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Bäcklund and Darboux Transformations for the Nonstationary Schrödinger Equation

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Abstract—Potentials of the nonstationary Schrödinger operator constructed by means of n recursive Bäcklund transformations are studied in detail. The corresponding Darboux transformations of the Jost solutions are introduced. We show that these solutions obey modified integral equations and present their analyticity properties. Generated transformations of the spectral data are derived.

1. INTRODUCTION

In this article we continue (see [1–5]) our investigation into the direct and inverse scattering transform for the nonstationary Schrödinger operator

$$L = i\partial_{x_2} + \partial_{x_1}^2 - u(x_1, x_2) \tag{1.1}$$

in the case when u is a real function with "ray" type behavior. More exactly, u is supposed to be rapidly decaying in all directions on the x-plane with the exception of a finite number of directions, where it has finite and nontrivial limits; i.e.,

$$u_{n,\pm}(x_1) = \lim_{x_2 \to \pm\infty} u(x_1 - 2\mu_n x_2, x_2), \qquad n = 1, 2, \dots, N, \tag{1.2}$$

for N real constants μ_n. The spectral theory of operator L with Potential of this class is interesting *per se* and because it is associated [6, 7] with the Kadomtsev–Petviashvili equation [8]—in its version called KP I—

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = 3u_{x_2x_2}. \tag{1.3}$$

The mentioned extension of the spectral theory of the nonstationary Schrödinger operator would provide possibility to extend correspondingly the class of solutions of the KP I equation, including the solutions with asymptotic one-dimensional behavior. On the other hand, such spectral theory is essential for investigation by the inverse scattering transform method of strongly localized soliton solutions and their interaction with the background for the Davey–Stewartson I equation, where operator (1.1) controls the behavior of the "boundaries at infinity" of an auxiliary function for this equation (see [9, 10] for details).

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The spectral theory of the operator (1.1) with the ray potential is essentially more involved for rapidly decaying potential. Indeed, in the standard case [11–18] one defines the Jost solution $\Phi(x,k)$ as the solution of the nonstationary Schrödinger equation,

$$(i\partial_{x_2} + \partial_{x_1}^2 - u(x))\phi(x,k) = 0, \tag{1.4}$$

that is analytic in the complex plane of the spectral parameter k, $k_\Im \neq 0$, and normalized at infinity by the condition that for the function

$$\chi(x,k) = e^{ikx_1 + ik^2x_2}\Phi(x,k) \tag{1.5}$$

we have

$$\lim_{k\to\infty,\ k_\Im\neq 0} \chi(x,k) = 1. \tag{1.6}$$

This function can be given as the solution of the integral equation [12]

$$\chi(x,k) = 1 + \int dx'\, G_0(x - x', k)u(x')\chi(x', k), \tag{1.7}$$

where $G_0(x,k)$ is the Green function

$$G_0(x,k) = \frac{\operatorname{sign} x_2}{2\pi i}\int d\alpha\, \theta(\alpha k_\Im x_2)e^{i\alpha x_1 - i\alpha(\alpha-2k)x_2}, \qquad k \in \mathbb{C}, \tag{1.8}$$

of zero potential,

$$(i\partial_{x_2} + \partial_{x_1}^2 - 2ik\partial_{x_1})G_0(x,k) = \delta(x). \tag{1.9}$$

The solvability of (1.7) under some small norm assumptions was proved in [19], and thanks to (1.7) it is easy to show that $\chi(x,k)$ has the asymptotic behavior

$$\lim_{|x_1|\to\infty} \chi(x,k) = 1 \tag{1.10}$$

on the x-plane, $k_\Im \neq 0$, independently of the direction. The Jost solution $\Phi(x,k)$ defined by (1.5) obeys [17, 18] the following normalization and completeness conditions:

$$\int dx_1\, \overline{\Phi(x_1, x_2, k+p)}\Phi(x_1, x_2, \bar{k}) = 2\pi\delta(p), \qquad p \in \mathbb{R}, \tag{1.11}$$

$$\int dk_\Re\, \overline{\Phi(x_1', x_2, k)}\Phi(x_1, x_2, \bar{k}) = 2\pi\delta(x_1 - x_1'). \tag{1.12}$$

The Jost solution is an analytic function of k in the complex plane, $k_\Im \neq 0$. It has finite limits on the real axis,

$$\Phi^{\pm}(x,k) = \lim_{k_\Im\to\pm 0} \Phi(x,k). \tag{1.13}$$

By (1.11) these boundary values obey the normalization conditions

$$\int dx_1\, \overline{\Phi^{\pm}(x_1, x_2, k)}\Phi^{\mp}(x_1, x_2, p) = 2\pi\delta(p-k), \qquad p, k \in \mathbb{R}. \tag{1.14}$$

The spectral data are introduced as a measure of departure from analyticity of the Jost solutions. In literature there exists a variety of definitions of spectral data. In what follows we use the one suggested in [14, 17, 18]:

$$\mathcal{F}(k,p) = \frac{1}{2\pi}\int dx_1'\, \overline{\Phi^{+}(x_1', x_2, k)}\Phi^{+}(x_1', x_2, p) - \delta(p-k), \qquad p, k \in \mathbb{R}. \tag{1.15}$$

It is easy to check that $\mathcal{F}(k,p)$ is independent of x_2. The condition of the reality of the potential u in (1.1) is equivalent to the self-adjointness of the integral operator with kernel $\mathcal{F}(k,p)$. Now by (1.12) we have

$$\Phi^+(x,k) = \Phi^-(x,k) + \int dp\, \Phi^-(x,p)\mathcal{F}(p,k). \tag{1.16}$$

Thus, the inverse scattering transform is the nonlocal Riemann–Hilbert problem of the construction of the function $\Phi(x,k)$ analytic in the upper and bottom half-planes with normalization (1.5), (1.6) and a discontinuity at the real axis given by (1.16).

It was mentioned in [16] that integral equation (1.7) cannot be applied in the case of a potential $u(x)$ not vanishing in all directions at large distances as the Green function is slowly decaying at space infinity. In [5] the following modification of this integral equation was suggested:

$$\chi(x,k) = 1 + \int_{-k_\Im\infty}^{x_1} dy_1 \int dx'\, \partial_{y_1} G_0(y_1 - x_1', x_2 - x_2', k)u(x')\chi(x',k), \tag{1.17}$$

where the order of operations is explicitly prescribed. Here and below we use the notations of the type $k_\Im\infty$ in the limits of the integrals to indicate the sign of infinity. If the solution of this equation exists and is bounded, then, like in the one-dimensional case,

$$\lim_{x_1\to -k_\Im\infty} \chi(x,k) = 1, \qquad k_\Im \neq 0, \tag{1.18}$$

while, in contrast to (1.10), it can be different from 1 in the opposite direction. This modified integral equation is applicable to the simplest case of a potential of type (1.2), i.e., to the case $u(x) = u(x_1)$, and it is trivial to check that it gives the standard (see [20]) one-dimensional equation for the Jost solution. Nevertheless, a full description of the solutions of equation (1.17) with potentials of the class (1.2) is absent until now. Only multiple pure soliton solutions were constructed in [21, 22]. It was shown[3] that solutions of this equation can have additional cuts in the complex plane of the spectral parameter [5]. Because of this we are studying here a special but rather wide subclass of potentials of type (1.2) that is obtained by applying recursively the so-called binary Bäcklund transformations [23] with a complex spectral parameter to a decaying potential. As we are interested in the spectral characteristics of potentials u having nontrivial limits, we also study the corresponding Darboux transformations furnishing the Jost solutions for the transformed potentials and their analytical properties as well as the transformations of the spectral data.

In [23] and [24], a rather simple and transparent method was proposed for performing binary Bäcklund transformations of the potential u and corresponding Darboux transformations. Let $\phi(x,k)$ be a solution of the nonstationary Schrödinger equation (1.4) with potential $u(x)$. Then the transformed potential is equal to

$$\tilde{u}(x) = u(x) - 2\partial_{x_1}^2 \ln \Delta(x), \tag{1.19}$$

where

$$\Delta(x) = \int^{x_1} dx_1'\, |\phi(x_1', x_2, \lambda)|^2. \tag{1.20}$$

The Darboux transform of $\phi(x,k)$,

$$\tilde{\phi}(x,k) = \phi(x,k) - \frac{\phi(x,\lambda)}{\Delta(x)} \int^{x_1} dx_1'\, \overline{\phi(x_1', x_2, \lambda)}\phi(x_1', x_2, k), \tag{1.21}$$

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solves the equation

$$(i\partial_{x_2} + \partial_{x_1}^2 - \tilde{u}(x))\tilde{\phi}(x,k) = 0 \tag{1.22}$$

with transformed potential. It is natural to expect that this transformation for complex parameter λ would supply an example of a potential of the type (1.2). The check of the fact that $\tilde{\phi}$ obeys equation (1.22) is based on the following identity for a pair of arbitrary solutions $f(x)$ and $g(x)$ of equation (1.22):

$$i\partial_{x_2}(\overline{f(x)}g(x)) = -\partial_{x_1}W(\overline{f(x)}, g(x)), \tag{1.23}$$

where Wronskian

$$W(\overline{f(x)}, g(x)) = \overline{f(x)}\partial_{x_1}g(x) - g(x)\partial_{x_1}\overline{f(x)} \tag{1.24}$$

was introduced.

In order to make equations (1.19)–(1.21) determined, it is necessary to substitute indefinite integrals by definite ones and to choose the constants of integration so that the potential $\tilde{u}$ is real and regular. In order to get transformations parametrized by constants and not by functions of x_2 obeying some differential equations, it is natural to choose semi-infinite limits of integrals in (1.20), (1.21). Then we can use the fact that the asymptotic with respect to x_1 behavior (1.10) is independent of x_2. On the other hand, the infinite limits of these integrals require an exact control of their convergence in the recursive procedure. In addition, one must write the solution $\tilde{\phi}$ as a linear combination of the Jost solution and a solution corresponding to the discrete spectrum. The way to build the correct recursive procedures for generating both solutions is suggested by the remark that $\phi(x,\lambda)\Delta(x)^{-1}$ is a solution of equation (1.22).

Thus, we start with a regular, rapidly decaying real potential $u(x)$ for which all above mentioned elements of the direct and inverse problems are given. In Section 2 we introduce an exact recursion procedure for an arbitrary number of Bäcklund and Darboux transformations both for the Jost solutions and solutions corresponding to the discrete spectrum. We formulate the conditions of the reality and regularity of the potentials constructed by these means and derive the transformations of spectral data (by analogy with [25], where the transformations of the continuous spectra were considered).

In Section 3 this recursion procedure is solved in terms of the original potential $u(x)$, its Jost solution $\Phi(x,k)$, and spectral data $\mathcal{F}(k,p)$. As a result, we get a potential depending on $(N+1)N$ complex parameters describing N solitons superimposed to a generic background. The necessary and sufficient conditions of regularity and reality of the constructed potential are formulated in terms of these parameters. In the case $u(x) \equiv 0$, our construction furnishes not only the multisoliton potentials obtained in [21], but also the potentials in the essentially more general class derived in [22]. Inclusion of such potentials in consideration essentially complicates the whole construction; however, they are essential for applications (see [9, 10]).

In Section 4 we consider the asymptotic behavior of the constructed potentials on the x-plane. We show that it is of the type (1.2) and that it essentially depends on the signs of the imaginary parts of parameters λ_j of the Bäcklund transformations. The latter essentially distinguishes the case of the two-dimensional nonstationary Schrödinger equation from the case of the one-dimensional stationary equation. In the future, we plan to apply these results to the study of the generic potentials of the type (1.2) by means of the formulation of the scattering problem on a nontrivial background suggested in [5].

2. RECURSION PROCEDURE

Formulas (1.19), (1.20), and (1.21) enable us to formulate the recursion procedure for composing an arbitrary number of binary Bäcklund transformations. Indeed, let $\phi_n(x,k)$ solve equation (1.4) with potential $u_n(x)$,

$$i\partial_{x_2}\phi_n(x,k)+\partial_{x_1}^2\phi_n(x,k)=u_n(x)\phi_n(x,k); \tag{2.1}$$

then we specify equation (1.21) for ϕ_{n+1} in the following way:

$$\phi_{n+1}(x,k)=\phi_n(x,k)-g_{n+1}(x)\left[B'_{n+1}(k)+\int\limits_{-(k_\Im+\lambda_{n+1\Im})\infty}^{x_1}dx'_1\,\overline{\phi_n(x'_1,x_2,\lambda_{n+1})}\phi_n(x'_1,x_2,k)\right], \tag{2.2}$$

where we introduced the notations

$$g_{n+1}(x)=\frac{\phi_n(x,\lambda_{n+1})}{\Delta_{n+1}(x)}, \tag{2.3}$$

$$\Delta_{n+1}(x)=c_{n+1}+\int\limits_{-\lambda_{n+1\Im}\infty}^{x_1}dx'_1\,|\phi_n(x'_1,x_2,\lambda_{n+1})|^2, \tag{2.4}$$

$$n=0,1,\dots,$$

and c_{n+1} and $B'_{n+1}(k)$ are an x-independent constant and a function of k, respectively. Then ϕ_{n+1} has to be a solution of the shifted $(n\to n+1)$ equation (2.1) with the potential

$$u_{n+1}(x)=u_n(x)-2\partial_{x_1}^2\ln\Delta_{n+1}(x). \tag{2.5}$$

In what follows, it is convenient to write all ϕ_n as sums of two solutions of equation (2.1),

$$\phi_n(x,k)=F_n(x,k)+f_n(x,k), \tag{2.6}$$

that are given by their recursion relations

$$F_{n+1}(x,k)=F_n(x,k)-g_{n+1}(x)\int\limits_{-(k_\Im+\lambda_{n+1\Im})\infty}^{x_1}dx'_1\,\overline{\phi_n(x'_1,x_2,\lambda_{n+1})}F_n(x'_1,x_2,k), \tag{2.7}$$

$$f_{n+1}(x,k)=f_n(x,k)-g_{n+1}(x)\left[B_{n+1}(k)+\int\limits_{-\lambda_{n+1\Im}\infty}^{x_1}dx'_1\overline{\phi_n(x'_1,x_2,\lambda_{n+1})}f_n(x'_1,x_2,k)\right], \tag{2.8}$$

$$n=0,1,\dots,$$

where functions $B_{n+1}(k)$ differ from $B'_{n+1}(k)$ because of the different bottom limits of integrals in equations (2.7) and (2.8). We also put

$$u_0(x)=u(x),\qquad F_0(x,k)=\Phi(x,k),\qquad f_0(x,k)=0. \tag{2.9}$$

The properties of all objects introduced are given in the following.

Theorem 2.1. *Let the potential $u(x)$ in (1.1) be real, regular, and rapidly decaying at space infinity and let $\Phi(x,k)$ be the Jost solution of this equation; i.e., let $\Phi(x,k)$ be defined by (1.5) and (1.7). Let also the sets of complex constants $\lambda_1, \lambda_2, \dots$ and real nonzero constants $c_1, c_2, \dots$ obey the following conditions:*

$$\lambda_{n\Im} \neq 0, \qquad n = 1, 2, \dots, \tag{2.10}$$

$$|\lambda_{1\Im}| > |\lambda_{2\Im}| > \dots, \tag{2.11}$$

$$\lambda_{n\Im} c_n > 0, \qquad n = 1, 2, \dots, \tag{2.12}$$

and let some functions $B_1(k), B_2(k), \dots$ of the complex parameter k be given. Then for all $n \geq 0$ the following statements hold true.

1. *The integrals in (2.4), (2.8), and (2.7) converge and define regular functions of x and k for $k_\Im \neq 0$, $k_\Im + \lambda_{j\Im} \neq 0$, $j = 1, 2, \dots, n$. Functions $\Delta_{n+1}(x)$ have no zeroes on the x-plane.*

2. *There exist nonzero limits*

$$\lim_{x_1 \to \pm k_\Im \infty} e^{ikx_1 + ik^2 x_2} F_n(x,k) = A_n(\pm, k) \tag{2.13}$$

that are independent of x_2 and obey the recursion relations

$$A_0(\pm, k) = 1, \tag{2.14}$$

$$A_{n+1}(\pm, k) = \left[1 + \theta(\pm k_\Im \lambda_{n+1\Im}) \frac{2i\lambda_{n+1\Im}}{\bar{\lambda}_{n+1} - k}\right] A_n(\pm, k). \tag{2.15}$$

3. *There exist finite nonzero limits*

$$\lim_{|x_1| \to \infty} e^{i\lambda_{n+1\Re} x_1 + |\lambda_{n+1\Im} x_1|} g_{n+1}(x) = \begin{cases} \dfrac{2\lambda_{n+1\Im}}{\overline{A_n(+, \lambda_{n+1})}} e^{-i\bar{\lambda}_{n+1}^2 x_2}, & x_1 \to +\lambda_{n+1\Im}\infty, \\[2ex] \dfrac{A_n(-, \lambda_{n+1})}{c_{n+1}} e^{-i\lambda_{n+1}^2 x_2}, & x_1 \to -\lambda_{n+1\Im}\infty, \end{cases} \tag{2.16}$$

4. *There exist finite nonzero (for $n \geq 1$) limits*

$$\lim_{|x_1| \to \infty} e^{i\lambda_{n\Re} x_1 + |\lambda_{n\Im} x_1|} f_n(x,k) = -\begin{cases} (B_n(k) + b_n(k)) \dfrac{2\lambda_{n\Im}}{\overline{A_{n-1}(+, \lambda_n)}} e^{-i\bar{\lambda}_n^2 x_2}, & x_1 \to +\lambda_{n\Im}\infty, \\[2ex] B_n(k) \dfrac{A_{n-1}(-, \lambda_n)}{c_n} e^{-i\lambda_n^2 x_2}, & x_1 \to -\lambda_{n\Im}\infty, \end{cases} \tag{2.17}$$

where the functions

$$b_n(k) = \operatorname{sign} \lambda_{n\Im} \int_{-\infty}^{+\infty} dx_1' \, \overline{\phi_{n-1}(x_1', x_2, \lambda_n)} f_{n-1}(x_1', x_2, k) \tag{2.18}$$

are x_2-independent.

5. *Functions $F_n(x,k)$, $g_n(x)$, $f_n(x,k)$, and $\phi_n(x,k)$ solve the nonstationary Schrödinger equation* (2.1) *with the potential*

$$u_n(x) = u(x) - 2\partial_{x_1}^2 \ln \prod_{j=1}^{n} \Delta_j(x). \tag{2.19}$$

6. *Function*

$$\Phi_n(x,k) = \frac{F_n(x,k)}{A_n(-,k)}, \qquad n = 1, 2, \dots, \tag{2.20}$$

is the Jost solution of equation (2.1) *with potential* (2.5); *i.e.,*

$$\chi_n(x,k) = e^{ikx_1 + ik^2 x_2} \Phi_n(x,k) \tag{2.21}$$

obeys the modified integral equations (1.17) *with $u_n(x)$ from* (2.19) *substituted for $u(x)$.*

The theorem is proved by induction on n. Therefore, it will sometimes be useful, when referring to a formula (#) depending on n, to make this dependence explicit by writing $(\#)_n$ and then $(\#)_{n+1}$ when the same formula is considered for the shifted $(n \to n+1)$ value.

We divide the proof into a sequence of lemmas.

Lemma 2.1. *Let for some $n \geq 1$ the functions F_n and f_n be regular functions of x and k, for $k_\Im \neq 0$, $k_\Im + \lambda_{j\Im} \neq 0$, $j = 1, 2, \dots, n$, and obey statements* 2 *and* 4 *of Theorem* 2.1, *or let they be given by* (2.9) *for $n = 0$. Let $B_n(k)$ be regular functions of k and let λ_n, λ_{n+1}, c_{n+1} obey* (2.10)–(2.12). *Then for ϕ_n defined in* (2.6) *there exist the limits*

$$\lim_{x_1 \to \pm k_\Im \infty} e^{ikx_1 + ik^2 x_2} \phi_n(x,k) = A_n(\pm,k), \qquad k \in \mathbb{C}, \quad |k_\Im| < |\lambda_{n\Im}|, \tag{2.22}$$

where A_n is given in (2.13) *and function $\Delta_{n+1}(x)$ determined by* (2.4) *exists, has no zeroes on the x-plane, and obeys the asymptotic behavior*

$$\Delta_{n+1}(x) \to \begin{cases} \dfrac{|A_n(+,\lambda_{n+1})|^2}{2\lambda_{n+1\Im}} e^{2\lambda_{n+1\Im}(x_1 + 2\lambda_{n+1\Re} x_2)}, & x_1 \to +\lambda_{n+1\Im}\infty, \\ c_{n+1}, & x_1 \to -\lambda_{n+1\Im}\infty, \end{cases} \tag{2.23}$$

where, in the case $n = 0$, in agreement with (2.6), $A_0 = 1$.

Proof. In the case $n = 0$ equality (2.22) directly follows from (1.7) and (2.9). If $n \geq 1$, then this asymptotic behavior of ϕ_n trivially follows from (2.6), (2.13), and (2.17). Thanks to this asymptotic behavior and taking into account that $|\lambda_{n\Im}| > |\lambda_{n+1\Im}| > 0$, we get the convergence of the integral in (2.4). Asymptotics (2.23) follows from (2.22). Taking into account that, by definition, Δ_{n+1} is a monotonous function of x_1 and that, thanks to (2.12), the signs of both limits in (2.23) coincide, we see that this function has no zeroes on the x-plane.

Lemma 2.2. *Under conditions of Lemma* 2.1, *function g_{n+1} defined by* (2.3) *is regular and has the asymptotics* (2.16).

Proof of this lemma follows directly from Lemma 2.1 as Δ_{n+1} has no zeroes and the asymptotics (2.22), (2.23) guarantee that g_{n+1} decays with a proper exponent for growing x. The exact values of the limits are also readily obtained from (2.22) and (2.23).

Lemma 2.3. *Under conditions of Lemma* 2.1, *the function f_{n+1} defined by* (2.8) *is regular and obeys the asymptotics* $(2.17)_{n+1}$.

Proof. The convergence of the integral in (2.8) and regularity of f_{n+1} follow directly from Lemma 2.1 if we take into account that by (2.11) $|\lambda_{n+1\Im}| < |\lambda_{n\Im}|$. In particular, thanks to this inequality and $(2.22)_n$ and $(2.17)_n$, the integrals $\int_{\pm\infty}^{x_1} dx_1' \, \overline{\phi_n(x_1', x_2, \lambda_{n+1})} f_n(x_1', x_2, k)$ are convergent. Then from (2.8) we get

$$e^{i\lambda_{n+1\Re}x_1+|\lambda_{n+1\Im}x_1|} f_{n+1}(x,k) = e^{i\lambda_{n+1\Re}x_1+|\lambda_{n+1\Im}x_1|} f_n(x,k)$$

$$- e^{i\lambda_{n+1\Re}x_1+|\lambda_{n+1\Im}x_1|} g_{n+1}(x) \left[B_{n+1}(k) + \int\limits_{-\lambda_{n+1\Im}\infty}^{x_1} dx_1' \, \overline{\phi_n(x_1', x_2, \lambda_{n+1})} f_n(x_1', x_2, k) \right].$$

The asymptotics $(2.17)_n$ guarantees that the first term in the r.h.s. goes to zero when $|x_1| \to \infty$ because of the condition $|\lambda_{n+1\Im}| < |\lambda_{n\Im}|$. The first factor of the second term has finite limits by Lemma 2.2, and thanks to the mentioned convergence of the integral in brackets we prove $(2.17)_{n+1}$ and $(2.18)_{n+1}$.

Lemma 2.4. *Under conditions of Lemma* 2.1, F_{n+1} *defined by* (2.7) *is a regular function obeying asymptotics* $(2.13)_{n+1}$, *where* A_{n+1} *is given in* (2.15).

Proof. The convergence of the integral in (2.7) follows from the asymptotic behaviors $(2.13)_n$ and $(2.22)_n$ and then the regularity of F_{n+1} follows from Lemma 2.2. In order to prove $(2.13)_{n+1}$, we consider the asymptotic behavior of the integral term:

$$e^{i(k-\bar\lambda_{n+1})x_1+i(k^2-\bar\lambda_{n+1}^2)x_2} \int\limits_{-(k_\Im+\lambda_{n+1\Im})\infty}^{x_1} dx_1' \, \overline{\phi_n(x_1', x_2, \lambda_{n+1})} F_n(x_1', x_2, k)$$

$$\to \overline{A_n(\pm\,\mathrm{sign}(k_\Im \lambda_{n+1\Im}), \lambda_{n+1})} \frac{A_n(\pm, k)}{i(\bar\lambda_{n+1} - k)}, \qquad x_1 \to \pm k_\Im \infty, \tag{2.24}$$

where $(2.13)_n$ and $(2.22)_n$ were used. Let us now write (2.7) as

$$e^{ikx_1+ik^2x_2} F_{n+1}(x,k) = e^{ikx_1+ik^2x_2} F_n(x,k)$$

$$- e^{\lambda_{n+1\Im}x_1 - |\lambda_{n+1\Im}x_1|} \left(e^{i\lambda_{n+1\Re}x_1+|\lambda_{n+1\Im}x_1|+i\bar\lambda_{n+1}^2 x_2} g_{n+1}(x) \right)$$

$$\times \left(e^{i(k-\bar\lambda_{n+1})x_1+i(k^2-\bar\lambda_{n+1}^2)x_2} \int\limits_{-(k_\Im+\lambda_{n+1\Im})\infty}^{x_1} dx_1' \, \overline{\phi_n(x_1', x_2, \lambda_{n+1})} F_n(x_1', x_2, k) \right).$$

In the first term of the r.h.s. we can use $(2.13)_n$. The second term is given as a product of three multipliers, each having a finite limit for x_1 going to infinity. Moreover, the first multiplier has a nonzero limit only for $x_1 \to \lambda_{n+1\Im}\infty$. Then $(2.15)_{n+1}$ follows from $(2.16)_n$ proved in Lemma 2.2 and (2.24).

Lemma 2.5. *If under the assumptions of Lemma* 2.1 *statement* 5 *of Theorem* 2.1 *is satisfied, then this statement is also valid for* $n \to n+1$ *with the potential* u_{n+1} *given in* $(2.19)_{n+1}$.

Proof. By the inductive hypothesis, ϕ_n obeys

$$(i\partial_{x_2} + \partial_{x_1}^2 - u_n(x))\phi_n(x,k) = 0 \tag{2.25}$$

and the same equation is valid for F_n, g_n, and f_n. Thus, by identity (1.23),

$$i\partial_{x_2} \int\limits_{-(k_\Im+\lambda_{n+1\Im})\infty}^{x_1} dx_1' \, \overline{\phi_n(x_1', x_2, \lambda_{n+1})} F_n(x_1', x_2, k) = -W(\overline{\phi_n(x, \lambda_{n+1})}, F_n(x,k)),$$

where the asymptotics (2.13) and (2.22) were taken into account. Analogously, from (2.4) we get

$$i\partial_{x_2}\Delta_{n+1}(x) = -W(\overline{\phi_n(x,\lambda_{n+1})}, \phi_n(x,\lambda_{n+1})). \tag{2.26}$$

Then, by (2.3) and (2.25)

$$(i\partial_{x_2} + \partial_{x_1}^2 - u_{n+1}(x))g_{n+1}(x) = 0 \tag{2.27}$$

with the potential $u_{n+1}(x)$ defined as in (2.5). Now, by (2.6), (2.7), and (2.8) it is easy to check that F_{n+1}, f_{n+1}, and ϕ_{n+1} solve (2.27).

Lemma 2.6. *If under the assumptions of Lemma* 2.1 *statement* 5 *of Theorem* 2.1 *is fulfilled, then statement* 6 *is valid for* $n \to n+1$.

Proof. We have to demonstrate that χ_{n+1} defined by (2.20) and (2.21) obeys integral equation (1.17) with u_{n+1} given in (2.5). Since by Lemma 2.5 F_{n+1} obeys differential equation (2.27), we can write

$$\int dx'\,\partial_{x_1}G_0(x-x',k)u_{n+1}(x')\chi_{n+1}(x',k) = \int dx'\,\partial_{x_1}G_0(x-x',k)(i\partial_{x'_2} + \partial_{x'_1}^2 - 2ik\partial_{x'_1})\chi_{n+1}(x',k).$$

Taking into account that the derivative ∂_{x_1} cancels the slowly decaying terms in the asymptotics of the Green function, we can integrate by parts and then use (1.9). Thus,

$$\begin{aligned} 1 &+ \int_{-k_\Im\infty}^{x_1} dy_1 \int dx'\,\partial_{y_1}G_0(y_1-x'_1, x_2-x'_2, k)u_{n+1}(x')\chi_{n+1}(x',k) \\ &= 1 + \int_{-k_\Im\infty}^{x_1} dy_1\,\partial_{y_1}\chi_{n+1}(y_1,x_2,k) = 1 + \chi_{n+1}(x,k) - \lim_{x_1\to -k_\Im\infty}\chi_{n+1}(x,k). \end{aligned}$$

This proves the lemma thanks to (2.13), (2.20), and (2.21). In the same way it is easy to show that $g_n(x)$ and $f_n(x,k)$ obey the corresponding homogeneous integral equation.

Proof of Theorem 2.1 now follows from Lemmas 2.1–2.6 by induction on n since thanks to (2.9) the theorem is valid for $n=0$.

Corollary 2.1. *We have scalar products*

$$\int dx'_1\,\overline{F_n(x'_1,x_2,k+p)}F_n(x'_1,x_2,\bar{k}) = 2\pi\delta(p), \qquad p\in\mathbb{R}, \tag{2.28}$$

$$\int dx'_1\,\overline{F_n(x'_1,x_2,k+p)}f_n(x'_1,x_2,\bar{k}) = 0. \tag{2.29}$$

Proof. By means of (2.7), we have

$$\begin{aligned} \overline{F_{n+1}(x,k+p)}F_{n+1}(x,\bar{k}) &= \overline{F_n(x,k+p)}F_n(x,\bar{k}) \\ -\,\partial_{x_1}\Bigg[\frac{1}{\Delta_{n+1}(x)} &\int_{-(k_\Im+\lambda_{n+1\Im})\infty}^{x_1} dx'_1\,\phi_n(x'_1,x_2,\lambda_{n+1})\overline{F_n(x'_1,x_2,k+p)} \\ &\times \int_{(k_\Im-\lambda_{n+1\Im})\infty}^{x_1} dx'_1\,\overline{\phi_n(x'_1,x_2,\lambda_{n+1})}F_n(x'_1,x_2,\bar{k})\Bigg]. \end{aligned}$$

It is easy to see that the integral of the last term is equal to zero and, therefore,

$$\int dx_1' \, \overline{F_{n+1}(x_1', x_2, k+p)} F_{n+1}(x_1', x_2, \bar{k}) = \int dx_1' \, \overline{F_n(x_1', x_2, k+p)} F_n(x_1', x_2, \bar{k}), \tag{2.30}$$

which proves (2.28) thanks to (1.11) and (2.9). The second equality is derived analogously.

Corollary 2.2. *$F_n(x,k)$ is an analytic function in the complex plane of k with possible discontinuity on the real axis and poles at points $k = \overline{\lambda}_j$, $j = 1, \ldots, n$.*

Proof. Indeed, by definition (2.7), we see that F_{n+1} inherits the analyticity of F_n and has an additional discontinuity at $k_\Im = -\lambda_{n+1\Im}$. Thanks to (2.7) and (2.28), (2.29), we have

$$F_{n+1}(x, k_\Re - i(\lambda_{n+1\Im} + 0)) - F_{n+1}(x, k_\Re - i(\lambda_{n+1\Im} - 0)) = 2\pi\delta(k_\Re - \lambda_{n+1\Re}) g_{n+1}(x). \tag{2.31}$$

Thus, $F_{n+1}(x,k)$ has an additional pole at $k = \bar{\lambda}_{n+1}$,

$$F_{n+1}(x,k) = -\frac{i g_{n+1}(x)}{k - \bar{\lambda}_{n+1}} + O(1), \qquad k \to \bar{\lambda}_{n+1}. \tag{2.32}$$

Corollary 2.3. *Because of* (2.14) *and* (2.15),

$$A_n(\pm, k) = \prod_{j=1}^{n} \left(\frac{k - \lambda_j}{k - \overline{\lambda}_j} \right)^{\theta(\pm k_\Im \lambda_{j\Im})}; \tag{2.33}$$

i.e., $A_n(-,k)$ is an analytic function discontinuous on the real axis of k that has simple poles at $k = \bar{\lambda}_j$, $j = 1, \ldots, n$.

Corollary 2.4. *Let us introduce the transmission coefficients*

$$a_n(k) = \frac{A_n(+,k)}{A_n(-,k)} = \prod_{j=1}^{n} \left(\frac{k - \lambda_j}{k - \overline{\lambda}_j} \right)^{\operatorname{sign} k_\Im \lambda_{j\Im}}. \tag{2.34}$$

Then, like in the one-dimensional case,

$$a_n(k) = \lim_{x_1 \to + k_\Im \infty} \chi_n(x, k). \tag{2.35}$$

This function is analytic in the upper and lower half-planes with a discontinuity on the real axis and has simple zeroes at points $k = \lambda_j$ and $k = \overline{\lambda}_j$, $j = 1, \ldots, n$.

Corollary 2.5. *$\Phi_n(x,k)$ is an analytic function in the complex plane of k with possible discontinuity on the real axis and*

$$\Phi_{n+1}(x, \bar{\lambda}_{n+1}) = -\frac{g_{n+1}(x)}{2\lambda_{n+1\Im}} \prod_{j=1}^{n} \left(\frac{\overline{\lambda}_{n+1} - \overline{\lambda}_j}{\overline{\lambda}_{n+1} - \lambda_j} \right)^{\theta(\lambda_{n+1\Im}\lambda_{j\Im})}. \tag{2.36}$$

Proof follows from Corollaries 2.2 and 2.3 since the singularities of $F_n(x,k)$ in the complex plane are compensated by normalization (2.20). Then (2.36) follows from (2.32).

Corollary 2.6. *Let the boundary values of $\Phi_n(x,k)$ and $a_n(k)$ on the real axis be defined by analogy with* (1.13). *Then for these values we have the relation*

$$\frac{\Phi_n^+(x,k)}{a_n^+(k)} = \Phi_n^-(x,k) + \int dp\, \Phi_n^-(x,p)\mathcal{F}_n(p,k). \tag{2.37}$$

(cf. (1.16)), *where the continuous part of the spectral data is given by*

$$\mathcal{F}_n(k,p) = \mathcal{F}(k,p)\prod_{j=1}^{n}\left(\frac{(k-\lambda_j)(p-\overline{\lambda}_j)}{(k-\overline{\lambda}_j)(p-\lambda_j)}\right)^{\theta(\lambda_{j\Im})}, \qquad k,p\in\mathbb{R}. \tag{2.38}$$

Proof. For the boundary values of $F_n(x,k)$ on the real axis by analogy with (2.30) we derive

$$\int dx_1'\, \overline{F_{n+1}^{\mp}(x_1',x_2,k)}F_{n+1}^{\pm}(x_1',x_2,p) = \int dx_1'\, \overline{F_n^{\mp}(x_1',x_2,k)}F_n^{\pm}(x_1',x_2,p)$$
$$= 2\pi\delta(k-p), \tag{2.39}$$

$$\int dx_1'\, \overline{F_{n+1}^{+}(x_1',x_2,k)}F_{n+1}^{+}(x_1',x_2,p) = \int dx_1'\, \overline{F_n^{+}(x_1',x_2,k)}F_n^{+}(x_1',x_2,p)$$
$$= 2\pi\mathcal{F}(k,p) + 2\pi\delta(k-p), \tag{2.40}$$

where (1.14) and (1.15) were used. Then (2.37) follows from definition (2.20) and (2.33).

3. RESOLUTION OF THE RECURSION RELATIONS

In order to resolve the recursion relations explicitly, we introduce

$$B_{l,m}(x) = \int\limits_{-(\lambda_{l\Im}+\lambda_{m\Im})\infty}^{x_1} dx_1'\, \overline{\Phi(x_1',x_2,\lambda_l)}\Phi(x_1',x_2,\lambda_m), \tag{3.1}$$

$$\beta_l(x,k) = \int\limits_{-(k_\Im+\lambda_{l\Im})\infty}^{x_1} dx_1'\, \overline{\Phi(x_1',x_2,\lambda_l)}\Phi(x_1',x_2,k), \tag{3.2}$$

$$l,m = 1,2,\ldots,$$

so that

$$B_{l,m}(x) = \beta_l(x,\lambda_m) = \overline{\beta_m(x,\lambda_l)}. \tag{3.3}$$

Let $B_n(x)$ denote the $n\times n$ matrix

$$B_n(x) = \|B_{l,m}(x)\|_{l,m=1,\ldots,n}, \tag{3.4}$$

and let us define the following row and two columns:

$$\Phi(x) = (\Phi(x,\lambda_1),\ldots,\Phi(x,\lambda_n)), \tag{3.5}$$

$$\beta(x,k) = (\beta_1(x,k),\ldots,\beta_n(x,k))^{\mathrm{T}}, \tag{3.6}$$

$$\gamma(x,k) = (\gamma_1(k),\ldots,\gamma_n(k))^{\mathrm{T}}, \tag{3.7}$$

where subscript T means transposition and $\gamma_n(k)$ are certain given functions such that the matrix

$$C_n = \|c_{l,m}\|_{l,m=1,\ldots,n}, \qquad c_{l,m} = \gamma_l(\lambda_m), \tag{3.8}$$

is Hermitian. Let us denote

$$A_n(x) = C_n + B_n(x), \tag{3.9}$$

that is also an Hermitian matrix by construction.

In order to formulate the conditions of regularity of the potential $u_n(x)$, we introduce also the matrices

$$C_n(y) = \|\{c_{l,m} | \, y\lambda_{l\Im} > 0, \; y\lambda_{m\Im} > 0\}\|_{l,m=1,\dots,n}, \tag{3.10}$$

where y is a certain real parameter, in fact, its sign. The sizes of these matrices can be less than $n \times n$, since they are constructed by removing from the matrix C_n those rows and columns that do not obey the inequalities in (3.10). These matrices are Hermitian also and, if all rows and columns are removed, we put $\det C_n(y) = 1$. Let also $\Lambda_n(y)$ denote a matrix with entries $-i(\overline{\lambda}_l - \lambda_m)^{-1}$ obeying the same properties as in (3.10). It is known that

$$\det \Lambda_n(y) = \prod_{l=1}^{n} (2\lambda_{l\Im})^{-\theta(y\lambda_{l\Im})} \prod_{\substack{l,m=1 \\ l \neq m}}^{n} \left| \frac{\lambda_l - \lambda_m}{\overline{\lambda}_l - \lambda_m} \right|^{\theta(y\lambda_{l\Im})\theta(y\lambda_{m\Im})}. \tag{3.11}$$

In these terms we impose the following conditions on the constants $c_{l,m}$:

$$\pm C_n(\pm) > 0 \tag{3.12}$$

and prove the following

Theorem 3.1. *Let conditions* (2.10), (2.11), *and* (3.12) *be fulfilled. Then for any* $n = 1, 2, \dots$ *and* x

$$\det A_n(x) \prod_{l=1}^{n} \lambda_{l\Im} > 0, \qquad \det A_0(x) = 1, \tag{3.13}$$

and the solutions of the recursion equations (2.5)–(2.9) *are given by the following relations:*

$$\Delta_n(x) = \frac{\det A_n(x)}{\det A_{n-1}(x)}, \tag{3.14}$$

$$F_n(x,k) = \frac{1}{\det A_n(x)} \begin{vmatrix} A_n(x) & \beta(x,k) \\ \Phi(x) & \Phi(x,k) \end{vmatrix}, \tag{3.15}$$

$$f_n(x,k) = \frac{1}{\det A_n(x)} \begin{vmatrix} A_n(x) & \gamma(k) \\ \Phi(x) & 0 \end{vmatrix}, \tag{3.16}$$

$$g_n(x) = \frac{-1}{\det A_n(x)} \begin{vmatrix} A_n(x) & e_n \\ \Phi(x) & 0 \end{vmatrix}, \qquad e_n = (0, \dots, 0, 1)^{\mathrm{T}}, \tag{3.17}$$

$$u_n(x) = u(x) - 2\partial_{x_1}^2 \ln \det A_n(x), \tag{3.18}$$

$$n = 1, 2, \dots .$$

Moreover, for the constants c_n *in* (2.4) *we have the relation*

$$c_{n+1} = \frac{\det C_{n+1}(\lambda_{n+1\Im})}{\det C_n(\lambda_{n+1\Im})}, \tag{3.19}$$

where the matrices $C_n(\pm)$ *are defined in* (3.10).

In order to prove that these relations resolve (2.5)–(2.9), we need to calculate all the integrals involved. By (2.6), (3.15), and (3.16),

$$\phi_n(x,k) = \frac{1}{\det A_n(x)} \begin{vmatrix} A_n(x) & \gamma(k)+\beta(x,k) \\ \Phi(x) & \Phi(x,k) \end{vmatrix}, \tag{3.20}$$

so that

$$\phi_n(x,\lambda_{n+1}) = \frac{1}{\det A_n(x)} \begin{vmatrix} A_n(x) & A_{*,n+1}(x) \\ \Phi(x) & \Phi(x,\lambda_{n+1}) \end{vmatrix}, \tag{3.21}$$

where we introduced the column

$$A_{*,n+1}(x) = (A_{1,n+1}(x),\dots,A_{n,n+1}(x))^{\mathrm{T}} \tag{3.22}$$

and used (3.3), (3.8), and (3.9) in order to write $\gamma_m(\lambda_{n+1}) + \beta_m(x,\lambda_{n+1}) = A_{m,n+1}(x)$. Let us also introduce the matrix

$$A_{n+1}(x,k) = \begin{pmatrix} A_n(x) & \gamma(k)+\beta(x,k) \\ A_{n+1,*}(x) & \beta_{n+1}(x,k) \end{pmatrix}, \tag{3.23}$$

where the row $A_{n+1,*}(x)$ is defined analogously to the column (3.22). Below, for any matrix A, we denote by $A^{(l,m)}$ the same matrix with the removed lth row and mth column, say,

$$A_n^{(l,m)}(x,k) = \|(A_n(x,k))_{i,j}\|_{\substack{i,j=1,\dots,n,\\ i\neq l,\ j\neq m}}. \tag{3.24}$$

Lemma 3.1. *Let Theorem* 3.1 *be valid for some n. Then*

$$\overline{\phi_n(x,\lambda_{n+1})}\phi_n(x,k) = \partial_{x_1}\frac{\det A_{n+1}(x,k)}{\det A_n(x)}. \tag{3.25}$$

Proof. Thanks to notation (3.23), we can write the expansions of (3.21) and (3.20) with respect to the last row as

$$\overline{\phi_n(x,\lambda_{n+1})} = \frac{1}{\det A_n(x)}\sum_{k=1}^{n+1}(-1)^{n+1+k}\overline{\Phi(x,\lambda_k)}\det A_{n+1}^{(k,n+1)}(x,k), \tag{3.26}$$

$$\phi_n(x,k) = \frac{1}{\det A_n(x)}\left[\sum_{l=1}^{n}(-1)^{n+l+1}\Phi(x,\lambda_l)\det A_{n+1}^{(n+1,l)}(x,k) + \Phi(x,k)\det A_{n+1}^{(n+1,n+1)}(x)\right]. \tag{3.27}$$

Then, taking into account (3.1), (3.3), (3.9), and (3.23), we get

$$\overline{\Phi(x,\lambda_m)}\Phi(x,\lambda_l) = \partial_{x_1}(A_{n+1}(x,k))_{m,l}, \quad l\le n,$$

and

$$\overline{\Phi(x,\lambda_m)}\Phi(x,k) = \partial_{x_1}(A_{n+1}(x,k))_{m,n+1}.$$

Thus,

$$\overline{\phi_n(x,\lambda_{n+1})}\phi_n(x,k) = \sum_{k,l=1}^{n+1}(-1)^{k+l}\frac{(\partial_{x_1}A_{n+1}(x,k))_{k,l}}{\det^2 A_n(x)}\det A_{n+1}^{(k,n+1)}(x,k)\det A_{n+1}^{(n+1,l)}(x,k),$$

and the statement of the lemma follows from the known property of determinants: If A_n, $n = 1, 2, \ldots$, are $n\times n$ matrices depending on some parameter and such that for every n the matrix A_n occupies the upper left corner of matrix A_{n+1}, then

$$(\det A_{n+1})' \det A_n - (\det A_n)' \det A_{n+1} = \sum_{k,l=1}^{n+1} (-1)^{k+l} A'_{k,l} \det A_{n+1}^{(k,n+1)} \det A_{n+1}^{(n+1,l)}, \tag{3.28}$$

where the primes denote the derivatives with respect to this parameter.

Lemma 3.2. *Let $A_n(x)$ be defined in (3.9). Then the leading asymptotic behavior of its determinant for $|x_1| \to \infty$ and x_2 fixed is given by*

$$\det A_n(x) = \det C_n(\mp) \det \Lambda_n(\pm) \prod_{j=1}^{n} \left|1 + e^{-i\lambda_j x_1 - i\lambda_j^2 x_2}\right|^2, \qquad x_1 \to \pm\infty, \tag{3.29}$$

where the matrices $C_n(\mp)$ and $\det \Lambda_n(\pm)$ are defined in (3.10) and (3.11).

Proof. Let us introduce the $n\times n$ diagonal matrix

$$D_n(x) = \operatorname{diag}\Big\{1 + e^{-i\lambda_j x_1 - i\lambda_j^2 x_2}\Big\}_{j=1}^{n}. \tag{3.30}$$

Then, taking into account that, thanks to (1.5), (1.10), and (3.2),

$$\begin{aligned} e^{i(k-\bar\lambda_l)x_1 + i(k^2 - \bar\lambda_l^2)x_2} \beta_l(x,k) &\to \frac{1}{i(\bar\lambda_l - k)}, \\ e^{i(\lambda_m - \bar\lambda_l)x_1 + i(\lambda_m^2 - \bar\lambda_l^2)x_2} B_{l,m}(x) &\to \frac{1}{i(\bar\lambda_l - \lambda_m)} \end{aligned} \tag{3.31}$$

for $|x_1| \to \infty$, we derive

$$\lim_{x_1 \to \pm\infty} \overline{D_n(x)}^{-1} A_n(x) D_n(x)^{-1} = \alpha_n(\pm), \tag{3.32}$$

where the $n\times n$ matrices $\alpha_n(\pm) = \|\alpha_{l,m}(\pm)\|_{l,m=1,\ldots,n}$, $\alpha_0 = 1$, are defined by their entries

$$\alpha_{l,m}(\pm) = \theta(\lambda_{l\Im}\lambda_{m\Im}) \left(c_{l,m}\theta(\mp\lambda_{l\Im}) + \frac{\theta(\pm\lambda_{l\Im})}{i(\bar\lambda_l - \lambda_m)} \right). \tag{3.33}$$

We see that, due to (3.33), only entries (l,m) obeying condition $\lambda_{l\Im}\lambda_{m\Im} > 0$ can give nontrivial limits. It is easy to notice that entries such that $\lambda_{l\Im}\lambda_{m\Im} < 0$ are decaying at least as $e^{-|\lambda_{n\Im} x_1|}$, i.e., by (2.11) as the lowest exponential involved in the matrix A_n.

Taking into account the block structure of the matrix $\alpha_n(\pm)$, we get by (3.12)

$$\det \alpha_n(\pm) = \det C_n(\mp) \det \Lambda_n(\pm),$$

which gives (3.29).

Lemma 3.3. *Let Theorem 3.1 be valid for some n and let $\lambda_1, \ldots, \lambda_{n+1}$ obey (2.11). Then for $\Delta_{n+1}(x)$ defined in (2.4) we have equality $(3.14)_{n+1}$ where the coefficient $c_{n+1,n+1}$ of the matrix $A_{n+1}(x)$ is given by*

$$c_{n+1,n+1} = c_{n+1} + C_{n+1,*}(\lambda_{n+1\Im}) C_n^{-1}(\lambda_{n+1\Im}) C_{*,n+1}(\lambda_{n+1\Im}). \tag{3.34}$$

Here $C_{n+1,*}(\lambda_{n+1\Im})$ and $C_{*,n+1}(\lambda_{n+1\Im})$ are the last row and column of $C_{n+1}(\lambda_{n+1\Im})$. Moreover, the matrix $A_{n+1}(x)$ obeys property $(3.13)_{n+1}$.

Proof. In order to calculate the integral in (2.4) we use (3.21), so $\gamma_m(\lambda_{n+1}) = c_{m,n+1}$. Then, by Lemma 3.1,

$$\overline{\phi_n(x,\lambda_{n+1})}\phi_n(x,\lambda_{n+1}) = \partial_{x_1}\frac{\det A_{n+1}(x)}{\det A_n(x)},$$

where we used the fact that in this case $\det A_{n+1}(x,\lambda_{n+1}) = \det A_{n+1}(x) - c_{n+1,n+1}\det A_n(x)$ by (3.23). Then, by Lemma 3.2, the integral in (2.4) is convergent, and

$$\Delta_{n+1}(x) = c_{n+1} + \frac{\det A_{n+1}(x)}{\det A_n(x)} - \frac{\det C_{n+1}(\lambda_{n+1\Im})\det \Lambda_{n+1}(-\lambda_{n+1\Im})}{\det C_n(\lambda_{n+1\Im})\det \Lambda_n(-\lambda_{n+1\Im})}.$$

By definition (3.11),

$$\Lambda_{n+1}(-\lambda_{n+1\Im}) = \Lambda_n(-\lambda_{n+1\Im}); \tag{3.35}$$

thus, the determinants of Λ cancel out and $(3.14)_{n+1}$ follows by (3.19). The latter one is equivalent to (3.34) thanks to the known property of the determinants of bordered matrices:

$$\frac{1}{\det C_n}\begin{vmatrix} C_n & C_{*,n+1} \\ C_{n+1,*} & c_{n+1,n+1}\end{vmatrix} = c_{n+1,n+1} - C_{n+1,*}C_n^{-1}C_{*,n+1}. \tag{3.36}$$

In order to prove $(3.13)_{n+1}$, let us mention first that, by (3.10),

$$C_{n+1}(-\lambda_{n+1\Im}) = C_n(-\lambda_{n+1\Im}). \tag{3.37}$$

Second, let v be an arbitrary n-column and v_{n+1} be an arbitrary complex scalar. Then

$$(v^\dagger,\overline{v}_{n+1})C_{n+1}(\lambda_{n+1\Im})\begin{pmatrix} v \\ v_{n+1}\end{pmatrix} = w^\dagger C_n(\lambda_{n+1\Im})w + |v_{n+1}|^2 c_{n+1}, \tag{3.38}$$

where (3.19) was used in the last term and we denoted

$$w = v + v_{n+1}C_n^{-1}(\lambda_{n+1\Im})C_{*,n+1}(\lambda_{n+1\Im}).$$

Thus, condition $(3.12)_{n+1}$ for one of the signs trivially follows from (3.37) and for the other sign is equivalent to the condition $c_{n+1}\lambda_{n+1\Im} > 0$, which, in its turn, by Theorem 2.1, is equivalent to the condition that $\Delta_{n+1}(x)$ has no zeroes on the x-plane and its sign is equal to the sign of $\lambda_{n+1\Im}$. Then, finally, $(3.13)_{n+1}$ follows from $(3.14)_{n+1}$ and the proof of the lemma is completed.

Lemma 3.4. *Let Theorem 3.1 be valid for some n and let $\lambda_1,\dots,\lambda_{n+1}$ obey (2.11). Then, for $g_{n+1}(x)$ defined in (2.3), we have equality $(3.17)_{n+1}$.*

Proof. By (2.3), (3.14), and (3.21),

$$g_{n+1}(x) = \frac{1}{\det A_{n+1}(x)}\begin{vmatrix} A_n(x) & A_{*,n+1}(x) \\ \Phi(x) & \Phi(x,\lambda_{n+1})\end{vmatrix} \tag{3.39}$$

that is nothing else but $(3.17)_{n+1}$ expanded with respect to the last column.

Lemma 3.5. *Let Theorem 3.1 be valid for some n and let $\lambda_1,\dots,\lambda_{n+1}$ obey (2.11). Then, for $F_{n+1}(x)$ defined in (2.7), we have equality $(3.15)_{n+1}$.*

Proof. We need to calculate the integral with F_n in (2.7). Thus, in order to use (3.25), we choose all $\gamma_l(k)$ equal to zero in (3.20) and, respectively, in (3.23). Then from Lemma 3.1 we get

$$F_{n+1}(x,k) = F_n(x,k) - g_{n+1}(x)\int\limits_{-(k_\Im+\lambda_{n+1\Im})\infty}^{x_1} dx_1'\,\partial_{x_1'}\frac{\det A_{n+1}(x_1',x_2,k)}{\det A_n(x_1',x_2)}. \tag{3.40}$$

By property (3.36), equations (3.23) and (3.30) imply that

$$\frac{\det A_{n+1}(x,k)}{\det A_n(x)}$$
$$= \beta_{n+1}(x,k) - \sum_{l,m=1}^{n} \frac{c_{n+1,l} + B_{n+1,l}(x)}{1+e^{-i\lambda_l x_1 - i\lambda_l^2 x_2}} \frac{\beta_m(x,k)}{1+e^{i\lambda_m x_1 + i\lambda_m^2 x_2}} \left(\left(\overline{D}_n^{-1}(x) A_n(x) D_n^{-1}(x)\right)^{-1}\right)_{l,m}.$$

We consider the limit $x_1 \to -(k_\Im + \lambda_{n+1\Im})\infty$ of the r.h.s. Thanks to the asymptotics (3.31), the first term goes to zero. By Lemma 3.2, the matrix $(\overline{D}_n^{-1} A_n D_n^{-1})^{-1}$ has a finite and nonzero limit. The entries of this matrix corresponding to $\lambda_{l\Im}\lambda_{m\Im} < 0$ exponentially decay as $e^{-|\lambda_{n\Im} x_1|}$ (see the proof for the inverse matrix in Lemma 3.2). Then, for the asymptotics of $\beta_m(x,k)$, we can write $e^{k_\Im x_1} = e^{(k_\Im + \lambda_{n+1\Im})x_1} e^{-\lambda_{n+1\Im} x_1}$, where the first factor decays by the sign of infinity, and the second factor is majorized by some of the decaying exponents by virtue of condition (2.11).

Now by means of $(3.15)_n$ and (3.39) we get from (3.40) that

$$F_{n+1}(x,k) = \frac{1}{\det A_n(x) \det A_{n+1}(x)} \begin{vmatrix} A_n(x) & Z(x,k) \\ \Phi(x) & z_{n+1}(x,k) \end{vmatrix}, \tag{3.41}$$

where we introduced the column $Z = (z_1, \dots, z_n)^{\mathrm{T}}$ and the entry z_{n+1} as follows:

$$Z = \beta(x,k) \det A_{n+1}(x) - A_{*,n+1}(x) \begin{vmatrix} A_n(x) & \beta(x,k) \\ A_{n+1,*}(x) & \beta_{n+1}(x,k) \end{vmatrix},$$
$$z_{n+1} = \Phi(x,k) \det A_{n+1}(x) - \Phi(x,\lambda_{n+1}) \begin{vmatrix} A_n(x) & \beta(x,k) \\ A_{n+1,*}(x) & \beta_{n+1}(x,k) \end{vmatrix}.$$

By means of (3.23) with $\gamma(k) = 0$, this can be rewritten as

$$z_j = (A_{n+2}(x,k))_{j,n+2} \det A_{n+2}^{(n+2,n+2)}(x,k) - (A_{n+2}(x,k))_{j,n+1} \det A_{n+2}^{(n+2,n+1)}(x,k),$$
$$z_{n+1} = \Phi(x,k) \det A_{n+2}^{(n+2,n+2)}(x,k) - \Phi(x,\lambda_{n+1}) \det A_{n+2}^{(n+2,n+1)}(x,k).$$

We see that the determinant in (3.41) is unchanged if all $z_1, \dots, z_{n+1}$ are replaced with

$$\tilde{z}_j = \sum_{l=1}^{n+2} (-1)^{n+l} (A_{n+2}(x,k))_{j,l} \det A_{n+2}^{(n+2,l)}(x,k), \qquad k = 1, \dots, n,$$
$$\tilde{z}_{n+1} = \Phi(x,k) \det A_{n+2}^{(n+2,n+2)}(x,k) + \sum_{l=1}^{n+1} (-1)^{n+l} \Phi(x,\lambda_l) \det A_{n+2}^{(n+2,l)}(x,k),$$

as all additional terms are just columns proportional to some other columns of the determinant in (3.41). On the other hand, $\tilde{z}_j$ for $j = 1, \dots, n$ are equal to the determinants of the matrix $A_{n+2}(x,k)$ with the row $n+2$ replaced by the jth row of the same matrix. So, all of them are equal to zero, and thus the determinant in (3.41) is equal to $\tilde{z}_{n+1} \det A_n(x)$. On the other hand, $\tilde{z}_{n+1}$ is the expansion of the determinant in $(3.15)_{n+1}$ with respect to the last row. The lemma is proved.

Lemma 3.6. *Let Theorem* 3.1 *be valid for some* n *and let* $\lambda_1, \dots, \lambda_{n+1}$ *obey* (2.11). *Then for* $f_{n+1}(x)$ *defined in* (2.8) *we have equality* $(3.16)_{n+1}$.

Proof. We need to calculate the integral with f_n in (2.8). Thus, we have to replace $\beta(x,k)$ and $\Phi(x,k)$ with zeros in the last column in (3.20). Correspondingly, Lemma 3.1 must be applied with the matrix $A_{n+1}(x,k)$ (3.23) with only γ-terms in the last column and zero on the bottom place. Using the same consideration as in Lemma 3.5, we prove that, for such matrix $A_{n+1}(x,k)$, the limit of $\det A_{n+1}(x,k) \det A_n^{-1}(x)$ for $x_1 \to -\lambda_{n+1\Im}\infty$ is finite, and then the integral in (2.8) is convergent. Continuing in this way, we prove $(3.16)_{n+1}$ and get a relation between $B_{n+1}(k)$ and $\gamma_{n+1}(k)$. We omit these details since the solutions $f_n(x,k)$ are not used in what follows.

Proof of Theorem 3.1 follows by induction on the basis of the lemmas proved above if we notice that, for $n=1$, the formulation of the theorem coincides with relations (2.3)–(2.8) for $n=0$ if we take into account (2.9) and condition (2.12) for $n=1$. Similarly, equation $(3.18)_{n+1}$ follows from (2.5) and $(3.18)_n$ thanks to $(3.14)_{n+1}$, i.e., thanks to Lemma 3.3. Let us emphasize that the reality and regularity of the potentials u_n for all n are equivalent to the conditions that the matrices C_n are Hermitian and obey property (3.12). These properties are independent of the original potential $u(x)$, so they coincide with the conditions given in [22], where the nondiagonal matrix C_n was introduced for the first time for the case $u(x) \equiv 0$.

Corollary 3.1.

$$\Phi_n(x,\lambda_m) = \sum_{l=1}^{n} d_{l,m} \Phi_n(x,\overline{\lambda}_l), \tag{3.42}$$

where we introduced the constants

$$d_{l,m} = 2c_{l,m}\lambda_{l\Im} \prod_{\substack{j=1\\ j\neq l}}^{n} \left(\frac{\overline{\lambda}_l - \lambda_j}{\overline{\lambda}_l - \overline{\lambda}_j}\right)^{\theta(\lambda_{l\Im}\lambda_{j\Im})} \prod_{j=1}^{n} \left(\frac{\lambda_m - \overline{\lambda}_j}{\lambda_m - \lambda_j}\right)^{\theta(-\lambda_{m\Im}\lambda_{j\Im})} . \tag{3.43}$$

Proof. Taking (1.11) into account, we use (3.2) and (3.6) in (3.15) in order to write

$$F_n(x, k_{\Re} - i\lambda_{l\Im} + i0) - F_n(x, k_{\Re} - i\lambda_{l\Im} - i0) = -2\pi\delta(k_{\Re} - \lambda_{l\Re})\varphi_l(x), \qquad l = 1,\dots,n,$$

where, by analogy with (3.17), we introduced

$$\varphi_l(x) = \frac{-1}{\det A_n(x)} \begin{vmatrix} A_n(x) & c_l \\ \Phi(x) & 0 \end{vmatrix}, \tag{3.44}$$

where $c_l = (0,\dots,0,1,0,\dots,0)^{\mathrm{T}}$ is a column with 1 in the lth place only. Thus, as we already know from Corollary 2.2, F_n has poles at points $k = \overline{\lambda}_l$, and

$$\operatorname*{res}_{k=\overline{\lambda}_l} F_n(x,k) = -i\varphi_l(x).$$

Then, by (2.20) and Corollary 2.3,

$$\Phi_n(x,\overline{\lambda}_l) = \frac{\varphi_l(x)}{2\lambda_{l\Im}} \prod_{\substack{j=1\\ j\neq l}}^{n} \left(\frac{\overline{\lambda}_l - \overline{\lambda}_j}{\overline{\lambda}_l - \lambda_j}\right)^{\theta(\lambda_{l\Im}\lambda_{j\Im})} .$$

On the other hand, directly by (3.15), again taking into account (2.20) and Corollary 2.3, we have

$$\Phi_n(x,\lambda_m) = \sum_{l=1}^{n} c_{l,m}\varphi_l(x) \prod_{j=1}^{n} \left(\frac{\lambda_m - \overline{\lambda}_j}{\lambda_m - \lambda_j}\right)^{\theta(-\lambda_{m\Im}\lambda_{j\Im})} ,$$

which proves the statement.

This corollary completes the formulation of the inverse problem for the Jost solution Φ_n since its discontinuity on the real axis was given by Corollary 2.6.

Corollary 3.2. *Condition* (2.11) *for the final formulas* (3.15) *and* (3.18) *can be omitted.*

Proof. Indeed, this condition was relevant only for the proof of the fact that these formulas obey the recursion procedure of Theorem 2.1. On the other hand, equations (3.12), (3.15), and (3.18) are invariant under any permutation of λ_j's obeying (2.10).

4. PROPERTIES OF POTENTIALS

We proved that potentials given by (3.18) are real and regular and these properties are equivalent to the Hermiticity of the matrix C_n defined in (3.8) and conditions (3.12). Here we demonstrate that these potentials belong to the class (1.2). For this purpose, we have to study their asymptotic behavior when x_1 is replaced with $x_1 - 2\mu x_2$ and $x_2 \to \infty$, where x_1 is fixed and μ is a parameter determining the direction of asymptotics on the x-plane. In the same way as in Lemma 3.2, it is easy to prove that the leading term of the potential is decaying for all $\mu \neq \lambda_{1\Re}, \dots, \lambda_{n\Re}$. Let now μ be equal to some $\lambda_{j\Re}$. Taking into account that the original potential is rapidly decaying, we have, by (3.18), the following expression for the corresponding limit:

$$\lim_{x_2\to\infty} u_n(x_1 - 2\lambda_{j\Re}x_2, x_2) = -2 \lim_{x_2\to\infty} \partial_{x_1}^2 \ln\det A_n(x_1 - 2\lambda_{j\Re}x_2, x_2). \tag{4.1}$$

By analogy with Lemma 3.2, we introduce the diagonal matrix

$$\Gamma_{l,m} = \delta_{l,m}\delta_{l,j} + \delta_{l,m}(1-\delta_{l,j})\left(1 + e^{-i\lambda_l x_1 - i(\lambda_l - \lambda_{j\Re})^2 x_2 - i\lambda_{j\Im}^2 x_2}\right).$$

It is obvious that we can replace matrix A_n with the matrix $\overline{\Gamma}^{-1}A_n\Gamma^{-1}$ in (4.1) without changing the value of the limit. In what follows we denote the transformations of the matrix A_n that do not affect the limit (4.1) by the sign $\simeq$. Let us introduce, by analogy with (3.33), the $(n-1)\times(n-1)$ matrix

$$\alpha(j,\pm) = \|\alpha_{l,m}(j,\pm)\|_{l,m=1,\dots,n,\ l,m\neq j}, \tag{4.2}$$

$$\begin{aligned}\alpha_{l,m}(j,\pm) = {} & c_{l,m}\theta(\mp\lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re}))\theta(\mp\lambda_{m\Im}(\lambda_{m\Re}-\lambda_{j\Re})) \\ & + \frac{\theta(\pm\lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re}))\theta(\pm\lambda_{m\Im}(\lambda_{m\Re}-\lambda_{j\Re}))}{i(\overline{\lambda}_l - \lambda_m)}.\end{aligned} \tag{4.3}$$

This matrix has a block structure, and, by analogy with (3.10)–(3.11), we introduce

$$C(j,x_2) = \|\{c_{l,m} \mid \lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re})x_2 > 0,\ \lambda_{m\Im}(\lambda_{m\Re}-\lambda_{j\Re})x_2 > 0\}\|, \tag{4.4}$$

$$\Lambda(j,x_2) = \|\{-i(\overline{\lambda}_l - \lambda_m)^{-1} \mid \lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re})x_2 > 0,\ \lambda_{m\Im}(\lambda_{m\Re}-\lambda_{j\Re})x_2 > 0\}\|, \tag{4.5}$$

$$l,m = 1,\dots,n, \qquad l,m\neq j,$$

and again we set the determinant of a matrix that has no entries equal to 1. Then

$$\det\alpha(j,\pm) = \det C(j,\mp)\det\Lambda(j,\pm).$$

For $\Lambda(j,\pm)$, we explicitly have

$$\det\Lambda(j,\pm)=\prod_{\substack{l=1\\ l\neq j}}^{n}(2\lambda_{l\Im})^{-\theta(\pm\lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re}))}$$

$$\times\prod_{\substack{l,m=1\\ l\neq m,\ l,m\neq j}}^{n}\left|\frac{\lambda_l-\lambda_m}{\overline{\lambda}_l-\lambda_m}\right|^{\theta(\pm\lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re}))\theta(\pm\lambda_{m\Im}(\lambda_{m\Re}-\lambda_{j\Re}))}, \tag{4.6}$$

and, thanks to the condition (3.12) of regularity, $\det\alpha(j,\pm)\neq 0$. Dividing $\det(\overline{\Gamma}^{-1}A_n\Gamma^{-1})$ by $\det\alpha$ and using the property (3.36) and asymptotics (3.31), we get

$$\lim_{x_2\to\pm\infty}\det A_n(x_1-2\lambda_{j\Re}x_2,x_2)$$

$$\simeq c_{j,j}+\frac{e^{2\lambda_{j\Im}x_1}}{2\lambda_{j\Im}}-\sum_{\substack{l,m=1\\ l,m\neq j}}^{n}\left[c_{j,l}\theta(\mp\lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re}))+\frac{e^{i\bar\lambda_j x_1}}{i(\overline{\lambda}_j-\lambda_l)}\theta(\pm\lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re}))\right]$$

$$\times(\alpha(j,\pm)^{-1})_{l,m}\left[c_{m,j}\theta(\mp\lambda_{m\Im}(\lambda_{m\Re}-\lambda_{j\Re}))+\frac{e^{-i\lambda_j x_1}}{i(\overline{\lambda}_m-\lambda_j)}\theta(\pm\lambda_{m\Im}(\lambda_{m\Re}-\lambda_{j\Re}))\right].$$

Thanks to (4.2) and (4.3), the elements of the matrix $(\alpha^{-1})_{l,m}$ are proportional to the corresponding θ-functions. Then, taking (4.4) and (4.5) into account, we can write

$$\lim_{x_2\to\pm\infty}\det A_n(x_1-2\lambda_{j\Re}x_2,x_2)$$

$$\simeq c_{j,j}-\sum_{\substack{l,m=1\\ l,m\neq j}}^{n}c_{j,l}\theta(\mp\lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re}))(C(j,\mp)^{-1})_{l,m}c_{m,j}\theta(\mp\lambda_{m\Im}(\lambda_{m\Re}-\lambda_{j\Re}))$$

$$+e^{2\lambda_{j\Im}x_1}\left[\frac{1}{2\lambda_{j\Im}}-\sum_{\substack{l,m=1\\ l,m\neq j}}^{n}\frac{\theta(\pm\lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re}))}{i(\overline{\lambda}_j-\lambda_l)}(\Lambda(j,\pm)^{-1})_{l,m}\frac{\theta(\pm\lambda_{m\Im}(\lambda_{m\Re}-\lambda_{j\Re}))}{i(\overline{\lambda}_m-\lambda_j)}\right].$$

Let us introduce now the matrices $\widehat{C}(j,\pm)$ constructed by removing from the matrix $C_n=\|c_{l,m}\|$ all rows and columns with numbers $l\neq j$ such that $\pm\lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re})<0$. Let the matrix $\widehat{\Lambda}(j,\pm)$ be the same as $\widehat{C}(j,\pm)$ with $c_{l,m}$ replaced by $-i(\overline{\lambda}_l-\lambda_m)^{-1}$. Then, again by (3.36), we get

$$\det A_n(x_1-2\lambda_{j\Re}x_2,x_2)\simeq\frac{\det\widehat{C}(j,\mp)}{\det C(j,\mp)}+e^{2\lambda_{j\Im}x_1}\frac{\det\widehat{\Lambda}(j,\pm)}{\det\Lambda(j,\pm)}.$$

By virtue of (4.6),

$$\frac{\det\widehat{\Lambda}(j,\pm)}{\det\Lambda(j,\pm)}=\frac{1}{2\lambda_{j\Im}}\prod_{\substack{l=1\\ l\neq j}}^{n}\left|\frac{\lambda_l-\lambda_j}{\overline{\lambda}_l-\lambda_j}\right|^{2\theta(\pm\lambda_{l\Im}(\lambda_{l\Re}-\lambda_{j\Re}))}, \tag{4.7}$$

and, using (4.1), we finally obtain

$$\lim_{x_2\to\pm\infty}u_n(x_1-2\lambda_{j\Re}x_2,x_2)=-\frac{2\lambda_{j\Im}^2}{\cosh^2(\lambda_{j\Im}x_1+\varepsilon_{j,\pm})}, \tag{4.8}$$

where

$$e^{2\varepsilon_{j,\pm}} = \frac{\det C(j,\mp)}{\det \widehat{C}(j,\mp)} \frac{\det \widehat{\Lambda}(j,\pm)}{\det \Lambda(j,\pm)}.$$

This proves that the potentials constructed by means of the binary Bäcklund transformations indeed give nontrivial examples of the class (1.2) since they are not decaying along the directions $x_1 + 2\lambda_{j\Re}x_2 = \mathrm{const}$. The two rays belonging to one direction are mutually shifted by

$$e^{2(\varepsilon_{j,+}-\varepsilon_{j,-})} = \frac{\det \widehat{C}(j,+)\det C(j,-)}{\det \widehat{C}(j,-)\det C(j,+)} \prod_{\substack{l=1\\ l\neq j}}^{n} \left| \frac{|\lambda_{j\Re}-\lambda_{l\Re}| - i(\lambda_{j\Im}\operatorname{sign}(\lambda_{l\Re}-\lambda_{j\Re}) - |\lambda_{l\Im}|)}{|\lambda_{j\Re}-\lambda_{l\Re}| - i(\lambda_{j\Im}\operatorname{sign}(\lambda_{l\Re}-\lambda_{j\Re}) + |\lambda_{l\Im}|)} \right|^2, \qquad (4.9)$$

where we used (4.7).

Our consideration was done under the assumption that λ_j's are in a generic situation, in particular, that their real parts are all different. In the situation when, say, $\lambda_{j\Re} = \lambda_{m\Re}$, the limit in (4.1) does not exist (for a generic matrix C_n) since there are oscillating terms for large x_2. Therefore, in this case, the potential does not belong to the class (1.2).

Here we considered only the leading asymptotic behavior of the potentials $u_n(x)$. More detailed investigation performed in [5] for the case $n = 1$ shows that, even in this simple situation, the asymptotic behavior gets some rational corrections in the higher terms. Moreover, these corrections depend on the sign of $\lambda_{1\Im}$ if the original potential $u(x)$ is nontrivial. This dependence on the signs of the imaginary parts of λ_j's is also obvious here. Indeed, the formulation of the inverse problem in (2.37) involves the spectral data $\mathcal{F}_n(k,p)$ that include only λ_j's with positive imaginary parts. Such dependence of the potential on these signs is a specific two-dimensional feature and, just as the nondiagonal matrix in (3.42), it has no analogue in the one-dimensional case. We plan to discuss these aspects in more detail in a forthcoming publication.

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