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Spectral Theory of the Nonstationary Schrödinger Equation with a Bidimensionally Perturbed One-Dimensional Potential

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Abstract—We derive and describe in detail the extension of the inverse scattering transform method to the case of linear spectral problems with potentials that do not decay in some space directions. Our presentation is based on the extended resolvent approach. As a basic example, we consider the nonstationary Schrödinger equation with a potential that is a perturbation of a generic one-dimensional potential by means of a decaying function of two variables. We give the corresponding modifications of the Jost solutions and the spectral data and derive their properties and characterization equations.

1. INTRODUCTION

A specific feature of $(2 + 1)$-dimensional integrable equations is that they can be considered as generalizations of $(1 + 1)$-dimensional integrable equations. Thus, the Kadomtsev–Petviashvili equation [1–3], in both its versions KPI and KPII, is known to be a $(2+1)$-dimensional generalization of the Korteweg–de Vries (KdV) equation. Moreover, the KP equation admits solutions that behave at the space infinity like solutions of the KdV equation. It is necessary to emphasize that such embedding of one-dimensional objects in two space dimensions is not a pure mathematical trick but has an essential physical relevance. Indeed, the KP equation was derived in [1] from the equations of the classical water wave problem (for an inviscid, incompressible, homogeneous fluid subject to a constant gravitational force) under the assumption of small amplitude and weak dispersion as well

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as under the hypothesis of small transverse variation of the wave motion. In the leading order, the resulting equation is linear, nondispersive, and one-dimensional. When the perturbation expansion is continued to the second order, the equation acquires inhomogeneous terms that represent weak nonlinearity, weak dispersion, and weak "two-dimensionality." Therefore, from the physical point of view, the KP equation must be considered to describe small two-dimensional perturbations of one-dimensional waves. In spite of the physical relevance of this fact, a general theory of the KP equation, including one-dimensional solutions, is far from being complete, as is also the case for other $(2+1)$-dimensional equations. This is due to the fact that the standard approach (see [4–8]) was developed for solutions that rapidly decay at the space infinity and, therefore, is not applicable to this case (see, e.g., [9]). Thus, the theory of the inverse scattering transform (IST) method for the related spectral problems (nonstationary Schrödinger and heat equations) must be extended to the "nonscattering" case of potentials that do not decay in some space directions and, therefore, to a case that inherits the properties of the Sturm–Liouville problem, which is the spectral problem for the underlying KdV equation.

In [9–18], a method called an extended resolvent approach was proposed as a way for constructing such a generalization of the IST. In particular, in [16–18], it was shown that this method, on the one hand, unifies all the known approaches to the IST techniques such as dressing transformations, nonlocal Riemann–Hilbert problems, $\overline{\partial}$-method, etc., and, on the other hand, enables one to consider operators of more generic type, say, with nontrivial asymptotic behavior at the space infinity.

In the present article, the extended resolvent approach is applied to the nonstationary Schrödinger operator

$$\mathcal{L}(x,\partial_x) = i\partial_{x_2} + \partial_{x_1}^2 - u(x), \qquad x=(x_1,x_2), \tag{1.1}$$

which is the linear problem associated with KPI [1]:

$$\left(u_t - 6uu_{x_1} + u_{x_1x_1x_1}\right)_{x_1} = 3u_{x_2x_2}. \tag{1.2}$$

The integrability of this equation was established a long time ago in [2, 3], and the IST for the case of rapidly decaying potentials was developed in [4–8]. At the same time, as we mentioned above, if $u_1(t,x_1)$ obeys the KdV equation, then

$$u(t,x_1,x_2) = u_1\left(t, x_1 + \mu x_2 + 3\mu^2 t\right) \tag{1.3}$$

solves (1.2) for an arbitrary constant $\mu \in \mathbb{R}$. This confirms that a natural class of solutions of KPI should include solutions that do not asymptotically decay in all directions on the x-plane but have one-dimensional rays with behavior of the type (1.3). Consequently, one has to consider a corresponding class of potentials u in (1.1). The basic example of such potentials is given by

$$u(x) = u_1(x_1) + u_2(x), \tag{1.4}$$

where $u_1(x_1)$ and $u_2(x)$ are rapidly decaying functions of their arguments. In this case, we have two rays or, more exactly, a line direction along which the potential u does not decay at the space infinity but goes to a finite one-dimensional limit potential of the form (1.3) with $\mu=0$. The latter is not a limitation since the generic case is reconstructed by means of the Galileo invariance of (1.2), according to which if $u(t,x)$ is a solution of KPI, then

$$\widetilde{u}(t,x_1,x_2) = u\left(t, x_1 + \mu x_2 + 3\mu^2 t, x_2 + 6\mu t\right) \tag{1.5}$$

also obeys the same equation.

The spectral theory for the simplest case of potentials (1.4), where $u_1(x_1)$ is the zero-time value of the one-soliton solution of the KdV equation, was developed in [19]. There, the direct problem was

studied by using a modified integral equation for the Jost solution, i.e., an equation that involves, as a background solution, the Jost solution of the one-soliton potential and the corresponding Green's function. It was shown that the Jost solution, in addition to the standard jump across the real axis of the complex plane of the spectral parameter k, also has a jump across a segment of the imaginary k-axis. In [19], an appropriately modified formulation of the direct and inverse problems was developed, even though some essential properties of the Jost solutions and relations between spectral data were stated without proof. The reason was that a technique based on the Jost solutions alone is not sufficient for studying these properties and especially for investigating relations between spectral data. This gap was filled in [18], where the problem was completely solved in the framework of the extended resolvent approach.

In the present article, we extend the results of [18] to the case of a generic (real, smooth, and rapidly decaying) one-dimensional potential u_1. The outline of the article is as follows. In Section 2, we present the basic notions of the resolvent approach and refer to [16–18] for additional details. In particular, we demonstrate that, in this framework, one can easily derive the known results for operator (1.1) with decaying potentials. In Section 3, we consider the embedding of the pure one-dimensional case, i.e., the case $u_2(x) \equiv 0$ in (1.4), in the theory of the two-dimensional operator (1.1). We derive the corresponding resolvent and Green's functions and describe their properties. In Section 4, we apply these results to the study of the resolvent, Green's functions, and Jost solutions of the operator (1.1) with a generic potential $u(x)$ as given in (1.4). In particular, in Subsection 4.3, we show that in order to describe the discontinuity of the Jost solutions on the imaginary axis, it is necessary to introduce an additional set of solutions, which we call auxiliary Jost solutions. Relations between these auxiliary solutions and the Jost ones are given in Subsection 4.3.3. This study enables us to derive in Subsection 4.3.4 a bilinear representation for the extended resolvent, the Green's function, and the advanced/retarded Green's functions. Relations between these Green's functions are given in Subsection 4.4. In Subsection 4.5, we derive relations between the Jost solutions on the real axis and the advanced/retarded solutions and introduce the corresponding spectral data. We present the properties of these spectral data and derive characterization equations for them. In Section 5, we summarize the main aspects of the spectral theory developed in the framework of the extended resolvent approach.

2. BASIC OBJECTS OF THE EXTENDED RESOLVENT APPROACH

2.1. Extension of differential operators and a resolvent. Let $\mathcal{A} = \mathcal{A}(x, \partial_x)$ denote a differential operator

$$\mathcal{A}(x, \partial_x) = \sum_n a_n(x)\partial_x^n, \qquad x = (x_1, x_2), \quad n = (n_1, n_2) \in \mathbb{N}^2, \tag{2.1.1}$$

where $a_n(x)$ are smooth bounded functions of x, and let us call

$$\mathcal{A}(x, \partial_x + q) = \sum_n a_n(x)(\partial_x + q)^n, \qquad q = (q_1, q_2) \in \mathbb{R}^2, \tag{2.1.2}$$

its *extension*. It is convenient to consider these operators as integral ones with the respective kernels

$$A(x, x') = \mathcal{A}(x, \partial_x)\delta(x - x') \tag{2.1.3}$$

and

$$A(x, x'; q) = \mathcal{A}(x, \partial_x + q)\delta(x - x') \equiv e^{-q(x-x')}A(x, x'), \tag{2.1.4}$$

where $\delta(x) = \delta(x_1)\delta(x_2)$ is the two-dimensional δ-function and $qx = q_1x_1 + q_2x_2$.

Below, we consider a space of operators $A(q)$ whose kernels $A(x,x';q)$ belong to the space $\mathcal{S}'$ of tempered distributions of the six real variables x, x', and q. The differential operators of the kind (2.1.3) and their extensions defined in (2.1.4) form a subclass of these operators. In this space, we define a composition $(AB)(q)$ of two generic (not necessarily differentiable) operators $A(q)$ and $B(q)$ with kernels $A(x,x';q)$ and $B(x,x';q)$ in a standard way as

$$(AB)(x,x';q) = \int dx''\, A(x,x'';q)\, B(x'',x';q). \tag{2.1.5}$$

Since the kernels are distributions, this composition is neither necessarily defined for all pairs of operators nor necessarily associative. On the space of these distributions, we define the operation of Hermitian conjugation $A^\dagger$ as

$$A^\dagger(x,x';q) = \overline{A(x',x;-q)}, \tag{2.1.6}$$

where the bar denotes complex conjugation and, in the case of matrix operators, the matrix on the right-hand side must be transposed. An operator A may possess a right and/or left inverse in the sense of the composition law (2.1.5), say, $AA^{-1} = I$ and/or $A^{-1}A = I$, where I is the unit operator in $\mathcal{S}'$,

$$I(x,x';q) = \delta(x - x'). \tag{2.1.7}$$

With any operator $A(q)$ with kernel $A(x,x';q)$, we associate its "hat kernel"

$$\widehat{A}(x,x';q) = e^{q(x-x')}A(x,x';q). \tag{2.1.8}$$

In general, such objects do not belong to the space $\mathcal{S}'(\mathbb{R}^6)$; however, one can still use their composition relation

$$(\widehat{AB})(x,x';q) = \int dx''\, \widehat{A}(x,x'';q)\, \widehat{B}(x'',x';q) \tag{2.1.9}$$

derived from (2.1.5). Notice that for any differential operator $\mathcal{A}(x,\partial_x)$ with a kernel $A(x,x')$, operation (2.1.8) is just the inversion of the extension procedure (2.1.4); therefore, in this case, $\widehat{A}(x,x';q) = A(x,x')$. Moreover, for any differential operator $A(q)$ and an arbitrary operator $B(q)$, we have

$$\begin{aligned}(\widehat{AB})(x,x';q) &= \mathcal{A}(x,\partial_x)\widehat{B}(x,x';q),\\ (\widehat{BA})(x,x';q) &= \mathcal{A}^{\mathrm{d}}(x',\partial_{x'})\widehat{B}(x,x';q),\end{aligned} \tag{2.1.10}$$

where $\mathcal{A}^{\mathrm{d}}$ is the operator dual to $\mathcal{A}$. In what follows, we will use the notation

$$\widehat{AB} = \overrightarrow{\mathcal{A}}\widehat{B}, \qquad \widehat{BA} = \widehat{B}\overleftarrow{\mathcal{A}} \tag{2.1.11}$$

for equalities of this type.

2.2. Resolvent approach in the case of a rapidly decaying potential.

2.2.1. *Definition of the resolvent.* In this subsection, we shortly describe the basic aspects of the resolvent approach as applied to the nonstationary Schrödinger operator (1.1) with a rapidly decaying potential $u(x)$. An extension $L(q)$ of the operator $\mathcal{L}(x,\partial_x)$ in (1.1) is given by

$$L(q) = L_0(q) - U, \tag{2.2.1}$$

where the bare operator L_0 has the kernel

$$L_0(x,x';q) = \bigl[i(\partial_{x_2} + q_2) + (\partial_{x_1} + q_1)^2\bigr]\delta(x - x') \tag{2.2.2}$$

and U is a multiplication operator with the kernel

$$U(x,x';q) = u(x)\delta(x-x'), \tag{2.2.3}$$

which is independent of q. Below, we always assume that $u(x)$ is real, which, by (2.1.6), is equivalent to the requirement that

$$L^\dagger = L. \tag{2.2.4}$$

The main object of our approach is the extended resolvent $M(q)$ of the operator $L(q)$, which is defined as the inverse of the operator L; that is, M satisfies

$$LM = ML = I. \tag{2.2.5}$$

In order to make such inversion uniquely defined, we impose a condition that the composition

$$M(q+s)M(q), \qquad s \in \mathbb{R}^2, \tag{2.2.6}$$

is a bounded function of s in the vicinity of $s = 0$.

Under the assumption that M exists and is unique, for a real potential $u(x)$, conditions (2.2.4) and (2.2.5) imply that the resolvent is self-adjoint in the sense of (2.1.6):

$$M^\dagger = M. \tag{2.2.7}$$

From (2.2.5), one can also derive that M obeys the asymptotics

$$M(q) = -\frac{iI}{q_2} + o(q_2^{-1}), \qquad q_2 \to \infty. \tag{2.2.8}$$

Given the resolvent M of L and the resolvent M' of another operator L' of the type (1.1), one can immediately derive from (2.2.5) the following analog of the Hilbert identity under the assumption of associativity:

$$M' - M = -M'(L'-L)M. \tag{2.2.9}$$

This identity plays a crucial role in the whole theory, as will be shown below and as was already shown, in particular, in [16–18].

In the case of the bare operator L_0 defined in (2.2.2), the extended resolvent M_0 is given by

$$M_0(x,x';q) = \frac{1}{(2\pi)^2}\int \frac{dp\, e^{-ip(x-x')}}{p_2 + iq_2 - (p_1 + iq_1)^2}, \qquad p = (p_1, p_2). \tag{2.2.10}$$

Notice that condition (2.2.6) guarantees the uniqueness of M_0 since it prevents the addition of left and right annihilators of L_0 like, for instance, $\delta(L_0)$ to M_0 in (2.2.10). After integration on the right-hand side of (2.2.10) with respect to p_2, we get

$$\widehat{M}_0(x,x';q) = \frac{1}{2\pi i}\int\limits_{\mathrm{k}_\Im = q_1} d\mathrm{k}_\Re\, \bigl[\theta(x_2 - x_2') - \theta\bigl(\ell_{2\Im}(\mathrm{k}) - q_2\bigr)\bigr]\Phi_0(x,\mathrm{k})\Psi_0(x',\mathrm{k}) \tag{2.2.11}$$

for the hat kernel. Here we denoted

$$\Phi_0(x,\mathrm{k}) = e^{-i\ell(\mathrm{k})x}, \qquad \Psi_0(x,\mathrm{k}) = e^{i\ell(\mathrm{k})x}, \tag{2.2.12}$$

where ℓ is the two-component vector

$$\ell(\mathrm{k}) = (\mathrm{k}, \mathrm{k}^2) \tag{2.2.13}$$

and $\mathbf{k} = \mathbf{k}_{\Re} + i\mathbf{k}_{\Im} \in \mathbb{C}$. In what follows, the complex (one-dimensional) parameter $\mathbf{k}$ will play the role of a spectral parameter. Here and in what follows, we use boldface letters to denote complex variables, preserving nonbold letters (like k) for their real parts; for example,

$$k \equiv \mathbf{k}_{\Re}. \tag{2.2.14}$$

Notice that unlike the hat kernel of the differential operator L_0, the hat kernel of its inverse nontrivially depends on the parameter q.

The functions $\Phi_0(x, \mathbf{k})$ and $\Psi_0(x, \mathbf{k})$, which naturally appeared in (2.2.11), satisfy the differential equations

$$\mathcal{L}_0(x, \partial_x)\Phi_0(x, \mathbf{k}) \equiv \big(i\partial_{x_2} + \partial_{x_1}^2\big)\Phi_0(x, \mathbf{k}) = 0, \tag{2.2.15}$$

$$\mathcal{L}_0^{\mathrm{d}}(x, \partial_x)\Psi_0(x, \mathbf{k}) \equiv \big(-i\partial_{x_2} + \partial_{x_1}^2\big)\Psi_0(x, \mathbf{k}) = 0; \tag{2.2.16}$$

i.e., they provide a solution to the nonstationary Schrödinger equation and its dual in the case of zero potential. Thus, they can be considered as the Jost solutions for this trivial case. Notice also that they obey the conjugation property

$$\overline{\Phi_0(x, \mathbf{k})} = \Psi_0(x, \overline{\mathbf{k}}). \tag{2.2.17}$$

Using notation (2.1.11), we can rewrite equation (2.2.5) in this case as

$$\overrightarrow{\mathcal{L}_0}\widehat{M}_0(q) = \widehat{M}_0(q)\overleftarrow{\mathcal{L}_0} = I, \tag{2.2.18}$$

which shows that the hat version of the extended resolvent $M_0(q)$ is a two-parametric set of Green's functions of (1.1) and its dual. By analogy with these notations, we rewrite (2.2.15) and (2.2.16) as

$$\overrightarrow{\mathcal{L}_0}\Phi_0(\mathbf{k}) = 0, \qquad \Psi_0(\mathbf{k})\overleftarrow{\mathcal{L}_0} = 0, \tag{2.2.19}$$

considering $\Phi_0(\mathbf{k})$ and $\Psi_0(\mathbf{k})$ as a "vector" and a "covector" with "components" labeled by the variable x.

By (2.2.11), one can directly verify that for $q \neq 0$,

$$\frac{\partial \widehat{M}_0(q)}{\partial q_1} = \frac{i}{\pi} \int\limits_{\mathbf{k}_{\Im}=q_1} d\mathbf{k}_{\Re}\, \overline{\mathbf{k}}\, \delta\big(\ell_{2\Im}(\mathbf{k}) - q_2\big)\, \Phi_0(\mathbf{k}) \otimes \Psi_0(\mathbf{k}), \tag{2.2.20}$$

$$\frac{\partial \widehat{M}_0(q)}{\partial q_2} = \frac{1}{2\pi i} \int\limits_{\mathbf{k}_{\Im}=q_1} d\mathbf{k}_{\Re}\, \delta\big(\ell_{2\Im}(\mathbf{k}) - q_2\big)\, \Phi_0(\mathbf{k}) \otimes \Psi_0(\mathbf{k}), \tag{2.2.21}$$

where, according to the "vector" interpretation of the Jost solutions, the direct product is defined in the standard way as an operator with the kernel

$$\big(\Phi_0(\mathbf{k}) \otimes \Psi_0(\mathbf{k})\big)(x, x') = \Phi_0(x, \mathbf{k})\Psi_0(x', \mathbf{k}). \tag{2.2.22}$$

2.2.2. *Resolvent and the Hilbert identity.* The resolvent of the dressed operator L (2.2.1) satisfying condition (2.2.6) can also be defined as a solution of the integral equations

$$M = M_0 + M_0 U M, \qquad M = M_0 + M U M_0, \tag{2.2.23}$$

which imply that M is the left and right inverse of L. Here we do not investigate the problem of finding necessary and sufficient conditions for the existence and uniqueness of a common solution

to both integral equations. It has already been shown (see, in particular, [16–18]) that under the assumption that equality (2.2.23) holds in the case of decaying potentials $u(x)$, one can use the Hilbert identity (2.2.9) for studying the properties of the resolvent and its reductions such as Green's functions, Jost solutions, and spectral data, and, finally, for deriving a bilinear expression for M in terms of the Jost solutions (see Subsection 2.2.4, where this result is briefly reported for further use). Then, thanks to the known results (see [8]) on the existence and uniqueness of the Jost solutions in the case of potentials satisfying a small-norm assumption, this bilinear representation proves the existence and uniqueness of the resolvent under the same conditions. In this paper, we face the problem of constructing the resolvent in the case of a much wider class of potentials, specifically, for the class given in (1.4), which does not satisfy any small-norm condition. Nevertheless, we show in Section 3 that the extended resolvent exists in this case, although the standard integral equation for the Jost solutions becomes senseless.

As the first example of using the Hilbert identity, let us choose $U' = U$ and let $L = L(q)$ and $L' = L(q')$. Then (2.2.9) can be rewritten as

$$M(q') - M(q) = -M(q')(L_0(q') - L_0(q))M(q) \tag{2.2.24}$$

or, in view of (2.2.5), as

$$M(q') - M(q) = M(q')L_0(q')(M_0(q') - M_0(q))L_0(q)M(q). \tag{2.2.25}$$

Then, using notation (2.1.11), we have

$$\frac{\partial \widehat{M}(q)}{\partial q_j} = \widehat{M}(q)\overleftarrow{\mathcal{L}}_0 \frac{\partial \widehat{M}_0(q)}{\partial q_j}\overrightarrow{\mathcal{L}}_0\widehat{M}(q), \qquad j = 1, 2, \tag{2.2.26}$$

and using (2.2.20) and (2.2.21) on the right-hand side, we get

$$\frac{\partial \widehat{M}(q)}{\partial q_1} = \frac{i}{\pi} \int\limits_{\mathbf{k}_\Im = q_1} d\mathbf{k}_\Re\, \overline{\mathbf{k}}\, \delta(\ell_{2\Im}(\mathbf{k}) - q_2)\, \Phi(\mathbf{k}) \otimes \Psi(\mathbf{k}), \tag{2.2.27}$$

$$\frac{\partial \widehat{M}(q)}{\partial q_2} = \frac{1}{2\pi i} \int\limits_{\mathbf{k}_\Im = q_1} d\mathbf{k}_\Re\, \delta(\ell_{2\Im}(\mathbf{k}) - q_2)\, \Phi(\mathbf{k}) \otimes \Psi(\mathbf{k}) \tag{2.2.28}$$

for $q_1 \neq 0$, where we introduced a "vector" $\Phi(\mathbf{k})$ and a "covector" $\Psi(\mathbf{k})$ defined as

$$\Phi(\mathbf{k}) = \mathcal{G}(\mathbf{k})\overleftarrow{\mathcal{L}}_0\Phi_0(\mathbf{k}), \tag{2.2.29}$$

$$\Psi(\mathbf{k}) = \Psi_0(\mathbf{k})\overrightarrow{\mathcal{L}}_0\mathcal{G}(\mathbf{k}), \tag{2.2.30}$$

where $\mathcal{G}(\mathbf{k})$ is a specific value of the resolvent itself:

$$\mathcal{G}(\mathbf{k}) = \widehat{M}(q)\big|_{q=\ell_\Im(\mathbf{k})}. \tag{2.2.31}$$

Note that one explicitly has

$$\Phi(x, \mathbf{k}) = \int dx' \left(\mathcal{L}_0^{\mathrm{d}}(x', \partial_{x'})\mathcal{G}(x, x', \mathbf{k})\right)\Phi_0(x', \mathbf{k}), \tag{2.2.32}$$

$$\Psi(x', \mathbf{k}) = \int dx\, \Psi_0(x, \mathbf{k})\, \mathcal{L}_0(x, \partial_x)\, \mathcal{G}(x, x', \mathbf{k}), \tag{2.2.33}$$

with

$$\mathcal{G}(x,x',\mathbf{k}) = \widehat{M}(x,x';q)\big|_{q=\ell_{\Im}(\mathbf{k})} \equiv \widehat{M}(x,x';\mathbf{k}_{\Im},2\mathbf{k}_{\Im}\mathbf{k}_{\Re}). \tag{2.2.34}$$

In view of (2.2.10), the kernel $M_0(x,x';q)$ is a tempered distribution in x and x' and a continuous function for all values of q except $q=0$, where it is discontinuous. Therefore, this point should be considered separately. For this purpose, we introduce specific notations for the following limits of the resolvent at this point:

$$\mathcal{G}_{\pm}(x,x') = \lim_{q_2\to\pm0}\lim_{q_1\to0}\widehat{M}(x,x';q), \tag{2.2.35}$$

where the limit $q_1\to0$ is independent of the sign. Expressing the Hilbert identity (2.2.25) in terms of the hat kernels (2.1.8) of the resolvent and passing to the above limits, we get

$$\mathcal{G}_+ - \mathcal{G}_- = \mathcal{G}_{\pm}\overleftarrow{\mathcal{L}}_0(\mathcal{G}_{0,+}-\mathcal{G}_{0,-})\overrightarrow{\mathcal{L}}_0\mathcal{G}_{\mp}. \tag{2.2.36}$$

Here $\mathcal{G}_{0,\pm}$ is defined in terms of $\widehat{M}_0$ in (2.2.11) by analogy with (2.2.35); using notation (2.2.14) for $\mathbf{k}_{\Re}$, we have

$$\mathcal{G}_{0,\pm}(x,x') = \pm\frac{\theta(\pm(x_2-x_2'))}{2\pi i}\int dk\,\Phi_0(x,k)\Psi_0(x',k), \tag{2.2.37}$$

so that

$$\mathcal{G}_{0,+}-\mathcal{G}_{0,-} = \frac{1}{2\pi i}\int dk\,\Phi_0(k)\otimes\Psi_0(k). \tag{2.2.38}$$

Inserting this relation into the right-hand side of (2.2.36), we derive the equality

$$\mathcal{G}_+-\mathcal{G}_- = \frac{1}{2\pi i}\int dk\,\Phi_{\pm}(k)\otimes\Psi_{\mp}(k), \tag{2.2.39}$$

where, by analogy with (2.2.29) and (2.2.30), the functions $\Phi_{\pm}(x,k)$ and $\Psi_{\pm}(x,k)$ are defined with the use of the same notation as

$$\Phi_{\pm}(k) = \mathcal{G}_{\pm}\overleftarrow{\mathcal{L}}_0\Phi_0(k), \qquad \Psi_{\pm}(k) = \Psi_0(k)\overrightarrow{\mathcal{L}}_0\mathcal{G}_{\pm}. \tag{2.2.40}$$

Next, we consider in detail the properties of all the objects introduced so far: $\mathcal{G}(\mathbf{k})$, $\Phi(\mathbf{k})$, $\Psi(\mathbf{k})$, $\mathcal{G}_{\pm}$, $\Phi_{\pm}(k)$, and $\Psi_{\pm}(k)$.

2.2.3. *Properties of the Green's function.* In view of (2.1.11), it is clear that $\mathcal{G}(\mathbf{k})$ defined in (2.2.31) is a Green's function of the operator $\mathcal{L}$ and depends on the complex parameter $\mathbf{k}$,

$$\overrightarrow{\mathcal{L}}\mathcal{G}(\mathbf{k}) = \mathcal{G}(\mathbf{k})\overleftarrow{\mathcal{L}} = I. \tag{2.2.41}$$

By virtue of (2.2.7) and (2.2.34), i.e., since $u(x)$ is real, we have

$$\overline{\mathcal{G}(x,x',\mathbf{k})} = \mathcal{G}(x',x,\overline{\mathbf{k}}). \tag{2.2.42}$$

Applying reduction (2.2.34) to equalities (2.2.23), we find that this function satisfies the integral equations

$$\mathcal{G}(\mathbf{k}) = \mathcal{G}_0(\mathbf{k}) + \mathcal{G}_0(\mathbf{k})U\mathcal{G}(\mathbf{k}), \qquad \mathcal{G}(\mathbf{k}) = \mathcal{G}_0(\mathbf{k}) + \mathcal{G}(\mathbf{k})U\mathcal{G}_0(\mathbf{k}), \tag{2.2.43}$$

where the Green's function $\mathcal{G}_0(\mathbf{k})$ of the operator $\mathcal{L}_0$ is defined by the general formula (2.2.31) in terms of M_0 and, by virtue of (2.2.11), equals

$$\mathcal{G}_0(x,x',\mathbf{k}) = \frac{1}{2\pi i}\int dp\,[\theta(x_2-x_2')-\theta(\mathbf{k}_{\Im}p)]\,\Phi_0(x,p+\mathbf{k})\Psi_0(x',p+\mathbf{k}). \tag{2.2.44}$$

Using (2.2.43), we can verify that the function

$$G(x,x',\mathbf{k}) = e^{i\ell(\mathbf{k})(x-x')}\mathcal{G}(x,x',\mathbf{k}) \tag{2.2.45}$$

is a bounded function of its arguments and that

$$\lim_{\mathbf{k}\to\infty} G(x,x',\mathbf{k}) = 0 \tag{2.2.46}$$

if the potential $u(x)$ decays sufficiently rapidly. This also follows from (2.2.8) if we take into account the definition of the hat kernel in (2.1.8) and equalities (2.2.13) and (2.2.34).

The function $\mathcal{G}(x,x',\mathbf{k})$ is a continuously differentiable function of $\mathbf{k}$ in the whole complex plane $\mathbb{C}$ except the real axis $\mathbf{k}_\Im = 0$. By (2.2.27) and (2.2.28), for $\mathbf{k}_\Im \neq 0$ we have

$$\frac{\partial\mathcal{G}(\mathbf{k})}{\partial\mathbf{k}_\Re} = \frac{\operatorname{sgn}\mathbf{k}_\Im}{2\pi i}\,\Phi(\mathbf{k})\otimes\Psi(\mathbf{k}), \qquad \frac{\partial\mathcal{G}(\mathbf{k})}{\partial\mathbf{k}_\Im} = \frac{\operatorname{sgn}\mathbf{k}_\Im}{2\pi}\,\Phi(\mathbf{k})\otimes\Psi(\mathbf{k}), \tag{2.2.47}$$

so that this Green's function is analytic in the complex domain,

$$\frac{\partial\mathcal{G}(\mathbf{k})}{\partial\overline{\mathbf{k}}} = 0, \qquad \mathbf{k}_\Im \neq 0, \tag{2.2.48}$$

and discontinuous on the real axis.

The properties of the functions $\mathcal{G}_\pm(x,x')$ in (2.2.35) follow from the properties of the resolvent. They are Green's functions of operator (1.1) and its dual; i.e.,

$$\overrightarrow{\mathcal{L}}\mathcal{G}_\pm = \mathcal{G}_\pm\overleftarrow{\mathcal{L}} = I. \tag{2.2.49}$$

They are referred to as advanced $(+)$ and retarded $(-)$ Green's functions. They obey the conjugation property

$$\overline{\mathcal{G}_\pm(x,x')} = \mathcal{G}_\mp(x',x) \tag{2.2.50}$$

and the integral equations

$$\mathcal{G}_\pm = \mathcal{G}_{0,\pm} + \mathcal{G}_{0,\pm}U\mathcal{G}_\pm, \qquad \mathcal{G}_\pm = \mathcal{G}_{0,\pm} + \mathcal{G}_\pm U\mathcal{G}_{0,\pm}, \tag{2.2.51}$$

where $\mathcal{G}_{0,\pm}$ is given in (2.2.37).

The limiting values of the Green's function $\mathcal{G}(\mathbf{k})$ on the real axis must be close, in a sense, to the advanced/retarded Green's functions $\mathcal{G}_\pm$. Indeed, introducing the notation

$$\mathcal{G}^\pm(x,x',k) = \mathcal{G}(x,x',k\pm i0), \qquad k\in\mathbb{R}, \tag{2.2.52}$$

we can see by (2.2.34) that they are given by the following limits of the resolvent:

$$\mathcal{G}^\pm(x,x',k) = \lim_{\varepsilon\to+0}\widehat{M}(x,x';\pm\varepsilon,\pm 2k\varepsilon); \tag{2.2.53}$$

thus, like $\mathcal{G}_\pm$ in (2.2.35), they correspond to some limit of $\widehat{M}(q)$ at $q=0$. As we mentioned when discussing (2.2.35), the resolvent is discontinuous at this point; therefore, these limits are different in general. In [16], we showed that the study of this discontinuity leads to the definition of spectral data. In Subsection 2.2.5, we shortly demonstrate this in a simple case of a decaying potential as a guide to the essentially more complicated case of potentials of the form (1.4).

In order to find relations between these Green's functions, we again start from (2.2.25). Passing to the limit (2.2.53) with respect to q' and (2.2.35) with respect to q and vice versa, we get

$$\mathcal{G}^{\sigma}(k) - \mathcal{G}_{\pm} = \big(\mathcal{G}_{\pm}\overleftarrow{\mathcal{L}}_0\big)\big(\mathcal{G}_0^{\sigma}(k) - \mathcal{G}_{0,\pm}\big)\big(\overrightarrow{\mathcal{L}}_0\mathcal{G}^{\sigma}(k)\big), \tag{2.2.54}$$

$$\mathcal{G}^{\sigma}(k) - \mathcal{G}_{\pm} = \big(\mathcal{G}^{\sigma}(k)\overleftarrow{\mathcal{L}}_0\big)\big(\mathcal{G}_0^{\sigma}(k) - \mathcal{G}_{0,\pm}\big)\big(\overrightarrow{\mathcal{L}}_0\mathcal{G}_{\pm}\big), \tag{2.2.55}$$

where $\sigma = +,-$. By (2.2.44),

$$\mathcal{G}_0^{\sigma}(x,x',k) = \frac{1}{2\pi i}\int dp\,\big[\theta(x_2 - x_2') - \theta(\sigma(p-k))\big]\,\Phi_0(x,p)\Psi_0(x',p). \tag{2.2.56}$$

Then, using (2.2.37) and definitions (2.2.40), from (2.2.54) and (2.2.55) we get

$$\mathcal{G}^{\sigma}(k) - \mathcal{G}_{\pm} = \mp\frac{1}{2\pi i}\int dp\,\theta(\mp\sigma(k-p))\Phi_{\pm}(p)\otimes\big(\Psi_0(p)\overrightarrow{\mathcal{L}}_0\mathcal{G}^{\sigma}(k)\big), \tag{2.2.57}$$

$$\mathcal{G}^{\sigma}(k) - \mathcal{G}_{\pm} = \mp\frac{1}{2\pi i}\int dp\,\theta(\mp\sigma(k-p))\big(\mathcal{G}^{\sigma}(k)\overleftarrow{\mathcal{L}}_0\Phi_0(p)\big)\otimes\Psi_{\pm}(p), \tag{2.2.58}$$

where $\sigma = +,-$ and, for the "vector" $\mathcal{G}^{\sigma}(k)\overleftarrow{\mathcal{L}}_0\Phi_0(p)$ and the "covector" $\Psi_0(p)\overrightarrow{\mathcal{L}}_0\mathcal{G}^{\sigma}(k)$, we used a shorthand notation analogous to that in (2.2.29) and (2.2.30). Finally, let us recall that a relation between $\mathcal{G}_+$ and $\mathcal{G}_-$ was given in (2.2.39).

2.2.4. *Jost and advanced/retarded solutions and a bilinear representation for the resolvent.* In view of (2.2.19) and (2.2.41), it follows directly from definitions (2.2.29) and (2.2.30) that the functions $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$ satisfy the nonstationary Schrödinger equation with the potential $u(x)$ and its dual, i.e.,

$$\overrightarrow{\mathcal{L}}\Phi(\mathbf{k}) = 0, \qquad \Psi(\mathbf{k})\overleftarrow{\mathcal{L}} = 0, \tag{2.2.59}$$

and in view of (2.2.42), these functions satisfy the conjugation property

$$\overline{\Phi(x,\mathbf{k})} = \Psi(x,\overline{\mathbf{k}}). \tag{2.2.60}$$

The integral equations for these functions

$$\Phi(\mathbf{k}) = \Phi_0(\mathbf{k}) + \mathcal{G}_0(\mathbf{k})U\Phi(\mathbf{k}), \tag{2.2.61}$$

$$\Psi(\mathbf{k}) = \Psi_0(\mathbf{k}) + \Psi(\mathbf{k})U\mathcal{G}_0(\mathbf{k}) \tag{2.2.62}$$

follow from (2.2.43). If we introduce the standard representation

$$\Phi(x,\mathbf{k}) = e^{-i\ell(\mathbf{k})x}\chi(x,\mathbf{k}), \qquad \Psi(x,\mathbf{k}) = e^{i\ell(\mathbf{k})x}\xi(x,\mathbf{k}), \tag{2.2.63}$$

we find that χ and ξ satisfy the equations

$$\big(i\partial_{x_2} + \partial_{x_1}^2 - 2i\mathbf{k}\partial_{x_1} - u(x)\big)\chi(x,\mathbf{k}) = 0, \tag{2.2.64}$$

$$\big(-i\partial_{x_2} + \partial_{x_1}^2 + 2i\mathbf{k}\partial_{x_1} - u(x)\big)\xi(x,\mathbf{k}) = 0 \tag{2.2.65}$$

and are normalized at infinity,

$$\lim_{\mathbf{k}\to\infty}\chi(x,\mathbf{k}) = \lim_{\mathbf{k}\to\infty}\xi(x,\mathbf{k}) = 1, \tag{2.2.66}$$

while the potential $u(x)$ is reconstructed by these functions as

$$u(x) = -2i \lim_{\mathbf{k}\to\infty} \mathbf{k}\partial_{x_1}\chi(x,\mathbf{k}) = 2i \lim_{\mathbf{k}\to\infty} \mathbf{k}\partial_{x_1}\xi(x,\mathbf{k}). \tag{2.2.67}$$

The solutions $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$ satisfy the orthogonality relation

$$\frac{1}{2\pi}\int dx_1\, \Psi(x,\mathbf{k}+p)\Phi(x,\mathbf{k}) = \delta(p), \qquad \mathbf{k}\in\mathbb{C}, \quad p\in\mathbb{R}, \tag{2.2.68}$$

which can also be derived from a proper reduction of the Hilbert identity (see [10]).

The properties of the functions $\Phi_\pm(x,k)$ and $\Psi_\pm(x,k)$ are derived from their definition (2.2.40) in a similar way. By (2.2.49), they also satisfy the nonstationary Schrödinger equation and its dual,

$$\overrightarrow{\mathcal{L}}\,\Phi_\pm(k) = 0, \qquad \Psi_\pm(k)\overleftarrow{\mathcal{L}} = 0, \tag{2.2.69}$$

where we again use notations similar to those in (2.2.19). The integral equations

$$\Phi_\pm(k) = \Phi_0(k) + \mathcal{G}_{0,\pm}U\Phi_\pm(k), \tag{2.2.70}$$

$$\Psi_\pm(k) = \Psi_0(k) + \Psi_\pm(k)U\mathcal{G}_{0,\pm} \tag{2.2.71}$$

and the conjugation property

$$\overline{\Phi_\pm(x,k)} = \Psi_\mp(x,k) \tag{2.2.72}$$

follow from definitions (2.2.40) and (2.2.50). Taking into account the form of the Green's functions $\mathcal{G}_\pm$ (see (2.2.37) or (2.2.78) below), we can check that the solutions $\Phi_\pm(x,k)$ and $\Psi_\pm(x,k)$ are the well-known advanced/retarded ones. They also satisfy an orthogonality relation, i.e.,

$$\frac{1}{2\pi}\int dx_1\, \Psi_\pm(x,k+p)\Phi_\mp(x,k) = \delta(p), \qquad k,p\in\mathbb{R}. \tag{2.2.73}$$

Above, the Jost solutions appeared as special reductions of the Green's function, i.e., of the resolvent itself. On the other hand, as was already mentioned in the introduction, the resolvent itself can be expressed in terms of these solutions. Indeed, recalling (2.2.8) and taking into account that χ and ξ in (2.2.63) are bounded functions of their arguments, we reconstruct $M(q)$ from (2.2.27) in the form

$$\widehat{M}(x,x';q) = \frac{1}{2\pi i}\int\limits_{\mathbf{k}_\Im = q_1} d\mathbf{k}_\Re \left[\theta(x_2-x_2') - \theta(\ell_{2\Im}(\mathbf{k}) - q_2)\right]\Phi(x,\mathbf{k})\Psi(x',\mathbf{k}), \tag{2.2.74}$$

which generalizes (2.2.11) to the case of nonzero potentials.

We have already mentioned (see (2.2.11)) that the resolvent $M_0(q)$ is a continuous function of q_1 when $q \neq 0$. By (2.2.23), the resolvent $M(q)$ has the same property, while the right-hand side of (2.2.74) involves the Jost solutions that are discontinuous at $\mathbf{k}_\Im = 0$. More exactly, let us introduce, by analogy with (2.2.52), a specific notation for the limiting values of the Jost solutions on the real axis, i.e.,

$$\Phi^\pm(x,k) = \Phi(x,k\pm i0), \qquad \Psi^\pm(x,k) = \Psi(x,k\pm i0). \tag{2.2.75}$$

Then the above-mentioned condition of continuity reads as

$$\int dk\, \Phi^+(k)\otimes\Psi^+(k) = \int dk\, \Phi^-(k)\otimes\Psi^-(k). \tag{2.2.76}$$

Representation (2.2.74) plays a crucial role in the resolvent approach since it enables us to express all objects of the spectral theory in terms of the Jost solutions. In particular, using (2.2.74), we obtain the following expression for the Green's function:

$$\mathcal{G}(x, x', \mathrm{k}) = \frac{1}{2\pi i} \int dp \, [\theta(x_2 - x_2') - \theta(\mathrm{k}_{\Im} p)] \, \Phi(x, p + \mathrm{k}) \Psi(x', p + \mathrm{k}), \tag{2.2.77}$$

which generalizes (2.2.44). Moreover, from (2.2.74) we can get the following expression for the advanced/retarded Green's functions in terms of the limiting values of the Jost solutions on the real axis:

$$\mathcal{G}_{\pm}(x, x') = \pm \frac{\theta(\pm(x_2 - x_2'))}{2\pi i} \int dk \, \Phi^{\sigma}(x, k) \Psi^{\sigma}(x', k), \tag{2.2.78}$$

where $\sigma = +$, and we used notation (2.2.75). Condition (2.2.76) guarantees that the functions $\mathcal{G}_{\pm}$ are independent of the choice of the sign σ on the right-hand side, as it must be.

Taking into account (2.1.8) and (2.2.8), we obtain the following important property of the Jost solutions from the bilinear representation (2.2.74):

$$\frac{1}{2\pi} \int\limits_{x_2' = x_2} d\mathrm{k}_{\Re} \, \Psi(x', \mathrm{k}) \Phi(x, \mathrm{k}) = \delta(x_1 - x_1'), \tag{2.2.79}$$

which can be considered a completeness relation.

2.2.5. *Relations between Jost and advanced/retarded solutions. Spectral data.* In the previous subsection, we derived bilinear representations (2.2.77) and (2.2.78) in terms of the Jost solutions for the Green's function and the advanced/retarded Green's functions, respectively. We also derived equations (2.2.57) and (2.2.58), which relate these Green's functions. Now, these results can be used for deriving relations between advanced/retarded and Jost solutions on the real axis and, then, for introducing the spectral data. Applying $\overleftarrow{\mathcal{L}_0} \Phi_0(k)$ to (2.2.57) from the right and $\Psi_0(k) \overrightarrow{\mathcal{L}_0}$ to (2.2.58) from the left, recalling definitions (2.2.29), (2.2.30), and (2.2.40), for $k, p \in \mathbb{R}$ and $\sigma = +, -$ we get

$$\Phi^{\sigma}(k) = \int dp \, \Phi_{\pm}(p) r_{\pm}^{\sigma}(p, k), \tag{2.2.80}$$

$$\Psi^{\sigma}(k) = \int dp \, \overline{r_{\mp}^{-\sigma}(p, k)} \Psi_{\pm}(p), \tag{2.2.81}$$

where

$$r_{\pm}^{\sigma}(p, k) = \delta(p - k) \mp \theta(\pm\sigma(p - k)) r^{\sigma}(p, k), \qquad p, k \in \mathbb{R}, \tag{2.2.82}$$

$$r^{\sigma}(p, k) = \frac{\Psi_0(p) \overrightarrow{\mathcal{L}_0} \mathcal{G}^{\sigma}(k) \overleftarrow{\mathcal{L}_0} \Phi_0(k)}{2\pi i} \equiv \frac{\Psi_0(p) \overrightarrow{\mathcal{L}_0} \Phi^{\sigma}(k)}{2\pi i}, \tag{2.2.83}$$

are the spectral data. When deriving (2.2.81), we used conjugation properties (2.2.17), (2.2.42), and (2.2.60) for the Green's function and Jost solutions. Recalling "vector" and "covector" notations (cf. (2.2.32), (2.2.33) and (2.2.29), (2.2.30)), we can write the following explicit expressions for the "expectation values" in the numerators of the previous equalities:

$$\Psi_0(p) \overrightarrow{\mathcal{L}_0} \mathcal{G}^{\sigma}(k) \overleftarrow{\mathcal{L}_0} \Phi_0(k) = \int dx \int dx' \, \Psi_0(x, p) \big(\mathcal{L}_0(x, \partial_x) \mathcal{L}_0^{\mathrm{d}}(x', \partial_{x'}) \mathcal{G}^{\sigma}(x, x', k)\big) \Phi_0(x', k), \tag{2.2.84}$$

$$\Psi_0(p) \overrightarrow{\mathcal{L}_0} \Phi^{\sigma}(k) = \int dx \, \Psi_0(x, p) \mathcal{L}_0(x, \partial_x) \Phi^{\sigma}(x, k), \tag{2.2.85}$$

which show that these expectation values depend only on p and k. The triangular operator (2.2.82) gives a standard representation for the spectral data [20]. Here we present it only to show how these data appear in a natural way in our approach and that they can be obtained by a successive reduction of the resolvent.

In order to get the advanced/retarded solutions in terms of the boundary values of the Jost solutions, we use the limiting values of (2.2.77) on the real axis:

$$\mathcal{G}^{\sigma}(x,x',k)=\frac{1}{2\pi i}\int dp\,\left[\theta(x_2-x_2')-\theta(\sigma(p-k))\right]\Phi^{\sigma}(x,p)\Psi^{\sigma}(x',p),\qquad k\in\mathbb{R}. \tag{2.2.86}$$

Then, by (2.2.78), we have

$$\mathcal{G}^{\sigma}(k)-\mathcal{G}_{\pm}=\mp\frac{1}{2\pi i}\int dp\,\theta(\pm\sigma(p-k))\Phi^{\sigma}(p)\otimes\Psi^{\sigma}(p),\qquad \sigma=+,-, \tag{2.2.87}$$

and in the same way as above, we readily derive

$$\Phi_{\pm}(k)=\int dp\,\Phi^{\sigma}(p)\overline{r_{\pm}^{-\sigma}(k,p)}, \tag{2.2.88}$$

$$\Psi_{\pm}(k)=\int dp\,r_{\mp}^{\sigma}(k,p)\Psi^{\sigma}(p). \tag{2.2.89}$$

Now, substituting these expressions into (2.2.80) and (2.2.81) and taking into account the orthogonality properties (2.2.68) and (2.2.73) of the Jost and advanced/retarded solutions, we derive that the spectral data satisfy the following characterization equations [20]:

$$\int dp\,\overline{r_{\pm}^{-\sigma}(p,k)}r_{\pm}^{\sigma}(p,k')=\delta(k-k'), \tag{2.2.90}$$

$$\int dp\,\overline{r_{\pm}^{-\sigma}(k',p)}r_{\pm}^{\sigma}(k,p)=\delta(k-k'), \tag{2.2.91}$$

$$\int dp\,\overline{r_{+}^{\sigma}(p,k)}r_{+}^{\sigma}(p,k')=\int dp\,\overline{r_{-}^{\sigma}(p,k)}r_{-}^{\sigma}(p,k'), \tag{2.2.92}$$

where $\sigma=+,-$. More explicitly, in view of (2.2.82), we have

$$\int_{k}^{k'} dp\,\overline{r^{-\sigma}(p,k)}r^{\sigma}(p,k')=-\sigma r^{\sigma}(k,k')-\sigma\overline{r^{-\sigma}(k',k)}, \tag{2.2.93}$$

$$\int_{k}^{k'} dp\,\overline{r^{-\sigma}(k',p)}r^{\sigma}(k,p)=-\sigma r^{\sigma}(k,k')-\sigma\overline{r^{-\sigma}(k',k)}, \tag{2.2.94}$$

$$\left(\int_{k'}^{+\infty}-\int_{-\infty}^{k}\right)dp\,\overline{r^{\sigma}(p,k)}r^{\sigma}(p,k')=\sigma r^{\sigma}(k,k')+\sigma\overline{r^{\sigma}(k',k)}. \tag{2.2.95}$$

Notice that the characterization equation (2.2.91) implies (2.2.76). In fact, substituting (2.2.80) and (2.2.81) into $\int dk\,\Phi^{\sigma}(k)\Psi^{\sigma}(k)$ and using (2.2.91), we get $\int dp\,\Phi_{\pm}\otimes\Psi_{\mp}$, which does not depend on σ. Notice also that the characterization equations (2.2.90) and (2.2.91) guarantee that once, for instance, $r_{\pm}^{+}$ is given, $r_{\pm}^{-}$ can be defined as the Hermitian conjugate of the inverse of the operator $r_{\pm}^{+}$. Due to this fact, equation (2.2.92) for $\sigma=-$ follows from the same equation for $\sigma=+$. Therefore, we have obtained a regularity condition, that is, the existence of the right and left inverses of the

operator $r_{\pm}^{+}$, and, strictly speaking, only one characterization equation, that is, (2.2.92) with $\sigma = +$, which can be considered a consequence of the reality of the potential $u(x)$. This remark shows that the set of admissible potentials and the set of admissible spectral data have the same functional arbitrariness.

If we introduce alternative spectral data

$$F^{\sigma}(k,k') = \int dp\, \overline{r_{\pm}^{-\sigma}(p,k)}\, r_{\pm}^{-\sigma}(p,k'), \tag{2.2.96}$$

we can express the discontinuity of the Jost solutions across the real axis by the following relations:

$$\Phi^{\sigma}(k) = \int dp\, \Phi^{-\sigma}(p) F^{-\sigma}(p,k), \qquad \Psi^{\sigma}(k) = \int dp\, F^{\sigma}(k,p)\Psi^{-\sigma}(p). \tag{2.2.97}$$

In view of (2.2.92), these spectral data are independent of the choice of the + or − signs on the right-hand side of (2.2.96). These spectral data satisfy the characterization equations

$$(F^{\sigma})^{\dagger} = F^{\sigma}, \qquad F^{-\sigma} = (F^{\sigma})^{-1}, \tag{2.2.98}$$

which should be supplemented with the requirement that F^{σ} can be decomposed into the product (2.2.96) of two triangular operators of the form (2.2.82). Again, the first equation in (2.2.98) is related to the reality requirement for $u(x)$, while the second equation is a regularity condition.

The inverse problem can be formulated in the standard way [4] by using the fact that the Jost solution $\Phi(x,\mathbf{k})$ is an analytic function of $\mathbf{k}$ for $\mathbf{k}_{\Im} \neq 0$, has discontinuity (2.2.97) on the real axis, and satisfies the normalization condition given by (2.2.63) and (2.2.66). In [13], we also demonstrated that the inverse problem can be formulated in terms of the resolvent itself. Here we omit this result for short, as well as many other results that follow from the extended resolvent approach like the introduction and the properties of the dressing operators [13], the derivation of the time evolution of the spectral data corresponding to the KPI equation, and the algorithmic construction of time evolutions compatible with a given linear problem [12].

3. THE CASE OF ONE-DIMENSIONAL POTENTIAL

3.1. Review of the standard theory of the Sturm–Liouville equation. In this section, we consider the spectral theory of operator (1.1) when the perturbation $u_2(x)$ in (1.4) is identically zero. But first we start with a short review of the standard theory of the Sturm–Liouville equation in the context of IST (see [21, 22]). We use subscript 1 to denote the Jost solutions of this equation. As we have seen in the previous section, the resolvent approach naturally involves the Jost solutions of the linear problem itself as well as solutions of the dual equation. Here, in spite of the self-duality of the Sturm–Liouville equation, we still introduce two Jost solutions

$$[\partial_{x_1}^2 - u_1(x_1) + \mathbf{k}^2]\Phi_1(x_1,\mathbf{k}) = 0, \qquad [\partial_{x_1}^2 - u_1(x_1) + \mathbf{k}^2]\Psi_1(x_1,\mathbf{k}) = 0 \tag{3.1.1}$$

that are defined as functions of the variable x_1 and the spectral parameter $\mathbf{k} \in \mathbb{C}$ by the relations

$$\Phi_1(x_1,\mathbf{k}) = e^{-i\mathbf{k}x_1}\chi_1(x_1,\mathbf{k}), \qquad \Psi_1(x_1,\mathbf{k}) = e^{i\mathbf{k}x_1}\xi_1(x_1,\mathbf{k}), \tag{3.1.2}$$

where χ_1 and ξ_1 satisfy the integral equations

$$\chi_1(x_1,\mathbf{k}) = 1 + \int\limits_{-\mathbf{k}_{\Im}\infty}^{x_1} dy_1 \frac{e^{2i\mathbf{k}(x_1-y_1)} - 1}{2i\mathbf{k}}\, u_1(y_1)\chi_1(y_1,\mathbf{k}), \tag{3.1.3}$$

$$\xi_1(x_1,\mathbf{k}) = 1 + \int\limits_{x_1}^{\mathbf{k}_{\Im}\infty} dy_1 \frac{e^{2i\mathbf{k}(y_1-x_1)} - 1}{2i\mathbf{k}}\, u_1(y_1)\xi_1(y_1,\mathbf{k}) \tag{3.1.4}$$

for any k ($\mathrm{k}_\Im \neq 0$) in the complex plane. In particular,

$$\lim_{\mathrm{k}\to\infty} \chi_1(x_1,\mathrm{k}) = \lim_{\mathrm{k}\to\infty} \xi_1(x_1,\mathrm{k}) = 1. \tag{3.1.5}$$

By virtue of (3.1.2), the reality of the potential u_1 and the self-duality of the Sturm–Liouville equation are equivalent to the symmetry properties

$$\overline{\Phi_1(x_1,\mathrm{k})} = \Psi_1(x_1,\overline{\mathrm{k}}), \qquad \Phi_1(x_1,\mathrm{k}) = \Psi_1(x_1,-\mathrm{k}). \tag{3.1.6}$$

The elements of the monodromy matrix are defined as

$$a(\mathrm{k}) = 1 - \frac{1}{2i\mathrm{k}} \int dy_1\, e^{i\mathrm{k}y_1} u_1(y_1)\Phi_1(y_1,\mathrm{k}), \qquad \mathrm{k}_\Im \geq 0, \tag{3.1.7}$$

$$b(k) = \frac{1}{2ik} \int dy_1\, e^{-iky_1} u_1(y_1)\Phi_1^{\pm}(y_1,k), \qquad k \in \mathbb{R}, \tag{3.1.8}$$

where $\Phi_1^{\pm}$ are the limiting values on the real axis from above and below (cf. the notation in (2.2.75)). It is well known [21, 22] that

$$\overline{a(\mathrm{k})} = a(-\overline{\mathrm{k}}), \qquad \overline{b(k)} = b(-k), \tag{3.1.9}$$

$$|a(k)|^2 - |b(k)|^2 = 1, \qquad k \in \mathbb{R}. \tag{3.1.10}$$

In general, $a(\mathrm{k})$ has a finite number of simple zeroes at some points $i\varkappa_j$ on the imaginary axis:

$$a(i\varkappa_j) = 0, \qquad \varkappa_j > 0, \qquad j = 1,2,\dots,N. \tag{3.1.11}$$

Another set of discrete spectral data, coefficients b_j, is defined by one of the following equalities:

$$\Phi_1(x_1,i\varkappa_j) = b_j\Phi_1(x_1,-i\varkappa_j), \qquad \Psi_1(x_1,-i\varkappa_j) = b_j\Psi_1(x_1,i\varkappa_j), \tag{3.1.12}$$

$j = 1,2,\dots,N$. These coefficients are real and nonzero. In what follows, it is convenient to use a transmission coefficient defined for any $\mathrm{k} \in \mathbb{C}$, $\mathrm{k}_\Im \neq 0$, as

$$t(\mathrm{k}) = \frac{\theta(\mathrm{k}_\Im)}{a(\mathrm{k})} + \frac{\theta(-\mathrm{k}_\Im)}{\overline{a(\overline{\mathrm{k}})}} \equiv \frac{\theta(\mathrm{k}_\Im)}{a(\mathrm{k})} + \frac{\theta(-\mathrm{k}_\Im)}{a(-\mathrm{k})}, \tag{3.1.13}$$

so that we have

$$t(\mathrm{k}) = t(-\mathrm{k}) = \overline{t(\overline{\mathrm{k}})} \tag{3.1.14}$$

and, for the limiting values on the real axis (again using the notation (2.2.75)),

$$t^{\pm}(k) = \frac{1}{a(\pm k)}, \qquad k \in \mathbb{R}. \tag{3.1.15}$$

Let us denote the residues of the transmission coefficient at the poles in the upper half-plane by

$$t_j = \operatorname*{res}_{\mathrm{k}=i\varkappa_j} t(\mathrm{k}). \tag{3.1.16}$$

They are purely imaginary,

$$\overline{t_j} = -t_j, \tag{3.1.17}$$

and by (3.1.14),

$$t(\mathrm{k}) = \pm \frac{t_j}{\mathrm{k} \mp i\varkappa_j} + o(1), \qquad \mathrm{k} \to \pm i\varkappa_j. \tag{3.1.18}$$

Moreover, it is known [21, 22] that

$$\operatorname{sgn}(it_j b_j) = -1. \tag{3.1.19}$$

The Jost solutions satisfy the orthogonality relations (cf. (2.2.68))

$$\frac{1}{2\pi}\int dx_1\, \Psi_1(x_1, \mathrm{k}+p)\Phi_1(x_1, \mathrm{k}) = \frac{\delta(p)}{t(\mathrm{k})}, \qquad p \in \mathbb{R}, \quad \mathrm{k} \in \mathbb{C}, \tag{3.1.20}$$

$$\int dx_1\, \Psi_1(x_1, i\varkappa_j)\Phi_1(x, \mathrm{k}) = \int dx_1\, \Psi_1(x_1, \mathrm{k})\Phi_1(x_1, i\varkappa_j) = 0, \qquad |\mathrm{k}_\Im| < \varkappa_j, \tag{3.1.21}$$

$$\int dx_1\, \Psi_1(x_1, i\varkappa_j)\Phi_1(x_1, i\varkappa_{j'}) = \frac{i\delta_{jj'}}{t_j}, \qquad j, j' = 1, \dots, N, \tag{3.1.22}$$

and the completeness relation (cf. (2.2.79))

$$\frac{1}{2\pi}\int d\mathrm{k}_\Re\, t(\mathrm{k})\Phi_1(x_1, \mathrm{k})\Psi_1(x_1', \mathrm{k}) - i\sum_{j=1}^{N} t_j\theta(\varkappa_j - |\mathrm{k}_\Im|)\Phi_1(x_1, i\varkappa_j)\Psi_1(x_1', i\varkappa_j) = \delta(x_1 - x_1'). \tag{3.1.23}$$

For the boundary values of the Jost solutions on the real axis, we have

$$t^{\pm}(k)\Phi_1^{\pm}(k) = \Phi_1^{\mp}(k) + r^{\pm}(k)\Phi_1^{\mp}(-k), \tag{3.1.24}$$

where we denoted

$$r^{\pm}(k) = \pm\frac{b(k)}{a(\pm k)}, \tag{3.1.25}$$

so that

$$\overline{r^{\pm}(k)} = r^{\pm}(-k). \tag{3.1.26}$$

It is known [22] that

$$t^{\pm}(0) = 0, \qquad r^{\pm}(0) = -1 \tag{3.1.27}$$

for a generic potential u_1.

The inverse problem is formulated as the problem of reconstructing the function of k equal to $t(\mathrm{k})\Phi_1(\mathrm{k})$ for $\mathrm{k}_\Im > 0$ and $\Phi_1(\mathrm{k})$ for $\mathrm{k}_\Im < 0$. This function is meromorphic in the upper half-plane with poles at $\mathrm{k} = i\varkappa_j$, $j = 1, \dots, N$, and analytic in the lower half-plane. Its discontinuity on the real axis is given by (3.1.24) with the upper sign, and its asymptotic behavior at infinity is fixed by equations (3.1.2) and (3.1.5). The formulation of the inverse problem is completed by equations (3.1.12).

3.2. Resolvent approach. Now, we use the objects introduced in Subsection 3.1 to describe the spectral properties of the operator

$$\mathcal{L}_1 = i\partial_{x_2} + \partial_{x_1}^2 - u_1(x_1), \qquad x = (x_1, x_2), \tag{3.2.1}$$

i.e., a special case of operator (1.1) when the perturbation $u_2(x)$ in (1.4) is identically zero. In other words, here we have a two-dimensional differential operator with a potential depending on

one variable only. For the Jost solutions (cf. (2.2.63)) of the nonstationary Schrödinger equation and its dual, such an embedding is trivial:

$$\varphi(x,\mathrm{k}) = e^{-i\ell(\mathrm{k})x}\chi_1(x_1,\mathrm{k}) \equiv e^{-i\mathrm{k}^2x_2}\Phi_1(x_1,\mathrm{k}), \tag{3.2.2}$$

$$\psi(x,\mathrm{k}) = e^{i\ell(\mathrm{k})x}\xi_1(x_1,\mathrm{k}) \equiv e^{i\mathrm{k}^2x_2}\Psi_1(x_1,\mathrm{k}), \tag{3.2.3}$$

and their properties follow directly from above. In particular, by virtue of (3.1.1), (3.2.2), and (3.2.3),

$$\overrightarrow{\mathcal{L}}_1\varphi(\mathrm{k}) = 0, \qquad \psi(\mathrm{k})\overleftarrow{\mathcal{L}}_1 = 0. \tag{3.2.4}$$

Due to (3.1.6), they obey a conjugation property like (2.2.60); i.e.,

$$\overline{\varphi(x,\mathrm{k})} = \psi(x,\overline{\mathrm{k}}), \tag{3.2.5}$$

so that for the boundary values on the real axis (defined by analogy with (2.2.52)), we have

$$\overline{\varphi^{\pm}(x,k)} = \psi^{\mp}(x,k), \qquad k\in\mathbb{R}. \tag{3.2.6}$$

We also have scalar products and a completeness relation that follow from (3.1.20)–(3.1.23):

$$\frac{1}{2\pi}\int dx_1\,\psi(x,\mathrm{k}+p)\varphi(x,\mathrm{k}) = \frac{\delta(p)}{t(\mathrm{k})}, \qquad p\in\mathbb{R}, \quad \mathrm{k}\in\mathbb{C}, \tag{3.2.7}$$

$$\int dx_1\,\psi(x,i\varkappa_j)\varphi(x,\mathrm{k}) = \int dx_1\,\psi(x,\mathrm{k})\varphi(x,i\varkappa_j) = 0, \qquad |\mathrm{k}_\Im| < \varkappa_j, \tag{3.2.8}$$

$$\int dx_1\psi(x,i\varkappa_j)\varphi(x,i\varkappa_{j'}) = \frac{i\delta_{jj'}}{t_j}, \tag{3.2.9}$$

$$\left[\frac{1}{2\pi}\int d\mathrm{k}_\Re\, t(\mathrm{k})\varphi(x,\mathrm{k})\psi(x',\mathrm{k}) - i\sum_{j=1}^{N} t_j\theta(\varkappa_j - |\mathrm{k}_\Im|)\varphi(x,i\varkappa_j)\psi(x',i\varkappa_j)\right]_{x_2'=x_2} = \delta(x_1 - x_1'); \tag{3.2.10}$$

in addition, in view of (3.1.6) and (3.1.12), we have

$$\varphi(x,i\varkappa_j) = b_j\varphi(x,-i\varkappa_j), \qquad \varphi(x,i\varkappa_j) = b_j\overline{\psi(x,i\varkappa_j)}. \tag{3.2.11}$$

It is necessary to mention that the solutions $\Phi_1(x_1,i\varkappa_j)$ are square-integrable functions of x_1, so the values $\mathrm{k}^2 = -\varkappa_j^2$ are eigenvalues of the Sturm–Liouville operator. On the contrary, the functions $\varphi(x,i\varkappa_j)$ are not square-integrable in both variables x_1 and x_2, so they belong to the continuous spectrum of operator (3.2.1).

On the other hand, the construction of the resolvent $M_1(q)$ of the extended differential operator

$$L_1(q) = L_0(q) - U_1, \qquad U_1(x,x';q) = u_1(x_1)\delta(x-x'), \tag{3.2.12}$$

where $L_0(q)$ is defined in (2.2.2), and of the Green's functions as well, is more difficult. The expression obtained for $M_1(q)$ in [18] for the case of a pure one-soliton potential can be generalized to the case of an arbitrary $u_1(x_1)$ by setting

$$\begin{aligned}\widehat{M}_1(x,x';q) = \frac{1}{2\pi i}\int\limits_{\mathrm{k}_\Im=q_1} d\mathrm{k}_\Re\,\left[\theta(x_2-x_2') - \theta(2\mathrm{k}_\Re\mathrm{k}_\Im - q_2)\right] t(\mathrm{k})\varphi(x,\mathrm{k})\psi(x',\mathrm{k})\\ - \sum_j \theta(\varkappa_j^2 - q_1^2)t_j\left[\theta(x_2-x_2') - \theta(-q_2)\right]\varphi(x,i\varkappa_j)\psi(x',i\varkappa_j),\end{aligned} \tag{3.2.13}$$

where the hat notation introduced in (2.1.8) is used. Indeed, using (3.2.2) and (3.2.10), one can verify that such $\widehat{M}_1$ satisfies

$$\overrightarrow{\mathcal{L}}_1\widehat{M}_1(q) = \widehat{M}_1(q)\overleftarrow{\mathcal{L}}_1 = I \tag{3.2.14}$$

and property (2.2.6). If we multiply (3.2.13) by $e^{q(x-x')}$, we get, according to (2.1.8), $M_1(x,x';q)$, and then both terms on the right-hand side are distributions belonging to S'. For the first term, this follows from the boundedness properties of χ_1 and ξ_1, and for the second term, from the fact that $e^{-q_1x_1}\varphi(x,i\varkappa_j)$ and $e^{q_1x'_1}\psi(x',i\varkappa_j)$ are bounded at the space infinity when $|q_1| < \varkappa_j$. The second term in (3.2.13) compensates the discontinuities of the first term at $q_1 = \pm\varkappa_j$ that are attributed to the pole singularities of $t(\mathrm{k})$. As in the decaying case, the resolvent has a discontinuity at $q_1 = 0$, but the second term introduces an additional discontinuity at $q_2 = 0$ for all $|q_1| < \max_j\{\varkappa_j\}$. Subtracting rational terms compensating the pole singularities due to $t(\mathrm{k})$ from the integrand of the first term, we get

$$\begin{aligned}\widehat{M}_{1,\mathrm{reg}}(x,x';q) = \frac{1}{2\pi i}\int\limits_{\mathrm{k}_{\Im}=q_1} d\mathrm{k}_{\Re}\,[\theta(x_2-x'_2)-\theta(2\mathrm{k}_{\Re}\mathrm{k}_{\Im}-q_2)]\\ \times\left\{t(\mathrm{k})\varphi(x,\mathrm{k})\psi(x',\mathrm{k}) - \sum_j \frac{2i\varkappa_j t_j}{\mathrm{k}^2+\varkappa_j^2}\varphi(x,i\varkappa_j)\psi(x',i\varkappa_j)\right\},\end{aligned} \tag{3.2.15}$$

which is a regular function of the variables q. Thus, we derive by direct calculations that $\widehat{M}_1(q)$ can be split into regular and singular parts as follows:

$$\widehat{M}_1(q) = \widehat{M}_{1,\mathrm{reg}}(q) + \sum_j \Gamma_j(q)\,\varphi(i\varkappa_j)\otimes\psi(i\varkappa_j), \tag{3.2.16}$$

where we introduced the x-independent functions

$$\Gamma_j(q) = \frac{t_j\,\mathrm{sgn}\,q_1}{2\pi i}\ln\frac{q_2+2iq_1(q_1-\varkappa_j)}{q_2+2iq_1(q_1+\varkappa_j)}, \qquad j=1,\dots,N, \tag{3.2.17}$$

where the logarithm has a cut along the negative part of the real axis of its argument. Thus, we see that the extended resolvent in the case of a one-dimensional potential acquires logarithmic singularities at the points $q = (\pm\varkappa_j, 0)$ on the q-plane with cuts along $q_2 = 0$ for $|q_1| \le \varkappa_j$, because, by (3.2.17),

$$\Gamma_j(q_1,+0) - \Gamma_j(q_1,-0) = -t_j\theta(\varkappa_j - |q_1|). \tag{3.2.18}$$

3.3. Properties of the resolvent and Green's function. For the discontinuity of the resolvent at $q_2 = 0$, from (3.2.13) or (3.2.18) we get

$$\widehat{M}_1(q_1,+0) - \widehat{M}_1(q_1,-0) = -\sum_j t_j\theta(\varkappa_j - |q_1|)\,\varphi(i\varkappa_j)\otimes\psi(i\varkappa_j). \tag{3.3.1}$$

For any other value of q, resolvent (3.2.13) has derivatives with respect to q of the form (2.2.20), (2.2.21). For instance,

$$\frac{\partial\widehat{M}_1(q)}{\partial q_2} = \frac{1}{2\pi i}\int\limits_{\mathrm{k}_{\Im}=q_1} d\mathrm{k}_{\Re}\,\delta\big(\ell_{2\Im}(\mathrm{k})-q_2\big)\,t(\mathrm{k})\varphi(\mathrm{k})\otimes\psi(\mathrm{k}), \tag{3.3.2}$$

which should be considered in the sense of distributions in the vicinity of the point $q_1 = 0$. Then, we can introduce the Green's function $\mathcal{G}_1(x, x', \mathrm{k})$ by using the same reduction of the resolvent as that used in (2.2.31). From (3.2.13), we get the bilinear representation

$$\mathcal{G}_1(x, x', \mathrm{k}) = \frac{1}{2\pi i} \int d\alpha \left[\theta(x_2 - x_2') - \theta(\mathrm{k}_\Im(\alpha - \mathrm{k}_\Re))\right] t(\alpha + i\mathrm{k}_\Im)\varphi(x, \alpha + i\mathrm{k}_\Im)\psi(x', \alpha + i\mathrm{k}_\Im)$$
$$- \sum_j t_j \theta(\varkappa_j - |\mathrm{k}_\Im|)\left[\theta(x_2 - x_2') - \theta(-\mathrm{k}_\Re \mathrm{k}_\Im)\right]\varphi(x, i\varkappa_j)\psi(x', i\varkappa_j), \tag{3.3.3}$$

which generalizes the Green's function used in [18] for the pure one-soliton potential to the case of a generic one-dimensional potential u_1. It is easy to check that this Green's function satisfies conjugation property (2.2.42) and that the function $G_1(x, x', \mathrm{k})$ defined as in (2.2.45) satisfies (2.2.46). Taking into account that the resolvent satisfies (2.2.23) with $U = U_1$, we find that the Green's function satisfies integral equations of the type (2.2.43); i.e.,

$$\mathcal{G}_1(\mathrm{k}) = \mathcal{G}_0(\mathrm{k}) + \mathcal{G}_0(\mathrm{k})U_1\mathcal{G}_1(\mathrm{k}), \qquad \mathcal{G}_1(\mathrm{k}) = \mathcal{G}_0(\mathrm{k}) + \mathcal{G}_1(\mathrm{k})U_1\mathcal{G}_0(\mathrm{k}); \tag{3.3.4}$$

it is analytic when $\mathrm{k}_\Re \mathrm{k}_\Im \neq 0$, and in this region, by analogy with (2.2.47), we have

$$\frac{\partial \mathcal{G}_1(\mathrm{k})}{\partial \mathrm{k}_\Im} = \frac{\operatorname{sgn} \mathrm{k}_\Im}{2\pi} t(\mathrm{k})\varphi(\mathrm{k}) \otimes \psi(\mathrm{k}). \tag{3.3.5}$$

The discontinuity across the imaginary axis is given by

$$\mathcal{G}_1(+0 + i\mathrm{k}_\Im) - \mathcal{G}_1(-0 + i\mathrm{k}_\Im) = -\operatorname{sgn} \mathrm{k}_\Im \sum_j t_j \theta(\varkappa_j - |\mathrm{k}_\Im|)\varphi(i\varkappa_j) \otimes \psi(i\varkappa_j). \tag{3.3.6}$$

Decomposition (3.2.16) gives the following decomposition of $\mathcal{G}_1(\mathrm{k})$ into regular and singular parts:

$$\mathcal{G}_1(\mathrm{k}) = \mathcal{G}_{1,\mathrm{reg}}(\mathrm{k}) + \sum_j \gamma_j(\mathrm{k})\varphi(i\varkappa_j) \otimes \psi(i\varkappa_j), \tag{3.3.7}$$

where

$$\gamma_j(\mathrm{k}) = \Gamma_j(\ell_\Im(\mathrm{k})) \equiv \frac{t_j \operatorname{sgn} \mathrm{k}_\Im}{2\pi i} \ln \frac{\mathrm{k} - i\varkappa_j}{\mathrm{k} + i\varkappa_j}. \tag{3.3.8}$$

This proves that in addition to the standard discontinuity on the real axis, the Green's function has a discontinuity on the imaginary axis when $|\mathrm{k}_\Im| < \max_j \varkappa_j$. Inside the quadrants $\mathrm{k}_\Re \mathrm{k}_\Im \neq 0$, the Green's function is continuous up to the boundaries. Note that since $\overrightarrow{\mathcal{L}}_1\varphi(i\varkappa_j) = 0$, the regular part $\mathcal{G}_{1,\mathrm{reg}}(\mathrm{k})$ is also a Green's function.

Equation (3.3.7) gives a decomposition of the function $G_1(x, x', \mathrm{k})$ defined as in (2.2.45) into a sum of one regular and N singular parts. Taking (2.2.13) into account, one can check that the jth singular part of this decomposition grows exponentially with x when $|\mathrm{k}_\Im| > \varkappa_j$ and $\mathrm{k}_\Re \neq 0$. Since $G_1(x, x', \mathrm{k})$ is bounded, the regular part also grows with x when $|\mathrm{k}_\Im| > \max_j \varkappa_j$ and $\mathrm{k}_\Re \neq 0$. This imposes essential limitations on the applicability of (3.3.7). In Subsection 4.3.2, we show how to overcome this difficulty by introducing N decompositions (4.3.12) that are suitable to describe the singular behavior of $G_1(x, x', \mathrm{k})$ separately in the vicinity of each point $\mathrm{k} = \pm i\varkappa_j$.

The Jost solutions can be given as reductions of the Green's function by analogy with equations (2.2.32) and (2.2.33), but in this case a regularization is necessary. Namely, we have

$$\varphi(x, \mathrm{k}) = \lim_{\varepsilon \to +0} \int dx' \,(\mathcal{G}_1(\mathrm{k})\overleftarrow{\mathcal{L}}_0)(x, x')e^{-i\ell(\mathrm{k})x' - i\varepsilon x_2'}, \tag{3.3.9}$$

$$\psi(x, \mathrm{k}) = \lim_{\varepsilon \to +0} \int dx' \, e^{i\ell(\mathrm{k})x' + i\varepsilon x_2'}(\overrightarrow{\mathcal{L}}_0\mathcal{G}_1(\mathrm{k}))(x', x), \tag{3.3.10}$$

where $\mathrm{k}_\Im \neq 0$. Introducing the bilinear representation (3.3.3) into (3.3.9) and (3.3.10), after direct but rather lengthy computations, we get an identity, thus checking the correctness of the chosen regularization.

The boundary values of the resolvent at discontinuity (3.3.1) with respect to q_2 when $q_1 \neq 0$ are expressed in terms of the Green's function, defined by (2.2.31), as

$$\lim_{q_2 \to \pm 0\mathrm{k}_\Im} \left(\widehat{M}_1(q)\big|_{q_1=\mathrm{k}_\Im} \right) = \mathcal{G}_1(\pm 0 + i\mathrm{k}_\Im). \tag{3.3.11}$$

Due to (3.3.3), these boundary values of the Green's function equal

$$\begin{aligned} \mathcal{G}_1(x, x', \pm 0 + i\mathrm{k}_\Im) = \frac{1}{2\pi i} \int d\alpha \left[\theta(x_2 - x_2') - \theta(\mathrm{k}_\Im \alpha)\right] t(\alpha + i\mathrm{k}_\Im) \varphi(x, \alpha + i\mathrm{k}_\Im) \psi(x', \alpha + i\mathrm{k}_\Im) \\ - \sum_j t_j \theta(\varkappa_j - |\mathrm{k}_\Im|) \left[\theta(x_2 - x_2') - \theta(\mp \mathrm{k}_\Im)\right] \varphi(x, i\varkappa_j) \psi(x', i\varkappa_j). \end{aligned} \tag{3.3.12}$$

Accordingly, decomposition (3.3.7) gives

$$\mathcal{G}_1(\pm 0 + i\mathrm{k}_\Im) = \mathcal{G}_{1,\mathrm{reg}}(i\mathrm{k}_\Im) + \sum_j \gamma_j(\pm 0 + i\mathrm{k}_\Im)\, \varphi(i\varkappa_j) \otimes \psi(i\varkappa_j), \tag{3.3.13}$$

where, by (3.3.8),

$$\gamma_j(\pm 0 + i\mathrm{k}_\Im) = \frac{t_j}{2\pi i} \ln \frac{|\mathrm{k}_\Im| - \varkappa_j \mp i0\mathrm{k}_\Im}{|\mathrm{k}_\Im| + \varkappa_j}. \tag{3.3.14}$$

3.4. Advanced/retarded Green's functions and solutions. For $q_1 = 0$, the kernel $M_1(q)$ is discontinuous at $q_2 = 0$. This means that we have to introduce advanced/retarded Green's functions as in (2.2.35). By (3.2.13), we get the following bilinear representation for these Green's functions in terms of the Jost solutions on the real axis:

$$\mathcal{G}_{1,\pm}(x, x') = \pm\theta(\pm(x_2 - x_2')) \left(\frac{1}{2\pi i} \int d\alpha\, t^\sigma(\alpha) \varphi^\sigma(x, \alpha) \psi^\sigma(x', \alpha) - \sum_j t_j \varphi(x, i\varkappa_j) \psi(x', i\varkappa_j) \right), \tag{3.4.1}$$

where $\sigma = +, -$ and where we used notation similar to (2.2.52) for the boundary values of the Jost solutions and $t(\mathrm{k})$ on the real k-axis. It is clear that the advanced/retarded Green's functions are independent of the choice of σ. Below, we will show this explicitly. The singular part of these Green's functions is equal to zero, as follows from (3.2.17). These Green's functions obey the conjugation property (2.2.50).

The advanced/retarded solutions should also be defined by relations analogous to (2.2.40) in the regular case; i.e.,

$$\varphi_\pm(x, k) = \int dx'\, (\mathcal{G}_{1,\pm} \overleftarrow{\mathcal{L}}_0)(x, x') e^{-i\ell(k)x'}, \tag{3.4.2}$$

$$\psi_\pm(x, k) = \int dx'\, e^{i\ell(k)x'} (\overrightarrow{\mathcal{L}}_0 \mathcal{G}_{1,\pm})(x', x). \tag{3.4.3}$$

Substituting (3.4.2) and (3.4.3) into the bilinear representation for $\mathcal{G}_{1,\pm}$, we obtain relations between Jost and advanced/retarded solutions. After rather lengthy computations, by analogy with (2.2.88), we obtain

$$\varphi_\pm(k) = \int dp\, \varphi^\sigma(p)\, \overline{r_\pm^{-\sigma}(p, k)}, \tag{3.4.4}$$

where

$$r^{\sigma}_{\pm}(p,k) = \delta(k-p)\big[\theta(\pm\sigma k) + \theta(\mp\sigma k)t^{\sigma}(k)\big] + \theta(\mp\sigma k)\delta(k+p)r^{\sigma}(k) \tag{3.4.5}$$

and $r^{\sigma}(k)$ is defined in (3.1.25).

Inverting (3.4.4) and using (3.4.5), we get

$$t^{\sigma}(k)\varphi^{\sigma}(k) = \int dp\,\varphi_{\pm}(p)r^{\sigma}_{\pm}(p,k), \tag{3.4.6}$$

which modifies (2.2.80). By (3.4.2) and (3.4.3), we also find that

$$\overline{\varphi_{\pm}(k)} = \psi_{\mp}(k), \tag{3.4.7}$$

which allows us to write the corresponding relations for dual solutions

$$t^{\sigma}(k)\psi^{\sigma}(k) = \int dp\,\overline{r^{-\sigma}_{\mp}(p,k)}\psi_{\pm}(p), \tag{3.4.8}$$

$$\psi_{\pm}(k) = \int dp\,r^{\sigma}_{\mp}(k,p)\psi^{\sigma}(p). \tag{3.4.9}$$

In particular, we get

$$\begin{aligned} t^{\sigma}(k)\varphi^{\sigma}(k)\otimes\psi^{\sigma}(k) &= \varphi_{\mp}(k)\otimes\psi_{\pm}(k) \\ &+ \sigma\theta(\pm\sigma k)b(k)\varphi_{\mp}(-k)\otimes\psi_{\pm}(k) - \sigma\theta(\mp\sigma k)b(-k)\varphi_{\mp}(k)\otimes\psi_{\pm}(-k), \end{aligned} \tag{3.4.10}$$

so that the second line cancels out when this expression is substituted into (3.4.1); this gives

$$\mathcal{G}_{1,\pm}(x,x') = \pm\theta(\pm(x_2-x'_2))\left(\frac{1}{2\pi i}\int d\alpha\,\varphi_{\mp}(x,\alpha)\psi_{\pm}(x',\alpha) - \sum_j t_j\varphi(x,i\varkappa_j)\psi(x',i\varkappa_j)\right), \tag{3.4.11}$$

which proves that (3.4.1) is independent of the sign $\sigma = +,-$ on the right-hand side. Equation (3.4.11) also demonstrates that, in spite of the regularity of the advanced/retarded Green's functions, the set of the advanced/retarded solutions is not complete because we have an additional sum on the right-hand side of (3.4.11). Taking into account (3.1.27) and the fact that $\varphi(k)$ is also a smooth function of k at $k=0$, we get

$$\varphi_{\pm}(0) = 0. \tag{3.4.12}$$

?

Formulas (3.4.4) and (3.4.6) imply characterization equations that modify (2.2.90)–(2.2.92) for the one-dimensional case:

$$\int dp\,\overline{r^{-\sigma}_{\pm}(p,k)}r^{\sigma}_{\pm}(p,k') = \delta(k-k')t^{\sigma}(k), \tag{3.4.13}$$

$$\int \frac{dp}{t^{\sigma}(p)}\,\overline{r^{-\sigma}_{\pm}(k',p)}r^{\sigma}_{\pm}(k,p) = \delta(k-k'), \tag{3.4.14}$$

$$\int dp\,\overline{r^{\sigma}_{+}(p,k)}r^{\sigma}_{+}(p,k') = \int dp\,\overline{r^{\sigma}_{-}(p,k)}r^{\sigma}_{-}(p,k'), \tag{3.4.15}$$

where $\sigma = +,-$. Introducing an operator T^{σ} with the kernel

$$T^{\sigma}(k,k') = \delta(k-k')t^{\sigma}(k), \qquad k,k'\in\mathbb{R}, \tag{3.4.16}$$

we can rewrite the above equalities in the operator form:

$$r_{\pm}^{-\sigma\dagger}r_{\pm}^{\sigma} = T^{\sigma}, \tag{3.4.17}$$

$$r_{\pm}^{\sigma}(T^{\sigma})^{-1}r_{\pm}^{-\sigma\dagger} = I, \tag{3.4.18}$$

$$r_{+}^{\sigma\dagger}r_{+}^{\sigma} = r_{-}^{\sigma\dagger}r_{-}^{\sigma}. \tag{3.4.19}$$

Defining, as in (2.2.96),

$$f^{\sigma} = r_{\pm}^{-\sigma\dagger}r_{\pm}^{-\sigma}, \tag{3.4.20}$$

which is independent of the sign $\pm$ due to (3.4.15), we have (cf. (2.2.97))

$$\varphi^{\sigma}T^{\sigma} = \varphi^{-\sigma}f^{-\sigma}, \qquad T^{\sigma}\psi^{\sigma} = f^{\sigma}\psi^{-\sigma}, \tag{3.4.21}$$

and f^{σ} satisfies the properties (cf. (2.2.98))

$$f^{\sigma\dagger} = f^{\sigma}, \qquad f^{-\sigma} = T^{-\sigma}(f^{\sigma})^{-1}T^{\sigma}. \tag{3.4.22}$$

Explicitly, taking into account (3.4.5), (3.1.15), (3.1.10), and (3.1.25), we have

$$f^{\sigma}(k',k) = \delta(k-k') + \delta(k+k')r^{-\sigma}(k). \tag{3.4.23}$$

Concluding this section, which is devoted to the reformulation of the spectral theory of the Sturm–Liouville equation in terms of the spectral theory of the two-dimensional differential operator (3.2.1), let us mention that, as in the case of decaying potentials, the bilinear representations (3.3.3) and (3.4.1) enable us to derive relations between the Jost and advanced/retarded solutions. Thus, using notations (2.2.53) for the boundary values of the Green's function (3.3.3), we get

$$\begin{aligned}\mathcal{G}_1^{\sigma}(x,x',k) = \frac{1}{2\pi i}\int d\alpha\,\bigl[\theta(x_2-x_2') - \theta(\sigma(\alpha-k))\bigr]\,t^{\sigma}(\alpha)\varphi^{\sigma}(x,\alpha)\psi^{\sigma}(x',\alpha)\\ - \sum_j t_j\bigl[\theta(x_2-x_2') - \theta(-\sigma k)\bigr]\varphi(x,i\varkappa_j)\psi(x',i\varkappa_j), \qquad k\in\mathbb{R}, \quad \sigma=+,-.\end{aligned} \tag{3.4.24}$$

These functions are finite for all k but discontinuous at $k=0$. By (3.4.1) and (3.4.24),

$$\mathcal{G}_1^{\sigma}(k) - \mathcal{G}_{1,\pm} = \mp\frac{1}{2\pi i}\int d\alpha\,\theta(\pm\sigma(\alpha-k))t^{\sigma}(\alpha)\varphi^{\sigma}(\alpha)\otimes\psi^{\sigma}(\alpha) \pm \theta(\mp\sigma k)\sum_j t_j\varphi(i\varkappa_j)\otimes\psi(i\varkappa_j). \tag{3.4.25}$$

For completeness, let us mention that

$$\mathcal{G}_{1,+} - \mathcal{G}_{1,-} = \frac{1}{2\pi i}\int d\alpha\,t^{\sigma}(\alpha)\varphi^{\sigma}(\alpha)\otimes\psi^{\sigma}(\alpha) - \sum_j t_j\varphi(i\varkappa_j)\otimes\psi(i\varkappa_j), \tag{3.4.26}$$

where, in view of (3.4.10), the first term can be rewritten in terms of advanced/retarded solutions (cf. (3.4.11)):

$$\mathcal{G}_{1,+} - \mathcal{G}_{1,-} = \frac{1}{2\pi i}\int d\alpha\,\varphi_{\pm}(\alpha)\otimes\psi_{\mp}(\alpha) - \sum_j t_j\varphi(i\varkappa_j)\otimes\psi(i\varkappa_j). \tag{3.4.27}$$

4. INVERSE SCATTERING TRANSFORM OF A TWO-DIMENSIONAL PERTURBATION OF A ONE-DIMENSIONAL POTENTIAL

4.1. Resolvent. Now, we start the investigation of the operator $\mathcal{L}$ (1.1) with a potential given by (1.4). For this purpose, we introduce an extension of this operator, $L(q)$, as in (2.2.1), and the inverse of this extension, i.e., its resolvent $M(q)$. The potential u in (1.4) can be considered as a two-dimensional perturbation of the one-dimensional potential u_1. Therefore, it is convenient to consider the resolvent $M(q)$ as a perturbation of the resolvent $M_1(q)$ and to apply, instead of the integral equations for $M(q)$ in (2.2.23), the integral equations

$$M(q) = M_1(q) + M_1(q)U_2M(q), \qquad M(q) = M_1(q) + M(q)U_2M_1(q), \tag{4.1.1}$$

where $U_2(x,x';q) = u_2(x)\delta(x-x')$. We choose the potential $u_2(x)$ to be an arbitrary function of two variables that is real, smooth, and rapidly decays on the plane. Moreover, we assume it to be sufficiently "small" to guarantee that solutions of both equations in (4.1.1) exist in $\mathcal{S}'(\mathbb{R}^6)$, are unique, and coincide. $M(x,x';q)$ inherits the properties of $M_1(x,x';q)$ and is a distribution with respect to the difference variable $x-x'$, a smooth function with respect to the variable $x+x'$, and a continuous function of q for $q \neq 0$ and $q_2 \neq 0$. As we saw in Section 2, an efficient tool for studying the properties of the resolvent is given by the Hilbert identity (2.2.9), which can also be represented in the form

$$M'(q') - M(q) = M'(q')L_1'(q')\big(M_1'(q') - M_1(q)\big)L_1(q)M(q), \tag{4.1.2}$$

which allows us to exploit the known properties of $M_1(q)$. By analogy with (2.2.26), we obtain the following equations for the derivatives of the hat kernel $\widehat{M}(q)$ of the resolvent:

$$\frac{\partial \widehat{M}(q)}{\partial q_j} = \widehat{M}(q)\overleftarrow{\mathcal{L}}_1\frac{\partial \widehat{M}_1(q)}{\partial q_j}\overrightarrow{\mathcal{L}}_1\widehat{M}(q), \qquad j=1,2, \quad q_2\neq 0.$$

Then, using (3.3.2), we get, for $q_2 \neq 0$, equalities of the form (2.2.27) and (2.2.28) for the derivatives of $\widehat{M}(q)$, i.e.,

$$\frac{\partial \widehat{M}(q)}{\partial q_1} = \frac{i}{\pi}\int\limits_{\mathbf{k}_{\Im}=q_1} d\mathbf{k}_{\Re}\,\overline{\mathbf{k}}\,\delta(\ell_{2\Im}(\mathbf{k})-q_2)\,t(\mathbf{k})\Phi(\mathbf{k})\otimes\Psi(\mathbf{k}), \tag{4.1.3}$$

$$\frac{\partial \widehat{M}(q)}{\partial q_2} = \frac{1}{2\pi i}\int\limits_{\mathbf{k}_{\Im}=q_1} d\mathbf{k}_{\Re}\,\delta(\ell_{2\Im}(\mathbf{k})-q_2)\,t(\mathbf{k})\Phi(\mathbf{k})\otimes\Psi(\mathbf{k}), \tag{4.1.4}$$

where now the Jost solutions are defined as (cf. (2.2.29) and (2.2.30))

$$\Phi(\mathbf{k}) = \mathcal{G}(\mathbf{k})\overleftarrow{\mathcal{L}}_1\varphi(\mathbf{k}), \tag{4.1.5}$$

$$\Psi(\mathbf{k}) = \psi(\mathbf{k})\overrightarrow{\mathcal{L}}_1\mathcal{G}(\mathbf{k}) \tag{4.1.6}$$

and the Green's function $\mathcal{G}(x,x',\mathbf{k})$ is defined by the same reduction of the resolvent as (2.2.31) in the decaying case.

In order to study the discontinuity of the resolvent at $q_2 = 0$, following the results of Section 3, we consider separately the cases $q_1 = 0$ and $q_1 \neq 0$. The boundary values of the resolvent in the first case, $\lim_{q_2\to\pm 0}\widehat{M}(q)|_{q_1=0}$, define the advanced/retarded Green's functions as in (2.2.35). On the other hand, in the case $q_1 \neq 0$, in view of (2.2.31), we have, by analogy with (3.3.11), the following equation for the boundary values of the resolvent:

$$\lim_{q_2\to\pm 0\mathbf{k}_{\Im}}\Big(\widehat{M}(q)\big|_{q_1=\mathbf{k}_{\Im}}\Big) = \mathcal{G}(\pm 0 + i\mathbf{k}_{\Im}). \tag{4.1.7}$$

Now, we investigate the properties of these Green's functions in detail.

4.2. Green's function $\mathcal{G}(\mathbf{k})$. By (4.1.1) and definition (2.2.31), the Green's function $\mathcal{G}(\mathbf{k})$ satisfies the integral equations

$$\mathcal{G}(\mathbf{k}) = \mathcal{G}_1(\mathbf{k}) + \mathcal{G}_1(\mathbf{k})U_2\mathcal{G}(\mathbf{k}), \qquad \mathcal{G}(\mathbf{k}) = \mathcal{G}_1(\mathbf{k}) + \mathcal{G}(\mathbf{k})U_2\mathcal{G}_1(\mathbf{k}). \tag{4.2.1}$$

Taking into account that $\mathcal{L} = \mathcal{L}_1 - U_2$, one can easily verify that $\mathcal{G}(\mathbf{k})$ satisfies the differential equations (2.2.41) and conjugation property (2.2.42). It also satisfies the integral equations (2.2.43), as follows from the substitution of the right-hand side of (3.3.4) for $\mathcal{G}_1(\mathbf{k})$ into (4.2.1). The analyticity properties of the Green's function $\mathcal{G}(\mathbf{k})$ are inherited from $\mathcal{G}_1(\mathbf{k})$. This means that this function is analytic in the region $\mathbf{k}_\Re \mathbf{k}_\Im \neq 0$; i.e., it satisfies (2.2.47) and (2.2.48) in this region and, in addition to the standard cut at $\mathbf{k}_\Im = 0$, has a cut at $\mathbf{k}_\Re = 0$ for $|\mathbf{k}_\Im| < \max_j \varkappa_j$. Like $\mathcal{G}_1(\mathbf{k})$, inside the quadrants $\mathbf{k}_\Re \mathbf{k}_\Im \neq 0$, the Green's function $\mathcal{G}(\mathbf{k})$ is continuous up to the boundaries. Using notation (2.2.52) for the boundary values of the Green's function on the real axis, we get

$$\lim_{\sigma \mathbf{k}_\Im \to +0} \mathcal{G}(\pm 0 + i\mathbf{k}_\Im) = \lim_{\pm k \to +0} \mathcal{G}^\sigma(k) \equiv \mathcal{G}^\sigma(\pm 0), \qquad \sigma = +, -. \tag{4.2.2}$$

As usual, in order to study the resolvent along the discontinuities, we use the Hilbert identity. Thus, by (4.1.2), we obtain the following expression for the discontinuity along $q_2 = 0$:

$$\lim_{q_2 \to +0} \widehat{M}(q) - \lim_{q_2 \to -0} \widehat{M}(q) = \left(\lim_{q_2 \to \pm 0} \widehat{M}(q)\overleftarrow{\mathcal{L}}_1\right)\left(\lim_{q_2 \to +0} \widehat{M}_1(q) - \lim_{q_2 \to -0} \widehat{M}_1(q)\right)\left(\overrightarrow{\mathcal{L}}_1 \lim_{q_2 \to \mp 0} \widehat{M}(q)\right).$$

In view of (4.1.7), this equation implies that the discontinuity of the Green's function across the imaginary axis is given by

$$\begin{aligned}\mathcal{G}(+0 + i\mathbf{k}_\Im) - \mathcal{G}(-0 + i\mathbf{k}_\Im) = -\operatorname{sgn} \mathbf{k}_\Im \sum_j t_j \theta(\varkappa_j - |\mathbf{k}_\Im|) \\ \times \left(\mathcal{G}(\pm 0 + i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1 \varphi(i\varkappa_j)\right) \otimes \left(\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1 \mathcal{G}(\mp 0 + i\mathbf{k}_\Im)\right),\end{aligned} \tag{4.2.3}$$

where the expressions in parentheses are understood in the sense of the "vector" and "covector" notations, as in (2.2.29) and (2.2.30), and the direct product is defined in (2.2.22). Applying the operation $\overleftarrow{\mathcal{L}}_1 \varphi(i\varkappa_m)\theta(\varkappa_m - |\mathbf{k}_\Im|)$ to (4.2.3) from the right, we get the following expression for the lower sign:

$$\theta(\varkappa_m - |\mathbf{k}_\Im|)\left(\mathcal{G}(+0 + i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1 \varphi(i\varkappa_m)\right) = \operatorname{sgn} \mathbf{k}_\Im \sum_j t_j \theta(\varkappa_j - |\mathbf{k}_\Im|)\left(\mathcal{G}(-0 + i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1 \varphi(i\varkappa_j)\right) A_{jm}(\mathbf{k}_\Im), \tag{4.2.4}$$

where we introduced the matrix

$$A_{lm}(\mathbf{k}_\Im) = \frac{\delta_{lm}}{t_l \operatorname{sgn} \mathbf{k}_\Im} - \theta(\min\{\varkappa_l, \varkappa_m\} - |\mathbf{k}_\Im|)\left(\psi(i\varkappa_l)\overrightarrow{\mathcal{L}}_1 \mathcal{G}(+0 + i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1 \varphi(i\varkappa_m)\right). \tag{4.2.5}$$

An explicit expression for the "expectation value" in parentheses is given by analogy with (2.2.84) and (2.2.85). Now, applying the operation $\overleftarrow{\mathcal{L}}_1 \varphi(i\varkappa_l)\theta(\varkappa_l - |\mathbf{k}_\Im|)$ to (4.2.3) from the right, for the upper sign we get

$$\theta(\varkappa_l - |\mathbf{k}_\Im|)\left(\mathcal{G}(-0 + i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1 \varphi(i\varkappa_l)\right) = \operatorname{sgn} \mathbf{k}_\Im \sum_m \frac{1}{t_l} \theta(\varkappa_m - |\mathbf{k}_\Im|)\left(\mathcal{G}(+0 + i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1 \varphi(i\varkappa_m)\right) B_{ml}(\mathbf{k}_\Im), \tag{4.2.6}$$

where

$$B_{ml}(\mathbf{k}_\Im) = \delta_{lm} t_l \operatorname{sgn} \mathbf{k}_\Im + t_l t_m \theta(\min\{\varkappa_l, \varkappa_m\} - |\mathbf{k}_\Im|)\left(\psi(i\varkappa_m)\overrightarrow{\mathcal{L}}_1 \mathcal{G}(-0 + i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1 \varphi(i\varkappa_l)\right). \tag{4.2.7}$$

A comparison of (4.2.4) with (4.2.6) implies that the matrix $A(\mathrm{k}_\Im)$ is invertible and that its inverse matrix is given by

$$A(\mathrm{k}_\Im)^{-1} = B(\mathrm{k}_\Im), \tag{4.2.8}$$

so that, under the assumption of the unique solvability of the integral equation for the Green's function, we get

$$\det A(\mathrm{k}_\Im) \neq 0. \tag{4.2.9}$$

By (3.1.17) and (3.2.11), this matrix satisfies the conjugation property

$$A(\mathrm{k}_\Im)^\dagger = BA(-\mathrm{k}_\Im)B^{-1}, \tag{4.2.10}$$

where we introduced the diagonal (Hermitian according to (3.1.17)) matrix

$$B = \operatorname{diag}\{b_1, \dots, b_N\}. \tag{4.2.11}$$

The limiting values A^σ of the matrix $A(\mathrm{k}_\Im)$ at $\mathrm{k}_\Im = \sigma 0$ ($\sigma = \pm$) are obtained by (4.2.2) and (4.2.5):

$$A^\sigma_{lm} = \frac{\sigma\delta_{lm}}{t_l} - \big(\psi(i\varkappa_l)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(+0)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_m)\big). \tag{4.2.12}$$

By (4.2.9), both these constant matrices are invertible and, in view of (4.2.10), satisfy the conjugation property

$$(A^{-\sigma})^\dagger = BA^\sigma B^{-1}. \tag{4.2.13}$$

Equation (4.2.3) suggests introducing functions $\Phi_j(x,\mathrm{k}_\Im)$ and $\Psi_j(x,\mathrm{k}_\Im)$ by means of the relations

$$\begin{aligned}\Phi_j(\mathrm{k}_\Im) &= \theta(\varkappa_j - |\mathrm{k}_\Im|)\mathcal{G}(+0+i\mathrm{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j),\\ \Psi_j(\mathrm{k}_\Im) &= \theta(\varkappa_j - |\mathrm{k}_\Im|)\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\mathcal{G}(+0+i\mathrm{k}_\Im).\end{aligned} \tag{4.2.14}$$

They can be considered as a generalization, depending on $\mathrm{k}_\Im$, of the functions $\varphi(i\varkappa_l)$ and $\psi(i\varkappa_l)$ to the case $u_2 \neq 0$. Notice that the θ-functions appearing in (4.2.14) are necessary to make $\Phi_j(\mathrm{k}_\Im)$ and $\Psi_j(\mathrm{k}_\Im)$ well-defined since, due to the exponential decreasing of $\varphi(i\varkappa_j)$ and $\psi(i\varkappa_j)$, the integrals on the right-hand side are convergent in the interval $|\mathrm{k}_\Im| \leq \varkappa_j$ but divergent outside this interval. In fact, the terms in the sum on the right-hand side of (4.2.3) are present only in these intervals.

Since $\mathcal{G}(+0+i\mathrm{k}_\Im)$ is a Green's function, these functions satisfy the equations

$$\overrightarrow{\mathcal{L}}\Phi_l(\mathrm{k}_\Im) = 0, \qquad \Psi_l(\mathrm{k}_\Im)\overleftarrow{\mathcal{L}} = 0; \tag{4.2.15}$$

in what follows, we call them auxiliary Jost solutions of the nonstationary Schrödinger equation. Due to the properties of the Green's function, these solutions are discontinuous at $\mathrm{k}_\Im = 0$, and due to (3.2.11), they satisfy the conjugation properties

$$\overline{\Phi_j(x,\mathrm{k}_\Im)} = b_j\Psi_j(x,-\mathrm{k}_\Im), \qquad \overline{\Psi_j(x,\mathrm{k}_\Im)} = \frac{\Phi_j(x,-\mathrm{k}_\Im)}{b_j}. \tag{4.2.16}$$

Using (4.2.1), we get the following integral equations for these auxiliary Jost solutions:

$$\begin{aligned}\Phi_l(\mathrm{k}_\Im) &= \varphi(i\varkappa_l) + \mathcal{G}_1(+0+i\mathrm{k}_\Im)U_2\Phi_l(\mathrm{k}_\Im),\\ \Psi_l(\mathrm{k}_\Im) &= \psi(i\varkappa_l) + \Psi_l(\mathrm{k}_\Im)U_2\mathcal{G}_1(+0+i\mathrm{k}_\Im).\end{aligned} \tag{4.2.17}$$

From (4.2.4), we now derive

$$\sum_m \theta(\varkappa_m - |\mathbf{k}_\Im|)\Phi_m(\mathbf{k}_\Im)\bigl(A(\mathbf{k}_\Im)^{-1}\bigr)_{mj} = (\operatorname{sgn}\mathbf{k}_\Im)t_j\theta(\varkappa_j - |\mathbf{k}_\Im|)\bigl(\mathcal{G}(-0+i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\bigr),$$

which, in view of the structure of (4.2.7), can be rewritten as

$$\theta(\varkappa_j - |\mathbf{k}_\Im|)\bigl(\mathcal{G}(-0+i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\bigr) = \frac{\operatorname{sgn}\mathbf{k}_\Im}{t_j}\theta(\varkappa_j - |\mathbf{k}_\Im|)\sum_m \Phi_m(\mathbf{k}_\Im)\bigl(A(\mathbf{k}_\Im)^{-1}\bigr)_{mj}. \quad (4.2.18)$$

This allows us to express the discontinuity of the Green's function in (4.2.3) (the lower sign) as follows:

$$\mathcal{G}(+0+i\mathbf{k}_\Im) - \mathcal{G}(-0+i\mathbf{k}_\Im) = -\sum_{l,m}\theta(\varkappa_l - |\mathbf{k}_\Im|)\Phi_l(\mathbf{k}_\Im)\bigl(A(\mathbf{k}_\Im)^{-1}\bigr)_{lm}\otimes\Psi_m(\mathbf{k}_\Im), \quad (4.2.19)$$

which demonstrates that this discontinuity is expressed in terms of the auxiliary Jost solutions $\Phi_j(x,\mathbf{k}_\Im)$ and $\Psi_j(x,\mathbf{k}_\Im)$.

4.3. Jost solutions.

4.3.1. *Discontinuity of the Jost solutions on the imaginary axis.* The functions $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$ introduced in (4.1.5) and (4.1.6) are Jost solutions of the nonstationary Schrödinger equation and its dual with potential (1.4). Differential equations (2.2.59) for the Jost solutions are obtained if we apply $\mathcal{L}$ to (4.1.5) and (4.1.6) and use (2.2.41) and (3.2.4). Also, recalling definitions (4.1.5) and (4.1.6), we derive the following integral equations from (4.2.1):

$$\Phi(\mathbf{k}) = \varphi(\mathbf{k}) + \mathcal{G}_1(\mathbf{k})U_2\Phi(\mathbf{k}), \qquad \Psi(\mathbf{k}) = \psi(\mathbf{k}) + \Psi(\mathbf{k})U_2\mathcal{G}_1(\mathbf{k}), \quad (4.3.1)$$

which generalize (2.2.61) and (2.2.62) and in which the Green's function $\mathcal{G}_1(\mathbf{k})$ of the operator $\mathcal{L}_1$ is given in (3.3.3). By (2.2.42), the Jost solutions satisfy the conjugation property (2.2.60). Introducing functions $\chi(x,\mathbf{k})$ and $\xi(x,\mathbf{k})$ by means of (2.2.63), one can prove that they are bounded at the k-infinity.

In view of the properties of the Green's function $\mathcal{G}(\mathbf{k})$, the Jost solutions are analytic functions of $\mathbf{k}\in\mathbb{C}$ when $\mathbf{k}_\Re\mathbf{k}_\Im \neq 0$ and, in general, they have the standard discontinuity on the real axis, $\mathbf{k}_\Im = 0$, and an additional discontinuity along the segment of the imaginary axis $\mathbf{k}_\Re = 0$, $|\mathbf{k}_\Im| \le \max_j \varkappa_j$. Let us consider $|\mathbf{k}_\Im| \neq \varkappa_j$ for all j. Then $\varphi(\mathbf{k})$ and $\psi(\mathbf{k})$ are continuous functions at $\mathbf{k}_\Re = 0$; therefore, passing to the limit as $\mathbf{k}_\Re \to \pm 0$ in (4.1.5) and using (4.2.19), we get the following relation for the discontinuity of the Jost solution $\Phi(\mathbf{k})$:

$$\Phi(x,+0+i\mathbf{k}_\Im) - \Phi(x,-0+i\mathbf{k}_\Im) = \sum_l \theta(\varkappa_l - |\mathbf{k}_\Im|)\Phi_l(x,\mathbf{k}_\Im)w_l(\mathbf{k}_\Im), \quad (4.3.2)$$

where

$$w_l(\mathbf{k}_\Im) = \sum_m \theta(\varkappa_m - |\mathbf{k}_\Im|)\bigl(A(\mathbf{k}_\Im)^{-1}\bigr)_{lm}\bigl(\Psi_m(\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\mathbf{k}_\Im)\bigr) \quad (4.3.3)$$

$$\equiv \sum_m \theta(\varkappa_m - |\mathbf{k}_\Im|)\bigl(A(\mathbf{k}_\Im)^{-1}\bigr)_{lm}\bigl(\psi(i\varkappa_m)\overrightarrow{\mathcal{L}}_1\mathcal{G}(+0+i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\mathbf{k}_\Im)\bigr) \quad (4.3.4)$$

$$\equiv \sum_m \theta(\varkappa_m - |\mathbf{k}_\Im|)\bigl(A(\mathbf{k}_\Im)^{-1}\bigr)_{lm}\bigl(\psi(i\varkappa_m)\overrightarrow{\mathcal{L}}_1\Phi(+0+i\mathbf{k}_\Im)\bigr) \quad (4.3.5)$$

$$\equiv t_l\operatorname{sgn}\mathbf{k}_\Im\bigl(\psi(i\varkappa_l)\overrightarrow{\mathcal{L}}_1\mathcal{G}(-0+i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\mathbf{k}_\Im)\bigr) \quad (4.3.6)$$

$$\equiv t_l\operatorname{sgn}\mathbf{k}_\Im\bigl(\psi(i\varkappa_l)\overrightarrow{\mathcal{L}}_1\Phi(-0+i\mathbf{k}_\Im)\bigr). \quad (4.3.7)$$

When deriving (4.3.4), we used (4.1.5), (4.1.6), (4.2.14), and (4.2.19). Note that by (4.3.4) and conjugation properties (2.2.42) and (2.2.60), we have

$$\begin{aligned}\overline{w_l(-\mathrm{k}_\Im)} &= \frac{1}{b_l}\sum_m \theta(\varkappa_m - |\mathrm{k}_\Im|)\big(\psi(i\mathrm{k}_\Im)\overrightarrow{\mathcal{L}}_1\mathcal{G}(+0+i\mathrm{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_m)\big)\big(A(\mathrm{k}_\Im)^{-1}\big)_{ml}\\ &\equiv \frac{1}{b_l}\sum_m \theta(\varkappa_m - |\mathrm{k}_\Im|)\big(\Psi(+0+i\mathrm{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_m)\big)\big(A(\mathrm{k}_\Im)^{-1}\big)_{ml}\\ &\equiv \frac{1}{b_l}\sum_m \theta(\varkappa_m - |\mathrm{k}_\Im|)\big(\psi(i\mathrm{k}_\Im)\overrightarrow{\mathcal{L}}_1\Phi_m(\mathrm{k}_\Im)\big)\big(A(\mathrm{k}_\Im)^{-1}\big)_{ml};\end{aligned}\tag{4.3.8}$$

then, using (4.2.6), we obtain

$$\overline{w_l(-\mathrm{k}_\Im)} = \frac{t_l}{b_l}\operatorname{sgn}\mathrm{k}_\Im\,\theta(\varkappa_l - |\mathrm{k}_\Im|)\big(\psi(i\mathrm{k}_\Im)\overrightarrow{\mathcal{L}}_1\mathcal{G}(-0+i\mathrm{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_l)\big).\tag{4.3.9}$$

Thus, by (4.1.6), we obtain the following analog of (4.3.2) for the discontinuity of the Jost solution $\Psi(\mathrm{k})$ of the dual equation:

$$\Psi(x,+0+i\mathrm{k}_\Im) - \Psi(x,-0+i\mathrm{k}_\Im) = \sum_l \theta(\varkappa_l - |\mathrm{k}_\Im|)\Psi_l(x,\mathrm{k}_\Im)\overline{w_l(-\mathrm{k}_\Im)}.\tag{4.3.10}$$

Equations (4.3.2) and (4.3.10) show that, unlike the discontinuity across the real axis, the discontinuity of the Jost solutions across the imaginary axis is given not in terms of these solutions but in terms of the auxiliary Jost solutions $\Phi_l(x,\mathrm{k}_\Im)$ and $\Psi_l(x,\mathrm{k}_\Im)$ introduced in (4.2.14).

4.3.2. *Behavior of Green's function, Jost solutions, and spectral data at the points $\pm i\varkappa_j$.* Our construction is based on the resolvent $M_1(q)$ and the Green's function $\mathcal{G}_1(\mathrm{k})$ of the one-dimensional potential, as follows from (4.1.1), (4.2.1), (4.3.1), and (4.2.17), in the sense that both these objects enter as kernels and inhomogeneous terms in the equations that define dressed objects, i.e., objects related to the perturbed potential (1.4). As shown in (3.2.16) and (3.3.7), both $M_1(q)$ and $\mathcal{G}_1(\mathrm{k})$ have logarithmic singularities at all points $q=(\pm\varkappa_j,0)$ or, respectively, at $\mathrm{k}=\pm i\varkappa_j$, $j=1,\dots,N$. It is clear that these singularities affect all objects related to the generic potential (1.4), and their behavior in the vicinity of each of these points must be studied separately. In this subsection, we first describe this behavior for the Green's function $\mathcal{G}(\mathrm{k})$. Then the behavior of the Jost solutions, auxiliary Jost solutions, and spectral data follows from their definitions (4.1.5), (4.1.6), (4.2.14), (4.2.5), and (4.3.3).

Let us note that decomposition (3.3.7) of the Green's function $\mathcal{G}_1(\mathrm{k})$ into a sum of regular and singular terms at the points $\mathrm{k}=\pm i\varkappa_j$ is not appropriate for using in integral equations of type (4.2.1), because both these terms, unlike $\mathcal{G}_1(\mathrm{k})$ itself, have bad behavior at the space infinity. Indeed, from the definition (2.2.31) of the Green's function and from (2.1.8), we deduce that the expression

$$e^{i\ell(\mathrm{k})(x-x')}\mathcal{G}_1(x,x',\mathrm{k}) \equiv e^{i\ell_\Re(\mathrm{k})(x-x')}M_1(x,x';\ell_\Im(\mathrm{k}))\tag{4.3.11}$$

is bounded on the x-plane due to the properties of the resolvent $M_1(q)$. However, multiplying the singular additional term in the decomposition of $\mathcal{G}_1(\mathrm{k})$ by the same exponential function and taking into account (3.2.2) and (3.2.3), we obtain

$$\sum_j \gamma_j(\mathrm{k})e^{i(\ell(\mathrm{k})-\ell(i\varkappa_j))(x-x')}\chi(x_1,i\varkappa_j)\xi(x_1',i\varkappa_j),$$

which grows exponentially in some directions at the space infinity and in some regions of k. Consequently, the regular part of the Green's function $\mathcal{G}_{1,\mathrm{reg}}(\mathrm{k})$ exhibits the same bad behavior.

Therefore, it is more convenient to introduce the Green's functions

$$\mathcal{G}_{1,j}(\mathbf{k}) = \mathcal{G}_1(\mathbf{k}) - \gamma_j(\mathbf{k})\varphi(\mathbf{k}) \otimes \psi(\mathbf{k}), \qquad j = 1, 2, \dots, N, \tag{4.3.12}$$

each of which is finite in the vicinity of the corresponding points $\mathbf{k} = \pm i\varkappa_j$ in view of (3.3.7) and, being multiplied by $e^{i\ell(\mathbf{k})(x-x')}$, remains bounded on the x-plane by virtue of (4.3.11), (3.2.2), and (3.2.3) and the boundedness of $\chi_1(x_1, \mathbf{k})$ and $\xi_1(x_1, \mathbf{k})$. Notice, however, that unlike $\mathcal{G}_{1,\mathrm{reg}}(\mathbf{k})$, the functions $\mathcal{G}_{1,j}(\mathbf{k})$ are discontinuous on the imaginary axis according to (3.3.6). We see that, due to (4.3.11) and (3.2.2), (3.2.3), this regularization satisfies property (4.3.11) as well. Now, we define new Green's functions of the nonstationary Schrödinger equation with potential (1.4) by means of the integral equations

$$\mathcal{G}_j(\mathbf{k}) = \mathcal{G}_{1,j}(\mathbf{k}) + \mathcal{G}_{1,j}(\mathbf{k})U_2\mathcal{G}_j(\mathbf{k}), \tag{4.3.13}$$

$$\mathcal{G}_j(\mathbf{k}) = \mathcal{G}_{1,j}(\mathbf{k}) + \mathcal{G}_j(\mathbf{k})U_2\mathcal{G}_{1,j}(\mathbf{k}). \tag{4.3.14}$$

The inhomogeneous terms and the kernels of these equations are finite in the vicinity of the points $\pm i\varkappa_j$; therefore, under the assumption of the unique solvability of these equations, the functions $\mathcal{G}_j(\mathbf{k})$ have the same property. Therefore, $\mathcal{G}_j(\mathbf{k})$ is a regularization of $\mathcal{G}(\mathbf{k})$ in the neighborhood of $\pm i\varkappa_j$. In order to express $\mathcal{G}(\mathbf{k})$ in terms of this regularization, we subtract (4.3.13) from (4.2.1) and, using (3.3.7), obtain

$$\mathcal{G}(\mathbf{k}) - \mathcal{G}_j(\mathbf{k}) = \gamma_j(\mathbf{k})\,\varphi(\mathbf{k}) \otimes \psi(\mathbf{k})[I + U_2\mathcal{G}(\mathbf{k})] + \mathcal{G}_{1,j}(\mathbf{k})U_2[\mathcal{G}(\mathbf{k}) - \mathcal{G}_j(\mathbf{k})].$$

Again, under the assumption of the unique solvability of (4.3.13), using the identities $I + U_2\mathcal{G}(\mathbf{k}) = \overrightarrow{\mathcal{L}}_1\mathcal{G}(\mathbf{k})$ and $I + \mathcal{G}(\mathbf{k})U_2 = \mathcal{G}(\mathbf{k})\overleftarrow{\mathcal{L}}_1$, we derive the following representation from these equalities:

$$\mathcal{G}(\mathbf{k}) = \mathcal{G}_j(\mathbf{k}) + \gamma_j(\mathbf{k})\,\widetilde{\Phi}_j(\mathbf{k}) \otimes \Psi(\mathbf{k}). \tag{4.3.15}$$

Here and in what follows, we use notations (4.1.5) and (4.1.6) for the Jost solutions and introduce, by analogy, the notations

$$\widetilde{\Phi}_j(\mathbf{k}) = \mathcal{G}_j(\mathbf{k})\overleftarrow{\mathcal{L}}_1\varphi(\mathbf{k}), \qquad \widetilde{\Psi}_j(\mathbf{k}) = \psi(\mathbf{k})\overrightarrow{\mathcal{L}}_1\mathcal{G}_j(\mathbf{k}). \tag{4.3.16}$$

The functions $\widetilde{\Phi}_j(\mathbf{k})$ and $\widetilde{\Psi}_j(\mathbf{k})$ are also solutions of the nonstationary Schrödinger equation and its dual,

$$\overrightarrow{\mathcal{L}}\,\widetilde{\Phi}_j(\mathbf{k}) = 0, \qquad \widetilde{\Psi}_j(\mathbf{k})\overleftarrow{\mathcal{L}} = 0, \tag{4.3.17}$$

with potential (1.4). These functions are bounded in the vicinity of the points $\pm i\varkappa_j$, while their limits at these points may depend on the sign of $\mathbf{k}_\Re$. The function

$$g_j(\mathbf{k}) = \psi(\mathbf{k})\overrightarrow{\mathcal{L}}_1\mathcal{G}_j(\mathbf{k})\overleftarrow{\mathcal{L}}_1\varphi(\mathbf{k}), \tag{4.3.18}$$

which can also be represented as

$$g_j(\mathbf{k}) = \widetilde{\Psi}_j(\mathbf{k})\overleftarrow{\mathcal{L}}_1\varphi(\mathbf{k}) \equiv \psi(\mathbf{k})\overrightarrow{\mathcal{L}}_1\widetilde{\Phi}_j(\mathbf{k}) \tag{4.3.19}$$

in view of (4.3.16), has the same properties. One can check that the regularized Green's function $\mathcal{G}_j(\mathbf{k})$ satisfies the conjugation property (2.2.42), so that

$$\overline{g_j(\mathbf{k})} = g_j(\overline{\mathbf{k}}). \tag{4.3.20}$$

Now, using definition (4.3.18) and applying $\psi(\mathrm{k})\overrightarrow{\mathcal{L}_1}$ to (4.3.15) from the left, we get

$$\Psi(\mathrm{k}) = \frac{\widetilde{\Psi}_j(\mathrm{k})}{1-\gamma_j(\mathrm{k})g_j(\mathrm{k})}. \tag{4.3.21}$$

Substituting this relation into (4.3.18), we get an expression for the Green's function in terms of objects that are finite in the vicinity of the points $\pm i\varkappa_j$:

$$\mathcal{G}(\mathrm{k}) = \mathcal{G}_j(\mathrm{k}) + \frac{\gamma_j(\mathrm{k})}{1-\gamma_j(\mathrm{k})g_j(\mathrm{k})}\,\widetilde{\Phi}_j(\mathrm{k})\otimes\widetilde{\Psi}_j(\mathrm{k}). \tag{4.3.22}$$

Now, applying the operation $\overleftarrow{\mathcal{L}}_1\varphi(\mathrm{k})$ to (4.3.22) from the right, we get an expression symmetric to (4.3.21), i.e.,

$$\Phi(\mathrm{k}) = \frac{\widetilde{\Phi}_j(\mathrm{k})}{1-\gamma_j(\mathrm{k})g_j(\mathrm{k})}, \tag{4.3.23}$$

and applying the same operation to (4.3.21), we get

$$\psi(\mathrm{k})\overrightarrow{\mathcal{L}}_1\mathcal{G}(\mathrm{k})\overleftarrow{\mathcal{L}}_1\varphi(\mathrm{k}) = \frac{g_j(\mathrm{k})}{1-\gamma_j(\mathrm{k})g_j(\mathrm{k})}. \tag{4.3.24}$$

Thus, we see that the behavior of the Green's function and of the Jost solutions in the vicinity of the points $\pm i\varkappa_j$ is determined by the behavior of the function $g_j(\mathrm{k})$ at these points. In particular, if

$$\lim_{\mathrm{k}\to i\varkappa_j} g_j(\mathrm{k}) = 0, \tag{4.3.25}$$

then the same is valid for the conjugate point by virtue of (4.3.20) and, by definition, this decay is faster than $1/\ln(\mathrm{k}\mp i\varkappa_j)$:

$$g_j(\mathrm{k}) = o(1/\ln(\mathrm{k}\mp i\varkappa_j)), \qquad \mathrm{k}\sim -i\varkappa_j. \tag{4.3.26}$$

Thus, by (3.3.8), $\lim_{\mathrm{k}\to\pm i\varkappa_j}\gamma_j(\mathrm{k})g_j(\mathrm{k}) = 0$, and we observe the same behavior as in the pure one-dimensional case; i.e., the Green's function has logarithmic singularities at the points $\pm i\varkappa_j$, the Jost solutions are bounded at these points, and their limits depend on the sign of $\mathrm{k}_{\Re}$. In this case, by (4.3.24), we have

$$\psi(\mathrm{k})\overrightarrow{\mathcal{L}}_1\mathcal{G}(\mathrm{k})\overleftarrow{\mathcal{L}}_1\varphi(\mathrm{k}) = o(1/\ln(\mathrm{k}\mp i\varkappa_j)), \qquad \mathrm{k}\sim \pm i\varkappa_j. \tag{4.3.27}$$

If, on the other hand,

$$g_j(\mathrm{k}) = O(1), \qquad \mathrm{k}\sim i\varkappa_j, \tag{4.3.28}$$

and the limit (which depends on the way of passing to the limit) is different from zero, then, by (4.3.20), the same is valid for the conjugate point and, by (4.3.22), we have

$$\mathcal{G}(\mathrm{k}) = \mathcal{G}_j(\mathrm{k}) - \frac{1}{g_j(\mathrm{k})}\,\widetilde{\Phi}_j(\mathrm{k})\otimes\widetilde{\Psi}_j(\mathrm{k}), \qquad \mathrm{k}\sim \pm i\varkappa_j. \tag{4.3.29}$$

Then (4.3.24) implies

$$\psi(\mathrm{k})\overrightarrow{\mathcal{L}}_1\mathcal{G}(\mathrm{k})\overleftarrow{\mathcal{L}}_1\varphi(\mathrm{k}) = -\frac{1}{\gamma_j(\mathrm{k})} + O\big((\ln|\mathrm{k}\mp i\varkappa_j|)^{-2}\big), \qquad \mathrm{k}\sim \pm i\varkappa_j. \tag{4.3.30}$$

and (4.3.23) and (4.3.21) yield

$$\Phi(\mathrm{k}) = o(1), \qquad \mathrm{k} \sim \pm i\varkappa_j, \tag{4.3.31}$$

$$\Psi(\mathrm{k}) = o(1), \qquad \mathrm{k} \sim \pm i\varkappa_j, \tag{4.3.32}$$

where the terms on the right-hand sides are on the order of $1/\ln(\mathrm{k} \mp i\varkappa_j)$.

We can consider condition (4.3.28) as a more general condition than (4.3.26); the latter can be considered as a limiting case of (4.3.28) that includes a particular case of a nonperturbed one-dimensional potential. Therefore, in what follows, we assume that condition (4.3.28) holds for all $j = 1, \ldots, N$. Then, for the auxiliary Jost solutions, we get the following relations from their definitions and (4.3.22):

$$\lim_{\mathrm{k}_\Im \to \pm\varkappa_j} \gamma_j(+0 + i\mathrm{k}_\Im)\Phi_j(\mathrm{k}_\Im) = -\frac{\widetilde{\Phi}_j(+0 + i\varkappa_j)}{g_j(+0 + i\varkappa_j)}, \tag{4.3.33}$$

$$\lim_{\mathrm{k}_\Im \to \pm\varkappa_j} \gamma_j(+0 + i\mathrm{k}_\Im)\Psi_j(\mathrm{k}_\Im) = -\frac{\widetilde{\Psi}_j(+0 + i\varkappa_j)}{g_j(+0 + i\varkappa_j)}. \tag{4.3.34}$$

In particular,

$$\Phi_j(\pm\varkappa_j) = \Psi_j(\pm\varkappa_j) = 0, \qquad j = 1, \ldots, N, \tag{4.3.35}$$

so that, when (4.3.28) holds, the behavior of the auxiliary Jost solutions is modified compared with the case (4.3.26), where such values are different from zero. In the same way, we find that the values $\Phi_m(\pm\varkappa_j)$ and $\Psi_m(\pm\varkappa_j)$ of the auxiliary solutions for m such that $\varkappa_m > \varkappa_j$ are finite and different from zero, while

$$\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\Phi_m(\pm\varkappa_j) = 0, \qquad \Psi_m(\pm\varkappa_j)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j) = 0. \tag{4.3.36}$$

The properties of the matrix elements of $A(\mathrm{k}_\Im)$ follow from (4.2.5) and (4.2.7), (4.2.8). Substituting the limiting values of (4.3.22) as $\mathrm{k}_\Re \to +0$ into (4.2.5), we get

$$\begin{aligned} A(\mathrm{k}_\Im)_{lm} = \frac{\delta_{lm}}{t_l \operatorname{sgn} \mathrm{k}_\Im} - \theta(\min\{\varkappa_l, \varkappa_m\} - |\mathrm{k}_\Im|)\Biggl(&\bigl(\psi(i\varkappa_l)\overrightarrow{\mathcal{L}}_1\mathcal{G}_j(+0 + i\mathrm{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_m)\bigr) \\ &+ \frac{\bigl(\psi(i\varkappa_l)\overrightarrow{\mathcal{L}}_1\widetilde{\Phi}_j(+0 + i\mathrm{k}_\Im)\bigr)\bigl(\widetilde{\Psi}_j(+0 + i\mathrm{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_m)\bigr)}{1 - \gamma_j(+0 + i\mathrm{k}_\Im)g_j(+0 + i\mathrm{k}_\Im)}\Biggr). \end{aligned} \tag{4.3.37}$$

Then, taking into account (4.3.16) and (4.3.36), we get

$$A(\pm\varkappa_j)_{jm} = A(\pm\varkappa_j)_{mj} = \pm\frac{\delta_{jm}}{t_j}, \tag{4.3.38}$$

which coincide with the values of $A_{jm}(\mathrm{k}_\Im)$ and $A_{mj}(\mathrm{k}_\Im)$ for $\mathrm{k}_\Im > \min\{\varkappa_j, \varkappa_m\}$. In the same way, we derive

$$\bigl(A(\pm\varkappa_j)^{-1}\bigr)_{jm} = \bigl(A(\pm\varkappa_j)^{-1}\bigr)_{mj} = \pm\delta_{jm}t_j. \tag{4.3.39}$$

The other matrix elements of the matrices $A(\mathrm{k}_\Im)$ and $A(\mathrm{k}_\Im)^{-1}$ have some finite limits at $\mathrm{k}_\Im = \pm\varkappa_j$.

Finally, we consider $w_l(\mathrm{k}_\Im)$. Choosing representation (4.3.7), we use the limit of (4.3.23) as $\mathrm{k}_\Re \to -0$, which gives

$$w_l(\mathrm{k}_\Im) = t_l \operatorname{sgn} \mathrm{k}_\Im \frac{\psi(i\varkappa_l)\overrightarrow{\mathcal{L}}_1\widetilde{\Phi}_j(-0 + i\mathrm{k}_\Im)}{1 - \gamma_j(-0 + i\mathrm{k}_\Im)g_j(-0 + i\mathrm{k}_\Im)} \tag{4.3.40}$$

inside the interval $|k_\Im| < \varkappa_l$. Thus, we see that under condition (4.3.28), $w_l(k_\Im)$ goes to zero as $1/\ln||k_\Im| - \varkappa_j|$ at all points $k_\Im = \pm\varkappa_j$ such that $\varkappa_j \le \varkappa_l$; i.e.,

$$w_l(\pm\varkappa_j) = 0 \qquad \text{for all } j \text{ and } l \text{ such that } \varkappa_j \le \varkappa_l. \tag{4.3.41}$$

4.3.3. *Auxiliary Jost solutions in terms of the Jost solutions on the imaginary axis.* The properties of the Jost solutions and spectral data derived in the previous subsections enable us to reconstruct the auxiliary Jost solutions in terms of the boundary values of the Jost solutions. Indeed, the derivatives of the Green's function (as follows from (4.1.3) and (4.1.4)) are given (cf. (2.2.47)) by

$$\frac{\partial \mathcal{G}(k)}{\partial k_\Re} = \frac{\operatorname{sgn} k_\Im}{2\pi i}\, t(k)\Phi(k)\otimes\Psi(k), \qquad \frac{\partial \mathcal{G}(k)}{\partial k_\Im} = \frac{\operatorname{sgn} k_\Im}{2\pi}\, t(k)\Phi(k)\otimes\Psi(k). \tag{4.3.42}$$

Passing to the limits as $k_\Re \to \pm 0$ in the second equality, we get

$$\mathcal{G}(\pm 0 + ik_\Im) = \mathcal{G}^\sigma(\pm 0) + \frac{\sigma}{2\pi}\int_0^{k_\Im} d\alpha\, t(i\alpha)\Phi(\pm 0 + i\alpha)\otimes\Psi(\pm 0 + i\alpha), \qquad \sigma = \operatorname{sgn} k_\Im, \tag{4.3.43}$$

where we used notation (4.2.2).

Differentiating (4.2.14) with respect to $k_\Im$ and taking into account (4.3.42), we have

$$\begin{aligned}\frac{\partial \Phi_j(k_\Im)}{\partial k_\Im} &= \frac{\operatorname{sgn} k_\Im}{2\pi} t(ik_\Im)\Phi(+0 + ik_\Im)\big(\Psi(+0 + ik_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\big) \\ &\equiv \frac{\operatorname{sgn} k_\Im}{2\pi} t(ik_\Im)\Phi(+0 + ik_\Im)\big(\psi(ik_\Im)\overrightarrow{\mathcal{L}}_1\mathcal{G}(+0 + ik_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\big).\end{aligned} \tag{4.3.44}$$

From (4.3.9), we obtain

$$\sum_l b_l\overline{w_l(-k_\Im)}A_{lj}(k_\Im) = \theta(\varkappa_j - |k_\Im|)\big(\Psi(+0 + ik_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\big),$$

which, being substituted into (4.3.44), gives

$$\theta(\varkappa_j - |k_\Im|)\frac{\partial \Phi_j(k_\Im)}{\partial k_\Im} = \frac{\operatorname{sgn} k_\Im}{2\pi} t(ik_\Im)\Phi(0 + ik_\Im)\sum_l b_l\overline{w_l(-k_\Im)}A_{lj}(k_\Im); \tag{4.3.45}$$

consequently,

$$\theta(\varkappa_j - |k_\Im|)\Phi_j(k_\Im) = \frac{\operatorname{sgn} k_\Im}{2\pi}\int_{\varkappa_j \operatorname{sgn} k_\Im}^{k_\Im} d\alpha\, t(i\alpha)\Phi(+0 + i\alpha)\sum_l b_l\overline{w_l(-\alpha)}A_{lj}(\alpha). \tag{4.3.46}$$

Now, we apply $\psi(i\varkappa_m)\overrightarrow{\mathcal{L}}_1$ to (4.3.46) from the left:

$$\psi(i\varkappa_m)\overrightarrow{\mathcal{L}}_1\Phi_j(k_\Im) = \frac{\operatorname{sgn} k_\Im}{2\pi}\int_{\varkappa_j \operatorname{sgn} k_\Im}^{k_\Im} d\alpha\, t(i\alpha)\big(\psi(i\varkappa_m)\overrightarrow{\mathcal{L}}_1\Phi(+0 + i\alpha)\big)\sum_l b_l\overline{w_l(-\alpha)}A_{lj}(\alpha),$$

which, being substituted into (4.2.5), gives

$$\begin{aligned}A_{mj}(k_\Im) = {}& \frac{\delta_{mj}}{t_m \operatorname{sgn} k_\Im} - \theta(\min\{\varkappa_m, \varkappa_j\} - |k_\Im|)\frac{\operatorname{sgn} k_\Im}{2\pi} \\ &\times \int_{\min\{\varkappa_j, \varkappa_m\}\operatorname{sgn} k_\Im}^{k_\Im} d\alpha\, t(i\alpha)\big(\psi(i\varkappa_m)\overrightarrow{\mathcal{L}}_1\Phi(+0 + i\alpha)\big)\sum_l b_l\overline{w_l(-\alpha)}A_{lj}(\alpha).\end{aligned} \tag{4.3.47}$$

Taking into account (4.3.5), we also obtain

$$\sum_{l'} w_{l'}(\mathrm{k}_\Im)A_{jl'}(\mathrm{k}_\Im) = \theta(\varkappa_m - |\mathrm{k}_\Im|)\psi(i\varkappa_j)\overrightarrow{\mathcal{L}_1}\Phi(+0 + i\mathrm{k}_\Im),$$

which, being substituted into (4.3.47), yields the following relation in view of (4.3.38):

$$A_{mj}(\mathrm{k}_\Im) = \frac{\delta_{mj}}{t_m \operatorname{sgn} \mathrm{k}_\Im} - \theta(\min\{\varkappa_m, \varkappa_j\} - |\mathrm{k}_\Im|)\frac{\operatorname{sgn} \mathrm{k}_\Im}{2\pi} \times \int\limits_{\min\{\varkappa_m,\varkappa_j\}\operatorname{sgn}\mathrm{k}_\Im}^{\mathrm{k}_\Im} d\alpha\, t(i\alpha)\sum_{l,l'} b_l \overline{w_l(-\alpha)}A_{lj}(\alpha)w_{l'}(\alpha)A_{ml'}(\alpha).$$

Alternatively, we can start directly from the expression for the derivative of $A_{mj}(\mathrm{k}_\Im)$:

$$\frac{\partial A_{mj}(\mathrm{k}_\Im)}{\partial \mathrm{k}_\Im} = -\theta(\min\{\varkappa_m, \varkappa_j\} - |\mathrm{k}_\Im|)\frac{\operatorname{sgn}\mathrm{k}_\Im}{2\pi}t(i\mathrm{k}_\Im)\sum_{l,l'} b_{l'}\overline{w_{l'}(-\mathrm{k}_\Im)}A_{l'j}(\mathrm{k}_\Im)w_l(\mathrm{k}_\Im)A_{ml}(\mathrm{k}_\Im).$$

Multiplying this equation by A^{-1} from both sides, we get

$$\frac{\partial (A(\mathrm{k}_\Im)^{-1})_{jm}}{\partial \mathrm{k}_\Im} = \theta(\min\{\varkappa_m, \varkappa_j\} - |\mathrm{k}_\Im|)\frac{\operatorname{sgn}\mathrm{k}_\Im}{2\pi}t(i\mathrm{k}_\Im)b_m w_j(\mathrm{k}_\Im)\overline{w_m(-\mathrm{k}_\Im)}$$

and, consequently,

$$\big(A(\mathrm{k}_\Im)^{-1}\big)_{jm} = t_m \operatorname{sgn}\mathrm{k}_\Im \delta_{jm} + \theta(\min\{\varkappa_m, \varkappa_j\} - |\mathrm{k}_\Im|)\frac{\operatorname{sgn}\mathrm{k}_\Im}{2\pi}b_m \times \int\limits_{\min\{\varkappa_m,\varkappa_j\}\operatorname{sgn}\mathrm{k}_\Im}^{\mathrm{k}_\Im} d\alpha\, t(i\alpha)w_j(\alpha)\overline{w_m(-\alpha)}. \tag{4.3.48}$$

One can easily verify that the properties of Jost solutions and spectral data described in Subsection 4.3.2 are compatible with representations (4.3.46) and (4.3.48).

Summarizing this study, it is worth emphasizing one of the main features of the resolvent approach, which consists in the existence of different representations for the spectral data, like (4.3.4) and (4.3.5), in terms of different solutions of the equation defined by operator (1.1) and its dual. This enables us to get relations between different spectral data. For instance, equation (4.3.48), which relates the spectral data $A_{jm}(\mathrm{k}_\Im)$ to $w_l(\mathrm{k}_\Im)$, cannot be derived if one deals with equations for the Jost solutions only (cf. [19]).

4.3.4. *Bilinear representation for the resolvent and Green's functions.* We have proved that for $q_2 \neq 0$, the derivative of the hat kernel (see notation (2.1.8)) of the resolvent with respect to q_2 satisfies (4.1.4). Taking into account the asymptotic behavior (2.2.8) and notations (4.1.7) for the limiting values at $q_2 = 0$, we obtain (again for the hat kernel)

$$\widehat{M}(x, x'; q) = \frac{1}{2\pi i}\int dk\,\big[\theta(x_2 - x_2') - \theta(\ell_{2\Im}(k + iq_1) - q_2)\big]t(k + iq_1)\Phi(x, k + iq_1)\Psi(x', k + iq_1) + \operatorname{sgn} q_1\big[\theta(x_2 - x_2') - \theta(-q_2)\big]\big[\mathcal{G}(x, x'; +0 + iq_1) - \mathcal{G}(x, x'; -0 + iq_1)\big].$$

Substituting here (4.2.19), we derive the following generalization of the bilinear representation (3.2.13) for the resolvent of the perturbed potential:

$$\widehat{M}(x,x';q)=\frac{1}{2\pi i}\int\limits_{\mathrm{k}_\Im=q_1} d\mathrm{k}_\Re\,\bigl[\theta(x_2-x_2')-\theta(2\mathrm{k}_\Re\mathrm{k}_\Im-q_2)\bigr]t(\mathrm{k})\Phi(x,\mathrm{k})\Psi(x',\mathrm{k})$$
$$-\operatorname{sgn}q_1\bigl[\theta(x_2-x_2')-\theta(-q_2)\bigr]\sum_{l,m}\theta\bigl(\min\{\varkappa_l,\varkappa_m\}-|q_1|\bigr)\bigl(A(q_1)^{-1}\bigr)_{lm}\Phi_l(x,q_1)\Psi_m(x',q_1). \tag{4.3.49}$$

If we multiply $\widehat{M}(x,x';q)$ by $e^{q(x-x')}$, we get, according to (2.1.8), the kernel $M(x,x';q)$ of the resolvent; then, both terms on the right-hand side become distributions belonging to $\mathcal{S}'$. For the first term, this follows from the properties of the functions χ and ξ pointed out in (2.2.63), and for the second term, this results from the fact that $e^{-q_1x_1}\Phi_l(x,q_1)$ and $e^{q_1x_1}\Psi_l(x,q_1)$ are bounded at the space infinity when $|q_1|<\varkappa_l$. Equations (4.1.3) and (4.1.4) for $q_1\neq 0$ follow from the analyticity of the Jost solutions and the derivatives of (4.3.46) and analogous equations for $\Psi_l(q_1)$. As in the derivation of (2.2.76), the absence of discontinuity at $q_1=0$ in the case of $q_2\neq 0$ is equivalent to the condition

$$\frac{1}{2\pi i}\int dk\,t^+(k)\Phi^+(k)\otimes\Psi^+(k)-\sum_{lm}(A^+)^{-1}_{lm}\Phi_l^+\otimes\Psi_m^+$$
$$=\frac{1}{2\pi i}\int dk\,t^-(k)\Phi^-(k)\otimes\Psi^-(k)+\sum_{lm}(A^-)^{-1}_{lm}\Phi_l^-\otimes\Psi_m^-, \tag{4.3.50}$$

where we used notation (2.2.75) for the limiting values of the Jost solutions on the real axis. Similarly, we denoted

$$\Phi_l^\pm(x)=\lim_{\pm\mathrm{k}_\Im\to+0}\Phi_l(x,\mathrm{k}_\Im),\qquad \Psi_l^\pm(x)=\lim_{\pm\mathrm{k}_\Im\to+0}\Psi_l(x,\mathrm{k}_\Im). \tag{4.3.51}$$

By virtue of (2.2.60) and (4.2.16), these limiting values satisfy the following conjugation properties:

$$\overline{\Phi^\pm(x,k)}=\Psi^\mp(x,k),\qquad \overline{\Phi_l^\pm(x)}=b_l\Psi_l^\mp(x),\qquad k\in\mathbb{R},\quad l=1,\dots,N. \tag{4.3.52}$$

By virtue of (2.2.34), the bilinear representation (4.3.49) for the resolvent leads to the following bilinear representation for the Green's function:

$$\mathcal{G}(x,x',\mathrm{k})=\frac{1}{2\pi i}\int dk'\,\bigl[\theta(x_2-x_2')-\theta(\mathrm{k}_\Im k')\bigr]t(k'+\mathrm{k})\Phi(x,k'+\mathrm{k})\Psi(x',k'+\mathrm{k})$$
$$-\operatorname{sgn}\mathrm{k}_\Im\bigl[\theta(x_2-x_2')-\theta(-\mathrm{k}_\Re\mathrm{k}_\Im)\bigr]\sum_{l,m}\theta\bigl(\min\{\varkappa_l,\varkappa_m\}-|\mathrm{k}_\Im|\bigr)\bigl(A(\mathrm{k}_\Im)^{-1}\bigr)_{lm}\Phi_l(x,\mathrm{k}_\Im)\Psi_m(x',\mathrm{k}_\Im),$$

which generalizes (3.3.3). Below, we will use this bilinear representation to derive relations between the Jost and advanced/retarded solutions.

4.4. Discontinuity of the resolvent at the point $q=0$.

4.4.1. *Advanced/retarded Green's functions and solutions.* Above, we investigated the behavior of the resolvent when at least one of the variables q_1 or q_2 is different from zero. As in the case of the decaying potential, the investigation of the behavior of the resolvent at $q=0$ leads to relations between Jost solutions on the real axis and spectral data for the case of perturbed generic one-dimensional potential (1.4). From a technical point of view, this study is quite close to that carried out in Section 2, so we omit here analogous details and mainly present modifications relevant to

the case under consideration. First, we introduce advanced/retarded Green's functions as specific limits of the resolvent in the same way as in (2.2.35). It is straightforward to prove that they satisfy equations (2.2.49) and (2.2.50) and integral equations (2.2.51). In order to determine the difference between these Green's functions, we use the Hilbert identity in the form (4.1.2), where we set $q'_1 = q_1 = 0$ and consider the limits $q'_2 \to \pm 0$ and $q_2 \to \mp 0$:

$$\mathcal{G}_+ - \mathcal{G}_- = \mathcal{G}_\pm \overleftarrow{\mathcal{L}}_1 (\mathcal{G}_{1,+} - \mathcal{G}_{1,-}) \overrightarrow{\mathcal{L}}_1 \mathcal{G}_\mp.$$

By (3.4.27), this yields the following representation for the difference:

$$\mathcal{G}_+ - \mathcal{G}_- = \frac{1}{2\pi i} \int dk\, \Phi_\pm(k) \otimes \Psi_\mp(k) - \sum_j t_j\, \Phi_{\pm,j} \otimes \Psi_{\mp,j}, \tag{4.4.1}$$

where the standard and auxiliary advanced/retarded solutions are defined (cf. (2.2.40)) as

$$\Phi_\pm(k) = \mathcal{G}_\pm \overleftarrow{\mathcal{L}}_1 \varphi_\pm(k), \qquad \Phi_{\pm,j} = \mathcal{G}_\pm \overleftarrow{\mathcal{L}}_1 \varphi(i\varkappa_j), \tag{4.4.2}$$

$$\Psi_\pm(k) = \psi_\pm(k) \overrightarrow{\mathcal{L}}_1 \mathcal{G}_\pm, \qquad \Psi_{\pm,j} = \psi(i\varkappa_j) \overrightarrow{\mathcal{L}}_1 \mathcal{G}_\pm. \tag{4.4.3}$$

One can check that the functions $\Phi_\pm(x,k)$, $\Psi_\pm(x,k)$, $\Phi_{\pm,j}(x)$, and $\Psi_{\pm,j}(x)$ satisfy the differential equations (2.2.69) and conjugation properties (2.2.72) and that

$$\overline{\Phi_{\pm,j}} = b_j \Psi_{\mp,j}. \tag{4.4.4}$$

By virtue of (2.2.35), the bilinear representation (4.3.49) for the resolvent gives the following representation for the advanced/retarded Green's functions in terms of the Jost solutions on the real axis:

$$\mathcal{G}_\pm(x,x') = \pm\theta(\pm(x_2 - x'_2)) \left(\frac{1}{2\pi i} \int dk\, t^\sigma(k) \Phi^\sigma(x,k) \Psi^\sigma(x',k) - \sigma \sum_{l,m} (A^\sigma)^{-1}_{lm} \Phi^\sigma_l(x) \Psi^\sigma_m(x') \right), \tag{4.4.5}$$

where we used notations (2.2.75) for the limiting values of the Jost solutions on the real axis and where there is no dependence on the sign $\sigma = +, -$ due to condition (4.3.50).

4.4.2. *Relations between Green's functions.* Another limiting procedure for the resolvent $M(q)$ at the point $q = 0$ is given by the limiting values of the Green's function $\mathcal{G}(\mathbf{k})$ on the real axis, as follows from (2.2.34). For these boundary values, we use notation (2.2.52). The difference between them and the advanced/retarded Green's functions, as in the case of the decaying potential, can be represented in two forms. The first one follows from passing to the limits as $\mathbf{k} \to k \pm i0$ in (4.3.53) and (4.4.5):

$$\mathcal{G}^\sigma(k) - \mathcal{G}_\pm = \mp\frac{1}{2\pi i} \int dk'\, \theta(\pm\sigma(k'-k))\, t^\sigma(k') \Phi^\sigma(k') \otimes \Psi^\sigma(k') \pm \sigma\theta(\mp\sigma k) \sum_{l,m} (A^\sigma)^{-1}_{lm} \Phi^\sigma_l \otimes \Psi^\sigma_m. \tag{4.4.6}$$

The second set of relations can be derived from the Hilbert identity (4.1.2), like relations (2.2.57) and (2.2.58). Taking into account definitions (2.2.53) and (2.2.35), we get

$$\mathcal{G}^\sigma(k) - \mathcal{G}_\pm = \mathcal{G}_\pm \overleftarrow{\mathcal{L}}_1 (\mathcal{G}^\sigma_1(k) - \mathcal{G}_{1,\pm}) \overrightarrow{\mathcal{L}}_1 \mathcal{G}^\sigma(k), \qquad k \in \mathbb{R},$$

so that we can use (3.4.25), (3.4.6), and (3.4.7) to derive

$$\mathcal{G}^\sigma(k) - \mathcal{G}_\pm = \mp\frac{1}{2\pi i}\int d\alpha\,\theta(\pm\sigma(\alpha-k))t^\sigma(\alpha)\mathcal{G}_\pm\overleftarrow{\mathcal{L}}_1\varphi^\sigma(\alpha)\otimes\psi^\sigma(\alpha)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(k)$$
$$\pm\,\theta(\mp\sigma k)\sum_j t_j\mathcal{G}_\pm\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\otimes\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(k),$$

$$\mathcal{G}^\sigma(k) - \mathcal{G}_\pm = \mp\frac{1}{2\pi i}\int d\alpha\,\theta(\pm\sigma(\alpha-k))t^\sigma(\alpha)\mathcal{G}^\sigma(k)\overleftarrow{\mathcal{L}}_1\varphi^\sigma(\alpha)\otimes\psi^\sigma(\alpha)\overrightarrow{\mathcal{L}}_1\mathcal{G}_\pm$$
$$\pm\,\theta(\mp\sigma k)\sum_j t_j\mathcal{G}^\sigma(k)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\otimes\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\mathcal{G}_\pm,$$

where, as always, $\sigma = +,-$. Now, in view of (3.4.6) and (4.4.2), the first equality can be rewritten as

$$\mathcal{G}^\sigma(k) - \mathcal{G}_\pm = \mp\frac{1}{2\pi i}\int d\alpha\,\theta(\pm\sigma(\alpha-k))\int dp\,\Phi_\pm(p)r^\sigma_\pm(p,\alpha)\otimes\psi^\sigma(\alpha)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(k)$$
$$\pm\,\theta(\mp\sigma k)\sum_j t_j\Phi_{\pm,j}\otimes\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(k); \tag{4.4.7}$$

similarly, by (3.4.8) and (4.4.3), from the second equality we get

$$\mathcal{G}^\sigma(k) - \mathcal{G}_\pm = \mp\frac{1}{2\pi i}\int d\alpha\,\theta(\pm\sigma(\alpha-k))\mathcal{G}^\sigma(k)\overleftarrow{\mathcal{L}}_1\varphi^\sigma(\alpha)\otimes\int dp\,\overline{r^{-\sigma}_\mp(p,\alpha)}\,\Psi_\pm(p)$$
$$\pm\,\theta(\mp\sigma k)\sum_j t_j\mathcal{G}^\sigma(k)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\otimes\Psi_{\pm,j}. \tag{4.4.8}$$

Note that (4.4.8) can also be derived from (4.4.7) by conjugation.

Relations (4.4.6) and (4.4.7), (4.4.8) can be used for obtaining relations between the Jost and the advanced/retarded solutions, for the introduction of spectral data, and for the derivation of characterization equations for them, which, in particular, guarantee condition (4.3.50).

4.5. Spectral data.

4.5.1. *Relation between the Jost and the advanced/retarded solutions.* We apply the operation $\overleftarrow{\mathcal{L}}_1\varphi^\sigma(k)$ to (4.4.7) from the right and the operation $\psi^\sigma(k)\overrightarrow{\mathcal{L}}_1$ to (4.4.8) from the left. Then, by (4.1.5) and (4.1.6) in the limiting cases and equalities (3.4.6), (4.4.2) and (3.4.8), (4.4.3), we get

$$t^\sigma(k)\Phi^\sigma(k) = \int dp\,\Phi_\pm(p)r^\sigma_\pm(p,k)$$
$$\mp\frac{1}{2\pi i}\int d\alpha\,\theta(\pm\sigma(\alpha-k))\int dp\,\Phi_\pm(p)r^\sigma_\pm(p,\alpha)t^\sigma(k)\big(\psi^\sigma(\alpha)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(k)\overleftarrow{\mathcal{L}}_1\varphi^\sigma(k)\big)$$
$$\pm\,\theta(\mp\sigma k)\sum_j t_j\Phi_{\pm,j}t^\sigma(k)\big(\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(k)\overleftarrow{\mathcal{L}}_1\varphi^\sigma(k)\big), \tag{4.5.1}$$

$$t^\sigma(k)\Psi^\sigma(k) = \int dp\,\overline{r^{-\sigma}_\mp(p,k)}\,\Psi_\pm(p)$$
$$\mp\frac{1}{2\pi i}\int d\alpha\,\theta(\pm\sigma(\alpha-k))t^\sigma(k)\big(\psi^\sigma(k)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(k)\overleftarrow{\mathcal{L}}_1\varphi^\sigma(\alpha)\big)\int dp\,\overline{r^{-\sigma}_\mp(p,\alpha)}\,\Psi_\pm(p)$$
$$\pm\,\theta(\mp\sigma k)\sum_j t_jt^\sigma(k)\big(\psi^\sigma(k)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(k)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\big)\Psi_{\pm,j}. \tag{4.5.2}$$

Next, we pass to the limit as $k \to +0$ in (4.4.7) and (4.4.8), apply $\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_m)$ and $\psi(i\varkappa_m)\overrightarrow{\mathcal{L}}_1$, and use (4.2.14), (4.4.2), and (4.4.3). In this way, we get the relations

$$\Phi^\sigma_m = \Phi_{\pm,m} \mp \frac{1}{2\pi i}\int d\alpha\,\theta(\pm\sigma\alpha)\int dp\,\Phi_\pm(p)r^\sigma_\pm(p,\alpha)\big(\psi^\sigma(\alpha)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(+0)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_m)\big)$$
$$\pm\,\theta(\mp\sigma)\sum_j t_j\Phi_{\pm,j}\big(\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(+0)\overleftarrow{\mathcal{L}}_1\varphi^\sigma(i\varkappa_m)\big), \tag{4.5.3}$$

$$\Psi^\sigma_m = \Psi_{\pm,m} \mp \frac{1}{2\pi i}\int d\alpha\,\theta(\pm\sigma\alpha)\big(\psi(i\varkappa_m)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(+0)\overleftarrow{\mathcal{L}}_1\varphi^\sigma(\alpha)\big)\int dp\,\overline{r^{-\sigma}_\mp(p,\alpha)}\,\Psi_\pm(p)$$
$$\pm\,\theta(\mp\sigma)\sum_j t_j\big(\psi(i\varkappa_m)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(+0)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\big)\Psi_{\pm,j}. \tag{4.5.4}$$

Thus, we can write $(\sigma = +,-)$

$$t^\sigma(k)\Phi^\sigma(k) = \int dp\,\Phi_\pm(p)\mathcal{R}^\sigma_\pm(p,k) + \sum_j \Phi_{\pm,j}\mathcal{R}^\sigma_{\pm,j}(k), \tag{4.5.5}$$

$$\Phi^\sigma_m = \int dp\,\Phi_\pm(p)\widehat{\mathcal{R}}^\sigma_{\pm,m}(p) + \theta(\pm\sigma)\Phi_{\pm,m} \mp \theta(\mp\sigma)\sum_j t_j\Phi_{\pm,j}A^\sigma_{jm}, \tag{4.5.6}$$

where we used (4.2.12) in the last line and the spectral data $\mathcal{R}^\sigma_\pm$ are given by

$$\mathcal{R}^\sigma_\pm(p,k) = \int dk'\,r^\sigma_\pm(p,k')R^\sigma_\pm(k',k) \tag{4.5.7}$$

or, in the operator form, by

$$\mathcal{R}^\sigma_\pm = r^\sigma_\pm R^\sigma_\pm, \tag{4.5.8}$$

where $R^\sigma_\pm$ is a triangular operator,

$$R^\sigma_\pm(p,k) = \delta(p-k) \mp \theta(\pm\sigma(p-k))R^\sigma(p,k) \tag{4.5.9}$$

with (cf. (2.2.82) and (2.2.83))

$$R^\sigma(p,k) = t^\sigma(k)\frac{\psi^\sigma(p)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(k)\overleftarrow{\mathcal{L}}_1\varphi^\sigma(k)}{2\pi i}. \tag{4.5.10}$$

Passing to the limit as $k_\Im \to \sigma 0$ in (4.1.5) and using (4.3.1), we obtain

$$R^\sigma(p,k) \equiv t^\sigma(k)\frac{\psi^\sigma(p)\overrightarrow{\mathcal{L}}_1\Phi^\sigma(k)}{2\pi i} \equiv t^\sigma(k)\frac{\psi^\sigma(p)U_2\Phi^\sigma(k)}{2\pi i}, \tag{4.5.11}$$

$$\mathcal{R}^\sigma_{\pm,j}(k) = \pm\theta(\mp\sigma k)\mathcal{R}^\sigma_j(k), \tag{4.5.12}$$

$$\mathcal{R}^\sigma_j(k) = t_j t^\sigma(k)\big(\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\mathcal{G}^\sigma(k)\overleftarrow{\mathcal{L}}_1\varphi^\sigma(k)\big) \tag{4.5.13}$$

$$\equiv t_j t^\sigma(k)\big(\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\Phi^\sigma(k)\big) \equiv t_j t^\sigma(k)\big(\psi(i\varkappa_j)U_2\Phi^\sigma(k)\big), \tag{4.5.14}$$

$$\widehat{\mathcal{R}}^\sigma_{\pm,m} = r^\sigma_\pm\widehat{R}^\sigma_{\pm,m}, \tag{4.5.15}$$

$$\widehat{R}^\sigma_{\pm,m}(k) = \mp\theta(\pm\sigma k)\widehat{R}^\sigma_m(k), \tag{4.5.16}$$

$$\widehat{R}^{\sigma}_{m}(k) = \frac{\psi^{\sigma}(k)\overrightarrow{\mathcal{L}}_1\mathcal{G}^{\sigma}(+0)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_m)}{2\pi i} \tag{4.5.17}$$

$$\equiv \frac{\psi^{\sigma}(k)\overrightarrow{\mathcal{L}}_1\Phi^{\sigma}_m}{2\pi i} \equiv \frac{\psi^{\sigma}(k)U_2\Phi^{\sigma}_m}{2\pi i}. \tag{4.5.18}$$

These relations generalize (2.2.80) and (3.4.6). By virtue of the conjugation properties, using (3.1.14), (3.2.6), and (3.2.11), we obtain

$$\overline{R^{\sigma}(p,k)} = -t^{-\sigma}(k)\frac{\psi^{-\sigma}(k)\overrightarrow{\mathcal{L}}_1\mathcal{G}^{-\sigma}(k)\overleftarrow{\mathcal{L}}_1\varphi^{-\sigma}(p)}{2\pi i} \tag{4.5.19}$$

$$\equiv -t^{-\sigma}(k)\frac{\Psi^{-\sigma}(k)\overleftarrow{\mathcal{L}}_1\varphi^{-\sigma}(p)}{2\pi i} \equiv -t^{-\sigma}(k)\frac{\Psi^{-\sigma}(k)U_2\varphi^{-\sigma}(p)}{2\pi i}, \tag{4.5.20}$$

$$\overline{\mathcal{R}^{\sigma}_j(k)} = -\frac{t_j}{b_j}\,t^{-\sigma}(k)\big(\psi^{-\sigma}(k)\overrightarrow{\mathcal{L}}_1\mathcal{G}^{-\sigma}(k)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\big) \tag{4.5.21}$$

$$\equiv -\frac{t_j}{b_j}\,t^{-\sigma}(k)\big(\Psi^{-\sigma}(k)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\big) \equiv -\frac{t_j}{b_j}\,t^{-\sigma}(k)\big(\Psi^{-\sigma}(k)U_2\varphi(i\varkappa_j)\big), \tag{4.5.22}$$

and

$$\overline{\widehat{R}^{\sigma}_m(k)} = -b_m\frac{\psi(i\varkappa_m)\overrightarrow{\mathcal{L}}_1\mathcal{G}^{-\sigma}(+0)\overleftarrow{\mathcal{L}}_1\varphi^{-\sigma}(k)}{2\pi i} \tag{4.5.23}$$

$$\equiv -b_m\frac{\Psi^{-\sigma}_m\overleftarrow{\mathcal{L}}_1\varphi^{-\sigma}(k)}{2\pi i} \equiv -b_m\frac{\Psi^{-\sigma}_m U_2\varphi^{-\sigma}(k)}{2\pi i}. \tag{4.5.24}$$

Note that in view of (4.2.19), we have

$$\mathcal{G}^{\sigma}(+0) - \mathcal{G}^{\sigma}(-0) = -\sum_{l,m}\Phi^{\sigma}_l(A^{\sigma})^{-1}_{lm}\otimes\Psi^{\sigma}_m, \tag{4.5.25}$$

so that $R^{\sigma}(p,k)/t^{\sigma}(k)$ and $\mathcal{R}^{\sigma}_j(k)/t^{\sigma}(k)$ are discontinuous at $k=0$. On the other hand, due to the first equality in (3.1.27), the functions $R^{\sigma}(p,k)$ and $\mathcal{R}^{\sigma}_j(k)$ behave as $|k|$ at $k=0$. This is valid for a general one-dimensional potential, in contrast, say, to a pure solitonic one.

Equations (4.5.5) and (4.5.6) give the boundary values of the Jost solutions in terms of the advanced/retarded ones. In order to get the inverse relations, we apply the same procedures as above to (4.4.6). Namely, applying $\overleftarrow{\mathcal{L}}_1\varphi^{\sigma}(k)$ to (4.4.6) from the right and using (3.4.6), we get (in terms of the notations introduced)

$$\frac{1}{t^{\sigma}(k)}\int dp\,\Phi_{\pm}(p)r^{\sigma}_{\pm}(p,k) = \Phi^{\sigma}(k) \pm \frac{1}{2\pi i}\int dk'\,\theta(\pm\sigma(k'-k))t^{\sigma}(k')\Phi^{\sigma}(k')\big(\Psi^{\sigma}(k')\overleftarrow{\mathcal{L}}_1\varphi^{\sigma}(k)\big)$$
$$\mp\sigma\theta(\mp\sigma k)\sum_{l,m}\Phi^{\sigma}_l(A^{\sigma})^{-1}_{lm}\big(\Psi^{\sigma}_m\overleftarrow{\mathcal{L}}_1\varphi^{\sigma}(k)\big).$$

Similarly, applying $\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)$ to the limit of (4.4.6) as $k\to+0$ from the right and using (4.4.2), we derive

$$\Phi_{\pm,j} = \Phi^{\sigma}_j \pm \frac{1}{2\pi i}\int dk'\,\theta(\pm\sigma k')t^{\sigma}(k')\Phi^{\sigma}(k')\big(\Psi^{\sigma}(k')\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\big)$$
$$\mp\sigma\theta(\mp\sigma)\sum_{l,m}\Phi^{\sigma}_l(A^{\sigma})^{-1}_{lm}\big(\Psi^{\sigma}_m\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\big).$$

Finally, taking into account (3.4.14), (4.5.20), and (4.5.24), we get

$$\Phi_{\pm}(k) = \int dk'\Phi^{\sigma}(k')\overline{\mathcal{R}_{\pm}^{-\sigma}(k,k')} - 2\pi i\sigma \sum_{l,m} \Phi_l^{\sigma}(A^{\sigma})_{lm}^{-1}\frac{1}{b_m}\overline{\mathcal{R}_{\pm,m}^{-\sigma}(k)}, \tag{4.5.26}$$

$$\Phi_{\pm,j} = -\frac{b_j}{2\pi i t_j}\int dk'\Phi^{\sigma}(k')\overline{\mathcal{R}_{\pm,j}^{-\sigma}(k')} + \theta(\pm\sigma)\Phi_j^{\sigma} \mp \frac{\theta(\mp\sigma)}{t_j}\sum_{l}\Phi_l^{\sigma}(A^{\sigma})_{lj}^{-1}. \tag{4.5.27}$$

4.5.2. *Relations between the Jost solutions on the real axis and the characterization equations for the spectral data.* In order to derive the discontinuity of the Jost solutions on the real axis, it is convenient to rewrite the relations between the Jost solutions and the retarded/advanced solutions in a more compact form. To this end, we introduce $(N+1)\times(N+1)$ matrices

$$S_{\alpha\beta}(k,k'), \qquad \alpha,\beta = 0,1,2,\dots,N, \tag{4.5.28}$$

with elements that depend on k and k' as follows:

$$S(k,k') = \begin{pmatrix} S_{00}(k,k') & S_{0j}(k) \\ S_{l0}(k') & S_{lj} \end{pmatrix}, \qquad l,j = 1,2,\dots,N. \tag{4.5.29}$$

Define a composition of such matrices as follows:

$$R_{\alpha\beta}(k,k') = \sum_{\lambda=0}^{N}(S_{\alpha\lambda} * T_{\lambda\beta})(k,k'), \tag{4.5.30}$$

where

$$(S_{\alpha 0} * T_{0\beta})(k,k') = \int dk''\, S_{\alpha 0}(k,k'')T_{0\beta}(k'',k'), \qquad \alpha,\beta = 0,1,2,\dots,N, \tag{4.5.31}$$

$$(S_{0m} * T_{m0})(k,k') = S_{0m}(k)T_{m0}(k'), \qquad m = 1,2,\dots,N, \tag{4.5.32}$$

$$(S_{lm} * T_{m0})(k,k') = S_{lm}T_{m0}(k'), \qquad l,m = 1,2,\dots,N, \tag{4.5.33}$$

$$(S_{0m} * T_{mj})(k,k') = S_{0m}(k)T_{mj}, \qquad j,m = 1,2,\dots,N, \tag{4.5.34}$$

$$(S_{lm} * T_{mj})(k,k') = S_{lm}T_{mj}, \qquad l,j,m = 1,2,\dots,N. \tag{4.5.35}$$

The unity matrix in the set of such matrices is given by

$$I(k,k') = \begin{pmatrix} \delta(k-k') & 0 \\ 0 & I \end{pmatrix}, \tag{4.5.36}$$

where the zeroes stand for the null N-vector and the null N-covector and I is the $N\times N$ unity matrix. The adjoint matrix is defined by

$$S_{\alpha\beta}^{\dagger}(k,k') = \overline{S_{\beta\alpha}(k',k)}. \tag{4.5.37}$$

Similarly, we introduce $(N+1)$-covectors of the form

$$v_{\alpha}(k), \qquad \alpha = 0,1,2,\dots,N, \tag{4.5.38}$$

with components that depend on k as follows:

$$v_0(k) \equiv v_0(k), \tag{4.5.39}$$

$$v_j(k) \equiv v_j. \tag{4.5.40}$$

Define by

$$(v * S)_0(k) = \int dk'\, v_0(k') S_{00}(k', k) + \sum_{m=1}^{N} S_{0m}(k) v_m, \tag{4.5.41}$$

$$(v * S)_j = \int dk'\, v_0(k') S_{0j}(k') + \sum_{m=1}^{N} v_m S_{mj} \tag{4.5.42}$$

a covector obtained by applying the matrix S to $\boldsymbol{v}$.

Now, let us introduce covectors whose components are the Jost and auxiliary Jost solutions and analogous covectors for the advanced/retarded solutions,

$$\boldsymbol{\Phi}^\sigma(k) = \left(\Phi^\sigma(k), \Phi_1^\sigma, \Phi_2^\sigma, \dots, \Phi_N^\sigma\right), \tag{4.5.43}$$

$$\boldsymbol{\Phi}_\pm = \left(\Phi_\pm(k), \Phi_{\pm,1}, \Phi_{\pm,2}, \dots, \Phi_{\pm,N}\right), \tag{4.5.44}$$

a transmission matrix

$$\boldsymbol{T}^\sigma = \begin{pmatrix} t^\sigma(k)\delta(k-k') & 0 \\ 0 & I \end{pmatrix}, \tag{4.5.45}$$

and a matrix of the spectral data,

$$\boldsymbol{\mathcal{R}}_\pm^\sigma(k, k') = \begin{pmatrix} \mathcal{R}_\pm^\sigma(k, k') & \widehat{\mathcal{R}}_{\pm,j}^\sigma(k) \\ \mathcal{R}_{\pm,l}^\sigma(k') & \mathcal{R}_{\pm,lj}^\sigma \end{pmatrix}, \tag{4.5.46}$$

where

$$\mathcal{R}_{\pm,lj}^\sigma = \theta(\pm\sigma)\delta_{lj} \mp \theta(\mp\sigma) t_l A_{lj}^\sigma. \tag{4.5.47}$$

Then equations (4.5.5) and (4.5.6) can be rewritten in a compact form as

$$\boldsymbol{\Phi}^\sigma * \boldsymbol{T}^\sigma = \boldsymbol{\Phi}_\pm * \boldsymbol{\mathcal{R}}_\pm^\sigma. \tag{4.5.48}$$

Let us introduce constant $(N+1) \times (N+1)$ matrices

$$\boldsymbol{W}^\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{\sigma}{2\pi i} b_l A_{lj}^\sigma \end{pmatrix}, \tag{4.5.49}$$

$$\boldsymbol{K} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2\pi i}\delta_{lj}\frac{b_j}{t_j} \end{pmatrix}. \tag{4.5.50}$$

Taking into account that

$$\overline{\mathcal{R}_{\pm,lj}^{-\sigma}} = \theta(\mp\sigma)\delta_{lj} \pm \theta(\pm\sigma) t_l \overline{A_{lj}^{-\sigma}} = \theta(\mp\sigma)\delta_{lj} \pm \theta(\pm\sigma) t_l \frac{b_j}{b_l} A_{jl}^\sigma \tag{4.5.51}$$

in view of (4.2.13), we find that by (4.5.49) and (4.5.50), equations (4.5.26) and (4.5.27) can be rewritten in a compact form as

$$\boldsymbol{\Phi}_\pm = \boldsymbol{\Phi}^\sigma * (\boldsymbol{W}^\sigma)^{-1} (\boldsymbol{\mathcal{R}}_\pm^{-\sigma})^\dagger \boldsymbol{K}. \tag{4.5.52}$$

Substituting (4.5.52) into (4.5.48), we derive a characterization equation

$$(\boldsymbol{\mathcal{R}}_\pm^{-\sigma})^\dagger \boldsymbol{K} * \boldsymbol{\mathcal{R}}_\pm^\sigma = \boldsymbol{W}^\sigma \boldsymbol{T}^\sigma, \tag{4.5.53}$$

and substituting (4.5.48) into (4.5.52), we obtain

$$\boldsymbol{\mathcal{R}}_\pm^\sigma * (\boldsymbol{W}^\sigma \boldsymbol{T}^\sigma)^{-1} * (\boldsymbol{\mathcal{R}}_\pm^{-\sigma})^\dagger \boldsymbol{K} = \boldsymbol{I}. \tag{4.5.54}$$

Substituting (4.5.52) into (4.5.48) with the opposite sign of σ, we derive the following equation for the discontinuity of the Jost solutions on the real axis:

$$\Phi^\sigma * T^\sigma = \Phi^{-\sigma}(W^{-\sigma})^{-1}\mathcal{F}^{-\sigma}, \tag{4.5.55}$$

where

$$\mathcal{F}^\sigma = (\mathcal{R}_\pm^{-\sigma})^\dagger K * \mathcal{R}_\pm^{-\sigma}. \tag{4.5.56}$$

Since the right-hand side is independent of the sign $\pm$, we get an additional characterization equation

$$(\mathcal{R}_-^{-\sigma})^\dagger K * \mathcal{R}_-^{-\sigma} = (\mathcal{R}_+^{-\sigma})^\dagger K * \mathcal{R}_+^{-\sigma}. \tag{4.5.57}$$

Characterization equations for the spectral data $\mathcal{F}^\sigma$ look simpler. In fact, noticing that $K^\dagger = K$, we have

$$\mathcal{F}^{\sigma\dagger} = \mathcal{F}^\sigma, \qquad (W^{-\sigma}T^{-\sigma})^{-1} * \mathcal{F}^{-\sigma} * (W^\sigma T^\sigma)^{-1} * \mathcal{F}^\sigma = I. \tag{4.5.58}$$

To write out formula (4.5.55) in a more explicit form, by analogy with the notation used for the spectral data $\mathcal{R}_\pm^\sigma$, we denote

$$\mathcal{F}^\sigma(k,k') = \begin{pmatrix} \mathcal{F}^\sigma(k,k') & \widehat{\mathcal{F}}_j^\sigma(k) \\ \mathcal{F}_l^\sigma(k') & \mathcal{F}_{lj}^\sigma \end{pmatrix}. \tag{4.5.59}$$

Then, from (4.5.55) we have

$$t^\sigma(k)\Phi^\sigma(k) = \int dp\, \Phi^{-\sigma}(p)\mathcal{F}^{-\sigma}(p,k) + 2\pi i\sigma \sum_{l,m} \Phi_l^{-\sigma}(A^{-\sigma})^{-1}_{lm}\frac{1}{b_m}\mathcal{F}_m^{-\sigma}(k), \tag{4.5.60}$$

$$\Phi_j^\sigma = \int dp\, \Phi^{-\sigma}(p)\widehat{\mathcal{F}}_j^{-\sigma}(p) + 2\pi i\sigma \sum_{l,m} \Phi_l^{-\sigma}(A^{-\sigma})^{-1}_{lm}\frac{1}{b_m}\mathcal{F}_{mj}^{-\sigma}, \tag{4.5.61}$$

where

$$\mathcal{F}^{-\sigma}(k,k') = \int dk''\, \overline{\mathcal{R}_\pm^\sigma(k'',k)}\mathcal{R}_\pm^\sigma(k'',k') - \sum_{m=1}^N \frac{b_m}{2\pi i t_m}\overline{\mathcal{R}_{\pm,m}^\sigma(k)}\mathcal{R}_{\pm,m}^\sigma(k'), \tag{4.5.62}$$

$$\mathcal{F}_l^{-\sigma}(k') = -\frac{\theta(\pm\sigma)b_l}{2\pi i t_l}\mathcal{R}_{\pm,l}^\sigma(k) \pm \frac{\theta(\mp\sigma)}{2\pi i}\sum_{m=1}^N b_l A_{lm}^{-\sigma}\mathcal{R}_{\pm,m}^\sigma(k) + \int dk''\, \overline{\widehat{\mathcal{R}}_{\pm,l}^\sigma(k'')}\mathcal{R}_\pm^\sigma(k'',k'), \tag{4.5.63}$$

$$\widehat{\mathcal{F}}_j^{-\sigma}(k) = -\frac{\theta(\pm\sigma)b_j}{2\pi i t_j}\overline{\mathcal{R}_{\pm,j}^\sigma(k)} \pm \sum_{m=1}^N \frac{b_m\theta(\mp\sigma)}{2\pi i}A_{mj}^\sigma\overline{\mathcal{R}_{\pm,m}^\sigma(k)} + \int dk'\, \overline{\mathcal{R}_\pm^\sigma(k',k)}\,\widehat{\mathcal{R}}_{\pm,j}^\sigma(k'), \tag{4.5.64}$$

$$\mathcal{F}_{lj}^{-\sigma} = -\frac{\theta(\pm\sigma)b_l}{2\pi i t_l}\delta_{lj} + \frac{\theta(\mp\sigma)}{2\pi i}\sum_{m=0}^N b_l A_{lm}^{-\sigma}t_m A_{mj}^\sigma + \int dk'\, \overline{\widehat{\mathcal{R}}_{\pm,l}^\sigma(k')}\widehat{\mathcal{R}}_{\pm,j}^\sigma(k'). \tag{4.5.65}$$

Relations (4.5.53) and (4.5.54) modify equations (2.2.90) and (2.2.91) for the decaying case and (3.4.13) and (3.4.14) for the purely one-dimensional case, respectively. The third set of characterization equations (4.5.57) for $\mathcal{R}$ generalizes equations (2.2.92) and (3.4.15). In fact, these equations are the most essential since they impose conditions on the spectral data with the same σ, in contrast to equations (4.5.53) and (4.5.54), which relate spectral data with different signs of σ.

Now, we want to have a deeper insight into the structure of the characterization matrix equations (4.5.53), (4.5.54), and (4.5.57). To this end, we write out explicit expressions for all the elements of the matrix and check, in particular, the role played by the characterization equations for the one-dimensional spectral data. The characterization equation (4.5.53) contains terms of the form $(\mathcal{R}_\pm^{-\sigma})^\dagger\mathcal{R}_\pm^\sigma$, $(\mathcal{R}_\pm^{-\sigma})^\dagger\widehat{\mathcal{R}}_{\pm,j}^\sigma$, $(\widehat{\mathcal{R}}_{\pm,l}^{-\sigma})^\dagger\mathcal{R}_\pm^\sigma$, and $(\widehat{\mathcal{R}}_{\pm,l}^{-\sigma})^\dagger\widehat{\mathcal{R}}_{\pm,j}^\sigma$. Recalling the definition of $\mathcal{R}_\pm^\sigma$ and $\widehat{\mathcal{R}}_{\pm,j}^\sigma$ in (4.5.8) and (4.5.15) and using the characterization equations (3.4.17), we get

$$(\mathcal{R}_\pm^{-\sigma})^\dagger\mathcal{R}_\pm^\sigma = (R_\pm^{-\sigma})^\dagger T^\sigma R_\pm^\sigma,$$

$$(\mathcal{R}_\pm^{-\sigma})^\dagger\widehat{\mathcal{R}}_{\pm,j}^\sigma = (R_\pm^{-\sigma})^\dagger T^\sigma \widehat{R}_{\pm,j}^\sigma,$$

$$(\widehat{\mathcal{R}}_{\pm,l}^{-\sigma})^\dagger\mathcal{R}_\pm^\sigma = (\widehat{R}_{\pm,l}^{-\sigma})^\dagger T^\sigma R_\pm^\sigma,$$

$$(\widehat{\mathcal{R}}_{\pm,l}^{-\sigma})^\dagger\widehat{\mathcal{R}}_{\pm,j}^\sigma = (\widehat{R}_{\pm,l}^{-\sigma})^\dagger T^\sigma \widehat{R}_{\pm,j}^\sigma.$$

Notice that due to the θ-function in the definition (4.5.16) of $\widehat{R}_{\pm,j}^\sigma$,

$$(\widehat{R}_{\pm,l}^{-\sigma})^\dagger T^\sigma \widehat{R}_{\pm,j}^\sigma = 0. \tag{4.5.66}$$

Then, using (4.5.66), we derive from the characterization equation (4.5.53) an identity and the following characterization equations that do not contain the continuous part of the one-dimensional spectral data:

$$(R_\pm^{-\sigma})^\dagger T^\sigma R_\pm^\sigma = T^\sigma + \frac{1}{2\pi i}\sum_{m=1}^{N}\frac{b_m}{t_m}\overline{\mathcal{R}_{\pm,m}^{-\sigma}}\otimes\mathcal{R}_{\pm,m}^\sigma, \tag{4.5.67}$$

$$(R_\pm^{-\sigma})^\dagger T^\sigma \widehat{R}_{\pm,j}^\sigma = \frac{1}{2\pi i}\sum_{m=1}^{N}\frac{b_m}{t_m}\overline{\mathcal{R}_{\pm,m}^{-\sigma}}\mathcal{R}_{\pm,mj}^\sigma, \tag{4.5.68}$$

$$(\widehat{R}_{\pm,l}^{-\sigma})^\dagger T^\sigma R_\pm^\sigma = \frac{1}{2\pi i}\sum_{m=1}^{N}\frac{b_m}{t_m}\overline{\mathcal{R}_{\pm,ml}^{-\sigma}}\mathcal{R}_{\pm,m}^\sigma, \tag{4.5.69}$$

where $l,j = 1,2,\dots,N$, the symbol $\otimes$ in the first equation is used in the sense of a direct product of a function depending only on k and a function depending only on k', and the product of operators is now meant in the usual sense. Note that the third equation is just the adjoint of the second one (with $\sigma\to-\sigma$).

Let us consider the second characterization equation (4.5.54). In this case, the one-dimensional continuous spectral data $r_\pm^\sigma$ and $(r_\pm^{-\sigma})^\dagger$ factorize. Applying $(r_\pm^{-\sigma})^\dagger$ from the left and $r_\pm^\sigma$ from the right to the respective data and using the characterization equations (3.4.17) again, we eliminate $r_\pm^\sigma$ and $(r_\pm^{-\sigma})^\dagger$. Then, from (4.5.54), we obtain an identity and the following characterization equations ($l,j = 1,2,\dots,N$):

$$R_\pm^\sigma(T^\sigma)^{-1}(R_\pm^{-\sigma})^\dagger = (T^\sigma)^{-1} + 2\pi i\sigma\sum_{m,n=1}^{N}\frac{1}{b_m}(A^\sigma)^{-1}_{nm}\widehat{R}_{\pm,n}^\sigma\otimes\overline{\widehat{R}_{\pm,m}^{-\sigma}}, \tag{4.5.70}$$

$$R_\pm^\sigma(T^\sigma)^{-1}(\mathcal{R}_{\pm,l}^{-\sigma})^\dagger = 2\pi i\sigma\sum_{m=1}^{N}\left[\frac{1}{b_m}\theta(\mp\sigma)(A^\sigma)^{-1}_{ml}\pm\theta(\pm\sigma)\frac{t_m}{b_m}\delta_{lm}\right]\widehat{R}_{\pm,m}^\sigma, \tag{4.5.71}$$

$$\mathcal{R}_{\pm,j}^\sigma(T^\sigma)^{-1}(R_\pm^{-\sigma})^\dagger = 2\pi i\sigma\sum_{m=1}^{N}\left[\frac{1}{b_m}\theta(\pm\sigma)(A^\sigma)^{-1}_{jm}\mp\theta(\mp\sigma)\frac{t_m}{b_m}\delta_{mj}\right]\overline{\widehat{R}_{\pm,m}^{-\sigma}}, \tag{4.5.72}$$

where, again, the third equation is the adjoint of the previous one (with $\sigma\to-\sigma$).

As far as the spectral data $\mathcal{F}^{-\sigma}$ are concerned, we have $(j,l=1,\dots,N)$

$$\mathcal{F}^{-\sigma}=(R^{\sigma}_{\pm})^{\dagger}f^{-\sigma}R^{\sigma}_{\pm}-\frac{1}{2\pi i}\sum_{l=1}^{N}\frac{b_l}{t_l}\overline{\mathcal{R}^{\sigma}_{\pm,l}}\otimes\mathcal{R}^{\sigma}_{\pm,l},$$

$$\widehat{\mathcal{F}}^{-\sigma}_{j}=(R^{\sigma}_{\pm})^{\dagger}f^{-\sigma}\widehat{R}^{\sigma}_{\pm,j}-\frac{1}{2\pi i}\sum_{l=1}^{N}\frac{b_l}{t_l}\overline{\mathcal{R}^{\sigma}_{\pm,l}}\mathcal{R}^{\sigma}_{\pm,lj},$$

$$\mathcal{F}^{-\sigma}_{l}=(\widehat{R}^{\sigma}_{\pm,l})^{\dagger}f^{-\sigma}R^{\sigma}_{\pm}-\frac{1}{2\pi i}\sum_{j=1}^{N}\frac{b_j}{t_j}\overline{\mathcal{R}^{\sigma}_{\pm,jl}}\mathcal{R}^{\sigma}_{\pm,j},$$

$$\mathcal{F}^{-\sigma}_{lj}=(\widehat{R}^{\sigma}_{\pm,l})^{\dagger}f^{-\sigma}\widehat{R}^{\sigma}_{\pm,j}-\frac{1}{2\pi i}\theta(\pm\sigma)\frac{b_l}{t_l}\delta_{lj}+\frac{1}{2\pi i}\sum_{m=1}^{N}\theta(\mp\sigma)b_l t_m A^{-\sigma}_{lm}A^{\sigma}_{mj},$$

which imply that the spectral data $\mathcal{F}^{-\sigma}$ can be obtained by "dressing" the spectral data $f^{-\sigma}$ of the one-dimensional potential u_1. The requirement that the right-hand sides of these equations are independent of the sign $\pm$ provides explicit expressions for the third characterization equation (4.5.57).

5. CONCLUSION

In many examples in [9–16], it was already shown that the main object of our approach, the extended resolvent, can generate by successive reductions all the tools of the IST method, i.e., Green's functions, Jost solutions, spectral data, and their characterization equations. In the present paper (see also [17, 18]), we have shown that, at the moment, the extended resolvent is the only available tool for describing the embedding of one-dimensional objects in a two-dimensional theory within the framework of integrable equations.

For the specific case considered here, i.e., for the nonstationary Schrödinger equation with a potential that is a two-dimensional perturbation of a one-dimensional potential, one can find, as was done in [19], specific analytic properties of the Jost solutions, which, in addition to the cut along the real axis of the spectral parameter, also have a cut along the imaginary axis. However, only the application of the extended resolvent approach made it possible to relate different spectral data, in particular, those that appear in Subsections 4.3.1 and 4.3.3, and to derive characterization equations for the spectral data.

The inverse problem can be formulated as a problem of constructing a function $\Phi(x,\mathbf{k})$ that is analytic in the complex domain of the spectral parameter $\mathbf{k}$ except the real and imaginary axes and satisfies the asymptotic condition (2.2.66), which follows from (2.2.63). The discontinuity on the real axis is given in (4.5.60) and in (4.5.61) for the auxiliary Jost solutions, while the formulation of the inverse problem is completed by equations (4.3.2) and (4.3.46). In a forthcoming publication, we will consider the inverse problem in detail and consider the time evolution of initial data of kind (1.4) that is generated by the KPI equation. The results of this study will be applied, in particular, to the description of the interaction of $(1+1)$-dimensional objects, like solitons, that are inherited by the KPI equation from the underlying KdV theory.

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