix}.\end{aligned}$$

Next we use the block form of $\mathbf{S}$ in (2.4) and that of $\mathbf{C}$ in (2.15), to express the quantities $\{\tilde{\mathbf{H}}, \tilde{h}, \tilde{\mathbf{m}}\}$ in terms of $\{\mathbf{H}, h, \mathbf{m}\}$ and $\{\bar{\mathbf{a}}, \mathbf{b}, \bar{\mathbf{b}}, a\}$ only, and replace these expressions for $\{\tilde{\mathbf{H}}, \tilde{h}, \tilde{\mathbf{m}}\}$ in the matrix $\mathbf{T}_+$ above. Substituting the resulting block forms of $\mathbf{T}_\pm$ into (2.14), we finally obtain for the evolution of each block of the scattering matrix $\mathbf{S}$ the following equations for $k \in \mathbb{R}$,

$$a_\tau = -i\left[\langle\gamma\rangle + \frac{\pi i}{4}g\gamma\right]a, \quad \gamma = \mathbf{b}^\dagger\hat{\mathbf{H}}\mathbf{b} + a^*\mathbf{m}^\dagger\mathbf{b} + a\,\mathbf{b}^\dagger\mathbf{m} \tag{2.17a}$$

$$\mathbf{b}_\tau = i\left[\langle\mathbf{D}\rangle + \frac{\pi i}{4}g\mathbf{D}\right]\mathbf{b} - \frac{\pi}{2}g\mathbf{m}\,a, \quad \mathbf{D} = \hat{\mathbf{H}} - \gamma\mathbf{I}_2 \tag{2.17b}$$

$$\begin{aligned}\bar{\mathbf{b}}_\tau^\dagger &= -i\bar{\mathbf{b}}^\dagger\left[\langle\bar{\mathbf{D}}\rangle - \frac{\pi i}{4}g\bar{\mathbf{D}}\right] + \frac{\pi}{2}g\mathbf{m}^\dagger\bar{\mathbf{a}},\\ \bar{\mathbf{D}} &= \bar{\mathbf{a}}^\dagger\hat{\mathbf{H}}\bar{\mathbf{a}} + \bar{\mathbf{b}}\mathbf{m}^\dagger\bar{\mathbf{a}} + \bar{\mathbf{a}}^\dagger\mathbf{m}\bar{\mathbf{b}}^\dagger\end{aligned} \tag{2.17c}$$

$$\bar{\mathbf{a}}_\tau = i\left[\left(\langle\hat{\mathbf{H}}\rangle - \frac{\pi i}{4}g\hat{\mathbf{H}}\right)\bar{\mathbf{a}} - \bar{\mathbf{a}}\left(\langle\bar{\mathbf{D}}\rangle - \frac{\pi i}{4}g\bar{\mathbf{D}}\right)\right], \tag{2.17d}$$

where $\hat{\mathbf{H}} = \mathbf{H} - h\mathbf{I}_2$. We will henceforth assume $\hat{\mathbf{H}}$ to be a positive definite 2×2 matrix based on the following physical considerations: It is evident from (2.12) that the block-diagonal part of $\mathbf{C}$ in (2.15), consisting of the 2×2 Hermitian matrix $\mathbf{H}$ and the real scalar h, is the same as the block-diagonal part of $\tilde{\boldsymbol{\rho}}_-$ which corresponds to the initial configuration of the 3-level atomic states in the optical medium. Hence the diagonal elements of $\hat{\mathbf{H}}$ are given by $[\hat{\mathbf{H}}]_{jj} = [\tilde{\boldsymbol{\rho}}_-]_{jj} - [\tilde{\boldsymbol{\rho}}_-]_{33}$, $j = 1, 2$, and the off-diagonal element $[\hat{\mathbf{H}}]_{12} = [\hat{\mathbf{H}}]_{21}^* = [\tilde{\boldsymbol{\rho}}_-]_{12}$. Since in most physical applications the optical medium is initially prepared such that the ground states $|1\rangle$ and $|2\rangle$ are more populated than the excited state $|3\rangle$ as $x \to -\infty$, it is reasonable to assume that $[\hat{\mathbf{H}}]_{jj} > 0$, $j = 1, 2$. Furthermore, the complex polarizability envelope function $[\tilde{\boldsymbol{\rho}}_-]_{12}$, which corresponds to the coupling between the ground states $|1\rangle$ and $|2\rangle$ via the two photon absorption process, is assumed to be small when the medium is initially prepared. Therefore, we can take $0 \le |[\hat{\mathbf{H}}]_{12}| \ll [\hat{\mathbf{H}}]_{jj}$, $j = 1, 2$, which implies that $\hat{\mathbf{H}}$ is positive definite.

Eqs. (2.17) are coupled, nonlinear, nonlocal evolution equations for the components of $\mathbf{S}(k,\tau)$ for a given Hermitian matrix $\mathbf{C}(k,\tau)$, and are valid for $k \in \mathbb{R}$. In general, these equations do not seem to be amenable to exact explicit solutions even for special, simplified choices of the matrix $\mathbf{C}$. This situation is not totally unexpected because even for the 2-level system, where $\mathbf{S}, \mathbf{T}_\pm$ are 2×2 matrices, one cannot solve the τ-evolution equations for the scattering coefficients explicitly, except in the case when $\mathbf{C}$ is a constant, diagonal 2×2 matrix as was shown in [6]. For the 3-level system, it does not appear that there is an obvious way to linearize the evolution equations (2.17) even when $\mathbf{C}$ is a constant, diagonal matrix. In spite of this difficulty, the IST procedure can still be carried out because it requires the τ-evolution information for only a subset of the scattering coefficients instead of the whole scattering matrix $\mathbf{S}(k)$. Indeed, the inverse problem (2.8) at any $\tau > 0$ can be solved only from the knowledge of the reflection coefficient $\mathbf{r}(k,\tau)$, the k_j's and the norming constants $\boldsymbol{\beta}_j(\tau)$. It turns out that the τ-evolution of these quantities is not difficult to find.

2.3.2. Evolution of the reflection coefficient

From (2.17a) and (2.17b), it follows that the reflection coefficient $\mathbf{r} = \mathbf{b}/a$ satisfies a linear, inhomogeneous equation, namely

$$\begin{aligned}\mathbf{r}_\tau &= i\mathscr{H}\mathbf{r} - \frac{\pi}{2}g\mathbf{m},\\ \mathscr{H}(k) &= \langle\hat{\mathbf{H}}\rangle + \frac{\pi i}{4}g\hat{\mathbf{H}} = \frac{1}{4}\int_{-\infty}^{\infty}\frac{g(\alpha)\hat{\mathbf{H}}(\tau,\alpha)}{\alpha-(k+i0)}d\alpha, \quad k\in\mathbb{R},\end{aligned} \tag{2.18}$$

where we note that $\mathscr{H}(k)$ admits analytic extension to the upper-half plane. A similar equation can be also derived for $\bar{\mathbf{r}}$ in (2.7) from (2.17c) and (2.17d), and it is consistent with the symmetry $\bar{\mathbf{r}}(k,\tau) = -\mathbf{r}(k,\tau)$. The formal solution of (2.18) is given by $\mathbf{r} = \mathbf{r}_h + \mathbf{r}_p$,

$$\begin{aligned}\mathbf{r}_h(k,\tau) &= U(k,\tau)U^{-1}(k,0)\mathbf{r}(k,0),\\ \mathbf{r}_p(k,\tau) &= -\frac{\pi}{2}g(k)\int_0^\tau U(k,\tau)U^{-1}(k,s)\mathbf{m}(k,s)\,ds,\end{aligned}$$

where $U(k,\tau)$ is a fundamental matrix of solutions for the homogeneous equation $\mathbf{r}_\tau = i\mathscr{H}\mathbf{r}$. If $\hat{\mathbf{H}}$ is independent of τ, then the fundamental matrix is the usual matrix exponential, i.e., $U(k,\tau) = \exp(i\mathscr{H}(k)\tau)$.

If $\mathbf{m} = 0$, then $\mathbf{r} = \mathbf{r}_h$ is the homogeneous solution which depends on the initial condition $\mathbf{r}(k,0)$. In addition, the norm $\|\mathbf{r}\|$ evolves according to

$$\partial_\tau\|\mathbf{r}\|^2 = -\frac{\pi g}{2}\mathbf{r}^\dagger\hat{\mathbf{H}}\mathbf{r},$$

which, together with the assumption that $\hat{\mathbf{H}}$ is positive definite, implies that the norm of the reflection coefficient decays with respect to τ at a rate proportional to the inhomogeneous broadening function $g(k)$, $k \in \mathbb{R}$, and $\|\mathbf{r}\| \to 0$ as $\tau \to \infty$. Thus the radiation associated with the continuous spectra of the initial optical pulse is absorbed by the medium as the pulse propagates through it, which is a well-known fact for the two-level system [6]. In fact, if $\hat{\mathbf{H}}$ is a constant matrix, then the decay is exponential. Furthermore, from the relations $\|\mathbf{b}\|^2 = \|\mathbf{r}\|^2(1+\|\mathbf{r}\|^2)^{-1}$, $|a|^2 = (1+\|\mathbf{r}\|^2)^{-1}$ it follows that $\mathbf{b} \to 0$ and $|a| \to 1$ as $\tau \to \infty$.

If at $\tau = 0$, the initial electric fields $\Omega_1(x,0)$, $\Omega_2(x,0)$ decay sufficiently fast, then $\mathbf{b}(k,0)$ is analytically extendable onto a strip $0 < \operatorname{Im} k < \delta$ of the upper-half plane. In this case, it follows that $\mathbf{r}(k,0)$ is (at most) meromorphic on the strip, with simple poles at a finite number of points k_j, $j = 1,2,\ldots,n$ corresponding to the zeros of $a(k,0)$, and the residue at each pole $k = k_j$ is given by the norming constant $\boldsymbol{\beta}_j(0)$. However, it is important to note that for $\tau > 0$, $\mathbf{r}_h(k,\tau)$ has identical pole-configuration as $\mathbf{r}(k,0)$ because $\mathcal{H}(k,\tau)$, and hence $U(k,\tau)$, is analytic for $\operatorname{Im} k > 0$. This means that the location of the pole $k = k_j$ does not evolve with τ; moreover, no new poles appear for $\tau > 0$. The solution $\boldsymbol{\Omega}(x,\tau)$ for this case corresponds to soliton pulses traveling in a radiation background.

When $\mathbf{m} \neq 0$, the inhomogeneous term in (2.18) forces the particular solution $\mathbf{r}_p(\tau)$. An interesting consequence is that even when the initial value $\mathbf{r}(k,0) = 0$, (2.18) yields a non-zero reflection coefficient $\mathbf{r}(k,\tau) = \mathbf{r}_p(k,\tau)$ for $\tau > 0$. Thus, for instance, even if the initial electric field $\boldsymbol{\Omega}(x,0) = 0$, the polarizability fluctuation of the medium due to $\mathbf{m}(k,\tau) \neq 0$ would spontaneously generate an electric field for $\tau > 0$ leading to the phenomenon of superfluorescence. For two-level system, the superfluorescence phenomenon was investigated using IST in [9,7,8]. Note also that the particular solution $\mathbf{r}_p(k,\tau)$ exhibits secular behavior if $\mathbf{m}$ is proportional to any solution of the homogeneous equation $\mathbf{r}_\tau = i\mathcal{H}\mathbf{r}$.

2.3.3. τ-dependence of $\tilde{\boldsymbol{\rho}}_+$

Once the initial atomic configuration of the medium $\tilde{\boldsymbol{\rho}}_-$ is prescribed by (2.12), the final atomic state of the 3-level optical medium as $x \to \infty$ is given by the matrix $\tilde{\boldsymbol{\rho}}_+$ in (2.13). Aside from the conjugation by the matrix $e^{i\alpha\mathbf{J}x}$, $\tilde{\boldsymbol{\rho}}_+$ is completely determined by $\mathbf{C}$ and the scattering matrix $\mathbf{S}$. However, due to the complicated τ-dependence (2.17) of the scattering coefficients, it is difficult to describe the precise behavior of $\tilde{\boldsymbol{\rho}}_+$ with respect to τ. Nonetheless we can make a few general observations regarding the final atomic configuration. Using the block form of $\mathbf{C}$ and $\mathbf{S}^{-1}\mathbf{C}\mathbf{S}$ given in (2.15) one can express $\tilde{\boldsymbol{\rho}}_+$ as

$$\tilde{\boldsymbol{\rho}}_+ = e^{i\alpha\mathbf{J}x}(\mathbf{S}^{-1}\mathbf{C}\mathbf{S})e^{-i\alpha\mathbf{J}x} = \begin{pmatrix} \tilde{\mathbf{H}}(\alpha,\tau) & \tilde{\mathbf{m}}(\alpha,\tau)e^{-2i\alpha x} \\ \tilde{\mathbf{m}}^\dagger(\alpha,\tau)e^{2i\alpha x} & \tilde{h}(\alpha,\tau) \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

If for simplicity we assume that $\mathbf{C}$ is block diagonal, i.e., $\mathbf{m} = 0$, then

$$\widehat{\tilde{\mathbf{H}}} := \tilde{\mathbf{H}} - \tilde{h}\mathbf{I}_2 = \bar{\mathbf{a}}^\dagger\hat{\mathbf{H}}\bar{\mathbf{a}} - (\mathbf{b}^\dagger\hat{\mathbf{H}}\mathbf{b})\mathbf{I}_2, \qquad \tilde{\mathbf{m}} = \bar{\mathbf{a}}^\dagger\hat{\mathbf{H}}\mathbf{b}.$$

The diagonal elements of $\widehat{\tilde{\mathbf{H}}}$ are given by $[\tilde{\boldsymbol{\rho}}_+]_{jj} - [\tilde{\boldsymbol{\rho}}_+]_{33}$, $j = 1,2$, which measure the difference of population between each of the ground states and the excited state in the medium as $x \to \infty$. The sum of the diagonal elements of $\widehat{\tilde{\mathbf{H}}}$ is given by $\operatorname{Tr}\widehat{\tilde{\mathbf{H}}} = \operatorname{Tr}\bar{\mathbf{a}}^\dagger\hat{\mathbf{H}}\bar{\mathbf{a}} - 2\mathbf{b}^\dagger\hat{\mathbf{H}}\mathbf{b}$. Using $\operatorname{Tr}\bar{\mathbf{a}}^\dagger\hat{\mathbf{H}}\bar{\mathbf{a}} = \operatorname{Tr}\bar{\mathbf{a}}\bar{\mathbf{a}}^\dagger\hat{\mathbf{H}} = \operatorname{Tr}(\mathbf{I}_2 - \mathbf{b}\mathbf{b}^\dagger)\hat{\mathbf{H}} = \operatorname{Tr}\hat{\mathbf{H}} - \mathbf{b}^\dagger\hat{\mathbf{H}}\mathbf{b}$, one obtains $\operatorname{Tr}\widehat{\tilde{\mathbf{H}}} = \operatorname{Tr}\hat{\mathbf{H}} - 3\mathbf{b}^\dagger\hat{\mathbf{H}}\mathbf{b} < \operatorname{Tr}\hat{\mathbf{H}}$ since $\hat{\mathbf{H}}$ is assumed to be positive definite. Therefore, $[\tilde{\boldsymbol{\rho}}_+]_{jj} - [\tilde{\boldsymbol{\rho}}_+]_{33} < [\tilde{\boldsymbol{\rho}}_-]_{jj} - [\tilde{\boldsymbol{\rho}}_-]_{33}$ for at least one of the ground states $|j\rangle$. This implies that energy is transferred from the optical pulse to the medium so that after the optical pulse has passed through the medium, a certain fraction of the atoms are rendered in the excited state due to the $|j\rangle \to |3\rangle$ transition. The required energy for the transition must come from the radiation part of the pulse since the soliton part will be transmitted through without loss of energy. Recall that when $\mathbf{m} = 0$ the reflection coefficient $\mathbf{r}(k)$ satisfies the homogeneous version of (2.18), and both $\mathbf{r}, \mathbf{b} \to 0$ as $\tau \to \infty$. Then $\mathbf{b}^\dagger\hat{\mathbf{H}}\mathbf{b} \to 0$ as well, due to the inequality $0 \le \mathbf{b}^\dagger\hat{\mathbf{H}}\mathbf{b} \le (\operatorname{Tr}\hat{\mathbf{H}})\|\mathbf{b}\|^2$, which is another consequence of the positive definiteness of $\hat{\mathbf{H}}$. Therefore, as radiation continues to be absorbed by the medium, the fraction of energy available to excite the atoms decrease with τ, and one finds that $\operatorname{Tr}\widehat{\tilde{\mathbf{H}}} \to \operatorname{Tr}\hat{\mathbf{H}}$ as $\tau \to \infty$.

Yet another significant feature is the fact that even though the medium is initially prepared such that $\mathbf{m}(\alpha,\tau) = 0$, the medium polarizability envelope $\tilde{\mathbf{m}}(\alpha,\tau)$ of $\tilde{\boldsymbol{\rho}}_+$ becomes non-zero as $x \to \infty$. Indeed, the squared norm of $\tilde{\mathbf{m}}$ can be computed as follows

$$\|\tilde{\mathbf{m}}\|^2 = \mathbf{b}^\dagger\hat{\mathbf{H}}\bar{\mathbf{a}}\bar{\mathbf{a}}^\dagger\hat{\mathbf{H}}\mathbf{b} = \mathbf{b}^\dagger\hat{\mathbf{H}}(\mathbf{I}_2 - \mathbf{b}\mathbf{b}^\dagger)\hat{\mathbf{H}}\mathbf{b} = \mathbf{b}^\dagger\hat{\mathbf{H}}^2\mathbf{b} - (\mathbf{b}^\dagger\hat{\mathbf{H}}\mathbf{b})^2,$$

where both terms in the last equality become vanishingly small as $\tau \to \infty$. But depending on the initial value $\mathbf{b}(\alpha,0)$, the norm $\tilde{\mathbf{m}}(\alpha,\tau)$ may initially grow, attain a maximum value at $\tau = \tau_0$ inside the medium, and then decay as $\tau \to \infty$. For instance, if we consider $\hat{\mathbf{H}} = c(\alpha)\mathbf{I}_2$, $c(\alpha) \in \mathbb{R}$, i.e., the medium is initially prepared to have the *same* population difference between each of the ground states and the excited state throughout the medium, then it is clear that the squared norm $\|\tilde{\mathbf{m}}\|^2 = c^2\|\mathbf{b}\|^2(1-\|\mathbf{b}\|^2)$ will attain a maximum value inside the medium if $\|\mathbf{b}\|$ starts with an initial value $\|\mathbf{b}(\alpha,0)\|^2 > 1/2$, and decreases monotonically with τ.

The evolution equations (2.17) are valid for the components of the scattering matrix $\mathbf{S}(k,\tau)$ for $k \in \mathbb{R}$. However, in order to derive the evolution equations for the remainder of the scattering data, namely, $\{k_j, \boldsymbol{\beta}_j(\tau)\}_{j=1}^n$, we need to find the τ-evolution of $a(k,\tau)$ for $\operatorname{Im} k > 0$. Recall that $a(k,\tau)$ is analytic with simple zeros $\{k_j\}_{j=1}^n$ in the upper-half plane. In what follows, we show that both Eqs. (2.17a) for $a(k,\tau)$ and (2.17d) for $\bar{\mathbf{a}}(k,\tau)$ have appropriate analytic continuations to the upper-half and lower-half k-planes, respectively.

2.3.4. Evolution of the norming constants

In order to determine the τ-evolution of the scattering coefficients that are analytic off the real k-axis, we need the τ-dependence of the eigenfunctions in their respective half-planes of analyticity. These can be calculated from (2.1b) after taking into account the boundary conditions (2.3a) and (2.3b), and are given by

$$\begin{aligned} \boldsymbol{\phi}_\tau &= \mathbf{T}\boldsymbol{\phi} - i\boldsymbol{\phi}\langle\mathbf{H}\rangle, \qquad & \bar{\boldsymbol{\phi}}_\tau &= \mathbf{T}\bar{\boldsymbol{\phi}} - i\langle h\rangle\bar{\boldsymbol{\phi}}, \\ \bar{\boldsymbol{\psi}}_\tau &= \mathbf{T}\bar{\boldsymbol{\psi}} - i\bar{\boldsymbol{\psi}}\langle\tilde{\mathbf{H}}\rangle, \qquad & \boldsymbol{\psi}_\tau &= \mathbf{T}\boldsymbol{\psi} - i\langle\tilde{h}\rangle\boldsymbol{\psi}, \end{aligned} \tag{2.19}$$

where $\tilde{h}$ are $\tilde{\mathbf{H}}$ are obtained from the second of (2.15). Below we outline the derivation of the equation for $\bar{\boldsymbol{\phi}}$, the rest follows similarly.

Recall that $\bar{\boldsymbol{\phi}}$ is analytic in the lower-half plane, and $\bar{\boldsymbol{\phi}} \sim \mathbf{e}_3 e^{ikx}$ as $x \to -\infty$ for all $\tau > 0$. So for $\bar{\boldsymbol{\phi}}$ the evolution equation (2.1b) needs to be modified such that $\bar{\boldsymbol{\phi}}_\tau \to 0$ as $x \to -\infty$. Using

$$\lim_{x\to\pm\infty}\int_{-\infty}^{\infty}\frac{f(\alpha)e^{\pm 2i\alpha x}}{\alpha - k}d\alpha = 0, \quad \operatorname{Im} k \lessgtr 0, \tag{2.20}$$

one can show that for $\operatorname{Im} k < 0$

$$\begin{aligned} \lim_{x\to-\infty}\mathbf{T}\bar{\boldsymbol{\phi}}e^{-ikx} &= \lim_{x\to-\infty}\frac{i}{4}\int\frac{g(\alpha)}{\alpha - k} \\ &\quad\times\begin{pmatrix}\mathbf{H} & \mathbf{m}\,e^{-2i\alpha x} \\ \mathbf{m}^\dagger\,e^{2i\alpha x} & h\end{pmatrix}\bar{\boldsymbol{\phi}}(x,k)e^{-ikx}d\alpha \\ &= \lim_{x\to-\infty}\frac{i}{4}\int\frac{g(\alpha)}{\alpha - k}\begin{pmatrix}\mathbf{m}e^{-2i\alpha x} \\ h\end{pmatrix}d\alpha = \begin{pmatrix}0 \\ i\langle h\rangle\end{pmatrix}. \end{aligned}$$

Therefore, from (2.1b) one obtains the evolution equation for the eigenfunction $\bar{\boldsymbol{\phi}}$ as given above in (2.19).

From (2.4) and the complex unitarity relation $\boldsymbol{\Phi}^\dagger(k^*)\boldsymbol{\Phi}(k) = \mathbf{I}_3$ it follows that $\mathbf{S}(k) = \boldsymbol{\Phi}^{-1}(k)\boldsymbol{\Psi}(k) = \boldsymbol{\Phi}^\dagger(k^*)\boldsymbol{\Psi}(k)$. From this one can show that the scattering coefficients $a, \bar{\mathbf{a}}$ can be expressed as bilinear combinations of the eigenfunctions:

$$a(k) = \bar{\boldsymbol{\phi}}^\dagger(x, k^*)\boldsymbol{\psi}(x, k), \quad \operatorname{Im} k \ge 0,$$
$$\bar{\mathbf{a}}(k) = \boldsymbol{\phi}^\dagger(x, k^*)\bar{\boldsymbol{\psi}}(x, k), \quad \operatorname{Im} k \le 0.$$

Therefore, from the evolution equations (2.19) and the fact that $\mathbf{T}^\dagger(k^*) = -\mathbf{T}(k)$ we obtain

$$a_\tau = i\langle h - \tilde{h}\rangle a, \quad \operatorname{Im} k > 0,$$
$$\bar{\mathbf{a}}_\tau = i\langle \mathbf{H}\rangle\bar{\mathbf{a}} - i\bar{\mathbf{a}}\langle\bar{\mathbf{a}}^\dagger\hat{\mathbf{H}}\bar{\mathbf{a}} + \bar{\mathbf{b}}\mathbf{m}^\dagger\bar{\mathbf{a}} + \bar{\mathbf{a}}^\dagger\mathbf{m}\bar{\mathbf{b}}^\dagger\rangle, \quad \operatorname{Im} k < 0. \tag{2.21}$$

Eq. (2.21) coincides with the analytic continuation of (2.17a) and (2.17d) for $a, \bar{\mathbf{a}}$. Taking into account the distributional identity $1/(p \pm i0) = 1/p \mp i\pi\delta(p)$ that holds for any $p \in \mathbb{R}$, it can be shown that as k approaches the real axis from the upper-half plane, the equation for $a(k)$ in (2.21) reduces to (2.17a), with the integral becoming a principal value integral. The same holds for the evolution equation for $\bar{\mathbf{a}}(k)$, which reduces to (2.17d) as k approaches the real axis from the lower-half plane.

The zeros of the scattering coefficient $a(k, \tau)$ correspond to the equation $a(k_j, \tau) = 0$, $j = 1, 2, \ldots, n$. Differentiating this equation implicitly and using the first equation of (2.21), one has

$$0 = \frac{da}{d\tau}(k_j, \tau) = i\langle h - \tilde{h}\rangle(k_j, \tau)a(k_j, \tau) + a'(k_j, \tau)\frac{dk_j}{d\tau} \Longrightarrow$$
$$\frac{dk_j}{d\tau} = 0,$$

where $a' := \frac{\partial a}{\partial k}$. Moreover, $a'(\tau, k_j) \neq 0$ because $k = k_j$ is a simple zero of a. Hence the location of the zero k_j of $a(k, \tau)$ is *independent* of τ.

Next we derive the evolution equation for the norming constant $\boldsymbol{\beta}_j = \boldsymbol{\eta}_j/a'(k_j)$. Evaluating the evolution equations for $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ in (2.19) at $k = k_j$, and taking into account the relation $\boldsymbol{\psi}(k_j) = \boldsymbol{\phi}(k_j)\boldsymbol{\eta}_j$ (cf. Section 2.2), one obtains

$$\partial_\tau \boldsymbol{\eta}_j = i\langle \mathbf{H} - \tilde{h}\mathbf{I}_2\rangle(k_j)\boldsymbol{\eta}_j.$$

Also, differentiating the equation for a in (2.21) with respect to k, then evaluating the resulting equation at $k = k_j$, and using $a(k_j, \tau) = 0$, one can show that

$$\partial_\tau a'(k_j) = i\langle h - \tilde{h}\rangle(k_j)a'(k_j).$$

Consequently, the evolution of the norming constant $\boldsymbol{\beta}_j$ is given by

$$\partial_\tau \boldsymbol{\beta}_j = i\langle\hat{\mathbf{H}}\rangle(k_j)\boldsymbol{\beta}_j. \tag{2.22}$$

Solutions to Eqs. (2.18), (2.22), together with the fact that the k_j's are τ-independent, provide the information necessary to solve the inverse problem (2.8) associated with the CMB equation for any $\tau > 0$. In the next section we solve the inverse problem corresponding to the soliton solutions of (1.1).

We remark that it is possible to obtain $a(k, \tau)$ and $\mathbf{b}(k, \tau)$ by solving only (2.18) for $\mathbf{r}(k, \tau)$, for $k \in \mathbb{R}$ in the following way. One can use the information on the analyticity properties of $a(k, \tau)$, its asymptotic behavior at large k and the location of its zeros (assuming they do not evolve in τ), to reconstruct the function via a scalar Riemann–Hilbert problem (see, for instance, [29]). One obtains

$$a(k, \tau) = \prod_{j=1}^{n} \frac{k - k_j}{k - k_j^*} \exp\left[\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{\log\left(1 + \|\mathbf{r}(\xi, \tau)\|^2\right)}{\xi - k}\, d\xi\right],$$
$$\operatorname{Im} k > 0.$$

Then one finds $\mathbf{b}$ via $\mathbf{b}(k, \tau) = a(k, \tau)\,\mathbf{r}(k, \tau)$. Note that by differentiating the above expression for a with respect to τ and using (2.18) for $\mathbf{r}(\xi, \tau)$, one can recover the evolution equation for $a(k, \tau)$ in (2.21).

2.4. Sharp-line limit

Here we briefly discuss the effects on the evolution equations of the scattering data in the limiting case where there is *no* inhomogeneous broadening. This case is known as the sharp-line limit whence the inhomogeneous broadening function $g(\alpha) \to \delta(\alpha - \alpha_0)$, the Dirac delta function centered around α_0. Without loss of generality, we take $\alpha_0 = 0$. For the two-level Maxwell–Bloch equations with inhomogeneous broadening, the sharp-line limit leads to the sine–Gordon equation for real optical pulses [5,30]. The sharp-line limit for the CMB equations in three-level optical media has been also considered by several authors and some exact solutions have been found (see e.g., [25,13]).

Setting $g(\alpha) = \delta(\alpha)$ in (1.2) and in the evolution operator $\mathbf{T}$ in (2.1b), yields

$$\tilde{\rho}_{av} = \tilde{\rho}(x, \tau, 0) \equiv \boldsymbol{\rho}(x, \tau), \qquad \mathbf{T} = -\frac{i\boldsymbol{\rho}}{4k},$$

and the coupling between the optical pulse and the three-level optical medium system in this case is described by the following system of CMB equations

$$\boldsymbol{\Omega}_\tau = \frac{1}{4}\left[\boldsymbol{\rho}, \mathbf{J}\right], \qquad \boldsymbol{\rho}_x = \left[\boldsymbol{\Omega}, \boldsymbol{\rho}\right].$$

Since we are considering localized pulses such that $\boldsymbol{\Omega} \to 0$ as $x \to \pm\infty$ for all τ, we must have that $\partial_\tau\boldsymbol{\Omega} \to 0$ as $x \to \pm\infty$. This condition along with the equation $\boldsymbol{\rho}_x = [\boldsymbol{\Omega}, \boldsymbol{\rho}]$, imposes a nontrivial constraint on the boundary values for the Bloch matrix $\boldsymbol{\rho}$, namely, one must have

$$\lim_{x\to\pm\infty}[\boldsymbol{\rho}, \mathbf{J}] = [\boldsymbol{\rho}_\pm, \mathbf{J}] = 0, \quad \boldsymbol{\rho}_\pm \equiv \lim_{x\to\pm\infty}\boldsymbol{\rho}(x, \tau).$$

The Bloch matrix $\boldsymbol{\rho}$ and its boundary values $\boldsymbol{\rho}_\pm$ can be calculated via the "squared" eigenfunction $\mathbf{F} = \boldsymbol{\Phi}\mathbf{C}\boldsymbol{\Phi}^{-1}$ in the same way as discussed previously in Section 2.2. Since $\mathbf{F}(k = 0)$ satisfies the same ordinary differential equation given above for $\boldsymbol{\rho}$, from Eqs. (2.11)–(2.13) at $\alpha = 0$ one obtains

$$\boldsymbol{\rho}(x, \tau) = \boldsymbol{\Phi}(x, \tau, 0)\mathbf{C}_0\boldsymbol{\Phi}^{-1}(x, \tau, 0), \qquad \boldsymbol{\rho}_- = \mathbf{C}_0,$$
$$\boldsymbol{\rho}_+ = \mathbf{S}_0^{-1}\mathbf{C}_0\mathbf{S}_0,$$

where $\mathbf{C}_0 = \mathbf{C}(\tau, \alpha = 0)$, $\mathbf{C}_0^\dagger = \mathbf{C}_0$, and $\mathbf{S}_0 = \mathbf{S}(k = 0, \tau)$, $\mathbf{S}_0^\dagger = \mathbf{S}_0^{-1}$. Moreover, from the constraint $[\boldsymbol{\rho}_\pm, \mathbf{J}] = 0$ it follows that both $\mathbf{C}_0$ and $\mathbf{S}_0^{-1}\mathbf{C}_0\mathbf{S}_0$ must commute with $\mathbf{J}$, i.e., they must be block-diagonal. However, if we express the matrices $\mathbf{C}_0$ and $\mathbf{S}_0$ in block form as

$$\mathbf{C}_0 = \begin{pmatrix} \mathbf{H}_0 & 0 \\ 0 & h_0 \end{pmatrix}, \qquad \mathbf{S}_0 = \begin{pmatrix} \bar{\mathbf{a}}_0 & \mathbf{b}_0 \\ \bar{\mathbf{b}}_0^\dagger & a_0 \end{pmatrix},$$

where $\mathbf{H}_0^\dagger = \mathbf{H}_0$, $h_0 \in \mathbb{R}$ depend on the initial atomic configurations of the optical medium, and $\bar{\mathbf{a}}_0, \mathbf{b}_0, \bar{\mathbf{b}}_0, a_0$ are the components of the scattering matrix $\mathbf{S}(k)$ at $k = 0$, then it is easy to verify that $\mathbf{S}_0^{-1}\mathbf{C}_0\mathbf{S}_0$ is block-diagonal if and only if $\mathbf{b}_0(\tau) = 0$. Thus, in order to obtain localized pulse solutions in the sharp-line limit via the IST formalism, the initial and boundary data need to be restricted such that (i) $\boldsymbol{\rho}_-(\tau)$ is block-diagonal, and (ii) $\mathbf{b}_0 = 0$ for all τ. Note that it follows from (2.5) that the remaining scattering coefficients at $k = 0$ satisfy $\bar{\mathbf{b}}_0 = 0$, $\bar{\mathbf{a}}_0^{-1} = \bar{\mathbf{a}}_0^\dagger$, $|a_0| = 1$. Hence $\mathbf{S}_0$ is a 3×3, block-diagonal, unitary matrix.

Next, we derive the evolution equation for the scattering matrix $\mathbf{S}(k, \tau)$ in the sharp-line limit starting from the evolution equations for the eigenfunctions, as was done in Section 2.3. The eigenfunctions evolve according to

$$\boldsymbol{\Phi}_\tau = -\frac{i}{4k}\left(\boldsymbol{\rho}\boldsymbol{\Phi} - \boldsymbol{\Phi}\boldsymbol{\rho}_-\right), \qquad \boldsymbol{\Psi}_\tau = -\frac{i}{4k}\left(\boldsymbol{\rho}\boldsymbol{\Psi} - \boldsymbol{\Psi}\boldsymbol{\rho}_+\right),$$

so that the boundary conditions (2.3a) and (2.3b) are preserved for all τ. Then using the relation $\boldsymbol{\Psi} = \boldsymbol{\Phi}\mathbf{S}$, one obtains for $k \in \mathbb{R}$,

$$\mathbf{S}_\tau = -\frac{i}{4k}\left(\boldsymbol{\rho}_-\mathbf{S} - \mathbf{S}\boldsymbol{\rho}_+\right),$$

with $\boldsymbol{\rho}_- = \mathbf{C}_0$ and $\boldsymbol{\rho}_+ = \mathbf{S}_0^{-1}\mathbf{C}_0\mathbf{S}_0$ as given above. Since $\mathbf{C}_0$ and $\mathbf{S}_0$ are both block-diagonal, the evolution equations for the components of the scattering matrix are readily calculated as well, and are as follows

$$\bar{\mathbf{a}}_\tau = -\frac{i}{4k}\left(\hat{\mathbf{H}}_0\bar{\mathbf{a}} - \bar{\mathbf{a}}(\bar{\mathbf{a}}_0^\dagger\hat{\mathbf{H}}_0\bar{\mathbf{a}}_0)\right), \qquad a_\tau = 0 \tag{2.23a}$$

$$\mathbf{b}_\tau = -\frac{i}{4k}\hat{\mathbf{H}}_0\mathbf{b}, \qquad \bar{\mathbf{b}}_\tau^\dagger = \frac{i}{4k}\bar{\mathbf{b}}^\dagger(\bar{\mathbf{a}}_0^\dagger\hat{\mathbf{H}}_0\bar{\mathbf{a}}_0), \tag{2.23b}$$

where $\hat{\mathbf{H}}_0 = \mathbf{H}_0 - h_0\mathbf{I}_2$. Therefore, $a(k)$ is independent of τ, consequently, the reflection coefficient $\mathbf{r} = \mathbf{b}/a$ satisfies the same evolution equation as $\mathbf{b}$ in (2.23b). One can also verify from the second equation in (2.23b) that $\bar{\mathbf{r}}$ in (2.7) satisfies the same equation as $\mathbf{r}$ consistent with the symmetry $\bar{\mathbf{r}} = -\mathbf{r}$. Moreover, $\|\mathbf{b}\|$, $\|\bar{\mathbf{b}}\|$ are independent of τ. When $\hat{\mathbf{H}}_0$ is independent of τ, the reflection coefficient $\mathbf{r}(k,\tau)$ can be expressed as

$$\mathbf{r}(k,\tau) = \exp\left(-\frac{i\hat{\mathbf{H}}_0}{4k}\tau\right)\mathbf{r}(k,0), \tag{2.24}$$

and has an essential singularity at $k = 0$, which is true even when $\hat{\mathbf{H}}_0 = \hat{\mathbf{H}}_0(\tau)$. When $k \neq 0$, $\mathbf{r}(k,\tau)$ exhibits oscillatory behavior since the eigenvalues of the Hermitian matrix $\hat{\mathbf{H}}_0$ are real. The norm $\|\mathbf{r}(k)\|$ remains constant, instead of decaying with respect to τ as in the inhomogeneous broadening case.

It is instructive to examine the passage to the sharp-line limit of the evolution equation (2.18) for the reflection coefficient by considering a simple model situation where $\mathbf{m} = 0$, $\hat{\mathbf{H}} = \hat{\mathbf{H}}(\alpha)$ i.e., independent of τ, and the inhomogeneous broadening function $g(\alpha)$ is a simple Lorentzian function of α

$$g(\alpha) = \frac{p}{\pi(\alpha^2 + p^2)}, \quad p > 0,$$

so that $g(\alpha) \to \delta(\alpha)$ as $p \to 0$. Then (2.18) reduces to its homogeneous version $\mathbf{r}_\tau = i\mathscr{H}\mathbf{r}$ where the matrix $\mathscr{H}$ can be explicitly obtained by evaluating the integral in (2.18) with the Lorentzian $g(\alpha)$. One gets

$$\mathbf{r}_\tau = -\frac{i}{4}\frac{\hat{\mathbf{H}}(-ip)}{k+ip}\mathbf{r} \Rightarrow \mathbf{r}(k,\tau) = \exp\Big(-\frac{i\hat{\mathbf{H}}(-ip)}{4(k+ip)}\tau\Big)\mathbf{r}(k,0).$$

If we assume that $\hat{\mathbf{H}}$ is analytic near $k = 0$, then for $k \neq 0$, one recovers (2.24) as $p \to 0$. On the other hand, when $k = 0$, and $p \to 0$, $\mathbf{r}(0,\tau)$ and *all* k-derivatives of $\mathbf{r}(k,\tau)$ at $k = 0$ vanish instantaneously (in k), for all $\tau > 0$, due to the fact that the eigenvalues of the positive definite matrix $\hat{\mathbf{H}}$ are positive. Then from the second equation in (2.5c), it follows that $\mathbf{b}_0(\tau) = \mathbf{b}(0,\tau) = 0$ and $|a_0(\tau)| = |a(0,\tau)| = 1$, consistent with the restrictions on the scattering data for the sharp-line limit obtained earlier in this subsection.

The evolution equations for the scattering data analytic in the upper and lower half-planes are also determined in the sharp-line limit by considering the evolution of the eigenfunctions analytic in the appropriate half-planes. It can be shown that the evolution equations for the matrix $\bar{\mathbf{a}}$ and the scalar a given by (2.23a) also hold for $\operatorname{Im} k < 0$ and $\operatorname{Im} k > 0$, respectively. Moreover, the evolution equation for the norming constant $\boldsymbol{\beta}_j$ corresponding to the eigenvalue k_j is given by

$$\partial_\tau\boldsymbol{\beta}_j = -\frac{i}{4k_j}\hat{\mathbf{H}}_0\boldsymbol{\beta}_j, \quad j = 1, 2, \ldots, n,$$

where $\hat{\mathbf{H}}_0$ is a Hermitian matrix with real eigenvalues. The symmetry conditions on the reflection coefficient and the norming constants for real solutions derived in Section 2.2 hold in the sharp-line limit as well.

3. Reflection-less potentials and soliton solutions

In this section we consider the inverse problem associated with a class of potentials for which the scattering problem (2.1a) consists of only a finite number of discrete zeros in the upper-half plane $\{k_j\}_{j=1}^n$ of $a(k,\tau)$, and the reflection coefficient $\mathbf{r}(k,\tau) = 0$ for all $k \in \mathbb{R}$ and $\tau \geq 0$. Such potentials are called reflection-less potentials and correspond to pure soliton solutions of the CMB equations (1.1). In this case, the inverse problem is purely linear algebraic, hence can be solved in closed form. Consequently, exact explicit expressions for $\boldsymbol{\Omega}$ and $\boldsymbol{\rho}$ corresponding to multi-soliton solutions can be found. It follows from the inhomogeneous equation (2.18) that in order for $\mathbf{r}(k,\tau) = 0$ to hold for all $\tau > 0$, one must have $\mathbf{r}(k,0) = 0$ and the forcing term $\mathbf{m}(k,\tau) = 0$. Consequently, the matrix $\mathbf{C}$ in (2.15), as well as the scattering matrix $\mathbf{S}$ must be block diagonal. Then the boundary values $\tilde{\boldsymbol{\rho}}_\pm$ in (2.12) and (2.13) also become block diagonal, and are given by

$$\begin{aligned}\tilde{\boldsymbol{\rho}}_-(\alpha,\tau) &= \mathbf{C} = \begin{pmatrix}\mathbf{H}(\alpha,\tau) & 0\\ 0 & h(\alpha,\tau)\end{pmatrix},\\ \tilde{\boldsymbol{\rho}}_+(\alpha,\tau) &= \begin{pmatrix}\left(\bar{\mathbf{a}}^\dagger\mathbf{H}\bar{\mathbf{a}}\right)(\alpha,\tau) & 0\\ 0 & h(\alpha,\tau)\end{pmatrix}, \quad \alpha \in \mathbb{R},\end{aligned} \tag{3.1}$$

since $\mathbf{C}$ commutes with $\mathbf{J}$. In particular, notice here that the off-diagonal components $\tilde{\mathbf{m}} = 0$ in the matrix $\tilde{\boldsymbol{\rho}}_+$, since both $\mathbf{m} = 0$ and $\mathbf{b} = 0$ in the reflection-less case. Thus, $\tilde{\boldsymbol{\rho}}_+$ is determined only in terms of $\tilde{\boldsymbol{\rho}}_-$ and the scattering coefficient $\bar{\mathbf{a}}$. However, recall that $\tilde{\mathbf{m}} \neq 0$ in the general case (see Section 2.3.3), when $\mathbf{b} \neq 0$.

The inverse problem for the eigenfunctions in the reflection-less case, i.e., Eq. (2.8) with $\mathbf{r}(\xi) = 0$ takes the form

$$\begin{aligned}\bar{\boldsymbol{\phi}}(x,k)e^{-ikx} &= \mathbf{e}_3 + \sum_{\ell=1}^n \frac{\boldsymbol{\phi}(x,k_\ell)\boldsymbol{\beta}_\ell e^{-ik_\ell x}}{k-k_\ell},\\ \boldsymbol{\phi}(x,k)e^{ikx} &= (\mathbf{e}_1\,\mathbf{e}_2) - \sum_{\ell=1}^n \frac{\bar{\boldsymbol{\phi}}(x,k_\ell^*)\boldsymbol{\beta}_\ell^\dagger e^{ik_\ell^* x}}{k-k_\ell^*}.\end{aligned} \tag{3.2}$$

If one sets $k = k_j^*$ and $k = k_j$, respectively, in the above equations for $\bar{\boldsymbol{\phi}}$ and $\boldsymbol{\phi}$, then Eqs. (3.2) for the inverse problem reduce to a linear algebraic system for the unknowns $\mathbf{M}_j(x) \equiv \boldsymbol{\phi}(x,k_j)e^{ik_jx}$ and $\bar{\mathbf{M}}_j(x) \equiv \bar{\boldsymbol{\phi}}(x,k_j^*)e^{-ik_j^*x}$, namely,

$$\bar{\mathbf{M}}_j(x) = \mathbf{e}_3 + \sum_{\ell=1}^n \frac{\mathbf{M}_\ell(x)\boldsymbol{\beta}_\ell e^{-2ik_\ell x}}{k_j^* - k_\ell},$$

$$\mathbf{M}_j(x) = (\mathbf{e}_1\,\mathbf{e}_2) - \sum_{\ell=1}^n \frac{\bar{\mathbf{M}}_\ell(x)\boldsymbol{\beta}_\ell^\dagger e^{2ik_\ell^* x}}{k_j - k_\ell^*}.$$

Replacing the solutions of the linear system back into (3.2) gives the solution of the inverse problem for the eigenfunctions $\boldsymbol{\phi}(x,k)$ and $\bar{\boldsymbol{\phi}}(x,k)$ in terms of $\mathbf{M}_j$, $\bar{\mathbf{M}}_j$ as

$$\begin{aligned}\mu(x,k) &= \boldsymbol{\Phi}(x,k)e^{-ik\mathbf{J}x}\\ &= \mathbf{I}_3 + \sum_{j=1}^n\left(-\frac{\bar{\mathbf{M}}_j(x)\tilde{\boldsymbol{\beta}}_j^\dagger(x)}{k-k_j^*}, \frac{\mathbf{M}_j(x)\tilde{\boldsymbol{\beta}}_j(x)}{k-k_j}\right),\end{aligned} \tag{3.3}$$

where we have defined $\tilde{\boldsymbol{\beta}}_j(x) := \boldsymbol{\beta}_j e^{-2ik_j x}$. The τ-dependence of $\mathbf{M}_j$, $\bar{\mathbf{M}}_j$, which enters only through the norming constants $\boldsymbol{\beta}_j(\tau)$, is suppressed here and below for brevity. In order to solve for, say, $\bar{\mathbf{M}}_j$ one introduces the $n \times n$ matrices

$$\mathbf{K}_{ij} = \frac{1}{k_i^* - k_j}, \qquad \mathbf{B}_{ij} = \frac{\tilde{\boldsymbol{\beta}}_j^\dagger \tilde{\boldsymbol{\beta}}_i}{k_i - k_j^*},$$

and can write

$$\sum_{j=1}^n (\mathbf{I}_n + \mathbf{KB})_{ij}\, \bar{\mathbf{M}}_j = \mathbf{e}_3 + (\mathbf{e}_1\, \mathbf{e}_2) \sum_{j=1}^n \mathbf{K}_{ij} \tilde{\boldsymbol{\beta}}_j.$$

Multiplying the above equation by $\mathbf{K}^{-1}$, and using the well-known relations for the Cauchy matrix $\mathbf{K}$

$$\left(\mathbf{K}^{-1}\right)_{ij} = \frac{a_i a_j^*}{k_i - k_j^*}, \quad a_i = \frac{\prod_{j=1}^n (k_i - k_j^*)}{\prod_{j\neq i} (k_i - k_j)}, \tag{3.4}$$

$$\sum_{j=1}^n \left(\mathbf{K}^{-1}\right)_{ij} = -a_i,$$

one obtains the following compact equation for $\bar{\mathbf{M}}_j(x)$

$$\sum_{j=1}^n \tilde{\mathbf{K}}_{ij} \bar{\mathbf{M}}_j = \mathbf{v}_i, \quad \mathbf{v}_i = \left(\tilde{\boldsymbol{\beta}}_i, -a_i\right)^T, \tag{3.5}$$

where $\tilde{\mathbf{K}} = \mathbf{K}^{-1} + \mathbf{B}$ is a generalized Cauchy matrix of the form

$$\tilde{\mathbf{K}}_{ij} = \frac{a_i a_j^* + \tilde{\boldsymbol{\beta}}_j^\dagger \tilde{\boldsymbol{\beta}}_i}{k_i - k_j^*} = \frac{\mathbf{v}_j^\dagger \mathbf{v}_i}{k_i - k_j^*}.$$

Eq. (3.5) is a vector-valued system of linear equations, i.e., a linear algebraic system for each component of the 3-component column vector $\bar{\mathbf{M}}_j = \left(\bar{M}_{j1}, \bar{M}_{j2}, \bar{M}_{j3}\right)^T$. From the solution of (3.5), one can also construct the 3-component column vector $\mathbf{M}_j \tilde{\boldsymbol{\beta}}_j$ as follows

$$\mathbf{M}_j \tilde{\boldsymbol{\beta}}_j = (\mathbf{e}_1\, \mathbf{e}_2)\, \tilde{\boldsymbol{\beta}}_j - \sum_{\ell=1}^n \mathbf{B}_{j\ell} \bar{\mathbf{M}}_\ell = \sum_{\ell=1}^n (\mathbf{K}^{-1})_{j\ell} \bar{\mathbf{M}}_\ell + a_j \mathbf{e}_3,$$

which is obtained from the coupled linear system of equations for $\mathbf{M}_j$, $\bar{\mathbf{M}}_j$ given earlier. Thus, the eigenfunction in (3.3) is completely determined by the solution of (3.5).

3.1. *n*-soliton solution

As discussed in Section 2.2, the potential $\boldsymbol{\Omega}$ can be reconstructed by means of the large k-asymptotics of $\mu(k)$ which in the reflection-less case is given by (3.3). More precisely, $\boldsymbol{\Omega}$ is given by the formulas in (2.9) where μ_1 is the coefficient of the k^{-1}-term in the asymptotic expansion of (3.3). This yields

$$(\Omega_1,\ \Omega_2)^T = 4i \sum_{j=1}^n \bar{M}_{j3}^* \tilde{\boldsymbol{\beta}}_j = -4i \sum_{j,\ell=1}^n \left(\tilde{\mathbf{K}}^*\right)^{-1}_{j\ell} a_\ell^* \tilde{\boldsymbol{\beta}}_j,$$

where $\tilde{\mathbf{K}}^*$ is the complex conjugate of the generalized Cauchy matrix $\tilde{\mathbf{K}}$ defined above. Using Cramer's formula, the above expression can be written as the ratio of two determinants,

$$\Omega_s = 4i \frac{\begin{vmatrix} \tilde{\mathbf{K}}^* & \mathbf{a}^* \\ \tilde{\boldsymbol{\beta}}^{(s)} & 0 \end{vmatrix}}{|\tilde{\mathbf{K}}^*|}, \quad s = 1, 2, \tag{3.6}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)^T$ is a column vector whose components are given in (3.4), and the row vector $\tilde{\boldsymbol{\beta}}^{(s)}$ is defined by $\tilde{\boldsymbol{\beta}}^{(s)} = (\tilde{\boldsymbol{\beta}}_1^{(s)}, \tilde{\boldsymbol{\beta}}_2^{(s)}, \ldots, \tilde{\boldsymbol{\beta}}_n^{(s)})$ where $\tilde{\boldsymbol{\beta}}_j^{(s)}$ denotes the sth component of each 2-component vector $\tilde{\boldsymbol{\beta}}_j$.

The reflection-less potential $\boldsymbol{\Omega}$ given by (3.6) corresponds to the electric field envelope for the n-soliton solution for the CMB equation (1.1). It is parametrized by the discrete eigenvalues and the norming constants: $\{k_j, \boldsymbol{\beta}_j\}_{j=1}^n$, and, in particular, its dependence on x and τ is solely through the quantities $\tilde{\boldsymbol{\beta}}_j(x,\tau) = \boldsymbol{\beta}_j(\tau) e^{-2ik_j x}$. Furthermore, the electric field envelopes $\Omega_s(x,\tau)$ given by (3.6) are regular for all $x \in (-\infty, \infty)$ and $\tau \geq 0$ because the determinant $|\tilde{\mathbf{K}}^*| \neq 0$. The last assertion follows from the fact that for any non-zero vector $y \in \mathbb{C}^n$, the inner product

$$y^\dagger \tilde{\mathbf{K}}^* y = \sum_{\ell,j=1}^n \frac{y_\ell^* \mathbf{v}_\ell^\dagger y_j \mathbf{v}_j}{k_\ell^* - k_j} = -i \sum_{\ell,j=1}^n y_\ell^* \mathbf{v}_\ell^\dagger y_j \mathbf{v}_j \int_0^\infty e^{-i(k_\ell^* - k_j)\xi} d\xi$$
$$= -i \left| \sum_{j=1}^n y_j \mathbf{v}_j \int_0^\infty e^{ik_j \xi} d\xi \right|^2 \neq 0,$$

where the integral converges since $\operatorname{Im} k_j > 0$. Hence, none of the eigenvalues of the matrix $\tilde{\mathbf{K}}^*$ is zero.

The norming constant $\boldsymbol{\beta}_j(\tau)$, which gives rise to the τ-dependence of the potential $\boldsymbol{\Omega}(x,\tau)$ in (3.6), satisfies the linear evolution equation (2.22). The 2×2 Hermitian matrix $\hat{\mathbf{H}} = \mathbf{H} - h\mathbf{I}_2$ in (2.22) is given by the boundary condition $\tilde{\boldsymbol{\rho}}_-$ in (3.1) corresponding to the initial atomic configuration of the 3-level medium. In the following we assume that $\hat{\mathbf{H}}$ is independent of τ, i.e. $\hat{\mathbf{H}} = \hat{\mathbf{H}}(\alpha)$, and is such that the complex matrix $\langle\hat{\mathbf{H}}\rangle(k_j)$ is diagonalizable for each $j = 1, 2, \ldots, n$. Then the solution of (2.22) is given by

$$\boldsymbol{\beta}_j(\tau) = c_j^{(1)} e^{i\lambda_j^{(1)}\tau} \mathbf{v}_j^{(1)} + c_j^{(2)} e^{i\lambda_j^{(2)}\tau} \mathbf{v}_j^{(2)}, \tag{3.7}$$

where $\mathbf{v}_j^{(\ell)}$ is the eigenvector (with $\|\mathbf{v}_j^{(\ell)}\| = 1$) of $\langle\hat{\mathbf{H}}\rangle(k_j)$ associated to the eigenvalue $\lambda_j^{(\ell)}$, $\ell = 1, 2$, and $c_j^{(1)}$, $c_j^{(2)}$ are complex constants related to the initial value $\boldsymbol{\beta}_j(0)$ of the norming constants. Moreover, the complex eigenvalues $\lambda_j^{(\ell)}$ can be taken such that without loss of generality

$$\operatorname{Im} \lambda_j^{(1)} \leq \operatorname{Im} \lambda_j^{(2)}, \quad j = 1, \ldots, n. \tag{3.8}$$

If $\hat{\mathbf{H}}$ is independent of α, then the eigenvalues of $\langle\hat{\mathbf{H}}\rangle(k_j)$ are proportional to the eigenvalues of $\hat{\mathbf{H}}$, and the eigenvectors are the same as those of $\hat{\mathbf{H}}$. The soliton solutions in the case when $\hat{\mathbf{H}}$ is a constant diagonal matrix have been studied in [26].

Real solitons: The conditions under which the electric field components $\Omega_1(x,\tau)$, $\Omega_2(x,\tau)$ and the Bloch matrix $\tilde{\rho}(x,\tau)$ are real were discussed in Section 2.2. For pure n-soliton solutions the scattering data consist of p complex eigenvalue pairs $(k_j, -k_j^*)$ and q purely imaginary eigenvalues $i\eta_j$ together with the corresponding norming constants which must satisfy the symmetry relations

$$\boldsymbol{\beta}|_{k=-k_j^*}(\tau) = -\boldsymbol{\beta}_j^*(\tau), \qquad \boldsymbol{\beta}|_{k=i\eta_j}(\tau) = i\mathbf{b}_j(\tau), \quad \mathbf{b}_j \in \mathbb{R}^2$$

for all $\tau \geq 0$. If the norming constants are initially chosen such that they satisfy the above conditions at $\tau = 0$, it must still be ensured that the symmetry relations are preserved by the evolution equation (2.22) of the norming constants. This imposes certain conditions on the choice of the matrix $\hat{\mathbf{H}}(\alpha)$ as shown below. When $\hat{\mathbf{H}}$ is independent of τ the solution of (2.22) can be formally presented as $\boldsymbol{\beta}_j(\tau) = \exp\left(i\tau\langle\hat{\mathbf{H}}\rangle(k_j)\right)\boldsymbol{\beta}_j(0)$. If $\boldsymbol{\beta}|_{k=-k_j^*}(0) = -\boldsymbol{\beta}_j^*(0)$, then

$\boldsymbol{\beta}|_{k=-k_j^*}(\tau) = -\boldsymbol{\beta}_j^*(\tau)$ holds for all $\tau > 0$ if and only if

$$\langle\hat{\mathbf{H}}\rangle(-k_j^*) = -\langle\hat{\mathbf{H}}\rangle^*(k_j) \Rightarrow$$
$$\int \frac{g(\alpha)\hat{\mathbf{H}}(\alpha)}{k_j-\alpha}d\alpha = \int \frac{g(\alpha)\hat{\mathbf{H}}^T(\alpha)}{k_j+\alpha}d\alpha, \tag{3.9}$$

where the last equality follows after complex conjugation and using $\hat{\mathbf{H}}^\dagger(\alpha) = \hat{\mathbf{H}}(\alpha)$. By equating real and imaginary parts of various matrix elements, it is easy to verify that (3.9) holds under the following conditions: (i) the distribution function $g(\alpha)$ is an even function of α; (ii) the diagonal elements of $\hat{\mathbf{H}}(\alpha)$ are even functions of α; and (iii) the real part of the off-diagonal elements of $\hat{\mathbf{H}}(\alpha)$ is even while the imaginary part of the off-diagonal elements of $\hat{\mathbf{H}}(\alpha)$ is an odd function of α. These conditions are sufficient but not necessary, and are automatically satisfied when, for instance, $\hat{\mathbf{H}}(\alpha)$ is a constant diagonal, or a constant real symmetric matrix. The above argument holds in the case of purely imaginary eigenvalues $k = i\eta_j$ as well, whence the corresponding norming constants $\boldsymbol{\beta}(i\eta_j,\tau)$ remain purely imaginary for all $\tau > 0$ if and only if (3.9) holds. In addition, (3.9) implies that the matrix $\langle\hat{\mathbf{H}}\rangle(i\eta_j)$ is a purely imaginary matrix, and the above mentioned conditions are also sufficient to ensure this.

3.2. One-soliton solution

Let us now consider the case of one discrete eigenvalue for the scattering problem, $k_1 = \xi + i\eta$ with $\eta > 0$. Let $\boldsymbol{\beta}(\tau) = c^{(1)}e^{i\lambda^{(1)}\tau}\mathbf{v}^{(1)} + c^{(2)}e^{i\lambda^{(2)}\tau}\mathbf{v}^{(2)}$ denote the associated norming constant, where $\lambda^{(1)},\lambda^{(2)}$ are the eigenvalues of the matrix $\langle\hat{\mathbf{H}}\rangle(k_1)$ and $\mathbf{v}^{(1)},\mathbf{v}^{(2)}$ are the corresponding eigenvectors. One can write $\tilde{\boldsymbol{\beta}}(x,\tau) = \boldsymbol{\beta}(\tau)e^{-2ik_1x} = e^{i(\lambda\tau-2k_1x)}\mathbf{p}(\tau)$ where λ is either $\lambda^{(1)}$ or $\lambda^{(2)}$, and $\mathbf{p}(\tau)$ is defined accordingly, depending on the cases to be discussed below. Thus we have,

$$\tilde{\boldsymbol{\beta}}(x,\tau) = 2\eta e^{-2i\xi x + i\mathrm{Re}\,\lambda\tau}\exp[2\eta x - \mathrm{Im}\,\lambda\tau + \delta(\tau)]\hat{\mathbf{p}}(\tau),$$
$$\delta(\tau) = \ln\frac{\|\mathbf{p}(\tau)\|}{2\eta}, \tag{3.10}$$

where the unit vector $\hat{\mathbf{p}} = \mathbf{p}/\|\mathbf{p}\|$. Then from (3.6) with $n = 1$, the one-soliton solution for the optical pulses can be expressed in the form

$$(\Omega_1(x,\tau),\Omega_2(x,\tau))^T$$
$$= 4i\eta e^{-2i\xi x + i\mathrm{Re}\,\lambda\tau}\mathrm{sech}\,[2\eta x - \mathrm{Im}\,\lambda\tau + \delta(\tau)]\,\hat{\mathbf{p}}(\tau). \tag{3.11}$$

Recall that the constants $c^{(1)}, c^{(2)}$ depend on the initial value $\boldsymbol{\beta}(0)$ of the norming constant and that the eigenvalues that determine the time dependence of the norming constant $\boldsymbol{\beta}(\tau)$ have been ordered such that $\mathrm{Im}\,\lambda^{(1)} \le \mathrm{Im}\,\lambda^{(2)}$. Then one can distinguish several cases depending on the initial condition $\boldsymbol{\beta}(0)$.

Case (i). If $c^{(1)} \neq 0$, i.e., if the initial value of the norming constant $\boldsymbol{\beta}(0)$ does not coincide with the eigenvector $\mathbf{v}^{(2)}$ of $\langle\hat{\mathbf{H}}\rangle(k_1)$, then $\lambda = \lambda^{(1)}$ in (3.10) and (3.11). In addition,

$$\mathbf{p}(\tau) = c^{(1)}\mathbf{v}^{(1)} + c^{(2)}e^{i(\lambda^{(2)}-\lambda^{(1)})\tau}\mathbf{v}^{(2)},$$

where $e^{i\left(\lambda^{(2)}-\lambda^{(1)}\right)\tau}$ decays exponentially for $\tau > 0$, provided $\mathrm{Im}\,\lambda^{(1)} < \mathrm{Im}\,\lambda^{(2)}$. In this case, both the soliton velocity and the polarization vector $\mathbf{p}(\tau)$ depend on the propagation distance τ along the optical medium in contrast to the VNLS one-soliton solution where the soliton velocity and polarization are constant. As $\tau \to \infty$, $e^{i(\lambda^{(2)}-\lambda^{(1)})\tau} \to 0$, then the one-soliton solution has the asymptotic form

$$\left(\Omega_1,\ \Omega_2\right)^T \sim 4i\eta e^{-2i\xi x + i\mathrm{Re}\,\lambda^{(1)}\tau + i\arg c^{(1)}}$$
$$\times\,\mathrm{sech}\left[2\eta x - \mathrm{Im}\,\lambda^{(1)}\tau + \ln\frac{|c^{(1)}|}{2\eta}\right]\mathbf{v}^{(1)},$$

which shows that the asymptotic value of the soliton polarization vector coincides with the eigenvector $\mathbf{v}^{(1)}$ of $\langle\hat{\mathbf{H}}\rangle(k_1)$. Recall that the coordinates (x,τ) are related to the normalized time t and the normalized propagation distance z by $x = t - z/c$ and $\tau = z$. In these physical coordinates, as $z \to \infty$, the soliton travels to the right with the velocity

$$v_+ = c\left(1 + \frac{\mathrm{Im}\,\lambda^{(1)}c}{2\eta}\right)^{-1},$$

which is less than the light speed c in the medium when $\mathrm{Im}\,\lambda^{(1)} > 0$, and increases with the soliton amplitude parameter η. Note that if the initial norming constant $\boldsymbol{\beta}(0)$ is precisely along the eigenvector $\mathbf{v}^{(1)}$, so that $c^{(2)} = 0$, then the asymptotic form above becomes the exact one-soliton solution for all $\tau \ge 0$. In this case, the solution is a single sech profile traveling with a constant speed v_+ and a constant polarization vector $\mathbf{v}^{(1)}$.

Case (ii). If $0 < |c^{(1)}| \ll |c^{(2)}|$, then one can define a characteristic length along the optical medium by $\tau_0 = \left(\mathrm{Im}\,\lambda^{(2)} - \mathrm{Im}\,\lambda^{(1)}\right)^{-1}\ln|c^{(2)}/c^{(1)}|$ such that the one-soliton solution behaves differently from that in Case (i) for $\tau < \tau_0$. In this case one sets $\lambda = \lambda^{(2)}$ in (3.10) and (3.11). Then the polarization vector becomes

$$\mathbf{p}(\tau) = (c^{(1)}\mathbf{v}^{(1)}e^{i(\lambda^{(1)}-\lambda^{(2)})\tau} + c^{(2)}\mathbf{v}^{(2)}).$$

For $\tau \ll \tau_0$, this solution has the asymptotic form

$$\left(\Omega_1,\ \Omega_2\right)^T \sim 4i\eta e^{-2i\xi x + i\mathrm{Re}\,\lambda^{(2)}\tau + i\arg c^{(2)}}$$
$$\times\,\mathrm{sech}\left[2\eta x - \mathrm{Im}\,\lambda^{(2)}\tau + \ln\frac{|c^{(2)}|}{2\eta}\right]\mathbf{v}^{(2)},$$

which is a traveling wave moving with a velocity v_- whose expression is obtained by replacing $\lambda^{(1)}$ by $\lambda^{(2)}$ in the expression for v_+ above. Thus, the soliton velocity and polarization switch from v_-, $\mathbf{v}^{(2)}$ for $\tau \ll \tau_0$ to v_+, $\mathbf{v}^{(1)}$ for $\tau \gg \tau_0$. If however, $\boldsymbol{\beta}(0)$ is proportional to the eigenvector $\mathbf{v}^{(2)}$, then $c^{(1)} = 0$, and the above asymptotic solution becomes the exact one-soliton solution for all $\tau > 0$.

Case (iii). If the eigenvalues of $\langle\hat{\mathbf{H}}\rangle(k_1)$ are such that $\mathrm{Im}\,\lambda^{(1)} = \mathrm{Im}\,\lambda^{(2)} = \omega$, then from (3.10)

$$\tilde{\boldsymbol{\beta}}(x,\tau) = 2\eta e^{-2i\xi x + i\mathrm{Re}\,\lambda^{(1)}\tau}\exp[2\eta x - \omega\tau + \delta(\tau)]\hat{\mathbf{p}},$$
$$\mathbf{p}(\tau) = c^{(1)}\mathbf{v}^{(1)} + c^{(2)}e^{i(\mathrm{Re}\,\lambda^{(2)}-\mathrm{Re}\,\lambda^{(1)})\tau}\mathbf{v}^{(2)},$$

and $\delta(\tau)$ is defined the same way as in (3.10). In this case the one-soliton solution is given by

$$\left(\Omega_1,\ \Omega_2\right)^T = 4i\eta e^{-2i\xi x}\mathrm{sech}[2\eta x - \omega\tau + \delta(\tau)]\,\hat{\mathbf{p}}.$$

Therefore when $c^{(1)} \neq 0$ and $c^{(2)} \neq 0$, the soliton peak oscillates around the line $2\eta x - \omega\tau = 0$ in the (x,τ)-plane. In this case, the soliton polarization vector $\mathbf{p}$ is also a periodic function of τ with period $2\pi\left(\mathrm{Re}\,\lambda^{(2)} - \mathrm{Re}\,\lambda^{(1)}\right)^{-1}$, where we take $\mathrm{Re}\,\lambda^{(1)} < \mathrm{Re}\,\lambda^{(2)}$ without loss of generality.

Figs. 2 and 3 show plots of the magnitudes of the electric field envelopes for the exact one-soliton solutions corresponding to the discrete eigenvalue $k_1 = 1 + i/2$. In all the plots, the inhomogeneous broadening function is assumed to be a Lorentzian distribution

$$g(\alpha) = \frac{1}{\pi(\alpha^2+1)}.$$

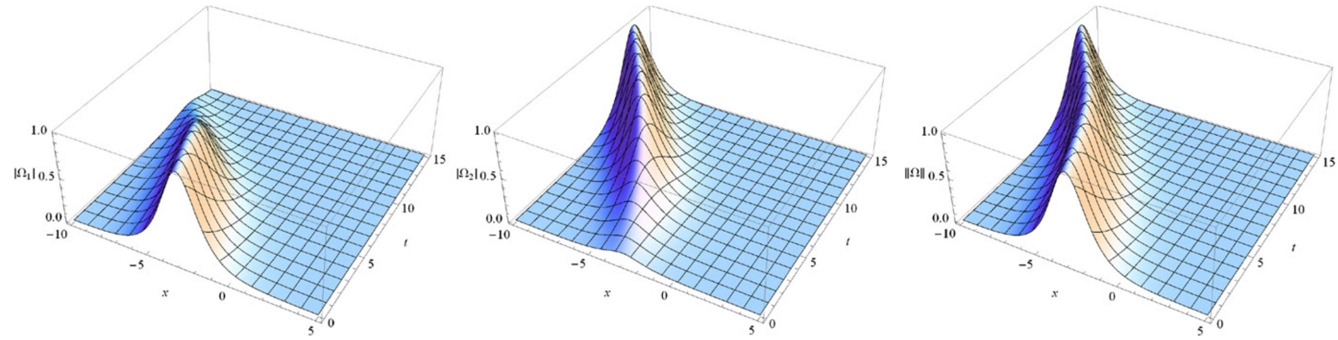

Fig. 2. One-soliton electric field envelope $\boldsymbol{\Omega}$ with Lorentzian inhomogeneous broadening and diagonal $\hat{\mathbf{H}}$. Left: modulus of the first component $|\Omega_1(x,\tau)|$. Center: modulus of the second component $|\Omega_2(x,\tau)|$. Right: norm $\|\boldsymbol{\Omega}(x,\tau)\|$.

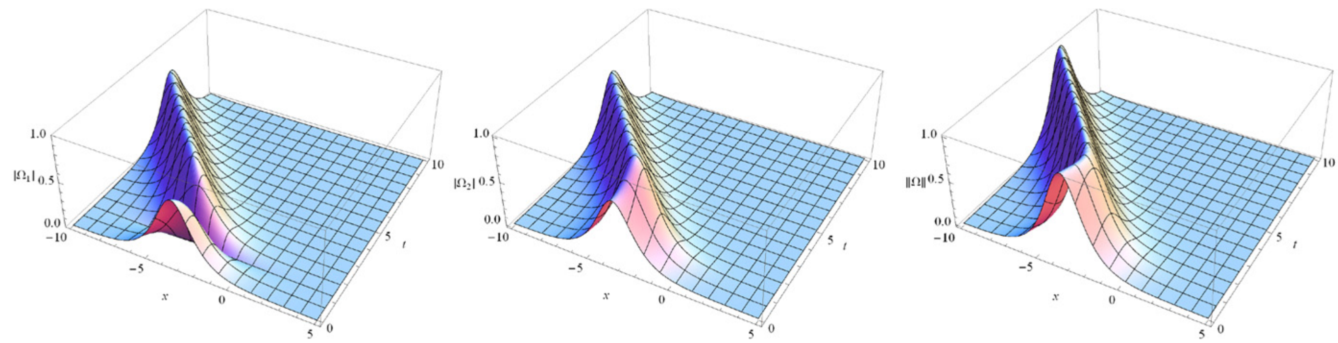

Fig. 3. Same plots as in Fig. 2 with $\hat{\mathbf{H}} = \begin{pmatrix} \pi & -8i \\ 8i & \pi \end{pmatrix}$.

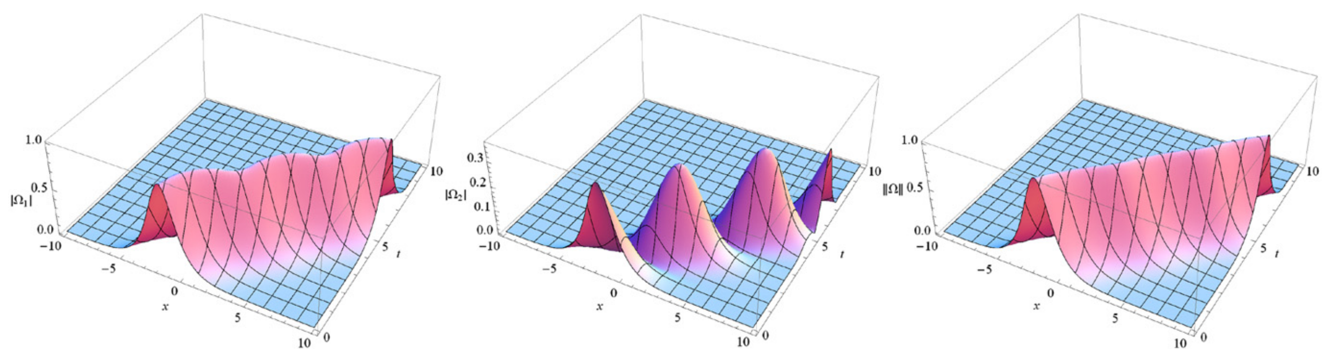

Fig. 4. Same plots as in Fig. 2. Here the matrix $\langle \hat{\mathbf{H}} \rangle (k_1) = \begin{pmatrix} \pi+2i & 1 \\ 1 & 1+2i \end{pmatrix}$ and $k_1 = 1 + i/2$.

The matrix $\hat{\mathbf{H}}$ describing the initial preparation of the medium is chosen to be independent of α. In Fig. 2 it is $\hat{\mathbf{H}} = \mathrm{diag}(\pi,\ 2\pi)$, and $\hat{\mathbf{H}} = \begin{pmatrix} \pi & -8i \\ 8i & \pi \end{pmatrix}$ in Fig. 3. In both cases $c^{(1)} = -2i$, $c^{(2)} = i$. In Fig. 4 we plot a one-soliton solution corresponding to the case where the matrix $\langle \hat{\mathbf{H}} \rangle (k_1)$ has two distinct eigenvalues with the same imaginary parts. The discrete eigenvalue is still $k_1 = 1 + i/2$, while $\langle \hat{\mathbf{H}} \rangle (k_1) = \begin{pmatrix} \pi + 2i & 1 \\ 1 & 1 + 2i \end{pmatrix}$, and for the initial value of the norming constant in (3.7) we have chosen $c^{(1)} = 1 - 2i$, $c^{(2)} = i$.

3.2.1. *Real one-soliton solution*

The scattering data for real soliton solutions were discussed earlier in Section 3.1 (see also Section 2.2). For real one-soliton solutions k_1 must be purely imaginary i.e., $k_1 = i\eta,\ \eta > 0$, and the associated norming constant $\boldsymbol{\beta}(\tau)$ must be purely imaginary for $\tau \geq 0$. Therefore, β must initially be chosen to be purely imaginary, and in addition, the matrix $\langle \hat{\mathbf{H}} \rangle (i\eta)$ must also be pure imaginary. The latter condition follows from (3.9) and guarantees that $\boldsymbol{\beta}(\tau)$ stays pure imaginary for all $\tau > 0$. A set of sufficient conditions for $\langle \hat{\mathbf{H}} \rangle (i\eta)$ to be a purely imaginary matrix is listed in Section 3.1 following Eq. (3.9).

There are two distinct types of real one-soliton solutions that can be obtained from (3.11) depending on the two distinct possibilities for the eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$ of the pure imaginary matrix $\langle \hat{\mathbf{H}} \rangle (i\eta)$.

Case (i). Pure imaginary eigenvalues: $\lambda^{(1)} = i\omega_1$, $\lambda^{(2)} = i\omega_2$, ω_1, $\omega_2 \in \mathbb{R}$, and without loss of generality $\omega_1 < \omega_2$. The corresponding eigenvectors $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ are however real. Then using (3.7) one can write

$$\boldsymbol{\beta}(\tau) = i(c_1 e^{-\omega_1 \tau} \mathbf{v}^{(1)} + c_2 e^{-\omega_2 \tau} \mathbf{v}^{(2)}),$$

for real constants c_1, c_2 which depend on the pure imaginary initial condition $\boldsymbol{\beta}(0)$. With $\xi = \mathrm{Re}\, k_1 = 0$ and $\boldsymbol{\beta}(\tau)$ as in above, it is easily verified that the one-soliton solution in (3.11) is indeed real, and its asymptotic behavior has the same essential features as the complex one-soliton solution (except of course the complex phase) corresponding to $\mathrm{Im}\, \lambda^{(1)} < \mathrm{Im}\, \lambda^{(2)}$.

Case (ii). $\lambda^{(1)} = i\lambda$, $\lambda^{(2)} = i\lambda^*$, $\lambda \in \mathbb{C}$. The associated eigenvectors are complex conjugate of each other, i.e., $\mathbf{v}^{(1)} = \mathbf{v}$ and $\mathbf{v}^{(2)} = \mathbf{v}^*$.

Then the solution from (3.7) can be expressed as

$$\boldsymbol{\beta}(\tau) = c\, e^{i\lambda\tau}\mathbf{v} - c^* e^{-i\lambda^*\tau}\mathbf{v}^* = e^{-\mathrm{Im}\,\lambda\tau}(c\, e^{i\mathrm{Re}\,\lambda\tau}\mathbf{v} - c^* e^{-i\mathrm{Re}\,\lambda\tau}\mathbf{v}^*)$$

where c is a complex constant determined by $\boldsymbol{\beta}(0)$. Since $\boldsymbol{\beta}(\tau)$ is pure imaginary, the corresponding one-soliton solution obtained from (3.11) is real. In this case the soliton polarization vector is a periodic function of τ with period $2\pi/\mathrm{Re}\,\lambda$, unlike Case (i) where the soliton polarization vector approaches a constant as $\tau \to \infty$.

3.2.2. One-soliton density matrix $\tilde{\rho}$

In the reflection-less case, the Bloch density matrix $\tilde{\boldsymbol{\rho}} = \boldsymbol{\Phi}\mathbf{C}\boldsymbol{\Phi}^{-1}$ can be written down explicitly in terms of the n-soliton eigenfunction. Since the matrix $\mathbf{C}$ is block diagonal in this case, it commutes with $\mathbf{J}$ so that $\mathbf{C} = \tilde{\boldsymbol{\rho}}_-$ (cf. (3.1)), and one can express $\tilde{\boldsymbol{\rho}}$ as

$$\tilde{\boldsymbol{\rho}}(x,\tau,\alpha) = \mu(x,\tau,\alpha)\tilde{\boldsymbol{\rho}}_-(\alpha,\tau)\mu^\dagger(x,\tau,\alpha),$$

with μ given as in (3.3). Here we consider in some detail the one-soliton density matrix when the eigenfunction μ has only one discrete eigenvalue $k = k_1$. When $n = 1$, one can solve for $\mathbf{M}_1\tilde{\boldsymbol{\beta}}_1$ and $\bar{\mathbf{M}}_1$ from the coupled linear system given above (3.3), and obtain the one-soliton eigenfunction from (3.3) as follows

$$\mu(x,\tau,k) = \mathbf{I}_3 - \frac{2i\eta}{4\eta^2 + \|\tilde{\boldsymbol{\beta}}_1\|^2}\begin{pmatrix} \dfrac{\tilde{\boldsymbol{\beta}}_1\tilde{\boldsymbol{\beta}}_1^\dagger}{k-k_1^*} & \dfrac{2i\eta\tilde{\boldsymbol{\beta}}_1}{k-k_1} \\ -\dfrac{2i\eta\tilde{\boldsymbol{\beta}}_1^\dagger}{k-k_1^*} & -\dfrac{\|\tilde{\boldsymbol{\beta}}_1\|^2}{k-k_1}\end{pmatrix}. \quad (3.12)$$

The corresponding probability density matrix $\tilde{\boldsymbol{\rho}}(x,\tau,\alpha)$ can be expressed in block form as

$$\tilde{\boldsymbol{\rho}}(x,\tau,\alpha) = \tilde{\boldsymbol{\rho}}_-(\alpha,\tau) + \begin{pmatrix}\tilde{\boldsymbol{\rho}}_u & \tilde{\boldsymbol{\rho}}_r \\ \tilde{\boldsymbol{\rho}}_r^\dagger & \tilde{\rho}_0\end{pmatrix},$$

$$\tilde{\boldsymbol{\rho}}_-(\alpha,\tau) = \begin{pmatrix}\mathbf{H} & 0 \\ 0 & h\end{pmatrix}. \quad (3.13a)$$

Setting $\zeta(x,\tau) = 2\eta x - \mathrm{Im}\,\lambda\tau + \delta(\tau)$ and $\theta(x,\tau) = -2\xi x + \mathrm{Re}\,\lambda\tau$, the components of $\tilde{\boldsymbol{\rho}}$ are given by

$$\tilde{\boldsymbol{\rho}}_u = i\eta(1+\tanh\zeta)\Big(\frac{\check{\mathbf{H}}\hat{\mathbf{p}}\hat{\mathbf{p}}^\dagger}{\alpha-k_1} - \frac{\hat{\mathbf{p}}\hat{\mathbf{p}}^\dagger\check{\mathbf{H}}}{\alpha-k_1^*}\Big),$$

$$\tilde{\rho}_0 = \eta^2\mathrm{sech}^2\zeta\,\frac{\hat{\mathbf{p}}^\dagger\hat{\mathbf{H}}\hat{\mathbf{p}}}{(\alpha-k_1)(\alpha-k_1^*)},$$

$$\tilde{\boldsymbol{\rho}}_r = -\eta e^{i\theta}\mathrm{sech}\,\zeta\Big(\frac{\check{\mathbf{H}}}{\alpha-k_1} + \frac{(1+\tanh\zeta)}{2}\frac{\hat{\mathbf{p}}^\dagger\hat{\mathbf{H}}\hat{\mathbf{p}}}{\alpha-k_1^*}\Big)\hat{\mathbf{p}},$$

$$\check{\mathbf{H}} = \hat{\mathbf{H}} - \frac{1+\tanh\zeta}{2}(\hat{\mathbf{p}}^\dagger\hat{\mathbf{H}}\hat{\mathbf{p}})\mathbf{I}_2, \quad (3.13b)$$

where $\hat{\mathbf{H}} = \mathbf{H} - h\mathbf{I}_2$ as before, and in (3.13b) the expression from (3.10) for $\tilde{\boldsymbol{\beta}}_1$ has been used. The unit vector $\hat{\mathbf{p}}$ is the polarization vector of the soliton pulse $\boldsymbol{\Omega}$.

Eqs. (3.13a) and (3.13b) give the density matrix for the three-level optical medium in the presence of a one-soliton pulse and no radiation. Note that $\tilde{\boldsymbol{\rho}} \to \tilde{\boldsymbol{\rho}}_-$ as $\zeta \to -\infty$ for a fixed τ, implying that the medium is at the state of its initial preparation long before the pulse arrives at a given location in the medium. On the other hand, long after the pulse has passed a given location τ in the medium, i.e., as $\zeta \to \infty$, the density matrix $\tilde{\boldsymbol{\rho}} \to \tilde{\boldsymbol{\rho}}_- + \mathrm{diag}(\Delta\tilde{\boldsymbol{\rho}}, 0)$, where

$$\Delta\tilde{\boldsymbol{\rho}} = \lim_{\zeta\to\infty}\tilde{\boldsymbol{\rho}}_u = 2i\eta\Big(\frac{\check{\mathbf{H}}_+\hat{\mathbf{p}}\hat{\mathbf{p}}^\dagger}{\alpha-k_1} - \frac{\hat{\mathbf{p}}\hat{\mathbf{p}}^\dagger\check{\mathbf{H}}_+}{\alpha-k_1^*}\Big),$$

$$\check{\mathbf{H}}_+ = \hat{\mathbf{H}} - (\hat{\mathbf{p}}^\dagger\hat{\mathbf{H}}\hat{\mathbf{p}})\mathbf{I}_2.$$

In particular, note that $[\tilde{\boldsymbol{\rho}}]_{33} \to [\tilde{\boldsymbol{\rho}}_-]_{33} = h$ since $\tilde{\rho}_0 \downarrow 0$ as $\zeta \to \infty$, indicating that the excited state atomic population increases, at first, due to transfer of energy from the optical pulse to the medium, then decays to its initial value long after the pulse has passed. But the atomic population at each of the ground states does change due to two-photon absorption during the optical pulse propagation through the medium as $[\Delta\tilde{\boldsymbol{\rho}}]_{ij} \neq 0,\ i,j = 1,2$. However, the *sum* of the ground state populations tends to its initial value as $\zeta \to \infty$. The latter fact is due to the vanishing of $\mathrm{Tr}(\Delta\tilde{\boldsymbol{\rho}})$ which follows from $\hat{\mathbf{p}}^\dagger\check{\mathbf{H}}_+\hat{\mathbf{p}} = 0$. Furthermore, at the location of the pulse center $\zeta = 0$, one has $\tilde{\rho}_0 = -\mathrm{Tr}(\tilde{\boldsymbol{\rho}}_u) = \eta^2(\hat{\mathbf{p}}^\dagger\hat{\mathbf{H}}\hat{\mathbf{p}})/|\alpha-k_1|^2 > 0$. Therefore, as the soliton pulse propagates, it excites the medium causing $|j\rangle \to |3\rangle$ transitions of the atomic states. Subsequently, energy is re-absorbed by the pulse as the population level of the excited state $|3\rangle$, as well as the sum of populations of the ground states $|j\rangle$, $j = 1,2$ tend to settle back to their previous values. This results in a lossless propagation of the one-soliton pulse through the optical medium which renders itself transparent to the soliton. Finally, note that even if the initial material polarization $\mathbf{m}$ must be zero (cf. (3.1)) for the reflection-less case, the soliton pulse does induce a material polarization $\tilde{\boldsymbol{\rho}}_r \neq 0$ during its passage through the medium. The medium polarization is not completely aligned with the soliton polarization $\mathbf{p}$ but differs by an off-phase component $\hat{\mathbf{H}}\mathbf{p}$ as seen from (3.13b). As noted earlier in this subsection, the soliton polarization $\mathbf{p} \to \mathbf{v}^{(j)}$, $j = 1,2$ as $\tau \to \infty$, where $\mathbf{v}^{(j)}$ are the eigenvectors of the matrix $\langle\hat{\mathbf{H}}\rangle(k_1)$. If, in addition, $\hat{\mathbf{H}}$ is independent of α, then the eigenvectors of $\hat{\mathbf{H}}$ and $\langle\hat{\mathbf{H}}\rangle(k_1)$ are the same, consequently the off-phase component $\hat{\mathbf{H}}\mathbf{p}$ becomes in-phase, in the limit $\tau \to \infty$. In this case, the material polarization $\tilde{\boldsymbol{\rho}}_r$ gets oriented closer to the soliton polarization direction as the pulse travels further along the medium.

Next, we briefly describe the situation when the density matrix can be expressed as $\tilde{\boldsymbol{\rho}} = \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger$ where the vector $\boldsymbol{\gamma} = (\gamma_1,\gamma_2,\gamma_3)^T$ represents the complex probability amplitudes for the atomic level occupation. This form of a rank one, Hermitian density matrix arises in applications such as matched pulse propagation in an absorbing medium (see e.g. [25]), where one works with the probability amplitude vector $\boldsymbol{\gamma}$ rather than the density matrix $\tilde{\boldsymbol{\rho}}$. In order for the reflection-less density matrix $\tilde{\boldsymbol{\rho}} = \mu\mathbf{C}\mu^\dagger$ to be rank one, the matrix $\mathbf{C}(\alpha,\tau)$ must also be of rank one, i.e., $\mathbf{C} = \mathbf{c}\mathbf{c}^\dagger$ for some complex 3-vector $\mathbf{c}(\alpha,\tau)$. Then one has $\boldsymbol{\gamma} = \mu\mathbf{c}$. Furthermore, the block diagonal form of $\mathbf{C}$ in (3.1) implies that either (i) $\mathbf{c} = (0,0,c_3)^T$, or (ii) $\mathbf{c} = (c_1,c_2,0)^T$. For the first case, we obtain using the one-soliton eigenfunction μ in (3.12) and the decomposition of $\tilde{\boldsymbol{\beta}}_1$ in (3.10) that

$$\gamma = \mu\mathbf{c} = \begin{pmatrix}0\\0\\c_3\end{pmatrix} + \frac{\eta c_3}{k-k_1}\begin{pmatrix}e^{i\theta}\mathrm{sech}\,\zeta\,\hat{\mathbf{p}}\\ i(1+\tanh\zeta)\end{pmatrix},$$

where $\hat{\mathbf{p}}$ is the two-component soliton polarization (unit) vector and ζ,θ were defined earlier (above (3.13b)). In this case, the medium is initially prepared so that the atoms are all in the excited state $|3\rangle$. The one-soliton pulse initially triggers transitions to the ground states $|j\rangle$, $j = 1,2$ as seen above, and induces a material dipole polarization along the direction of the soliton polarization vector $\hat{\mathbf{p}}$. But as $\zeta \to \infty$, $\boldsymbol{\gamma} \to (0,0,\hat{c}_3)$ where $\hat{c}_3 = c_3(k-k_1)/(k-k_1^*)$, which implies that the atoms tend to return to the excited state $|3\rangle$ long after the pulse has passed through a given location in the medium, where the dipole polarization also tends to zero. Note also that since $|\hat{c}_3| = |c_3|$ the complex probability amplitude of state $|3\rangle$ changes only by a phase factor. However, the population *density* of each atomic state approaches its initial value as $\zeta \to \infty$, so that the soliton propagates without any energy loss to the medium. For case (ii), we find that

$$\boldsymbol{\gamma} = \mathbf{c} - \eta \frac{\hat{\mathbf{p}}^\dagger \tilde{\mathbf{c}}}{k - k_1^*} \begin{pmatrix} i\tanh\zeta\,\hat{\mathbf{p}} \\ e^{-i\theta}\operatorname{sech}\zeta \end{pmatrix}, \quad \tilde{\mathbf{c}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

In this case, $\boldsymbol{\gamma} \to \mathbf{c}$ as $\zeta \to -\infty$, i.e., the ground states are populated initially. Then the electric field of the soliton pulse induces transition to the excited state, and as $\zeta \to \infty$, the atoms tend to return to the ground states, i.e., $\boldsymbol{\gamma} \to \hat{\mathbf{c}}$ where

$$\hat{\mathbf{c}} = \mathbf{c} - \frac{2i\eta}{k - k_1^*} \begin{pmatrix} (\mathbf{p}^\dagger \tilde{\mathbf{c}})\mathbf{p} \\ 0 \end{pmatrix}.$$

The transition from $\mathbf{c}$ to $\hat{\mathbf{c}}$ as the pulse propagates through the medium implies that the probability amplitudes of the ground states undergo a unitary transformation since $\|\hat{\mathbf{c}}\| = \|\mathbf{c}\|$. This fact also ensures that the soliton pulse propagation is lossless as in case (i).

3.3. Two-soliton solutions and polarization shift

A pure two-soliton solution corresponds to two distinct discrete eigenvalues $k_j = \xi_j + i\eta_j$, and associated norming constants $\boldsymbol{\beta}_j$, $j = 1, 2$. Then from (3.6) the two-soliton solution is of the form

$$(\Omega_1(x,\tau),\ \Omega_2(x,\tau))^T = -4i\,\frac{\tilde{\mathbf{K}}_1\tilde{\boldsymbol{\beta}}_1 + \tilde{\mathbf{K}}_2\tilde{\boldsymbol{\beta}}_2}{|\tilde{\mathbf{K}}^*|}, \tag{3.14}$$

where $|\tilde{\mathbf{K}}^*|$ is the determinant of the 2×2 generalized Cauchy matrix

$$\tilde{\mathbf{K}}^* = \begin{pmatrix} \dfrac{|a_1|^2 + \|\tilde{\boldsymbol{\beta}}_1\|^2}{-2i\eta_1} & \dfrac{a_1^* a_2 + \tilde{\boldsymbol{\beta}}_1^\dagger \tilde{\boldsymbol{\beta}}_2}{k_1^* - k_2} \\ \dfrac{a_2^* a_1 + \tilde{\boldsymbol{\beta}}_2^\dagger \tilde{\boldsymbol{\beta}}_1}{k_2^* - k_1} & \dfrac{|a_2|^2 + \|\tilde{\boldsymbol{\beta}}_2\|^2}{-2i\eta_2} \end{pmatrix},$$

$$a_1 = 2i\eta_1 \frac{k_1 - k_2^*}{k_1 - k_2},\ \ a_2 = 2i\eta_2 \frac{k_1^* - k_2}{k_1 - k_2},$$

$\tilde{\mathbf{K}}_1$ and $\tilde{\mathbf{K}}_2$ are the determinants obtained by replacing respectively, the first and the second column of $\tilde{\mathbf{K}}^*$ by $(a_1^*,\ a_2^*)^T$. Eq. (3.14) represents two interacting solitons in the sense that the asymptotic form of the solution before or after the interaction can be represented as a sum of two one-soliton solutions. Furthermore, as will be shown next, the polarization vectors associated with the asymptotic one-soliton states before the interaction are distinct from those after the interaction. Thus, in addition to the usual phase shift, the solitons also experience a polarization shift after the two-soliton collision process.

In order to analyze the soliton interaction dynamics, it is convenient to introduce the soliton phases θ_j, $j = 1, 2$ as follows

$$\tilde{\boldsymbol{\beta}}_j(x,\tau) = e^{-2ik_j x}\boldsymbol{\beta}_j(\tau) = e^{\theta_j - 2i\xi_j x}\frac{\boldsymbol{\beta}_j}{\|\boldsymbol{\beta}_j\|},$$

$$\theta_j(x,\tau) = 2\eta_j x + \ln\|\boldsymbol{\beta}_j(\tau)\|,$$

where $\boldsymbol{\beta}_j(\tau)$ is given in (3.7). The soliton interaction can be characterized by some $\tau = \tau_0 > 0$ when the soliton phases θ_j completely overlap, i.e., when

$$\Delta(\tau_0) = 0, \qquad \Delta(\tau) \equiv \frac{1}{2\eta_2}\ln\|\boldsymbol{\beta}_2(\tau)\| - \frac{1}{2\eta_1}\ln\|\boldsymbol{\beta}_1(\tau)\|.$$

We assume here that the initial conditions $\boldsymbol{\beta}_j(0)$ and $\langle\hat{\mathbf{H}}\rangle(k_j)$ are such that $\Delta(\tau_0) = 0$ holds for some $\tau_0 > 0$. In addition, $\Delta(t)$ is assumed to be a strictly monotonic function of τ in a sufficiently large neighborhood of τ_0 so that the solitons are well separated before and after the interaction. We assume first that $\Delta(\tau)$ is monotonically increasing, the decreasing case can be dealt with in a similar fashion. Next we derive the asymptotic form of the two-soliton solution under these assumptions for both $\tau \gg \tau_0$ and $\tau \ll \tau_0$.

Note that when $\tau \ll \tau_0$, $\Delta(\tau) = \theta_2/2\eta_2 - \theta_1/2\eta_1 \ll 0$, then there are two cases since $\eta_1, \eta_2 = O(1)$.

(i) $\theta_1 = O(1),\ \theta_2 \ll 0$, i.e., $\|\tilde{\boldsymbol{\beta}}_1\| = O(1),\ \|\tilde{\boldsymbol{\beta}}_2\| \ll \|\tilde{\boldsymbol{\beta}}_1\|$ as $\tau \ll \tau_0$. Then it follows from the expression of $\tilde{\mathbf{K}}^*$ above that

$$|\tilde{\mathbf{K}}^*| \sim \begin{vmatrix} \dfrac{|a_1|^2 + \|\tilde{\boldsymbol{\beta}}_1\|^2}{-2i\eta_1} & \dfrac{a_1^* a_2}{k_1^* - k_2} \\ \dfrac{a_2^* a_1}{k_2^* - k_1} & \dfrac{|a_2|^2}{-2i\eta_2} \end{vmatrix},$$

$$\tilde{\mathbf{K}}_1\tilde{\boldsymbol{\beta}}_1 \sim \begin{vmatrix} a_1^* & \dfrac{a_1^* a_2}{k_1^* - k_2} \\ a_2^* & \dfrac{|a_2|^2}{-2i\eta_2} \end{vmatrix}\tilde{\boldsymbol{\beta}}_1, \qquad \frac{\tilde{\mathbf{K}}_2}{|\tilde{\mathbf{K}}^*|}\|\tilde{\boldsymbol{\beta}}_2\| \ll 1,$$

from which one recovers after some calculation, the one-soliton formula (3.11) with ξ, η and $\boldsymbol{\beta}(\tau)$ replaced by ξ_1, η_1 and $\boldsymbol{\beta}_1(\tau)$.

(ii) $\theta_2 = O(1),\ \theta_1 \gg 0$, i.e., $\|\tilde{\boldsymbol{\beta}}_2\| = O(1),\ \|\tilde{\boldsymbol{\beta}}_1\| \gg \|\tilde{\boldsymbol{\beta}}_2\|$ as $\tau \ll \tau_0$. Then a similar calculation as in case (i) yields a one-soliton solution with parameters ξ_2 and η_2. But this solution has an additional phase χ, and the soliton polarization vector depends on both norming constants $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ (see below).

Combining cases (i) and (ii), the asymptotic form of the two-soliton solution as $\tau \ll \tau_0$ is given as

$$(\Omega_1(x,\tau), \Omega_2(x,\tau))^T \sim 4i\eta_1 e^{-2i\xi_1 x}\operatorname{sech}[2\eta_1 x + \delta_1(\tau)]\,\mathbf{p}_1^- + 4i\eta_2 e^{-2i\xi_2 x}\operatorname{sech}[2\eta_2 x + \delta_2(\tau) + \chi]\,\mathbf{p}_2^-,$$

with

$$\delta_j(\tau) = \ln\frac{\|\boldsymbol{\beta}_j(\tau)\|}{2\eta_j}, \quad j = 1, 2,$$

$$e^{2\chi} = \left|\frac{k_2 - k_1}{k_2 - k_1^*}\right|^2 \left\{1 - \frac{4\eta_1\eta_2}{|k_1^* - k_2|^2}|\hat{\boldsymbol{\beta}}_1^\dagger\hat{\boldsymbol{\beta}}_2|^2\right\}$$

$$\mathbf{p}_1^- = \hat{\boldsymbol{\beta}}_1, \qquad \mathbf{p}_2^- = e^{-\chi}\frac{k_2 - k_1}{k_2 - k_1^*}\left\{\hat{\boldsymbol{\beta}}_2 - \frac{2i\eta_1}{k_2 - k_1^*}(\hat{\boldsymbol{\beta}}_1^\dagger\hat{\boldsymbol{\beta}}_2)\hat{\boldsymbol{\beta}}_1\right\},$$

where $\hat{\boldsymbol{\beta}}_j$ is the unit vector along $\boldsymbol{\beta}_j$. Thus, asymptotically as $\tau \ll \tau_0$ the two-soliton optical pulse can be viewed as a superposition of two well-separated one-soliton pulses traveling along constant $\theta_j(x,\tau)$ curves, and $\mathbf{p}_1^-$ and $\mathbf{p}_2^-$ are the unit polarization vectors for the first and the second soliton. In fact, one can directly verify that $\|\mathbf{p}_2^-\| = 1$ from the above formulas.

Similarly when $\tau \gg \tau_0$, the soliton phase difference $\Delta(\tau) \gg 0$. Then the asymptotic behaviors of the solution in (3.14) arise by considering the cases (i) $\theta_2 = O(1),\ \theta_1 \ll 0$, i.e., $\|\tilde{\boldsymbol{\beta}}_2\| = O(1),\ \|\tilde{\boldsymbol{\beta}}_1\| \to 0$, and (ii) $\theta_1 = O(1),\ \theta_2 \gg 0$, i.e., $\|\tilde{\boldsymbol{\beta}}_1\| = O(1),\ \|\tilde{\boldsymbol{\beta}}_2\| \to \infty$ as $\tau \gg \tau_0$. The computations are similar to above, and the asymptotic form as $\tau \gg \tau_0$ of the two-soliton solution can be written as

$$(\Omega_1(x,\tau), \Omega_2(x,\tau))^T \sim 4i\eta_1 e^{-2i\xi_1 x}\operatorname{sech}[2\eta_1 x + \delta_1(\tau) + \chi]\,\mathbf{p}_1^+ + 4i\eta_2 e^{-2i\xi_2 x}\operatorname{sech}[2\eta_2 x + \delta_2(\tau)]\,\mathbf{p}_2^+,$$

where the expressions for $\delta_j(\tau)$ and χ are the same as above, and

$$\mathbf{p}_1^+ = e^{-\chi}\frac{k_2 - k_1}{k_2^* - k_1}\left\{\hat{\boldsymbol{\beta}}_1 + \frac{2i\eta_2}{k_2^* - k_1}(\hat{\boldsymbol{\beta}}_2^\dagger\hat{\boldsymbol{\beta}}_1)\hat{\boldsymbol{\beta}}_2\right\}, \qquad \mathbf{p}_2^+ = \hat{\boldsymbol{\beta}}_2,$$

with $\|\mathbf{p}_1^+\| = 1$. So for $\tau \gg \tau_0$, the solution (3.14) can once again be viewed as a sum of two well-separated one-soliton pulses. However, comparing the solutions for $\tau \ll \tau_0$ and $\tau \gg \tau_0$, one finds that the one-soliton pulses suffer an equal and opposite

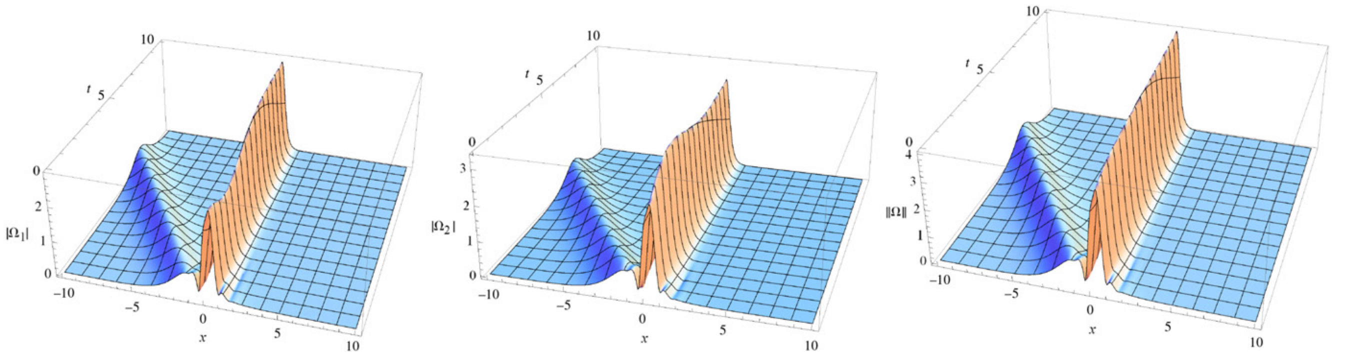

Fig. 5. Two-soliton electric field $\boldsymbol{\Omega}(x, \tau)$ with Lorentzian inhomogeneous broadening. Left: modulus of the first component $|\Omega_1(x, \tau)|$. Center: modulus of the second component $|\Omega_2(x, \tau)|$. Right: norm $\|\boldsymbol{\Omega}(x, \tau)\|$.

amount of phase shift χ, while the soliton polarization vectors undergo a transition: $\{\mathbf{p}_1^-, \mathbf{p}_2^-\} \to \{\mathbf{p}_1^+, \mathbf{p}_2^+\}$ due to the interaction process. Both the phase shift and the polarization shift depend nonlinearly on the norming constants $\boldsymbol{\beta}_j$, $j = 1, 2$. These results for the CMB equation are similar to those found by Manakov [31] for the VNLS equation, except for the fact that the τ-dependence of the norming constants $\boldsymbol{\beta}_j$ is more complicated in the CMB case. As a result, the asymptotic polarization vectors $\mathbf{p}_j^{\pm}$ and the phase shift χ depend non-trivially on τ for both $\tau \ll \tau_0$ and $\tau \gg \tau_0$, instead of being constants as one finds in the case VNLS equation as $|\tau| \to \infty$. The formulas for the soliton asymptotic states do simplify if one considers two-soliton interactions which take place well inside the optical medium, i.e., after the optical pulses have propagated through the fiber for a considerable length. Then one can assume that the interaction point $\tau_0 \gg 0$. In this situation, one finds from (3.7) that the norming constants behave as

$$\boldsymbol{\beta}_j(\tau) \sim c_j^{(1)} \mathbf{v}_j^{(1)} e^{i\lambda_j^{(1)}\tau}, \quad \text{for } 0 \ll \tau \ll \tau_0 \text{ and } \tau \gg \tau_0,$$

where we have taken $\operatorname{Im} \lambda_j^{(1)} < \operatorname{Im} \lambda_j^{(2)}$ as before so that $e^{i(\lambda_j^{(2)} - \lambda_j^{(1)})\tau} \to 0$ for $\tau \gg 0$. Introducing the functions for $j = 1, 2$,

$$\varphi_j(x, \tau) = -2\xi_j x + \operatorname{Re} \lambda_j^{(1)} \tau + \arg c_j^{(1)},$$

$$\vartheta_j(x, \tau) = 2\eta_j x - \operatorname{Im} \lambda_j^{(1)} \tau + \ln \frac{c_j^{(1)}}{2\eta_j},$$

the previous formulas for the asymptotic form of the two-soliton solution reduce to

$$(\Omega_1(x, \tau), \Omega_2(x, \tau))^T \sim 4i\eta_1 e^{i\varphi_1} \operatorname{sech} \vartheta_1 \tilde{\mathbf{p}}_1^- + 4i\eta_2 e^{i\varphi_2} \operatorname{sech}\left[\vartheta_2 + \tilde{\chi}\right] \tilde{\mathbf{p}}_2^-,$$

for $0 \ll \tau \ll \tau_0$, whereas for $\tau \gg \tau_0$,

$$(\Omega_1(x, \tau), \Omega_2(x, \tau))^T \sim 4i\eta_1 e^{i\varphi_1} \operatorname{sech}\left[\vartheta_1 + \tilde{\chi}\right] \tilde{\mathbf{p}}_1^+ + 4i\eta_2 e^{i\varphi_2} \operatorname{sech} \vartheta_2 \tilde{\mathbf{p}}_2^+.$$

In this case, the polarization vectors $\tilde{\mathbf{p}}_j^{\pm}$ and the phase shift $\tilde{\chi}$ are τ-independent, and can be recovered from the previous expressions for $\mathbf{p}_j^{\pm}$ and χ by simply replacing $\hat{\boldsymbol{\beta}}_j(\tau)$ with $\mathbf{v}_j^{(1)}$. Note that the overlap function $\Delta(\tau)$ in this case takes the simple form

$$\Delta(\tau) \sim \frac{\operatorname{Im} \lambda_1^{(1)} \tau - \ln |c_1|}{2\eta_1} - \frac{\operatorname{Im} \lambda_2^{(1)} \tau - \ln |c_2|}{2\eta_2},$$

and will increase monotonically with τ provided that $\operatorname{Im} \lambda_1^{(1)}/2\eta_1 > \operatorname{Im} \lambda_2^{(1)}/2\eta_2$. This implies that in the laboratory coordinate frame, soliton 1 (with parameters k_1, $\boldsymbol{\beta}_1$) is traveling slower than soliton 2, since the quantity $\operatorname{Im} \lambda_j^{(1)}/2\eta_j$ is the *inverse* velocity parameter. If instead $\Delta(\tau)$ is monotonically decreasing in the neighborhood of the interaction point $\tau = \tau_0$, then soliton 2 will be the slower soliton. Alternatively, one can always assume $\Delta(\tau)$ to be a monotonically increasing function near $\tau = \tau_0$ by simply labeling the faster soliton as soliton 2.

In Fig. 5 we plot a typical two-soliton solution, with discrete eigenvalues $k_1 = 1 + i/2$ and $k_2 = 5 + i$. The matrix $\hat{\mathbf{H}} = \begin{pmatrix} \pi & -8i \\ 8i & \pi \end{pmatrix}$ is chosen to be independent of α, and the norming constants in (3.7) have $c_1^{(1)} = -2i$, $c_1^{(2)} = i$, $c_2^{(1)} = -i$, $c_2^{(2)} = i/2$.

3.4. Scattering matrix for pure n-soliton solutions

We showed earlier in this section that the solution of the linear algebraic system (3.5) allows us to obtain the explicit form of the eigenfunction in (3.3). From the asymptotic behavior of the eigenfunction determined in (3.3) as $x \to \infty$, one can derive an explicit form of the scattering matrix $\mathbf{S}(k)$ in the reflectionless case, which depends only on the discrete eigenvalues and the corresponding norming constants.

In order to compute the asymptotics of the eigenfunctions, one needs to solve (3.5) in the large x limit. For that purpose, it is convenient to rewrite the linear system of Eqs. (3.5) by noting that the x-dependence appears only through $\tilde{\boldsymbol{\beta}}_j$ and $\tilde{\boldsymbol{\beta}}_j^{\dagger}$. Introducing the diagonal matrix $\mathbf{D}(x) = \operatorname{diag}(e^{-2ik_1 x}, \ldots, e^{-2ik_n x})$, and multiplying (3.5) by $\mathbf{D}^{-1}$ from the left, one obtains

$$\sum_{j=1}^{n} \left(\mathbf{D}^{-1}\mathbf{K}^{-1}(\mathbf{D}^{\dagger})^{-1} + \mathbf{B}_0\right)_{\ell j} \mathbf{x}_j = \hat{\mathbf{v}}_\ell, \quad \hat{\mathbf{v}}_\ell = \left(\boldsymbol{\beta}_\ell, \, -a_\ell e^{2ik_\ell x}\right)^T,$$

where we have used the following

$$\tilde{\mathbf{K}} = \mathbf{K}^{-1} + \mathbf{B} = \mathbf{K}^{-1} + \mathbf{D}\mathbf{B}_0\mathbf{D}^{\dagger}, \qquad (\mathbf{B}_0)_{ij} = \frac{\boldsymbol{\beta}_j^{\dagger}\boldsymbol{\beta}_i}{k_i - k_j^*},$$

$$\mathbf{x}_j = e^{2ik_j^* x} \bar{\mathbf{M}}_j.$$

The matrix $\mathbf{D}^{-1}$ decays exponentially as $x \to \infty$ since $e^{2ik_j x} \to 0$ when $\operatorname{Im} k_j > 0$, then the solution $\mathbf{x}_j \to \left(\mathbf{u}_j, 0\right)^T$ where $\mathbf{u}_j$ is a solution of the linear system which does *not* depend on x

$$\sum_{j=1}^{n} (\mathbf{B}_0)_{\ell j} \mathbf{u}_j = \boldsymbol{\beta}_\ell \Rightarrow \mathbf{u}_i = \sum_{j=1}^{n} \left(\mathbf{B}_0^{-1}\right)_{ij} \boldsymbol{\beta}_j.$$

Consequently, one has $\bar{\mathbf{M}}_j \to 0$ and $\bar{\mathbf{M}}_j \tilde{\boldsymbol{\beta}}_j^{\dagger} \to \mathbf{u}_j \boldsymbol{\beta}_j$ as $x \to \infty$. Furthermore, from the formula for $\mathbf{M}_j \tilde{\boldsymbol{\beta}}_j$ derived below (3.5), one also finds that as $x \to \infty$, $\mathbf{M}_j \tilde{\boldsymbol{\beta}}_j \to a_j \mathbf{e}_3$. Replacing the limiting values of $\bar{\mathbf{M}}_j \tilde{\boldsymbol{\beta}}_j^{\dagger}$ and $\mathbf{M}_j \tilde{\boldsymbol{\beta}}_j$ in (3.3) then gives $\lim_{x\to\infty} \mu(x, k)$, which is block diagonal, therefore commutes with the matrix $\mathbf{J}$. One can then use

(2.10) to recover $\mathbf{S}(k) = \lim_{x\to\infty} \mu^\dagger(x, k^*)$ whose off-diagonal blocks are zero, which is consistent with the fact that $\mathbf{b}(k) = \bar{\mathbf{b}}(k) = 0$ in the reflection-less case, while for the diagonal blocks of $\mathbf{S}(k)$ one obtains

$$\bar{\mathbf{a}}(k) = \mathbf{I}_2 - \sum_{i=1}^{n} \frac{\boldsymbol{\beta}_i \mathbf{u}_i^\dagger}{k - k_i}, \qquad a(k) = 1 + \sum_{i=1}^{n} \frac{a_i}{k - k_i^*} = \prod_{j=1}^{n} \frac{k - k_j}{k - k_j^*}, \tag{3.15}$$

where a_i are defined in (3.4). The last equality for $a(k)$ in (3.15) can be easily verified by calculating the residue at each pole $k = k_j^*$.

Observe that for the reflection-less case, the 2×2 matrix $\bar{\mathbf{a}}(k)$ has simple poles at $k = k_j$ in the upper-half plane in addition to being analytic in the lower-half plane, and $\bar{\mathbf{a}}(k) \to \mathbf{I}_2$ as $k \to \infty$. Furthermore, by a straightforward calculation presented in Appendix B.1, one can directly verify from the form given in (3.15) that the matrix $\bar{\mathbf{a}}(k)$ satisfies $\bar{\mathbf{a}}^\dagger(k^*)\bar{\mathbf{a}}(k) = \mathbf{I}_2$. The scalar scattering coefficient $a(k)$ is independent of τ, which is consistent with its evolution equation in (2.21), since $\tilde{h} = h$ when $\mathbf{b} = 0$. Moreover, $a(k)$ is just a product of simple zeros at $k = k_j$ and simple poles at $k = k_j^*$ as in the VNLS case. An important application for the exact formula for $\bar{\mathbf{a}}(k)$ given by (3.15) is that for a pure n-soliton solution the final atomic configuration $\tilde{\boldsymbol{\rho}}_+(\alpha, \tau)$, $\alpha \in \mathbb{R}$ of the 3-level optical medium can be explicitly determined using Eq. (3.1) for a given initial state $\tilde{\boldsymbol{\rho}}_-(\alpha, \tau)$. It is worth emphasizing that even though one is able to find closed form expressions for $\mathbf{S}(k)$ in the reflection-less case, obtaining explicit expressions for the full scattering matrix $\mathbf{S}(k)$ in the generic situation in terms of the scattering data $\mathbf{r}(k)$, $\{k_j, \boldsymbol{\beta}_j\}_{j=1}^n$ is a nontrivial task which is still an open problem. We refer the reader to Appendix C where an approximate solution to this problem is discussed for small $\|\mathbf{r}\|$.

For $n = 1$, one recovers from (3.15) the expressions for the matrix and scalar coefficients corresponding to the one-soliton solution

$$\bar{\mathbf{a}}(k) = \mathbf{I}_2 + \frac{k_1 - k_1^*}{k - k_1} \frac{\boldsymbol{\beta}_1 \boldsymbol{\beta}_1^\dagger}{\|\boldsymbol{\beta}_1\|^2}, \qquad a(k) = \frac{k - k_1}{k - k_1^*}. \tag{3.16}$$

It is easy to verify using the formula $\det(\mathbf{I}_2 + A) = 1 + \mathrm{Tr}(A)$ for a rank one matrix A that $\det \bar{\mathbf{a}}(k)$ has a simple pole at $k = k_1$ and a simple zero at $k = k_1^*$, and that the relation (2.6) holds. For the n-soliton case, Eq. (2.6) is not obvious from the expressions for $\bar{\mathbf{a}}(k)$ and $a(k)$ given by (3.15). But it turns out that $\bar{\mathbf{a}}(k)$ in (3.15) can be represented as a product of one-soliton factors of the form given by (3.16), i.e.,

$$\bar{\mathbf{a}}(k) = \mathbf{I}_2 - \sum_{i=1}^{n} \frac{\boldsymbol{\beta}_i \mathbf{u}_i^\dagger}{k - k_i} = \prod_{j=1}^{n} \boldsymbol{\ell}_j(k), \qquad \boldsymbol{\ell}_j(k) = \mathbf{I}_2 + \frac{k_j - k_j^*}{k - k_j} \frac{\mathbf{v}_j \mathbf{v}_j^\dagger}{\|\mathbf{v}_j\|^2}, \qquad \boldsymbol{\ell}_j^{-1}(k) = \boldsymbol{\ell}_j^\dagger(k^*), \tag{3.17}$$

from which the relation $\det \bar{\mathbf{a}}(k) = a^*(k^*)$ in (2.6) follows immediately. The Blaschke factors $\boldsymbol{\ell}_j$ in the factorization (3.17) of $\bar{\mathbf{a}}(k)$ are unique even though the vectors $\mathbf{v}_j$ are defined up to a rescaling: $\mathbf{v}_j \to c_j \mathbf{v}_j$, $c_j \in \mathbb{C}$. It will be shown in Appendix B.2 that the $\mathbf{v}_j$'s can be recursively defined by

$$\mathbf{v}_1 = \boldsymbol{\beta}_1, \qquad \mathbf{v}_j = \big(\boldsymbol{\ell}_1(k_j)\boldsymbol{\ell}_2(k_j) \cdots \boldsymbol{\ell}_{j-1}(k_j)\big)^{-1} \boldsymbol{\beta}_j, \quad j = 2, \ldots, n. \tag{3.18}$$

The above formula for the $\mathbf{v}_j$'s depends on the order in which the Blaschke factors are chosen in the matrix factorization of $\bar{\mathbf{a}}(k)$ because the factorization in (3.17) corresponds to only one of the $n!$ permutations in which the indices $\{1, 2, \ldots, n\}$ can be ordered. For instance, if instead of (3.17) one chooses the factorization

$$\bar{\mathbf{a}}(k) = \tilde{\boldsymbol{\ell}}_n(k)\tilde{\boldsymbol{\ell}}_{n-1}(k) \cdots \tilde{\boldsymbol{\ell}}_1(k), \qquad \tilde{\boldsymbol{\ell}}_j(k) = \mathbf{I}_2 + \frac{k_j - k_j^*}{k - k_j} \frac{\tilde{\mathbf{v}}_j \tilde{\mathbf{v}}_j^\dagger}{\|\tilde{\mathbf{v}}_j\|^2},$$

then the $\tilde{\mathbf{v}}_j$'s can be recursively given by

$$\tilde{\mathbf{v}}_n = \boldsymbol{\beta}_n, \qquad \tilde{\mathbf{v}}_j = \big(\tilde{\boldsymbol{\ell}}_n(k_j)\tilde{\boldsymbol{\ell}}_{n-1}(k_j) \cdots \tilde{\boldsymbol{\ell}}_{j+1}(k_j)\big)^{-1} \boldsymbol{\beta}_j, \quad j = n - 1, \ldots, 1.$$

The Blaschke product in terms of the $\tilde{\boldsymbol{\ell}}_n(k)$'s is a re-factorization of the matrix $\bar{\mathbf{a}}(k)$ factored in (3.17). One can express the Blaschke factors $\boldsymbol{\ell}_n(k)$'s in (3.17) in terms of the $\tilde{\boldsymbol{\ell}}_n(k)$'s from the following formula relating the $\mathbf{v}_j$'s to the $\tilde{\mathbf{v}}_j$'s for $j = 1, 2, \ldots, n$

$$\widehat{\mathbf{v}}_j = c_j \big(\boldsymbol{\ell}_1(k_j) \cdots \boldsymbol{\ell}_{j-1}(k_j)\big)^{-1} \big(\tilde{\boldsymbol{\ell}}_n(k_j)\tilde{\boldsymbol{\ell}}_{n-1}(k_j) \cdots \tilde{\boldsymbol{\ell}}_{j+1}(k_j)\big) \widehat{\tilde{\mathbf{v}}}_j, \qquad c_j \in \mathbb{C}, \tag{3.19}$$

where $\widehat{\mathbf{v}}_j$ and $\widehat{\tilde{\mathbf{v}}}_j$ are unit vectors corresponding to $\mathbf{v}_j$ and $\tilde{\mathbf{v}}_j$, respectively, and c_j is a suitable normalization constant. In (3.19), we also adopt the convention that $\big(\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{j-1}\big) = \mathbf{I}_2$ for $j = 1$, and $\big(\tilde{\boldsymbol{\ell}}_n \tilde{\boldsymbol{\ell}}_{n-1} \cdots \tilde{\boldsymbol{\ell}}_{j+1}\big) = \mathbf{I}_2$ for $j = n$.

The Blaschke product formula for $\bar{\mathbf{a}}(k)$ and its re-factorization play an important role in matrix soliton equations such as the VNLS and matrix KdV equations because the re-factorization formula determines the asymptotic value of the final polarization vectors for the individual solitons after collision in an n-soliton interaction. For the matrix KdV equation, the connection between soliton polarization shifts after a 2-soliton interaction and a matrix re-factorization problem, as well as the underlying relationship with Yang–Baxter maps, were elucidated in a series of papers [32–34] by Veselov and co-workers. The VNLS case was studied by Manakov [31] (see also [35] for the N-component case), who postulated a formula for the soliton polarizations after an n-soliton collision in terms of the soliton polarizations before the collision assuming the solitons are well separated before and after the collision process. The Manakov formula is essentially the same as (3.19) above, and originates simply from the re-factorization of the transmission matrix $\bar{\mathbf{a}}(k)$. Manakov verified his formula for $n = 2$. In the general case, the validity of his formula can be inferred (after some work) from the results of Tsuchida [36], who gave explicit formulas for soliton polarizations from the asymptotics of the exact n-soliton solution of the VNLS equation. Since the form of the n-soliton potential $\boldsymbol{\Omega}$ in (3.6) for the CMB equation is the same as that of the VNLS case, the results of [36] also hold for this case. However, it is worth pointing out that while in previous works the soliton polarization shifts were related to the re-factorization of an ad-hoc matrix, here we have shown that in both the VNLS and CMB cases this matrix is precisely the reflection-less transmission matrix $\bar{\mathbf{a}}(k)$.

It is instructive to consider the factorization of the transmission matrix $\bar{\mathbf{a}}(k)$ for $n = 2$, and compare it with the 2-soliton polarization shift results obtained in Section 3.1. In this case, one has

$$\bar{\mathbf{a}}(k) = \tilde{\boldsymbol{\ell}}_2(k)\tilde{\boldsymbol{\ell}}_1(k) = \boldsymbol{\ell}_1(k)\boldsymbol{\ell}_2(k), \qquad \tilde{\boldsymbol{\ell}}_j(k) = \mathbf{I}_2 + \frac{k_j - k_j^*}{k - k_j} \widehat{\tilde{\mathbf{v}}}_j \widehat{\tilde{\mathbf{v}}}_j^\dagger,$$

$$\boldsymbol{\ell}_j(k) = \mathbf{I}_2 + \frac{k_j - k_j^*}{k - k_j} \widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j^\dagger,$$

for $j = 1, 2$, and from the formulas given above, the unit vectors $\widehat{\widehat{\mathbf{v}}}_j$ and $\widehat{\mathbf{v}}_j$ are given by

$$\widehat{\widehat{\mathbf{v}}}_2 = \widehat{\boldsymbol{\beta}}_2, \qquad \widehat{\widehat{\mathbf{v}}}_1 = \frac{1}{\hat{\chi}}\left\{\widehat{\boldsymbol{\beta}}_1 - \frac{k_2 - k_2^*}{k_1 - k_2^*}\left(\widehat{\boldsymbol{\beta}}_2^\dagger \widehat{\boldsymbol{\beta}}_1\right)\widehat{\boldsymbol{\beta}}_2\right\},$$

$$\widehat{\mathbf{v}}_1 = \widehat{\boldsymbol{\beta}}_1, \qquad \widehat{\mathbf{v}}_2 = \frac{1}{\hat{\chi}}\left\{\widehat{\boldsymbol{\beta}}_2 - \frac{k_1 - k_1^*}{k_2 - k_1^*}\left(\widehat{\boldsymbol{\beta}}_1^\dagger \widehat{\boldsymbol{\beta}}_2\right)\widehat{\boldsymbol{\beta}}_1\right\},$$

where the normalization constant $\hat{\chi}$ is such that

$$\hat{\chi}^2 = 1 + \frac{(k_1 - k_1^*)(k_2 - k_2^*)}{|k_1 - k_2^*|^2}\left|\widehat{\boldsymbol{\beta}}_1^\dagger \widehat{\boldsymbol{\beta}}_2\right|^2.$$

Let us now consider a 2-soliton collision where the faster soliton 2 comes from behind, interacts with the slower soliton 1, and then overtakes it. In this scenario, the soliton centers are ordered as (2, 1) before collision ($\tau \ll \tau_0$), and as (1, 2) after collision ($\tau \gg \tau_0$). Then from the asymptotic analysis of the exact 2-soliton solution in Section 3.3 the soliton polarization vectors before collision are given by $\mathbf{p}_1^- = \widehat{\mathbf{v}}_1$, $\mathbf{p}_2^- = e^{i\phi_2}\widehat{\mathbf{v}}_2$ asymptotically as $\tau \ll \tau_0$, and after the collision they are $\mathbf{p}_1^+ = e^{i\phi_1}\widehat{\widehat{\mathbf{v}}}_1$, $\mathbf{p}_2^+ = \widehat{\widehat{\mathbf{v}}}_2$ asymptotically as $\tau \gg \tau_0$. The phases ϕ_1, ϕ_2 can be easily found, but that is not essential since the unit vectors in the Blaschke factors are defined up to an exponential phase anyway. Thus, $\bar{\mathbf{a}}(k) = \boldsymbol{\ell}_1(k)\boldsymbol{\ell}_2(k)$ corresponds to the factorization before collision, whereas $\bar{\mathbf{a}}(k) = \tilde{\boldsymbol{\ell}}_2(k)\tilde{\boldsymbol{\ell}}_1(k)$ corresponds to the factorization after collision. Note that the order of the Blaschke factors in each factorization is *opposite* to that of the soliton centers, which are ordered according to the Blaschke factors in the factorization of the inverse $\bar{\mathbf{a}}^\dagger(k^*)$, instead.

We now briefly discuss the τ-evolution of the transmission matrix $\bar{\mathbf{a}}(k, \tau)$ in the reflection-less case. Setting $\mathbf{m} = \bar{\mathbf{b}} = 0$ in (2.21), and using the relation $\bar{\mathbf{a}}^\dagger(\alpha, \tau) = \bar{\mathbf{a}}^{-1}(\alpha, \tau)$ for $\alpha \in \mathbb{R}$ which follows from (2.5c) with $\bar{\mathbf{b}} = 0$, one finds that $\bar{\mathbf{a}}(k, \tau)$ satisfies

$$\bar{\mathbf{a}}_\tau = i\langle \mathbf{H}\rangle \bar{\mathbf{a}} - i\bar{\mathbf{a}}\langle \bar{\mathbf{a}}^{-1}\hat{\mathbf{H}}\bar{\mathbf{a}}\rangle.$$

From the expression of $\bar{\mathbf{a}}(k, \tau)$ in (3.15) it is clear that the τ-dependence only appears through that of the norming constant $\boldsymbol{\beta}_i(\tau)$ which satisfies the evolution equation (2.22). Indeed, it is possible to verify the fact that if $\boldsymbol{\beta}_i(\tau)$ satisfies (2.22), then the reflection-less transmission matrix $\bar{\mathbf{a}}(k, \tau)$ satisfies the above evolution equation. The details of this calculation are given in Appendix B.3.

4. Conclusion

In this article we have studied the CMB equations by means of the inverse scattering transform. We have discussed the inverse scattering formulation of this (1 + 1)-dimensional problem in detail. In particular, we have shown that in the presence of inhomogeneous broadening, the scattering data evolve in a non-trivial fashion unlike most of the well-known (1 + 1)-dimensional integrable equations. However, it turns out that the evolution of the reflection coefficient and norming constants, which comprise the essential set of scattering data, is relatively simple. In the reflection-less case, we have obtained explicit expressions for the n-soliton solution, and studied one- and two-soliton solutions in more detail. A simple expression for the transmission matrix $\bar{\mathbf{a}}(k)$ corresponding to the pure n-soliton solutions is derived, which we believe is new for the CMB and VNLS equations. The soliton polarization shifts before and after the collision are related, respectively, to the factorization and re-factorization of $\bar{\mathbf{a}}(k)$. The solution of the inverse problem with reflection coefficient $\mathbf{r} \neq 0$ is a non-trivial issue which we have highlighted in Appendix C, where the approximate form of the scattering matrix $\mathbf{S}(k)$ is computed by solving (2.8) via a Born approximation assuming small $\|\mathbf{r}\|$. However, we have not considered issues such as the effect of small radiation on the n-soliton solution and its long-time asymptotics, which are worth investigating, but beyond the scope of this article.

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Appendix A. A note on the norming constant

Here we derive the relation from Section 2.2

$$\left(\bar{\boldsymbol{\psi}}\bar{\mathbf{a}}^{-1}\det\bar{\mathbf{a}}\right)(k_j^*) = -\bar{\boldsymbol{\phi}}(k_j^*)\boldsymbol{\eta}_j^\dagger,$$

where $\boldsymbol{\eta}_j$ is a two-component vector satisfying $\boldsymbol{\psi}(k_j) = \boldsymbol{\phi}(k_j)\boldsymbol{\eta}_j$, and k_j is a discrete eigenvalue in the upper-half plane such that $a(k_j) = 0$. It is convenient to define the matrices $\mathbf{P}(k) \equiv (\boldsymbol{\phi}\boldsymbol{\psi})(k)$ whose first two columns are same as those of $\boldsymbol{\Phi}(k)$ and the third column is the same as that of $\boldsymbol{\Psi}(k)$, and $\bar{\mathbf{P}}(k) \equiv \left(\bar{\boldsymbol{\psi}}\bar{\boldsymbol{\phi}}\right)(k)$ whose first two columns are same as those of $\boldsymbol{\Psi}(k)$ and the third column is the same as that of $\boldsymbol{\Phi}(k)$. Then it is possible to verify that

$$\mathbf{P}^\dagger(k^*)\bar{\mathbf{P}}(k) = \begin{pmatrix} \bar{\mathbf{a}}(k) & 0 \\ 0 & a^*(k^*) \end{pmatrix} \equiv \mathbf{A}(k) \tag{A.1}$$

by using (2.4) and the symmetries $\boldsymbol{\Phi}^{-1}(k) = \boldsymbol{\Phi}^\dagger(k^*)$, $\boldsymbol{\Psi}^{-1}(k) = \boldsymbol{\Psi}^\dagger(k^*)$ of the eigenfunctions, which imply that $\mathbf{S}(k) = \boldsymbol{\Phi}^{-1}(k)\boldsymbol{\Psi}(k) = \boldsymbol{\Phi}^\dagger(k^*)\boldsymbol{\Psi}(k)$. Notice that the matrices $\mathbf{P}^\dagger(k^*)$, $\bar{\mathbf{P}}(k)$, and $\mathbf{A}(k)$ in (A.1) are analytic when k is in the lower-half plane, and in particular $\mathbf{A}(k)$ is of rank one at $k = k_j^*$ since both $\det\bar{\mathbf{a}}(k_j^*)$ and $a^*(k_j)$ vanish. In Section 2.2 it was shown that $\det\bar{\mathbf{P}}(k)$ has a simple zero at $k = k_j^*$, hence the right null space of $\bar{\mathbf{P}}(k_j^*)$ is one-dimensional. If v_j is a right null vector of $\bar{\mathbf{P}}(k_j^*)$, then from (A.1) it follows that $\mathbf{A}(k_j^*)v_j = 0$ as well. Moreover, $v_j = (\tilde{v}_j, m_j)^T$ where the two-component vector $\tilde{v}_j$ satisfies $\bar{\mathbf{a}}(k_j^*)\tilde{v}_j = 0$. On the other hand, if $\mathbf{c}(k) = \bar{\mathbf{a}}^{-1}(k)\det\bar{\mathbf{a}}(k)$ is the adjoint matrix of $\bar{\mathbf{a}}(k)$, then $\bar{\mathbf{a}}(k_j^*)\mathbf{c}(k_j^*) = \det\bar{\mathbf{a}}(k_j^*)\mathbf{I}_2 = 0$ implying that the *columns* of $\mathbf{c}(k_j^*)$ are right null vectors of $\bar{\mathbf{a}}(k_j^*)$. Of course, the two columns of $\mathbf{c}(k_j^*)$ are proportional, since $\det\mathbf{c}(k_j^*) = \det\bar{\mathbf{a}}(k_j^*) = 0$. Then one can choose right null vectors $v_j = (\tilde{v}_j, m_j)^T$ and $w_j = (\tilde{w}_j, n_j)^T$ of $\bar{\mathbf{P}}(k_j^*)$ such that $\tilde{v}_j$ and $\tilde{w}_j$ are, respectively, the first and second column of the adjoint matrix $\mathbf{c}(k_j^*)$. Therefore, one has $\bar{\mathbf{P}}(k_j^*)(v_j, w_j) = 0$ which can be expressed as

$$\bar{\mathbf{P}}(k_j^*)\begin{pmatrix} \mathbf{c}(k_j^*) \\ d_j^\dagger \end{pmatrix} = 0 \Leftrightarrow$$

$$\left(\bar{\boldsymbol{\psi}}\bar{\mathbf{a}}^{-1}\det\bar{\mathbf{a}}\right)(k_j^*) = -\bar{\boldsymbol{\phi}}(k_j^*)d_j^\dagger, \quad d_j^\dagger \equiv (m_j, n_j).$$

Since the right null space of $\bar{\mathbf{P}}(k_j^*)$ is one-dimensional, the vectors v_j and w_j are proportional implying that the row vector $d_j^\dagger \equiv (m_j, n_j)$ is proportional to each row of the rank one matrix $\mathbf{c}(k_j^*)$. Next, we show that the vector d_j can be chosen to equal $\boldsymbol{\eta}_j$ where

$$\boldsymbol{\psi}(k_j) = \boldsymbol{\phi}(k_j)\boldsymbol{\eta}_j \Leftrightarrow \mathbf{P}(k_j)\begin{pmatrix} \boldsymbol{\eta}_j \\ -1 \end{pmatrix} = 0.$$

The Hermitian conjugate of the equation above, yields

$$(\boldsymbol{\eta}_j^\dagger, -1)\mathbf{P}^\dagger(k_j) = 0,$$

which implies that $\boldsymbol{\eta}_j^\dagger$ is then a left null vector of $\bar{\mathbf{a}}(k_j^*)$ due to (A.1). But one also has the identity $\mathbf{c}(k_j^*)\bar{\mathbf{a}}(k_j^*) = 0$, which means that the *rows* of $\mathbf{c}(k_j^*)$ are left null vectors of $\bar{\mathbf{a}}(k_j^*)$ as well. The left null space of $\bar{\mathbf{a}}(k_j^*)$ is one-dimensional because $\det \bar{\mathbf{a}}(k)$ has a simple zero at $k = k_j^*$. Therefore, like $d_j^\dagger$, $\boldsymbol{\eta}_j^\dagger$ is also proportional to each row of $\mathbf{c}(k_j^*)$. Hence one has $d_j = \alpha_j \boldsymbol{\eta}_j$, $\alpha_j \in \mathbb{C}$. If one solves (2.8) for $\boldsymbol{\Phi}$ with the norming constants d_j and $\boldsymbol{\eta}_j$ and recovers the potential $\boldsymbol{\Omega}$ as described in Section 2.2, then the symmetry $\boldsymbol{\Omega}^\dagger = -\boldsymbol{\Omega}$ implies that $\alpha_j = 1$, i.e., $d_j = \boldsymbol{\eta}_j$ for all j.

Appendix B. Properties of the reflection-less transmission matrix $\bar{\mathbf{a}}(\boldsymbol{k})$

In this appendix we derive some properties as well as verify the τ-evolution equation of the reflection-less transmission matrix $\bar{\mathbf{a}}(k)$ given by (3.15) in Section 3.4.

B.1. Symmetry of $\bar{\mathbf{a}}(k)$

From (2.5a) and (2.5c) in Section 2.1 it follows that in the reflection-less case with $\mathbf{b} = \bar{\mathbf{b}} = 0$, the transmission matrix $\bar{\mathbf{a}}(k)$ satisfies the complex unitarity condition

$$\bar{\mathbf{a}}^\dagger(k^*) = \bar{\mathbf{a}}^{-1}(k).$$

Here we show that the transmission matrix $\bar{\mathbf{a}}(k)$ derived in (3.15) satisfies this condition. This result will be used afterwards in deriving the factorization properties and the τ-evolution equation for $\bar{\mathbf{a}}(k)$.

Since $\bar{\mathbf{a}}(k)$ is a 2×2 matrix, it suffices to show that $\bar{\mathbf{a}}^\dagger(k^*)\bar{\mathbf{a}}(k) = \mathbf{I}_2$. Using (3.15) the product $\bar{\mathbf{a}}^\dagger(k^*)\bar{\mathbf{a}}(k)$ can be expressed as

$$\left(\mathbf{I}_2 - \sum_{j=1}^{n} \frac{\mathbf{u}_j \beta_j^\dagger}{k - k_j^*}\right)\left(\mathbf{I}_2 - \sum_{i=1}^{n} \frac{\boldsymbol{\beta}_i \mathbf{u}_i^\dagger}{k - k_i}\right) = \mathbf{I}_2 + \sum_{i=1}^{n} \frac{R_i}{k - k_i} + \sum_{i=1}^{n} \frac{\bar{R}_i}{k - k_i^*},$$

where R_i and $\bar{R}_i$ are the residues at the poles $k = k_i$ and $k = k_i^*$, respectively. Computing the residue at $k = k_i$ one obtains

$$R_i = -\boldsymbol{\beta}_i \mathbf{u}_i^\dagger + \sum_{j=1}^{n} \frac{\mathbf{u}_j \boldsymbol{\beta}_j^\dagger \boldsymbol{\beta}_i \mathbf{u}_i^\dagger}{k_i - k_j^*} = -\boldsymbol{\beta}_i \mathbf{u}_i^\dagger + \sum_{j=1}^{n} (\mathbf{B}_0)_{ij} \mathbf{u}_j \mathbf{u}_i^\dagger,$$

where the matrix $\mathbf{B}_0$ was introduced in Section 3.2. From the definition of the vector $\mathbf{u}_j$ given above (3.15), one readily sees that the second term on the right hand side of the last equality for R_i is just $\boldsymbol{\beta}_i \mathbf{u}_i^\dagger$. Hence, $R_i = 0$. In a similar way, one can show that $\bar{R}_i = 0$. Therefore, $\bar{\mathbf{a}}^\dagger(k^*)\bar{\mathbf{a}}(k) = \mathbf{I}_2$.

B.2. Factorization of $\bar{\mathbf{a}}(k)$

Here we derive Eqs. (3.18) through (3.19) related to the Blaschke factorization (3.17) of $\bar{\mathbf{a}}(k)$.

At first, we factorize $\bar{\mathbf{a}}(k)$ as

$$\bar{\mathbf{a}}(k) = \mathbf{I}_2 - \sum_{i=1}^{n} \frac{\boldsymbol{\beta}_i \mathbf{u}_i^\dagger}{k - k_i} = \prod_{j=1}^{n} \boldsymbol{\ell}_j(k), \tag{B.1}$$

where the Blaschke factor $\boldsymbol{\ell}_j(k)$ is given by

$$\boldsymbol{\ell}_j(k) = \mathbf{I}_2 + \frac{k_j - k_j^*}{k - k_j} \frac{\mathbf{x}_j \mathbf{y}_j^\dagger}{\mathbf{y}_j^\dagger \mathbf{x}_j}, \qquad \det \boldsymbol{\ell}_j(k) = \frac{k - k_j^*}{k - k_j},$$

$$\boldsymbol{\ell}_j^{-1}(k) = \mathbf{I}_2 - \frac{k_j - k_j^*}{k - k_j^*} \frac{\mathbf{x}_j \mathbf{y}_j^\dagger}{\mathbf{y}_j^\dagger \mathbf{x}_j}.$$

It is clear from the above factorization and the expression for $\det \boldsymbol{\ell}_j(k)$ that (2.6) holds for the matrix $\bar{\mathbf{a}}(k)$ in (3.15). The vectors $\mathbf{x}_i, \mathbf{y}_i$ can be determined by equating the residues at each pole $k = k_i$ of the two expressions for $\bar{\mathbf{a}}(k)$ in (B.1), as well as equating the residues at each pole $k = k_i^*$ of the corresponding expressions for $\bar{\mathbf{a}}^{-1}(k) = \bar{\mathbf{a}}^\dagger(k^*)$. The residue at $k = k_1$ yields

$$-\boldsymbol{\beta}_1 \mathbf{u}_1^\dagger = (k_1 - k_1^*) \frac{\mathbf{x}_1 \mathbf{y}_1^\dagger}{\mathbf{y}_1^\dagger \mathbf{x}_1} (\boldsymbol{\ell}_2 \boldsymbol{\ell}_3 \cdots \boldsymbol{\ell}_n)(k_1).$$

The image of the rank-one operator on the left hand side of the above expression is spanned by the vector $\boldsymbol{\beta}_1$ while the image of the rank-one operator on the right hand side is spanned by the vector $\mathbf{x}_1$. Hence, $\mathbf{x}_1 = c_1 \boldsymbol{\beta}_1$, $c_1 \in \mathbb{C}$. The residue of $\bar{\mathbf{a}}^{-1}(k)$ at $k = k_1^*$ is given by

$$-\mathbf{u}_1 \boldsymbol{\beta}_1^\dagger = -(k_1 - k_1^*)(\boldsymbol{\ell}_2 \boldsymbol{\ell}_3 \cdots \boldsymbol{\ell}_n)^{-1}(k_1^*) \frac{\mathbf{x}_1 \mathbf{y}_1^\dagger}{\mathbf{y}_1^\dagger \mathbf{x}_1}.$$

Taking the Hermitian conjugate of the above, one finds that the image spaces of the resulting rank-one operators on the left and right hand sides are spanned by the vectors $\boldsymbol{\beta}_1$ and $\mathbf{y}_1$, respectively. Hence, $\mathbf{y}_1 = d_1 \boldsymbol{\beta}_1$, $d_1 \in \mathbb{C}$. From the expressions of $\mathbf{x}_1$ and $\mathbf{y}_1$ in terms of $\boldsymbol{\beta}_1$, one easily concludes that

$$\frac{\mathbf{x}_1 \mathbf{y}_1^\dagger}{\mathbf{y}_1^\dagger \mathbf{x}_1} = \frac{\mathbf{v}_1 \mathbf{v}_1^\dagger}{\|\mathbf{v}_1\|^2}, \quad \mathbf{v}_1 = \boldsymbol{\beta}_1,$$

which gives the first equality in (3.18). Now by induction suppose (3.18) holds for $j = 1, 2, \ldots, i-1$, $i \geq 2$. Then equating the residue of $\bar{\mathbf{a}}(k)$ at $k = k_i$ yields

$$-\boldsymbol{\beta}_i \mathbf{u}_i^\dagger = (k_i - k_i^*)(\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{i-1})(k_i) \frac{\mathbf{x}_i \mathbf{y}_i^\dagger}{\mathbf{y}_i^\dagger \mathbf{x}_i} (\boldsymbol{\ell}_{i+1} \cdots \boldsymbol{\ell}_n)(k_i),$$

which implies that $\mathbf{x}_i = c_i (\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{i-1})^{-1}(k_i) \boldsymbol{\beta}_i$, $c_i \in \mathbb{C}$ by considering the image of each of the rank-one operators above. In a similar fashion, by equating the residue of $\bar{\mathbf{a}}(k)$ at $k = k_i$, and taking the Hermitian conjugate of the resulting equation, one obtains

$$-\boldsymbol{\beta}_i \mathbf{u}_i^\dagger = (k_i - k_i^*)(\boldsymbol{\ell}_{i-1}^{-1} \cdots \boldsymbol{\ell}_1^{-1})^\dagger(k_i^*) \frac{\mathbf{y}_i \mathbf{x}_i^\dagger}{\mathbf{x}_i^\dagger \mathbf{y}_i} (\boldsymbol{\ell}_n^{-1} \cdots \boldsymbol{\ell}_{i+1}^{-1})^\dagger(k_i^*),$$

which leads to $\mathbf{y}_i = d_i (\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{i-1})^\dagger(k_i^*) \boldsymbol{\beta}_i$, $d_i \in \mathbb{C}$. But by the induction hypothesis, for $j = 1, 2, \ldots i-1$, the $\boldsymbol{\ell}_j$s have the form as in (3.17) so that $\boldsymbol{\ell}_j^{-1}(k) = \boldsymbol{\ell}_j^\dagger(k^*)$ holds. Therefore, $\mathbf{y}_i = d_i (\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{i-1})^{-1}(k_i) \boldsymbol{\beta}_i$, $d_i \in \mathbb{C}$, and then one has

$$\frac{\mathbf{x}_i \mathbf{y}_i^\dagger}{\mathbf{y}_i^\dagger \mathbf{x}_i} = \frac{\mathbf{v}_i \mathbf{v}_i^\dagger}{\|\mathbf{v}_i\|^2}, \quad \mathbf{v}_i = (\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_{i-1})^{-1}(k_i) \boldsymbol{\beta}_i,$$

proving (3.18).

The recursion formula for the vectors $\tilde{\mathbf{v}}_j$ arising in the refactorization $\bar{\mathbf{a}}(k) = \tilde{\boldsymbol{\ell}}_n(k) \tilde{\boldsymbol{\ell}}_{n-1}(k) \cdots \tilde{\boldsymbol{\ell}}_1(k)$ given below (3.18) can be proved in a similar way as above, we skip the details. Lastly, if we set $\bar{\mathbf{a}}(k) = \boldsymbol{\ell}_1(k)\boldsymbol{\ell}_2(k) \cdots \boldsymbol{\ell}_n(k) = \tilde{\boldsymbol{\ell}}_n(k) \tilde{\boldsymbol{\ell}}_{n-1}(k) \cdots \tilde{\boldsymbol{\ell}}_1(k)$, and equate the residues at $k = k_i$ and $k = k_i^*$, $i = 1, 2, \ldots, n$, then as before, a similar argument involving the image spaces of rank-one projectors leads to (3.19). Again, we omit the details.

B.3. Evolution of $\bar{\mathbf{a}}(k, \tau)$

In this subsection we show that the τ-evolution of $\bar{\mathbf{a}}(k)$ follows from the evolution equation (2.22) of the norming constants. Let us first re-write the evolution equation of the reflection-less transmission matrix $\bar{\mathbf{a}}(k)$ given at the end of Section 3.4 in the form

$$\bar{\mathbf{a}}^{-1} \bar{\mathbf{a}}_\tau = i \bar{\mathbf{a}}^{-1} \langle \hat{\mathbf{H}} \rangle \bar{\mathbf{a}} - i \langle \bar{\mathbf{a}}^{-1} \hat{\mathbf{H}} \bar{\mathbf{a}} \rangle. \tag{B.2}$$

Note that since $\bar{\mathbf{a}}(k)$ in (3.15) is a meromorphic function defined on the entire complex plane with poles in the upper-half plane, the evolution

equation (B.2) can be extended to the whole plane even though Eq. (2.21) for $\bar{\mathbf{a}}(k)$ is valid only on $\operatorname{Im} k < 0$. We first consider $\bar{\mathbf{a}}(k)$ for the one-soliton case given by (3.16), and re-express it as

$$\bar{\mathbf{a}}(k) = \mathbf{I}_2 + \frac{k_1 - k_1^*}{k - k_1}\mathbf{P}_1, \quad \mathbf{P}_1 = \frac{\boldsymbol{\beta}_1\boldsymbol{\beta}_1^\dagger}{\|\boldsymbol{\beta}_1\|^2},$$

where $\boldsymbol{\beta}_1$ is the norming constant associated to the discrete eigenvalue k_1. Using the relation $\mathbf{P}_1^\dagger = \mathbf{P}_1$ one can express the quantity $(\bar{\mathbf{a}}^{-1}\hat{\mathbf{H}}\bar{\mathbf{a}})(\alpha)$ for $\alpha \in \mathbb{R}$ in partial fraction form as follows

$$(\bar{\mathbf{a}}^{-1}\hat{\mathbf{H}}\bar{\mathbf{a}})(\alpha) = \hat{\mathbf{H}}(\alpha) + \frac{k_1 - k_1^*}{\alpha - k_1}(\mathbf{I}_2 - \mathbf{P}_1)\hat{\mathbf{H}}(\alpha)\mathbf{P}_1 - \frac{k_1 - k_1^*}{\alpha - k_1^*}\mathbf{P}_1\hat{\mathbf{H}}(\alpha)(\mathbf{I}_2 - \mathbf{P}_1).$$

Then one can readily compute

$$\langle\bar{\mathbf{a}}^{-1}\hat{\mathbf{H}}\bar{\mathbf{a}}\rangle(k) = \langle\hat{\mathbf{H}}\rangle(k) + \frac{k_1 - k_1^*}{k - k_1}(\mathbf{I}_2 - \mathbf{P}_1)\langle\hat{\mathbf{H}}(k) - \hat{\mathbf{H}}(k_1)\rangle\mathbf{P}_1 - \frac{k_1 - k_1^*}{k - k_1^*}\mathbf{P}_1\langle\hat{\mathbf{H}}(k) - \hat{\mathbf{H}}(k_1^*)\rangle(\mathbf{I}_2 - \mathbf{P}_1),$$

which shows that the pole singularities of $\langle\bar{\mathbf{a}}^{-1}\hat{\mathbf{H}}\bar{\mathbf{a}}\rangle(k)$ are removable since the residues at $k = k_1$ and $k = k_1^*$ vanish. Substituting the expression for the one-soliton $\bar{\mathbf{a}}$ into the evolution equation (B.2), one finds that the left hand side consists of only terms that have simple poles at $k = k_1$ and $k = k_1^*$ arising from $\bar{\mathbf{a}}(k)$ and $\bar{\mathbf{a}}^{-1}(k) = \bar{\mathbf{a}}^\dagger(k^*)$, whereas the right hand side of the equation contains terms that are regular in k, as well as simple poles. For consistency, the regular terms on the right hand side must cancel, which can also be explicitly checked by direct calculation. Hence the remaining terms in Eq. (B.2) are only simple poles at $k = k_1$ and $k = k_1^*$. Equating the residues from both sides of the equation for the pole at $k = k_1$, yields

$$(\mathbf{I}_2 - \mathbf{P}_1)\partial_\tau\mathbf{P}_1 = i(\mathbf{I}_2 - \mathbf{P}_1)\langle\hat{\mathbf{H}}(k_1)\rangle\mathbf{P}_1.$$

Multiplying the above equation by $\boldsymbol{\beta}_1$, and making use of the identities $\mathbf{P}_1\boldsymbol{\beta}_1 = \boldsymbol{\beta}_1$, $(\mathbf{I}_2 - \mathbf{P}_1)^2 = \mathbf{I}_2 - \mathbf{P}_1$ and the relation $\partial_\tau(\mathbf{P}_1\boldsymbol{\beta}_1) = \partial_\tau\boldsymbol{\beta} \Rightarrow (\partial_\tau\mathbf{P}_1)\boldsymbol{\beta}_1 = (\mathbf{I}_2 - \mathbf{P}_1)\partial_\tau\boldsymbol{\beta}_1$, one finally arrives at the equation

$$(\mathbf{I}_2 - \mathbf{P}_1)\left[\partial_\tau\boldsymbol{\beta}_1 - i\hat{\mathbf{H}}(k_1)\boldsymbol{\beta}_1\right] = 0,$$

which is satisfied by virtue of the evolution equation (2.22) for the norming constant. Similarly, equating the residues for the pole at $k = k_1^*$ leads to the Hermitian conjugate of Eq. (2.22). Therefore, we have shown that if (2.22) holds for $\boldsymbol{\beta}_1(\tau)$, then the one-soliton $\bar{\mathbf{a}}(k, \tau)$ satisfies the evolution equation (B.2).

The above argument can be generalized by induction to the transmission matrix $\bar{\mathbf{a}}(k)$ corresponding to an arbitrary number of solitons. Indeed, let us denote by $\bar{\mathbf{a}}_n(k)$ the transmission matrix corresponding to a pure n-soliton solution. Then, according to the product representation (3.17), one has $\bar{\mathbf{a}}_{n+1}(k) = \bar{\mathbf{a}}_n(k)\,\boldsymbol{\ell}_{n+1}(k)$ with the Blaschke factors defined as in (3.17) and the vectors $\mathbf{v}_j$, $j = 1, 2, \ldots, n+1$ given by (3.18). Substituting $\bar{\mathbf{a}}_{n+1}(k)$ into the evolution equation (B.2), and assuming that it holds for $\bar{\mathbf{a}}_n(k)$, one obtains the following equation for $\boldsymbol{\ell}_{n+1}$:

$$\boldsymbol{\ell}_{n+1}^{-1}\partial_\tau\boldsymbol{\ell}_{n+1} = i\,\boldsymbol{\ell}_{n+1}^{-1}\langle\bar{\mathbf{a}}_n^{-1}\hat{\mathbf{H}}\,\bar{\mathbf{a}}_n\rangle\boldsymbol{\ell}_{n+1} - i\,\langle\boldsymbol{\ell}_{n+1}^{-1}\bar{\mathbf{a}}_n^{-1}\hat{\mathbf{H}}\,\bar{\mathbf{a}}_n\,\boldsymbol{\ell}_{n+1}\rangle. \quad \text{(B.3)}$$

Hence $\bar{\mathbf{a}}_{n+1}(k)$ satisfies (B.2) if and only if $\boldsymbol{\ell}_{n+1}$ evolves according to (B.3).

Next, by using a partial fraction decomposition as in the one-soliton case, one can show that the terms $\langle\bar{\mathbf{a}}_n^{-1}\hat{\mathbf{H}}\,\bar{\mathbf{a}}_n\rangle(k)$ and $\langle\boldsymbol{\ell}_{n+1}^{-1}\bar{\mathbf{a}}_n^{-1}\hat{\mathbf{H}}\,\bar{\mathbf{a}}_n\,\boldsymbol{\ell}_{n+1}\rangle(k)$ in (B.3) are regular in k since the singularities at $k = k_j$ and $k = k_j^*$ for $j = 1, 2, \ldots, n$ (and also the singularities at $k = k_{n+1}, k_{n+1}^*$ for the second term) are removable. Furthermore, it can be explicitly checked that all regular terms from the right hand side of (B.3) cancel out, leaving only simple pole terms at $k = k_{n+1}$ and $k = k_{n+1}^*$ on both sides of the equation. Equating the residues of (B.3) at $k = k_{n+1}$, yields

$$(\mathbf{I}_2 - \mathbf{P}_{n+1})\,\partial_\tau\mathbf{P}_{n+1} = i\,(\mathbf{I}_2 - \mathbf{P}_{n+1})\,\langle\bar{\mathbf{a}}_n^{-1}\hat{\mathbf{H}}\,\bar{\mathbf{a}}_n\rangle(k_{n+1})\mathbf{P}_{n+1},$$

$$\mathbf{P}_{n+1} = \frac{\mathbf{v}_{n+1}\mathbf{v}_{n+1}^\dagger}{\mathbf{v}_{n+1}^\dagger\mathbf{v}_{n+1}}.$$

Then, multiplying both sides of the previous equation from the right by the vector $\mathbf{v}_{n+1}$, and using the identities $(\mathbf{I}_2 - \mathbf{P}_{n+1})^2 = \mathbf{I}_2 - \mathbf{P}_{n+1}$, $\mathbf{P}_{n+1}\mathbf{v}_{n+1} = \mathbf{v}_{n+1} \Rightarrow (\partial_\tau\mathbf{P}_{n+1})\mathbf{v}_{n+1} = (\mathbf{I}_2 - \mathbf{P}_{n+1})\,\partial_\tau\mathbf{v}_{n+1}$, one obtains

$$(\mathbf{I}_2 - \mathbf{P}_{n+1})\,\partial_\tau\mathbf{v}_{n+1} = i\,(\mathbf{I}_2 - \mathbf{P}_{n+1})\,\langle\bar{\mathbf{a}}_n^{-1}\hat{\mathbf{H}}\,\bar{\mathbf{a}}_n\rangle(k_{n+1})\mathbf{v}_{n+1}.$$

Now recall that according to (3.18), $\mathbf{v}_{n+1} = \bar{\mathbf{a}}_n^{-1}(k_{n+1})\boldsymbol{\beta}_{n+1}$, and therefore

$$\partial_\tau\mathbf{v}_{n+1} = \bar{\mathbf{a}}_n^{-1}(k_{n+1})\left(\partial_\tau\boldsymbol{\beta}_{n+1} - \partial_\tau\bar{\mathbf{a}}_n(k_{n+1})\mathbf{v}_{n+1}\right).$$

Substituting this expression for $\partial_\tau\mathbf{v}_{n+1}$ into the preceding equation, and taking into account Eq. (B.2) for $\bar{\mathbf{a}}_n$ at $k = k_{n+1}$, one finally obtains

$$(\mathbf{I}_2 - \mathbf{P}_{n+1})\,\bar{\mathbf{a}}_n^{-1}(k_{n+1})\left[\partial_\tau\boldsymbol{\beta}_{n+1} - i\langle\hat{\mathbf{H}}\rangle(k_{n+1})\boldsymbol{\beta}_{n+1}\right] = 0.$$

Equating the residues at $k = k_{n+1}^*$ on both sides of (B.3), yields the complex conjugate of the above equation. Thus, if the norming constant $\boldsymbol{\beta}_{n+1}$ satisfies (2.22), then $\boldsymbol{\ell}_{n+1}(k)$ evolves according to (B.3), so that the transmission matrix $\bar{\mathbf{a}}_{n+1}(k)$ satisfies the evolution equation (B.2).

Appendix C. One soliton superimposed to small radiation

In this appendix we include some preliminary results on how to find the solution of the inverse problem (2.8) and the corresponding scattering data $\mathbf{S}(k)$ in the case when the solution is not purely solitonic but the reflection coefficient $\mathbf{r}(k) \neq 0$ and its norm $\|\mathbf{r}(k)\|$ is assumed to be small for all $k \in \mathbb{R}$. The effect of radiation on the optical pulses propagating through a three-level medium is worth investigating for a few reasons. First, as explained in Section 2.3.3, while the solitonic content of the electromagnetic pulse is transmitted through the medium without any energy loss, it is in fact the radiative part of the pulse that is responsible for the excitation of energy levels in the medium. Secondly, the radiation $\mathbf{r}(k, \tau)$ is intrinsically coupled to the medium polarizability envelopes via the evolution equation (2.18). In fact, even small fluctuations around zero in the medium polarizability coefficient $\mathbf{m}$ in the initial preparation of the medium will result in $\mathbf{r}(k, \tau) \neq 0$ for $\tau > 0$, in spite of having $\mathbf{r}(k, 0) \equiv 0$. Another important issue is the interaction of radiation with soliton pulses since the soliton parameters e.g., the soliton center and the polarization vector are modified due to the interaction after the pulses are well separated from the dispersive waves. However, a comprehensive treatment of all these important issues is beyond the scope of this paper, so we limit our discussions to finding the leading order corrections to the scattering data for a soliton solution in the presence of small radiation. For simplicity, we consider the case of one-soliton, i.e., when $a(k)$ has a single zero k_1 in the upper-half plane and norming constant $\boldsymbol{\beta}_1$, in addition to small reflection.

The main idea is to solve for the eigenfunctions in (2.8) by a Born approximation. First, the integro-algebraic system (2.8) is iterated to obtain a system of equations for $\boldsymbol{\phi}(k)$, $\bar{\boldsymbol{\phi}}(k)$, up to second order in the reflection coefficient $\mathbf{r}(k)$. This system is expressed in terms of the bound state eigenfunctions $\boldsymbol{\phi}(k_1)$, $\bar{\boldsymbol{\phi}}(k_1^*)$. Evaluating the expression for $\boldsymbol{\phi}(k)$ at $k = k_1$, and that of $\bar{\boldsymbol{\phi}}(k)$ at $k = k_1^*$, yields a coupled system of linear equations for the bound states $\boldsymbol{\phi}(k_1)$, $\bar{\boldsymbol{\phi}}(k_1^*)$, which can then be solved up to second order in $\mathbf{r}$. Next, plugging the solutions for $\boldsymbol{\phi}(k_1)$, $\bar{\boldsymbol{\phi}}(k_1^*)$ back in the original system for $\boldsymbol{\phi}(k)$, $\bar{\boldsymbol{\phi}}(k)$ leads to the expressions given below for the quantities $\mathbf{M}(x, \tau, k) = \boldsymbol{\phi}(x, \tau, k)e^{ikx}$ and $\bar{\mathbf{M}}(x, \tau, k) = \bar{\boldsymbol{\phi}}(x, \tau, k)e^{-ikx}$ in terms of k_1, $\boldsymbol{\beta}_1$, $\mathbf{r}(k)$ only, and up to second order in $\mathbf{r}(k)$. We obtain

$$\begin{aligned}\bar{\mathbf{M}}(k) = \mathbf{e}_3 \Bigg\{ & 1 + A_2(k) - \frac{1}{\Delta}\Bigg[\Big(1 + \Big(\mathbf{A}_1^\dagger(k_1^*) + \mathbf{B}_1^\dagger(k_1^*)\Big)\tilde{\boldsymbol{\beta}}_1\Big) \\ & \times \tilde{\boldsymbol{\beta}}_1^\dagger \mathbf{A}_1(k) + \frac{\|\tilde{\boldsymbol{\beta}}_1\|^2}{2i\eta_1} B_2(k)\Bigg] - \frac{1}{k-k_1}\frac{1}{\Delta} \\ & \times \Bigg[(1 + A_2(k_1^*))\frac{\|\tilde{\boldsymbol{\beta}}_1\|^2}{2i\eta_1} + (1 + \tilde{\boldsymbol{\beta}}_1^\dagger \mathbf{A}_1(k_1^*)) \\ & \times \mathbf{B}_1^\dagger(k_1^*)\tilde{\boldsymbol{\beta}}_1 + \tilde{\boldsymbol{\beta}}_1^\dagger \mathbf{D}_2(k_1)\tilde{\boldsymbol{\beta}}_1\Bigg]\Bigg\} + (\mathbf{e}_1, \mathbf{e}_2) \\ & \times \Bigg\{\mathbf{B}_1(k) + \frac{1}{\Delta}\Bigg[\frac{\tilde{\boldsymbol{\beta}}_1\tilde{\boldsymbol{\beta}}_1^\dagger}{2i\eta_1}\mathbf{A}_1(k) - \mathbf{B}_1(k_1^*)\tilde{\boldsymbol{\beta}}_1^\dagger \mathbf{A}_1(k) + \tilde{\boldsymbol{\beta}}_1 B_2(k)\Bigg] \\ & + \frac{1}{k-k_1}\frac{1}{\Delta}\Bigg[\tilde{\boldsymbol{\beta}}_1 - \mathbf{B}_1(k_1^*)\frac{\|\tilde{\boldsymbol{\beta}}_1\|^2}{2i\eta_1} \\ & + \tilde{\boldsymbol{\beta}}_1\tilde{\boldsymbol{\beta}}_1^\dagger \mathbf{A}_1(k_1^*) + \mathbf{C}_2(k_1)\tilde{\boldsymbol{\beta}}_1\Bigg]\Bigg\} + \text{h.o.t.,}\end{aligned} \tag{C.1a}$$

$$\begin{aligned}\mathbf{M}(k) = \mathbf{e}_3 \Bigg\{ & -\mathbf{B}_1^\dagger(k^*) + \frac{1}{\Delta}\Bigg[\frac{\|\tilde{\boldsymbol{\beta}}_1\|^2}{2i\eta_1}\mathbf{A}_1^\dagger(k^*) \\ & + \mathbf{B}_1^\dagger(k_1^*)\tilde{\boldsymbol{\beta}}_1 \mathbf{A}_1^\dagger(k^*) - \tilde{\boldsymbol{\beta}}_1^\dagger \mathbf{D}_2(k)\Bigg] - \frac{1}{k-k_1^*}\frac{1}{\Delta} \\ & \times \Bigg[\Big(1 + A_2(k_1^*)\Big)\tilde{\boldsymbol{\beta}}_1^\dagger + \mathbf{B}_1^\dagger(k_1^*)\frac{\tilde{\boldsymbol{\beta}}_1\tilde{\boldsymbol{\beta}}_1^\dagger}{2i\eta_1} + \mathbf{A}_1^\dagger(k_1^*)\tilde{\boldsymbol{\beta}}_1\tilde{\boldsymbol{\beta}}_1^\dagger\Bigg]\Bigg\} \\ & + (\mathbf{e}_1, \mathbf{e}_2)\Bigg\{\mathbf{I}_2 + \mathbf{C}_2(k) - \frac{1}{\Delta}\Bigg[\Big(1 + \tilde{\boldsymbol{\beta}}_1^\dagger \mathbf{A}_1(k_1^*)\Big)\tilde{\boldsymbol{\beta}}_1 \mathbf{A}_1^\dagger(k^*) \\ & - \mathbf{B}_1(k_1^*)\mathbf{A}_1^\dagger(k^*)\frac{\|\tilde{\boldsymbol{\beta}}_1\|^2}{2i\eta_1} - \frac{\tilde{\boldsymbol{\beta}}_1\tilde{\boldsymbol{\beta}}_1^\dagger}{2i\eta_1}\mathbf{D}_2(k)\Bigg] \\ & - \frac{1}{k-k_1^*}\frac{1}{\Delta}\Bigg[\mathbf{B}_1(k_1^*)\tilde{\boldsymbol{\beta}}_1^\dagger\Big(1 + \mathbf{A}_1^\dagger(k_1^*)\tilde{\boldsymbol{\beta}}_1\Big) \\ & - \Big(\mathbf{I}_2 + \mathbf{C}_2(k_1)\Big)\frac{\tilde{\boldsymbol{\beta}}_1\tilde{\boldsymbol{\beta}}_1^\dagger}{2i\eta_1} + \tilde{\boldsymbol{\beta}}_1\tilde{\boldsymbol{\beta}}_1^\dagger B_2(k_1^*)\Bigg]\Bigg\} + \text{h.o.t.}\end{aligned} \tag{C.1b}$$

where

$$\begin{aligned}\Delta = {} & 1 + \frac{\|\tilde{\boldsymbol{\beta}}_1\|^2}{4\eta_1^2} + \tilde{\boldsymbol{\beta}}_1^\dagger \mathbf{A}_1(k_1^*) + \mathbf{A}_1^\dagger(k_1^*)\tilde{\beta}_1 + |\tilde{\boldsymbol{\beta}}_1^\dagger \mathbf{A}_1(k_1^*)|^2 \\ & - \frac{\tilde{\boldsymbol{\beta}}_1^\dagger\Big(\mathbf{D}_2(k_1) - B_2(k_1^*)\mathbf{I}_2\Big)\tilde{\boldsymbol{\beta}}_1}{2i\eta_1},\end{aligned}$$

$\tilde{\boldsymbol{\beta}}_1 = \boldsymbol{\beta}_1 e^{-2ik_j x}$, h.o.t. denotes higher order terms in $\mathbf{r}$, and the x, τ-dependence have been omitted for brevity. The coefficients in (C.1) are given by:

$$\mathbf{A}_1(x, \tau, k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{r}(\alpha, \tau)e^{-2i\alpha x}}{\left(\alpha - k_1^*\right)(\alpha - (k - i0))}d\alpha,$$

$$\mathbf{B}_1(x, \tau, k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{r}(\alpha, \tau)e^{-2i\alpha x}}{\alpha - (k - i0)}d\alpha,$$

$$\begin{aligned}A_2(x, \tau, k) = {} & \frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\alpha' \\ & \times \frac{\mathbf{r}^\dagger(\alpha', \tau)\mathbf{r}(\alpha, \tau)\,e^{2i(\alpha' - \alpha)x}}{(\alpha' - (\alpha + i0))(\alpha - (k - i0))},\end{aligned}$$

$$\begin{aligned}B_2(x, \tau, k) = {} & \frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\alpha' \\ & \times \frac{\mathbf{r}^\dagger(\alpha', \tau)\,\mathbf{r}(\alpha, \tau)\,e^{2i(\alpha' - \alpha)x}}{(\alpha' - k_1)(\alpha' - (\alpha + i0))(\alpha - (k - i0))},\end{aligned}$$

$$\begin{aligned}\mathbf{C}_2(x, \tau, k) = {} & \frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\alpha' \\ & \times \frac{\mathbf{r}(\alpha', \tau)\,\mathbf{r}^\dagger(\alpha, \tau)\,e^{-2i(\alpha' - \alpha)x}}{(\alpha' - (\alpha - i0))(\alpha - (k + i0))},\end{aligned}$$

$$\begin{aligned}\mathbf{D}_2(x, \tau, k) = {} & \frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\alpha' \\ & \times \frac{\mathbf{r}(\alpha', \tau)\,\mathbf{r}^\dagger(\alpha, \tau)\,e^{-2i(\alpha' - \alpha)x}}{\left(\alpha' - k_1^*\right)(\alpha' - (\alpha - i0))(\alpha - (k + i0))}.\end{aligned}$$

We have retained up to quadratic terms in $\mathbf{r}$ above, which is necessary to obtain the leading order corrections to the entries of the scattering matrix $\mathbf{S}(k)$, as we show below. Note that the two sets $\{\mathbf{A}_1(k), \mathbf{B}_1(k), A_2(k), B_2(k)\}$ and $\{\mathbf{C}_2(k), \mathbf{D}_2(k)\}$ admit analytic continuations to the lower and upper half planes, respectively. Taking the Hermitian conjugates of the above coefficients switches $k - i0$ to $k + i0$ and vice versa, if $k \in \mathbb{R}$. But when $k \in \mathbb{C}$, one also needs to switch k and k^*. For example, the Hermitian conjugate of $\mathbf{B}_1(k)$ which is analytic when $\operatorname{Im} k < 0$, is defined as

$$\mathbf{B}_1^\dagger(x, \tau, k^*) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{r}^\dagger(\alpha, \tau)e^{2i\alpha x}}{\alpha - (k + i0)}d\alpha,$$

and it is analytic when $\operatorname{Im} k > 0$.

By taking the limit as $x \to \infty$ of the expressions for $\mathbf{M}$, $\bar{\mathbf{M}}$ in (C.1) and using Eq. (2.10), one obtains the components of the scattering matrix $\mathbf{S}(k)$ accurate up to second order in $\mathbf{r}(k)$. In order to derive the expression for $\mathbf{S}(k)$, one first needs to determine the asymptotics of the coefficients defined above as $x \to +\infty$. Taking into account (2.16) and (2.20), one obtains that

$$\mathbf{A}_1(x, \tau, k) \underset{x \to +\infty}{\sim} \begin{cases} -\dfrac{\mathbf{r}(k, \tau)}{k - k_1^*}e^{-2ikx} & k \in \mathbb{R} \\ 0 & \operatorname{Im} k < 0 \end{cases}$$

$$\mathbf{B}_1(x, \tau, k) \underset{x \to +\infty}{\sim} \begin{cases} -\mathbf{r}(k, \tau)e^{-2ikx} & k \in \mathbb{R} \\ 0 & \operatorname{Im} k < 0, \end{cases}$$

while for any k with $\operatorname{Im} k \le 0$

$$\lim_{x \to +\infty} A_2(x, \tau, k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{r}^\dagger(\alpha, \tau)\mathbf{r}(\alpha, \tau)}{\alpha - (k - i0)}d\alpha =: \tilde{A}_2(k, \tau)$$

$$\begin{aligned}\lim_{x \to +\infty} B_2(x, \tau, k) & = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{r}^\dagger(\alpha, \tau)\mathbf{r}(\alpha, \tau)}{(\alpha - k_1)(\alpha - (k - i0))}d\alpha \\ & =: \tilde{B}_2(k, \tau),\end{aligned}$$

and for any k with $\operatorname{Im} k \geq 0$

$$\lim_{x\to+\infty} \mathbf{C}_2(x,\tau,k) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbf{r}(\alpha,\tau)\mathbf{r}^\dagger(\alpha,\tau)}{\alpha-(k+i0)}d\alpha =: \tilde{\mathbf{C}}_2(k,\tau)$$

$$\lim_{x\to+\infty} \mathbf{D}_2(x,\tau,k) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbf{r}(\alpha,\tau)\mathbf{r}^\dagger(\alpha,\tau)}{\left(\alpha-k_1^*\right)(\alpha-(k+i0))}d\alpha =: \tilde{\mathbf{D}}_2(k,\tau).$$

In addition, we deduce that in (C.1) the coefficients $\mathbf{A}_1(k_1^*) = \mathbf{B}_1(k_1^*) = 0$ as $x \to \infty$ by the Riemann–Lebesgue lemma assuming $\mathbf{r}(\alpha,\tau)$ decays sufficiently fast for all $\tau > 0$. Finally, the following expressions for the scattering coefficients are derived from (C.1) in the limit as $x \to \infty$

$$a(k) = 1 + \tilde{A}_2^*(k^*) - 2i\eta_1 \tilde{B}_2^*(k^*) - \frac{2i\eta_1}{k-k_1^*}\left(1 + \tilde{A}_2^*(k_1^*) - 2i\eta_1\tilde{B}_2^*(k_1^*)\right) + \text{h.o.t.}$$

$$\bar{\mathbf{a}}(k) = \mathbf{I}_2 + \tilde{\mathbf{C}}_2^\dagger(k^*) + 2i\eta_1\tilde{\mathbf{D}}_2^\dagger(k^*)\hat{\boldsymbol{\beta}}_1\hat{\boldsymbol{\beta}}_1^\dagger + \frac{2i\eta_1}{k-k_1}\hat{\boldsymbol{\beta}}_1\hat{\boldsymbol{\beta}}_1^\dagger \times \left[(1+2i\eta_1\hat{\boldsymbol{\beta}}_1^\dagger\tilde{\mathbf{D}}_2^\dagger(k_1)\hat{\boldsymbol{\beta}}_1)\mathbf{I}_2 + \tilde{\mathbf{C}}_2^\dagger(k_1)\right] + \text{h.o.t.},$$

and for $k \in \mathbb{R}$,

$$\mathbf{b}(k) = \left(1 - \frac{2i\eta_1}{k-k_1^*}\right)\mathbf{r}(k) + \text{h.o.t.},$$

$$\bar{\mathbf{b}}(k) = -\left(\mathbf{I}_2 - \frac{2i\eta_1}{k-k_1^*}\hat{\boldsymbol{\beta}}_1\hat{\boldsymbol{\beta}}_1^\dagger\right)\mathbf{r}(k) + \text{h.o.t.},$$

where $\hat{\boldsymbol{\beta}}_1$ is the unit vector along $\boldsymbol{\beta}_1$. Note that the leading order corrections to the scattering coefficients a and $\bar{\mathbf{a}}$ are quadratic in the reflection coefficient, while they are linear in $\mathbf{r}$ for the coefficients $\mathbf{b}$ and $\bar{\mathbf{b}}$ which do not have any quadratic corrections. From the definition of the second order coefficients $\tilde{A}_2(k), \tilde{\mathbf{B}}_2(k), \tilde{\mathbf{C}}_2(k), \tilde{\mathbf{D}}_2(k)$, one can also verify that $a(k)$ and $\bar{\mathbf{a}}(k)$ are analytic in the upper and lower half planes, respectively. Furthermore, it is easy to see that up to higher order terms, $a(k_1) = 0$, and that $\hat{\boldsymbol{\beta}}_1$ is a null vector for $\bar{\mathbf{a}}(k_1^*)$ so that $\det \bar{\mathbf{a}}(k_1^*) = 0$.

The procedure outlined in this appendix gives a recursive scheme for obtaining the scattering matrix $\mathbf{S}(k)$ in terms of the *essential* scattering data $\mathbf{r}(k)$, $\{k_i, \boldsymbol{\beta}_i\}_{i=1}^n$. The above expression for $\bar{\mathbf{a}}(k)$ is particularly relevant, since it provides an approximate solution to the matrix Riemann–Hilbert problem of determining $\bar{\mathbf{a}}(k)$ that is analytic in $\operatorname{Im} k < 0$ from the following data: (i) its large-k asymptotic behavior; (ii) zero of $\det \bar{\mathbf{a}}(k)$ at $k = k_1^*$ and norming constant $\boldsymbol{\beta}_1$; and (iii) the value of $\bar{\mathbf{a}}(k)^\dagger\bar{\mathbf{a}}(k)$ for $k \in \mathbb{R}$, related to $\mathbf{r}(k)$ through (2.5) and (2.7).

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