Theoretical and Mathematical Physics, Vol. 116, No. 1, 1998

TOWARDS AN INVERSE SCATTERING THEORY FOR TWO-DIMENSIONAL NONDECAYING POTENTIALS

M. Boiti,[1] F. Pempinelli,[1] A. K. Pogrebkov,[2] and B. Prinari[1]

The inverse scattering method is considered for the nonstationary Schrödinger equation with the potential $u(x_1, x_2)$ nondecaying in a finite number of directions in the x plane. The general resolvent approach, which is particularly convenient for this problem, is tested using a potential that is the Bäcklund transformation of an arbitrary decaying potential and that describes a soliton superimposed on an arbitrary background. In this example, the resolvent, Jost solutions, and spectral data are explicitly constructed, and their properties are analyzed. The characterization equations satisfied by the spectral data are derived, and the unique solution of the inverse problem is obtained. The asymptotic potential behavior at large distances is also studied in detail. The obtained resolvent is used in a dressing procedure to show that with more general nondecaying potentials, the Jost solutions may have an additional cut in the spectral-parameter complex domain. The necessary and sufficient condition for the absence of this additional cut is formulated.

Contents

[1]Dipartimento di Fisica dell'Università and Sezione INFN, Lecce, Italy, e-mail: boiti@le.infn.it, pempi@le.infn.it, prinari@le.infn.it.
[2]Steklov Mathematical Institute, Russian Academy of Sciences, Moscow, Russia, e-mail: pogreb@mi.ras.ru.

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 116, No. 1, pp. 3–53, July, 1998. Original article submitted December 15, 1997.

1. Introduction

Since the early 1970s [1, 2], the nonstationary Schrödinger equation

$$(i\partial_{x_2} + \partial_{x_1}^2 - u)\Phi = 0 \tag{1.1}$$

is known to be a linear spectral problem associated with the Kadomtsev–Petviashvili equation [3] in the form (KPI)

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = 3u_{x_2x_2}, \tag{1.2}$$

where the function $u = u(x_1, x_2, t)$ is real. Being a natural generalization of the Korteweg–de Vries equation, Eq. (1.2) is considered the prototype of $(2+1)$-dimensional integrable equations. In fact, the KPI equation (together with KPII) was used by several authors as a laboratory for experimenting with new theoretical tools for handling the specific problems of multidimensional integrable models. (For a general overview of the subject and a rich bibliography, see [4, 5]).

The construction of a complete theory for the spectral transform of the potential $u(x_1, x_2)$ proved rather difficult. The real breakthrough was the discovery that the problem can be reduced to a nonlocal Riemann–Hilbert problem [6, 7]. The additional difficulties that arise in resolving the Cauchy problem for the KPI equation, because it is not strictly evolutional, were studied successively in [8–10].

The characterization equations for the spectral data of a potential $u(x_1, x_2)$ vanishing at infinity were formulated in [11]. The extension of the spectral transform to a potential approaching zero at large distances in every direction except a finite number was faced in [12]. However, only a potential asymptotically symmetric with respect to the origin was considered, excluding, in particular, the presence of solitons. As regards solitons, the pure N-soliton solution was given in [13, 14], where it was shown that the solitons of the KP equations are not localized in the plane but have a wavelike profile at large distances. The problem of the spectral transform of N solitons on an arbitrary decaying background was left unsolved. Up to now, only the problem of computing the spectral transform of an N-soliton solution with parameters "perturbed" by a dressing procedure has been solved [15].

Our aim is to investigate the spectral characteristics of Eq. (1.1) for potentials u having the nontrivial limits

$$u_{n,\pm}(x_1, x_2) = \lim_{x_2' \to \pm\infty} u\big(x_1 + 2\mu_n(x_2 - x_2'), x_2'\big), \qquad n = 1, 2, \ldots, N, \tag{1.3}$$

for N real constants μ_n and otherwise vanishing at large distances. In the general case, these asymptotic one-dimensional potentials must be free in the sense that their spectral transforms (related to the stationary Schrödinger equation) can have both a discrete and a continuous part.

The possibility of managing this class of solutions of the KPI equation using the inverse scattering method is very relevant mathematically since up to now, the usual methods of functional analysis (see, e.g., [16]) have not been able to handle this case. On the other hand, such an extension of the admissible potentials in (1.1) is crucial in applications since solitons in two dimensions are, in general, not localized in the space. In the case of the Davey–Stewartson equation, for which localized solitons were discovered, the inverse scattering theory of the KPI equation [17, 18] with a potential of type (1.3) governs the evolution of the boundaries.

The construction of the spectral theory for problem (1.1) for potentials satisfying the general property (1.3) is very involved. To overcome the difficulties, a new mathematical object, depending on two complex spectral parameters, called the extended resolvent (or resolvent for short) was introduced in [19–22]. This extended resolvent, via a reduction procedure, can be used to generate all relevant quantities in the inverse scattering theory, i.e., the Green's functions, Jost solutions, and spectral data, and the relations between them. It can be considered a generalization of the usual resolvent of the nonstationary Schrödinger operator. We are convinced that the extended resolvent is the effective foundation for the proper extension of the inverse scattering method. In [22, 23], general results for potentials of type (1.3) were obtained by making some (partially implicit) assumptions about the potential. Here, the general resolvent approach is presented in Sec. 2, where after reviewing the case of rapidly decaying potentials, we show how to modify the integral equations defining the Jost solution, the spectral data, and their characterization equations for nondecaying potentials.

Currently, we cannot precisely define the subclass of potentials of type (1.3) that satisfy the assumptions in Sec. 2; moreover, there is evidence that the structural properties of the spectral data can be even more complicated in the general case.

The aim of this paper is correspondingly twofold. On the one hand, we want to show that this subclass is neither void nor trivial, and on the other hand, we want to develop the necessary tools for exploring the spectral properties of the most general potentials of type (1.3).

Potentials belonging to this special subclass can be obtained via Bäcklund transformations of rapidly decaying potentials. We have a rather simple, transparent method for performing these transformations in [24], but we also need to control the analytic properties of the Jost solutions and the transformation and properties of the spectral data. Therefore, we need a significant reformulation of the method. This was done in [25] for Bäcklund transformations involving only the continuous spectrum. In Sec. 3, following the same lines, we consider a potential $\tilde{u}$ obtained à la Bäcklund by superimposing a soliton on a decaying smooth potential u. Using the so-called binary Bäcklund transformations and their Darboux version, we obtain the extended resolvent, Jost solutions, and spectral data for the potential $\tilde{u}$ and thoroughly study their properties. We show that these objects completely satisfy the scheme formulated in Sec. 2. In particular, we prove that the Jost solutions satisfy the modified integral equations and are piecewise analytic in the spectral parameter with a discontinuity at the real axis. We express the spectral data of $\tilde{u}$ in terms of the spectral data of the background u and the soliton parameters, and show that they satisfy the modified characterization equations of Sec. 2. In the spectral-parameter complex domain, the extended resolvent for $\tilde{u}$ has an additional cut in comparison with the case of a decaying potential.

At the end of Sec. 3, we present the asymptotic behavior of the potential $\tilde{u}$ at large distances in the x plane. We show that this potential indeed satisfies condition (1.3) with $N = 1$ and $u_{1,+}(x_1, x_2) \equiv u_{1,-}(x_1, x_2) \equiv u_1(x_1 + 2\mu x_2)$, where $u_1(x_1)$ is the standard one-dimensional soliton potential of the stationary Schrödinger equation and $\mu = \lambda_{\Re}$ with λ being the discrete spectrum value corresponding to the soliton. We also prove that the difference $\tilde{u}(x_1, x_2) - u_1(x_1 + 2\mu x_2)$ in the limit (1.3) with $\mu_n = \mu$ decays as $1/x_2$. This corresponds to the bending effect on the wave solutions described in [3]. Moreover, the asymptotic behavior at large distances of the difference $\tilde{u} - u$ is asymmetric with respect to the direction $x_1 + 2\mu x_2 = \text{const}$. More precisely, in the directions of the half plane to the left of $x_1 + 2\mu x_2 = \text{const}$, this difference decays exponentially, and in the other half-plane, it behaves as $(1/x_2)^3$ (or vice versa according to the sign of $\lambda_{\Im}$). We give conditions that improve these asymptotic behaviors.

The potential $\tilde{u}$ constructed in Sec. 3 is, of course, far from being a general potential that satisfies condition (1.3), even in the case $N = 1$. A more general example can be obtained by perturbing $\tilde{u}$, i.e., by considering the potential $\tilde{u} + u_2$, where u_2 is a rapidly decaying two-dimensional potential. In Sec. 4, we sketch this approach using a special version of dressing adapted to the extended resolvent scheme [26]. First, we show that for a general u_2, the additional cut of the extended resolvent (and of the corresponding Green's function), derived in Sec. 3 and specific to the two-dimensional case, leads to a cut of the Jost solution connecting points λ and $\bar{\lambda}$ in the spectral-parameter plane. Such an additional cut was discovered in the special case of a two-dimensional perturbation of a pure one-dimensional soliton in [27]. Here, we prove the existence of this cut in a more general case. We also formulate the necessary and sufficient restriction on the perturbing function u_2 for the absence of such a cut.

These results suggest that for a general potential satisfying (1.3), additional spectral data must be defined to describe the additional singularity of the Jost solution, and their characterization equations must be found. In Sec. 4, we discuss how the properties of these new quantities can be studied in two steps. First, one builds the resolvent of a special potential as a multisoliton potential or as an arbitrary decaying one-dimensional potential embedded in two dimensions. Then, one studies the potential dressed with a rapidly decaying two-dimensional potential. It is clear that the generalization from $N = 1$ to $N = 2$ in (1.3) can be significantly involved. The solution of these problems is deferred to a future work.

2. The general resolvent approach

In developing the two-dimensional inverse scattering theory, new problems appear which are intrinsically related to the multidimensionality of the theory. A new approach for solving them was proposed and developed in [12, 19–22]. It was called the resolvent approach because the principal tool used is a sort of extension to complex values of the resolvent of the principal Lax operator defining the scattering problem. We present the main features of this approach in the following.

2.1. Main definitions

The main mathematical objects of the theory are defined in the Schwartz space of distributions depending on two real and two complex variables, $p = (p_1, p_2)$ and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2)$ respectively. For brevity, this space is denoted by $\mathcal{S}'_{p,\mathbf{q}}$. The complex variable $\mathbf{q}$ plays the role of the spectral variable in the theory; the boldface font denotes that it is complex.

For any $A \in \mathcal{S}'_{p,\mathbf{q}}$, i.e., for any distribution $A(p;\mathbf{q})$, we define its Hermitian conjugate $A^\dagger$ in this space by

$$A^\dagger(p;\mathbf{q}) = \overline{A(-p; p+\overline{\mathbf{q}})} \tag{2.1}$$

and the distribution $A^{(\mathbf{s})}$ shifted in $\mathbf{q}$ by

$$A^{(\mathbf{s})}(p;\mathbf{q}) = A(p;\mathbf{q}+\mathbf{s}). \tag{2.2}$$

For any two elements A and B, we define their composition AB by the "shifted" convolution

$$(AB)(p;\mathbf{q}) = \int dp'\, A(p-p';\mathbf{q}+p')B(p';\mathbf{q}), \tag{2.3}$$

if the integral exists in the sense of distributions. This operation has the standard transitivity and associativity (when the composition is defined, of course) and also satisfies

$$(AB)^\dagger = B^\dagger A^\dagger, \qquad (AB)^{(\mathbf{s})} = A^{(\mathbf{s})}B^{(\mathbf{s})}. \tag{2.4}$$

We embed the "image" of a differential operator in the space $\mathcal{S}'_{p,\mathbf{q}}$ by the following rule: if $\mathcal{A}(x;\partial_x)$ is a differential operator

$$\mathcal{A}(x;\partial_x) = \sum_{n_1,n_2} a_{n_1,n_2}(x)\partial_{x_1}^{n_1}\partial_{x_2}^{n_2}, \qquad x = (x_1, x_2), \tag{2.5}$$

with the kernel $\mathcal{A}(x;y) = \mathcal{A}(x;\partial_x)\delta(x-y)$, then its image $A(p;\mathbf{q})$ is

$$A(p;\mathbf{q}) = \frac{1}{(2\pi)^2}\int dx \int dy\, e^{i(p+\mathbf{q})x}\mathcal{A}(x,y)e^{-i\mathbf{q}y}, \tag{2.6}$$

where px and $\mathbf{q}x$ denote scalar products (for instance, $px = p_1x_1 + p_2x_2$). Explicitly, we have

$$A(p;\mathbf{q}) = \sum_{n_1,n_2} a_{n_1,n_2}(p)(-i\mathbf{q}_1)^{n_1}(-i\mathbf{q}_2)^{n_2}, \tag{2.7}$$

where

$$a_{n_1,n_2}(p) = \frac{1}{(2\pi)^2}\int dx\, e^{ipx}\, a_{n_1,n_2}(x) \tag{2.8}$$

are the Fourier transforms of the coefficients in (2.5).

It is easy to verify that this embedding is consistent with the usual operations defined on differential operators. In fact, the image of the product of two differential operators is the composition (as defined in (2.3)) of their images, and the Hermitian conjugate of an image (as defined in (2.1)) is the image of the standard Hermitian conjugation of the differential operator, i.e., of the differential operator

$$\mathcal{A}^{\dagger}(x;\partial_x) = \sum_{n_1,n_2} (-1)^{n_1+n_2} \partial_{x_1}^{n_1} \partial_{x_2}^{n_2} \overline{a_{n_1,n_2}(x)}.$$

If the images of $i\partial_{x_j}$, $j = 1, 2$, are denoted by

$$Q_j(p, \mathbf{q}) = \mathbf{q}_j \delta(p), \qquad j = 1, 2, \tag{2.9}$$

then the image of the spectral operator in (1.1) is given by

$$L = L_0 - v, \tag{2.10}$$

where

$$L_0 = Q_2 - Q_1^2 \tag{2.11}$$

is the image of the differential operator (1.1) with potential $u = 0$ and v is the image of the potential u, i.e., the Fourier transform of u,

$$v(p) = \frac{1}{(2\pi)^2} \int dx\, e^{ipx} u(x). \tag{2.12}$$

The main object in our approach is the extended resolvent $M(p; \mathbf{q})$ that is the inverse of the differential operator L with respect to the composition defined in (2.3), i.e.,

$$ML = LM = I, \tag{2.13}$$

where

$$I(p; \mathbf{q}) = \delta(p) \tag{2.14}$$

is the image of unity.

To clarify the meaning of the introduced objects, we perform the inverse Fourier transform of relation (2.6) with respect to variables p and $\mathbf{q}_{\Re}$,

$$\mathcal{A}(x, y; \mathbf{q}_{\Im}) = \frac{1}{(2\pi)^2} \int dp \int d\mathbf{q}_{\Re}\, e^{-i(p+\mathbf{q}_{\Re})x} A(p; \mathbf{q}) e^{i\mathbf{q}_{\Re} y}, \tag{2.15}$$

and obtain

$$\mathcal{A}(x, y; \mathbf{q}_{\Im}) = \mathcal{A}(x; \partial_x + \mathbf{q}_{\Im}) \delta(x - y). \tag{2.16}$$

Therefore, the image $A(p; \mathbf{q})$ of the differential operator $\mathcal{A}(x; \partial_x)$ can be considered the Fourier transform of the kernel of the differential operator obtained from $\mathcal{A}(x; \partial_x)$ by adding $\mathbf{q}_{\Im}$ to the partial derivative ∂_x.

Then $M(p; \mathbf{q})$ is the Fourier transform of the fundamental solution of the operator obtained from L by the shift $\partial_x \to \partial_x + \mathbf{q}_{\Im}$. In the literature, such a shift is widely used as an intermediate step for obtaining a regularized fundamental solution in the limit $\mathbf{q}_{\Im} \to 0$. In contrast, it is essential in our approach to treat arbitrary values of $\mathbf{q}_{\Im}$ and then of $\mathbf{q}$, and to study, for instance, the analytic properties of $M(p; \mathbf{q})$ with respect to $\mathbf{q}$.

We show in the following that this additional freedom in the complex plane can be used to derive all relevant quantities, including the Jost solutions and the spectral data, from the extended resolvent $M(p; \mathbf{q})$ by using specific reduction procedures. Just this fact makes the study of their properties and their interrelations easier and more transparent.

For the inversion of L in (2.13) to be well defined and unique, we require that the resolvent $M(p; \mathbf{q})$ for a general $\mathbf{q}_{\Im}$ is a tempered distribution in the variables p and a locally integrable function with respect to the variables $\mathbf{q}_\Re$. Under these conditions, the inverse of a nondegenerate operator with constant coefficients exists and is unique. In particular, the inverse of L_0 (see (2.11)) exists and is equal to

$$M_0(p;\mathbf{q}) = \frac{\delta(p)}{\mathbf{q}_2 - \mathbf{q}_1^2}. \tag{2.17}$$

This inverse is unique because the condition of local integrability excludes the possibility of additional terms proportional to $\delta(p)\delta(\mathbf{q}_2 - \mathbf{q}_1^2)$.

The two equations in (2.13), which define M as right and left inverses of L, are then equivalent to the two integral equations

$$M = M_0 + M_0 v M, \qquad M = M_0 + M v M_0. \tag{2.18}$$

We expect that the existence and uniqueness of their solution can be studied with the standard methods of functional analysis. The main advantage over the standard integral equations defining the Jost solution is that the integrals in the right-hand sides of (2.18) are also well defined for potentials u of form (1.3), and we therefore have a well-defined quantity, as we show in the following, that can be used to define the Jost solutions for nondecaying potentials.

In this paper, we consider the special operator in (2.10), but the above scheme remains unchanged for any Lax operator L. The reality condition for the potential $u(x)$ in (1.1) is equivalent to the condition that both L and M are Hermitian in the sense of (2.1),

$$L^\dagger = L, \qquad M^\dagger = M. \tag{2.19}$$

The main technical tool in the study of the resolvent, as in the standard spectral theory of operators, is the Hilbert identity. If M and M' are the corresponding inverses of L and L', then

$$M' - M = -M'(L' - L)M. \tag{2.20}$$

This identity and its reduced versions in different forms are crucial in the theory. For instance, the analytic properties of the Jost solutions and the characterization equations for the spectral data can be derived from this Hilbert identity.

2.2. Jost and advanced/retarded solutions for decaying potentials

We first consider $v(p)$ that are Fourier transforms of smooth potentials $u(x)$ decaying sufficiently rapidly at large distances. For brevity, we call these potentials regular.

Because of (2.5) and (2.7), the image of a differential operator is a polynomial in $\mathbf{q}$, and images are therefore analytic functions of $\mathbf{q}$. This means that the extended resolvent $M(p;\mathbf{q})$ defined by (2.13) is almost analytic in a sense: the departure from analyticity of M is given by the $\bar\partial$-derivatives

$$(\bar\partial_j M)(p;\mathbf{q}) = \frac{\partial M(p;\mathbf{q})}{\partial \bar{\mathbf{q}}_j}, \qquad j = 1,2. \tag{2.21}$$

We can use the Hilbert identity (2.20) to calculate these derivatives, where we choose the shifted operator $L^{(\mathbf{s})}$ (see (2.2)) as L'. Then $M' = M^{(\mathbf{s})}$ and we have $M^{(\mathbf{s})} - M = -M^{(\mathbf{s})}(L^{(\mathbf{s})} - L)M$ instead of (2.20). From (2.2), (2.10), and (2.11), it follows that

$$L^{(\mathbf{s})} - L = L_0^{(\mathbf{s})} - L_0, \tag{2.22}$$

which goes to zero when $\mathbf{s} \to 0$. In contrast, we can see from (2.18) that M has multipliers M_0 from the left and the right and from (2.17) that the product $M_0^{(\mathbf{s})} M_0$ has no limit in $\mathcal{S}'_{p,\mathbf{q}}$ when $\mathbf{s} \to 0$. To avoid this indeterminacy, we rewrite (2.20) (with (2.22) taken into account) in terms of the less singular (truncated) quantities ML_0 and L_0M as

$$M^{(\mathbf{s}_2)} - M = (ML_0)^{(\mathbf{s}_2)}(M_0^{(\mathbf{s}_2)} - M_0)(L_0 M). \tag{2.23}$$

Then, for the $\bar{\partial}$-derivative with respect to $\mathbf{q}_2$ when $\mathbf{q}_{1\Im} \neq 0$, it follows from (2.17) that

$$\bar{\partial}_2 M = \pi|\nu\rangle\delta(L_0)\langle\omega|, \tag{2.24}$$

where $\delta(L_0)(p;\mathbf{q}) = \delta(p)\delta(\mathbf{q}_2 - \mathbf{q}_1^2)$ and we introduce $|\nu\rangle$, $\langle\omega| \in \mathcal{S}'_{p,\mathbf{q}}$ as

$$|\nu\rangle(p;\mathbf{q}) = (ML_0)(p;\mathbf{q}_1,\mathbf{q}_1^2), \tag{2.25}$$

$$\langle\omega|(p;\mathbf{q}) = (L_0 M)\big(p;\mathbf{q}_1, -p_2 + (p_1+\mathbf{q}_1)^2\big). \tag{2.26}$$

Thus, we can see that the nonanalytic part of $M(p;\mathbf{q})$ is completely determined by $|\nu\rangle$ and $\langle\omega|$, special reductions of ML_0 and L_0M depending on only the spectral variable $\mathbf{q}_1$. Their Fourier transforms in p are related to the standard Jost solutions. To show this, we introduce

$$\chi(x,\mathbf{q}_1) = \int dp\, e^{-ipx}|\nu\rangle(p;\mathbf{q}_1), \qquad \xi(x,\mathbf{q}_1) = \int dp\, e^{-ipx}\langle\omega|(p;\mathbf{q}_1 - p_1) \tag{2.27}$$

and let $\ell(\mathbf{k})$ be the special two-dimensional vector

$$\ell(\mathbf{k}) = (\mathbf{k},\mathbf{k}^2). \tag{2.28}$$

Then the functions

$$\Phi(x,\mathbf{k}) = e^{-i\ell(\mathbf{k})x}\chi(x,\mathbf{k}), \qquad \Psi(x,\mathbf{k}) = e^{i\ell(\mathbf{k})x}\xi(x,\mathbf{k}) \tag{2.29}$$

are the standard Jost solutions of the spectral problem (1.1) and of its dual respectively. The spectral parameter $\mathbf{q}_1$ has been renamed $\mathbf{k}$ in (2.29) to conform with the more traditional notation. That Φ and Ψ are the Jost solutions can be verified directly or by deriving the integral equations satisfied by $|\nu\rangle$ and $\langle\omega|$. By means of reductions (2.25) and (2.26) of integral equations (2.18), we obtain

$$|\nu\rangle(p;\mathbf{q}) = \delta(p) + \frac{\big(v|\nu\rangle\big)(p;\mathbf{q})}{\mathcal{P}(p;\mathbf{q}_1)}, \tag{2.30}$$

$$\langle\omega|(p;\mathbf{q}) = \delta(p) - \frac{\big(\langle\omega|v\big)(p;\mathbf{q})}{\mathcal{P}(p;\mathbf{q}_1)}, \tag{2.31}$$

where

$$\mathcal{P}(p;\mathbf{q}_1) = p_2 - p_1(p_1 + 2\mathbf{q}_1). \tag{2.32}$$

Then, performing transformation (2.27), we derive the integral equations in the x space

$$\chi(x,\mathbf{k}) = 1 + \int dx'\, G_0(x-x',\mathbf{k})u(x')\chi(x',\mathbf{k}), \tag{2.33}$$

$$\xi(x,\mathbf{k}) = 1 + \int dx'\, G_0(x'-x,\mathbf{k})u(x')\xi(x',\mathbf{k}), \tag{2.34}$$

where

$$G_0(x-x',\mathbf{k}) = \frac{1}{(2\pi)^2}\int dp \int dp'\, e^{-ipx+ip'x'} M_0\big(p-p';p'+\ell(\mathbf{k})\big). \tag{2.35}$$

They are the standard integral equations (cf. [6, 7]) defining the Jost solution and its dual. In fact,

$$G_0(x,\mathbf{k}) = \frac{\operatorname{sgn} x_2}{2\pi i}\int dp_1\, \theta(-p_1\mathbf{k}_{\Im}x_2)e^{-ip_1x_1 - ip_1(p_1+2\mathbf{k})x_2} \tag{2.36}$$

is the Green's function of the nonstationary Schrödinger operator with $u = 0$, i.e., it satisfies the differential equation

$$(i\partial_{x_2} + \partial_{x_1}^2 - 2i\mathbf{k}\partial_{x_1})G_0(x - x'; \mathbf{k}) = \delta(x - x') \tag{2.37}$$

and is analytic in $\mathbf{k}$ when $\mathbf{k}_\Im \neq 0$.

We stress that this result is general and that the Green's function $G(x, x', \mathbf{k})$ of the operator L, Eq. (2.10), with an arbitrary potential v is given in terms of $M(p; \mathbf{q})$ by

$$G(x, x', \mathbf{k}) = \frac{1}{(2\pi)^2} \int dp \int dp' \, e^{-ipx+ip'x'} G(p, p', \mathbf{k}), \tag{2.38}$$

$$G(p, p', \mathbf{k}) = M\big(p - p'; p' + \ell(\mathbf{k})\big). \tag{2.39}$$

Finally, we note that for $\mathbf{k}_\Im \neq 0$, Eq. (2.39) can be used to reconstruct the resolvent from the Green's function. In fact, we have

$$M(p; \mathbf{q}) = G\big(p + \mathbf{q}_\Re - \Re\ell(\mathbf{k}), \mathbf{q}_\Re - \Re\ell(\mathbf{k}), \mathbf{k}\big), \tag{2.40}$$

where

$$\mathbf{k} = \frac{\mathbf{q}_{2\Im}}{2\mathbf{q}_{1\Im}} + i\mathbf{q}_{1\Im}. \tag{2.41}$$

However, this property is specific to the spectral problem considered, Eq. (1.1). It is not true for the heat equation, for instance.

A two-dimensional inverse scattering theory could be built using only the Green's functions, following [28], for instance. However, the previous remark suggests that the Green's function, in general, contains less information than the resolvent and that the Hilbert identities written for the Green's functions are model dependent. In contrast, the Hilbert identity for the resolvent M in form (2.20) does not depend on the special L being considered. Moreover, a direct computation shows that the nonanalytic part of the Green's function cannot be directly expressed in terms of the Jost solutions as is done in (2.24).

From the integral equations for the Jost solutions, the following asymptotic behaviors at large $\mathbf{q}_1$, which allow the potential to be reconstructed from the Jost solutions, can be easily deduced:

$$|\nu\rangle(p, \mathbf{q}) = \delta(p) + O(1/\mathbf{q}_1), \qquad \langle\omega|(p, \mathbf{q}) = \delta(p) + O(1/\mathbf{q}_1), \tag{2.42}$$

and

$$p_1|\nu\rangle(p, \mathbf{q}) = -\frac{1}{2\mathbf{q}_1} v(p) + O(1/\mathbf{q}_1^2), \qquad p_1\langle\omega|(p, \mathbf{q}) = \frac{1}{2\mathbf{q}_1} v(p) + O(1/\mathbf{q}_1^2). \tag{2.43}$$

The resolvent can also be used to naturally obtain the so-called advanced/retarded solutions of the nonstationary Schrödinger equation, which are relevant in the theory (see [6, 19–22, 29]). Above, we considered the nonanalytic part of the resolvent $M(p; \mathbf{q})$ with respect to $\mathbf{q}_2$ in the case $\mathbf{q}_{1\Im} \neq 0$. We now take $\mathbf{q}_{1\Im} = 0$ and compute the $\bar{\partial}_2$-derivative of M. It depends on two other reductions of the resolvent,

$$|\nu_{a/r}\rangle(p; \mathbf{q}) = \lim_{\varepsilon \to +0} (ML_0)(p; \mathbf{q}_{1\Re}, \mathbf{q}_{1\Re}^2 \pm i\varepsilon), \tag{2.44}$$

$$\langle\omega_{a/r}|(p; \mathbf{q}) = \lim_{\varepsilon \to +0} (L_0 M)\big(p; \mathbf{q}_{1\Re}, -p_2 + (p_1 + \mathbf{q}_{1\Re})^2 \pm i\varepsilon\big). \tag{2.45}$$

These solutions depend on only the real parameter $\mathbf{q}_{1\Re}$ and have no analytic continuation in the complex domain. Their integral equations follow directly from proper reductions of (2.18) and have the forms

$$|\nu_{a/r}\rangle(p; \mathbf{q}) = \delta(p) + \frac{\big(v|\nu_{a/r}\rangle\big)(p; \mathbf{q})}{p_2 - p_1(p_1 + 2\mathbf{q}_{1\Re}) \pm i0}, \tag{2.46}$$

$$\langle\omega_{a/r}|(p; \mathbf{q}) = \delta(p) - \frac{\big(\langle\omega_{a/r}|v\big)(p; \mathbf{q})}{p_2 - p_1(p_1 + 2\mathbf{q}_{1\Re}) \mp i0}. \tag{2.47}$$

In the x space, introducing $\chi_{a/r}$ and $\xi_{a/r}$ (for real k) by analogy with (2.27), we obtain the integral equations

$$\chi_{a/r}(x,k) = 1 + \int dx'\, G_{0,a/r}(x-x',k)u(x')\chi_{a/r}(x',k), \tag{2.48}$$

$$\xi_{a/r}(x,k) = 1 + \int dx'\, G_{0,a/r}(x'-x,k)u(x')\xi_{a/r}(x',k), \tag{2.49}$$

where the Green's functions are given by

$$G_{0,a/r}(x,x',k) = \frac{1}{(2\pi)^2}\int dp \int dp'\, e^{-ipx+ip'x'} G_{0,a/r}(p,p',k), \tag{2.50}$$

$$G_{0,a/r}(p,p',k) = M_0(p-p'; p'_1 + k, p'_2 + k^2 \pm i0) \tag{2.51}$$

(cf. (2.38), (2.39), and (2.28)). In explicit form, we have

$$G_{0,a/r}(x,k) = -\frac{\theta(\pm x_2)}{2\sqrt{\pi|x_2|}} \exp\left(ix_1k + ix_2k^2 + i\frac{x_1^2}{4x_2} \pm i\frac{\pi}{4}\right). \tag{2.52}$$

We can thus see that the advanced/retarded Green's functions are also given as specific values of the resolvent. Of course, because these functions are determined only on the real axis, the resolvent cannot be reconstructed from them. In addition, (2.40) and (2.51) cannot be used to express them in terms of the Green's function of the Jost solutions because the variables in the right-hand side of (2.40) tend to infinity for the special reduction considered in (2.51).

Finally, we note that in the resolvent approach, all types of solutions naturally appear in dual pairs, like $|\nu\rangle$ and $\langle\omega|$ above, that are related to the Schrödinger spectral operator (1.1) and its dual. This duality is relevant in the following, and the bra and ket notation we have introduced helps in using it correctly. However, the reader must be advised that it has nothing to do with duality in the theory of vector spaces. Indeed, the solutions in our context are not vectors but elements of the space $\mathcal{S}'_{p,\mathbf{q}}$.

We can easily show, using their definitions and (2.19), that these dual pairs of the Jost and advanced/retarded solutions are related by the conjugation properties

$$|\nu\rangle^\dagger = \langle\omega|, \qquad |\nu_{a/r}\rangle^\dagger = \langle\omega_{r/a}|. \tag{2.53}$$

The Jost solutions satisfy the differential equations (in the p-space version)

$$L|\nu\rangle = |\nu\rangle L_0, \qquad \langle\omega|L = L_0\langle\omega|. \tag{2.54}$$

The advanced/retarded solutions satisfy the same equations, but only on the real axis $\mathbf{q}_\Im = 0$. Using relations (2.27) and (2.29), we can verify that Φ indeed satisfies (1.1).

Using a special reduced version of Hilbert identity (2.20), we can also derive the scalar products

$$\langle\omega|\nu\rangle = I, \qquad \langle\omega_{a/r}|\nu_{r/a}\rangle = I. \tag{2.55}$$

Finally, we note that equation (2.24) for the $\bar\partial_2$-derivative of M can be integrated using the Cauchy–Green formula. Taking into account that

$$\lim_{\mathbf{q}_2\to\infty} \mathbf{q}_2 M(p;\mathbf{q}) = \delta(p), \tag{2.56}$$

because of (2.10)–(2.13), we obtain a bilinear representation of the resolvent in terms of the Jost solutions,

$$M = |\nu\rangle M_0\langle\omega|. \tag{2.57}$$

From it, again using (2.56), we can prove the completeness relation

$$|\nu\rangle\langle\omega| = I. \tag{2.58}$$

Analogously, we can also obtain a bilinear representation of the resolvent at $\mathbf{q}_{1\Im} = 0$ in terms of the advanced/retarded solutions,

$$M = |\nu_{\mathrm{a/r}}\rangle M_0 \langle\omega_{\mathrm{r/a}}|. \tag{2.59}$$

Although this representation is valid only on the real $\mathbf{q}_1$ axis, we derive the completeness relation for the advanced/retarded solutions from it:

$$|\nu_{\mathrm{a/r}}\rangle\langle\omega_{\mathrm{r/a}}| = I. \tag{2.60}$$

Using (2.54) and (2.58), we derive a bilinear representation for the Schrödinger operator itself,

$$L = |\nu\rangle L_0 \langle\omega|. \tag{2.61}$$

Bilinear representations (2.57) and (2.61) can be considered the operator realization of the well-known dressing procedure [2]. They are significant in studying the analytic properties of the resolvent $M(p;\mathbf{q})$ with respect to the other complex variable $\mathbf{q}_1$. As we see in the following, we thus naturally consider special reductions of the second level of the resolvent that supply the spectral data of our problem (1.1). But we first consider how to modify the above construction for potentials of type (1.3).

2.3. Jost and advanced/retarded solutions for nondecaying potentials

In the following, the main mathematical objects (potential, resolvent, solutions, and spectral data) are marked by a tilde, to distinguish them from the corresponding objects in the decaying-potential case.

If the potential $\tilde{u}(x)$, as in (1.3), is not decaying in some directions, i.e., if $\tilde{v}$ is a singular distribution, it is known (see also below) that the standard integral equations (2.33) and (2.34) become meaningless, even in the simplest case $\tilde{u}(x) = u(x_1)$. On the other hand, integral equations (2.18) defining the resolvent are still well defined, as was shown in detail in [22]. For instance, for the one-dimensional potential $u(x_1)$, we obtain

$$\widetilde{M}(p;\mathbf{q}) = \delta(p_2) M_{\mathbf{q}_2}(p_1;\mathbf{q}_1), \tag{2.62}$$

where $M_{\mathbf{z}}(p_1;\mathbf{q}_1)$ is the resolvent of the one-dimensional Sturm–Liouville operator $\partial^2_{x_1} - u(x_1) + z$.

The resolvent properties, however, significantly change in the nondecaying-potential case. For instance, the reductions in (2.25) and (2.26) do not exist because $(ML_0)(p;\mathbf{q})$ and $(L_0M)(p;\mathbf{q})$ are discontinuous just at the points of reduction in this case. An accurate study (see [22]) shows that a specific path for approaching these limiting values must be chosen and that the Jost solutions in the nondecaying-potential case must be defined using the limiting procedures

$$|\tilde{\nu}\rangle(p;\mathbf{q}) = \lim_{\varepsilon\to+0} (\widetilde{M}L_0)(p;\mathbf{q}_1,\mathbf{q}_1^2+\varepsilon), \tag{2.63}$$

$$\langle\tilde{\omega}|(p;\mathbf{q}) = \lim_{\varepsilon\to+0} (L_0\widetilde{M})(p;\mathbf{q}_1,-p_2+(p_1+\mathbf{q}_1)^2+\varepsilon), \tag{2.64}$$

where ε is real and positive. Using the reduction procedure indicated in these formulas, we derive the integral equations

$$|\tilde{\nu}\rangle(p;\mathbf{q}) = \delta(p) + \frac{1}{p_1 + i0\mathbf{q}_{1\Im}} \left(\frac{p_1(\tilde{v}|\tilde{\nu}\rangle)(p;\mathbf{q})}{\mathcal{P}(p;\mathbf{q}_1)}\right), \tag{2.65}$$

$$\langle\tilde{\omega}|(p;\mathbf{q}) = \delta(p) - \frac{1}{p_1 - i0\mathbf{q}_{1\Im}} \left(\frac{p_1(\langle\tilde{\omega}|\tilde{v})(p;\mathbf{q})}{\mathcal{P}(p;\mathbf{q}_1)}\right) \tag{2.66}$$

from (2.18). Here, the quantity

$$\frac{p_1}{\mathcal{P}(p;\mathbf{q}_1)} \equiv \frac{p_1}{p_2 - p_1(p_1+2\mathbf{q}_1)}$$

(cf. (2.32)) must be considered as a whole object. This object admits well-defined reductions to the straight lines $p_2 = 2\mu_n p_1$. Therefore, this ratio can safely be multiplied by $\tilde{v}|\tilde{\nu}\rangle$ and $\langle\tilde{\omega}|\tilde{v}$ that have the union of such lines as a singular support. Thus, if the distributions are multiplied in the indicated order, the right-hand sides of these integral equations are well defined. For regular $\tilde{v}$, the brackets in the second term in the right-hand side can be opened, the distribution in front of them is cancelled by p_1 in the numerator, and integral equations (2.30) and (2.31) are recovered, as expected. We mention, however, that the construction based on the dressing procedure (see Sec. 4) demonstrates that the analytic properties of $|\tilde{\nu}\rangle$ and $\langle\tilde{\omega}|$ in the complex $\mathbf{q}_1$ plane could be more involved than in the decaying-potential case and that in addition to the traditional cut on the real axis, there could be other cuts related to the presence of solitons.

To make explicit the novelty of integral equations (2.65) and (2.66) in comparison with those for the usual potentials $u(x)$ vanishing at large distances, we rewrite them in the x space in terms of $\tilde{\chi}(x,\mathbf{k})$ and $\tilde{\xi}(x,\mathbf{k})$ defined in (2.27). Taking the Fourier transform of (2.65) and (2.66), we obtain

$$\tilde{\chi}(x,\mathbf{k}) = 1 + \int_{-\mathbf{k}_\Im\infty}^{x_1} dy_1 \int dx'\, \partial_{y_1} G_0(y_1 - x_1', x_2 - x_2', \mathbf{k})\tilde{u}(x')\tilde{\chi}(x',\mathbf{k}), \tag{2.67}$$

$$\tilde{\xi}(x,\mathbf{k}) = 1 + \int_{\mathbf{k}_\Im\infty}^{x_1} dy_1 \int dx'\, \partial_{y_1} G_0(x_1' - y_1, x_2' - x_2, \mathbf{k})\tilde{u}(x')\tilde{\xi}(x',\mathbf{k}), \tag{2.68}$$

where $G_0(x,\mathbf{k})$ is defined in (2.36). Because of (2.36), it is easy to show that

$$G_0(x,\mathbf{k}) = \frac{\operatorname{sgn}\mathbf{k}_\Im}{4\pi\mathbf{k}x_2} + o\big(|x_2|^{-1}\big), \qquad x_2 \to \infty, \tag{2.69}$$

and these integral equations, for potentials with nontrivial limits (1.3), therefore have well-defined kernels, because the integration over x_2' is performed after the differentiation with respect to y_1. In the one-dimensional case, i.e., where $\tilde{u}(x) = u(x_1)$, both $\tilde{\chi}(x,\mathbf{k})$ and $\tilde{\xi}(x,\mathbf{k})$ are independent of x_2. We can then integrate $\partial_{x_1} G_0$ over x_2' explicitly and obtain

$$\int dx_2'\, \partial_{x_1} G_0(x - x', \mathbf{k}) = \operatorname{sgn}\mathbf{k}_\Im \theta\big(\mathbf{k}_\Im(x_1 - x_1')\big)e^{2i\mathbf{k}(x_1 - x_1')}. \tag{2.70}$$

This reduces (2.67) to the standard integral equation for the Jost solution of the stationary one-dimensional Sturm–Liouville equation with the potential $u(x_1)$.

In contrast, limits (2.44) and (2.45) defining the advanced/retarded solutions are also well defined for nondecaying potentials, and integral equations (2.46)–(2.49) do not need any regularization because the Green's functions $G_{0,\mathrm{a}/r}$ are integrable with respect to x_2. In fact, from (2.52) we obtain

$$\int dx_2\, G_{0,\mathrm{a}/r}(x,k) = \frac{\mp i}{2|k|} e^{i(kx_1 \pm |kx_1|)}, \tag{2.71}$$

which is a well-defined distribution in x_1 for all $k \neq 0$. In the general case, the advanced/retarded solutions for a potential of type (1.3) have a zero at $k = 0$.

It can be verified that conjugation properties (2.53) as well as differential equations (2.54) are satisfied for both types of solutions. The asymptotic behavior of the Jost solutions given in (2.42) and (2.43) also remains valid. The scalar products for the advanced/retarded solutions are preserved in form (2.55), whereas the scalar product of the Jost solutions is modified,

$$\langle\widetilde{\omega}|\widetilde{\nu}\rangle = \widetilde{T}^{-1}, \tag{2.72}$$

where

$$\widetilde{T}^{-1}(p;\mathbf{q}) = \delta(p) - 2i\pi \operatorname{sgn}\mathbf{q}_{1\Im}\delta(p_1)\left(\frac{p_1(\tilde{v}|\widetilde{\nu}\rangle)(p;\mathbf{q}_1)}{\mathcal{P}(p;\mathbf{q}_1)}\right). \tag{2.73}$$

Here again, the distributions must be multiplied in the indicated order. If the potential $\tilde{v}$ is regular, the bracket in the second term in the right-hand side of (2.73) can be opened, and then $\widetilde{T} = I$. In the singular case, as was shown in [22], the term in brackets is proportional to $\delta(p_2)$; therefore, we can write

$$\widetilde{T}(p;\mathbf{q}) = \tilde{t}(\mathbf{q}_1)\delta(p), \tag{2.74}$$

where $\tilde{t}(\mathbf{q}_1)$ can be called the transmission coefficient. It is analytic in the upper and lower half-planes of $\mathbf{q}_1$ and has a discontinuity on the real axis. It is one of the spectral data that we describe in Sec. 2.5 and generalizes the traditional one-dimensional transmission coefficient to two dimensions.

Instead of considering the limit of $(\widetilde{ML_0})(p;\mathbf{q}_1,\mathbf{q}_1^2+\varepsilon)$ and $(\widetilde{L_0M})\big(p;\mathbf{q}_1,-p_2+(p_1+\mathbf{q}_1)^2+\varepsilon\big)$ for $\varepsilon\to+0$ in (2.63) and (2.64), we could consider the opposite limit $\varepsilon\to-0$. It easily follows that

$$\lim_{\varepsilon\to-0}(\widetilde{ML_0})(p;\mathbf{q}_1,\mathbf{q}_1^2+\varepsilon) = \big(|\widetilde{\nu}\rangle\widetilde{T}\big)(p;\mathbf{q}), \tag{2.75}$$

$$\lim_{\varepsilon\to-0}(\widetilde{L_0M})\big(p;\mathbf{q}_1,-p_2+(p_1+\mathbf{q}_1)^2+\varepsilon\big) = \big(\widetilde{T}\langle\widetilde{\omega}|\big)(p;\mathbf{q}). \tag{2.76}$$

The integral equations satisfied by $|\widetilde{\nu}\rangle\widetilde{T}$ and $\widetilde{T}\langle\widetilde{\omega}|$ differ from (2.65) and (2.66) only in the sign of the $i0$ terms in the denominators; in the x space, they differ from (2.67) and (2.68) only in the sign of the infinite limits of the integration with respect to y_1. Since $\widetilde{T}$ commutes with L_0 in the sense of composition (2.3), these reductions, as $|\widetilde{\nu}\rangle$ and $\langle\widetilde{\omega}|$, satisfy (2.54) and are respective solutions of spectral problem (1.1) and its dual.

We stress that in the nondecaying-potential case, the problem of obtaining a bilinear representation of type (2.57) for the resolvent as well as a completeness relation for the Jost solutions is essentially more complicated. In this paper, we treat this problem in the special case of a soliton on an arbitrary decaying background. The special superimposition obtained using a Bäcklund transformation and a more general superimposition obtained using a dressing are considered.

2.4. Spectral data for decaying potentials

We first consider a regular potential v. The Jost solutions are analytic functions of $\mathbf{q}_1$ with a cut along the real axis, and their values on the two sides of the real axis are denoted by $|\nu^\pm\rangle$, $\langle\omega^\pm| \in S'_{p,\mathbf{q}}$, where

$$|\nu^\pm\rangle(p;\mathbf{q}) = \lim_{\mathbf{q}_{1\Im}\to\pm0}|\nu\rangle(p;\mathbf{q}), \tag{2.77}$$

$$\langle\omega^\pm|(p;\mathbf{q}) = \lim_{\mathbf{q}_{1\Im}\to\pm0}\langle\omega|(p;\mathbf{q}). \tag{2.78}$$

Spectral data traditionally relate these values. It is, however, more convenient to define the spectral data as operators in the space $S'_{p,\mathbf{q}}$ that transform Jost solutions into advanced/retarded solutions. These operators can be computed using (2.57), the bilinear representation of the resolvent. Multiplying (2.57) by L_0 from the right and using (2.54) and (2.58), we obtain

$$ML_0 = I + |\nu\rangle M_0\langle\omega|v.$$

Computing special reductions (2.25) and (2.44) of this equation that respectively define the Jost and the advanced/retarded solutions, we obtain

$$|\nu^\sigma\rangle(p;\mathbf{q}) = \delta(p) + \int dp'\, \frac{|\nu^\sigma\rangle(p-p';\mathbf{q}+p')\big(\langle\omega^\sigma|v\big)(p';\mathbf{q})}{\mathcal{P}(p';\mathbf{q}_{1\Re}) - i\sigma p_1' 0}, \tag{2.79}$$

$$|\nu_{\mathrm{a}/r}\rangle(p;\mathbf{q}) = \delta(p) + \int dp'\, \frac{|\nu^\sigma\rangle(p-p';\mathbf{q}+p')\big(\langle\omega^\sigma|v\big)(p';\mathbf{q})}{\mathcal{P}(p';\mathbf{q}_{1\Re}) \pm i0}, \tag{2.80}$$

where $\sigma = +,-$ and $\mathcal{P}(p;\mathbf{q}_{1\Re})$ is given in (2.32). In fact, from the original integral equations defining the resolvent, Eq. (2.18), it follows that $M(p;\mathbf{q})$ for $\mathbf{q}_{2\Im} \neq 0$ is a continuous function of $\mathbf{q}_{1\Im}$ in a neighborhood of zero. Therefore, we can approach the real axis indifferently from above or below in evaluating (2.57) at $\mathbf{q}_{1\Im} = 0$, and we can thus choose either $\sigma = +$ or $\sigma = -$ in the right-hand side of (2.80). Analogous equations can be obtained for $\langle\omega^\sigma|$ and $\langle\omega_{\mathrm{a}/r}|$.

Considering the differences $|\nu^\sigma\rangle - |\nu_{\mathrm{a}/r}\rangle$ and $\langle\omega^\sigma| - \langle\omega_{\mathrm{a}/r}|$, we obtain

$$|\nu_{\mathrm{a}/r}\rangle = |\nu^\sigma\rangle(R_\pm^{-\sigma})^\dagger, \qquad \langle\omega_{\mathrm{a}/r}| = R_\mp^\sigma\langle\omega^\sigma|, \qquad \sigma = +,-, \tag{2.81}$$

where

$$R_\pm^\sigma(p;\mathbf{q}) = \delta(p) + r_\pm^\sigma(p;\mathbf{q}) \tag{2.82}$$

with

$$r_\pm^\sigma(p;\mathbf{q}) = \pm 2i\pi\theta(\pm\sigma p_1)\delta\big(\mathcal{P}(p;\mathbf{q}_{1\Re})\big)\big(v|\nu^\sigma\rangle\big)(p;\mathbf{q}). \tag{2.83}$$

Thus, the spectral data $R_\pm^\sigma$ are obtained as a special reduction of the second level of the resolvent. Using (2.12) and (2.27), we obtain

$$r_\pm^\sigma(p;\mathbf{q}) = \frac{\mp 1}{2i\pi}\theta(\pm\sigma p_1)\delta\big(\mathcal{P}(p;\mathbf{q}_{1\Re})\big)\int dx\, e^{ipx}u(x)\chi^\sigma(x,\mathbf{q}_{1\Re}) \tag{2.84}$$

in terms of quantities in the x space; Eq. (2.84) is the familiar definition of spectral data [9–11] in the decaying-potential case.

The characterization equations satisfied by spectral data (2.82) can be obtained by computing special reductions of the Hilbert identity satisfied by the resolvent or from the bilinear representation of the resolvent. They can be written in a particularly transparent form using the composition law introduced in (2.3):

$$(R_\pm^{-\sigma})^\dagger R_\pm^\sigma = I, \tag{2.85}$$

$$R_\pm^\sigma(R_\pm^{-\sigma})^\dagger = I, \tag{2.86}$$

$$(R_+^\sigma)^\dagger R_+^\sigma = (R_-^\sigma)^\dagger R_-^\sigma, \tag{2.87}$$

where again $\sigma = +,-$. The first relation enables us to invert (2.81),

$$|\nu^\sigma\rangle = |\nu_{\mathrm{a}/r}\rangle R_\pm^\sigma, \qquad \langle\omega^\sigma| = (R_\mp^{-\sigma})^\dagger\langle\omega_{\mathrm{a}/r}|. \tag{2.88}$$

Consequently, we can introduce the alternative spectral data

$$F^\sigma = (R_\pm^{-\sigma})^\dagger R_\pm^{-\sigma} \tag{2.89}$$

and relate the Jost solutions computed on the two sides of the real axis by

$$|\nu^\sigma\rangle = |\nu^{-\sigma}\rangle F^{-\sigma}, \qquad \langle\omega^\sigma| = F^\sigma\langle\omega^{-\sigma}|, \qquad \sigma = +,-. \tag{2.90}$$

These spectral data satisfy the characterization equations

$$(F^{\pm})^{\dagger} = F^{\pm}, \qquad F^{-} = (F^{+})^{-1}, \tag{2.91}$$

to which we must add the requirement that $F^{\pm}$ must be decomposable into the product (2.89) of two triangular operators (2.82) (see [6, 11, 12]).

Finally, we note that the quantity $v|\nu^{\sigma}\rangle$ entering the definition of the spectral data via use of the "differential" equation (2.54) satisfied by the Jost solution can be rewritten in the form

$$\big(v|\nu^{\sigma}\rangle\big)(p;\mathbf{q}) = \mathcal{P}(p;\mathbf{q}_{1\Re})|\nu^{\sigma}\rangle(p;\mathbf{q}), \tag{2.92}$$

explicitly showing that the spectral data are given as the "residuum" of the Jost solution on the surface $\mathcal{P}(p;\mathbf{q}_{1\Re}) = 0$. Of course, because of (2.53), the spectral data can also be written in terms of the dual Jost solution $\langle\omega|$. Because of integral equation (2.30) for the Jost solution, we also have

$$R^{\sigma}_{\pm}(p;\mathbf{q}) = |\nu^{\sigma}\rangle(p;\mathbf{q}) - \frac{\big(\mathcal{P}(p;\mathbf{q}_{1\Re})|\nu^{\sigma}\rangle(p;\mathbf{q})\big)}{\mathcal{P}(p;\mathbf{q}_{1\Re}) \pm i0}, \tag{2.93}$$

where the distributions must be multiplied in the indicated order.

In conclusion, we note that in the decaying-potential case, where the time is fixed, the resolvent approach discussed above adds nothing substantially new to the traditional approach. When the time evolution is turned on, however, the usefulness of the resolvent approach becomes evident. In fact, only in this framework (see [10] for details) was it possible to study, in the case of unconstrained rapidly decreasing initial data, the singular behavior of the KPI solution u and the Jost solutions and, at the initial time $t = 0$, the appearance of "constraints" on u for $t \neq 0$ and a tail slowly decreasing for $x_1 \to t\infty$.

2.5. Spectral data for nondecaying potentials

In the previous section, we described the case with decaying potentials u in detail to emphasize the necessary modifications for generalizing the definition of the spectral data to the case with nondecaying potentials $\tilde{u}$. The main problem is that a bilinear representation of type (2.57) for the resolvent is not yet known in the general case. In [23], by analogy with the alternative definition of spectral data in (2.93), it was argued that the spectral data can be defined as

$$\widetilde{R}^{\sigma}_{\pm}(p;\mathbf{q}) = \big(|\tilde{\nu}^{\sigma}\rangle\widetilde{T}^{\sigma}\big)(p;\mathbf{q}) - \frac{\big(\mathcal{P}(p;\mathbf{q}_{1\Re})\big(|\tilde{\nu}^{\sigma}\rangle\widetilde{T}^{\sigma}\big)(p;\mathbf{q})\big)}{\mathcal{P}(p;\mathbf{q}_{1\Re}) \pm i0}, \tag{2.94}$$

where $\widetilde{T}^{\pm}$ are the limiting values of $\widetilde{T}$ (cf. (2.73)) on the two sides of the real axis in the complex $\mathbf{q}_1$ plane. These limiting values are introduced to compensate possible singularities at $\mathbf{q}_{1\Re} = \mu_n$ due to the singularities of $\tilde{v}(p)$ along the lines $p_2 - 2\mu_n p_1 = 0$. Using (2.92), we can rewrite (2.94) as

$$\widetilde{R}^{\sigma}_{\pm}(p;\mathbf{q}) = |\tilde{\nu}^{\sigma}\rangle(p;\mathbf{q})\tilde{t}^{\sigma}(\mathbf{q}_1) - \frac{\big(\tilde{v}|\tilde{\nu}^{\sigma}\rangle\big)(p;\mathbf{q})\tilde{t}^{\sigma}(\mathbf{q}_1)}{\mathcal{P}(p;\mathbf{q}_{1\Re}) \pm i0}. \tag{2.95}$$

To obtain the characterization equations for the spectral data using the Hilbert identity, we must now use reductions (2.63) and (2.64) of the resolvent $\widetilde{M}$ instead of (2.25) and (2.26). This derivation requires precise knowledge of the singular behavior of the Jost and advanced/retarded solutions for potentials of type (1.3) and is not yet completed. In this paper, we first consider a potential $\tilde{u}(x)$ obtained by superimposing, via Bäcklund transformations, a soliton on a potential $u(x)$ rapidly decaying at infinity. In this case, we can compute the resolvent, the Jost solutions, and the spectral data of $\tilde{u}(x)$. The transformed Jost solutions are still piecewise analytic in the spectral complex plane with a cut along the real axis. Therefore, the spectral data $\widetilde{R}^{\sigma}_{\pm}$ of $\tilde{u}(x)$ can indeed be defined using (2.94), and we prove that they satisfy the modified characterization equations

$$(\widetilde{R}^{-\sigma}_{\mp})^{\dagger}\widetilde{R}^{\sigma}_{\pm} = \widetilde{T}^{\sigma}, \tag{2.96}$$

$$\widetilde{R}^{\sigma}_{\pm}(\widetilde{T}^{\sigma})^{-1}(\widetilde{R}^{-\sigma}_{\mp})^{\dagger} = I, \tag{2.97}$$

$$(\widetilde{R}^{\sigma}_{+})^{\dagger}\widetilde{R}^{\sigma}_{+} = (\widetilde{R}^{\sigma}_{-})^{\dagger}\widetilde{R}^{\sigma}_{-}. \tag{2.98}$$

Under definition (2.94), Eqs.(2.81) are preserved, but relations (2.88) become

$$|\tilde{\nu}^{\sigma}\rangle\widetilde{T}^{\sigma} = |\tilde{\nu}_{a/r}\rangle\widetilde{R}^{\sigma}_{\pm}, \qquad \widetilde{T}^{\sigma}\langle\tilde{\omega}^{\sigma}| = (\widetilde{R}^{-\sigma}_{\mp})^{\dagger}\langle\tilde{\omega}_{a/r}|. \tag{2.99}$$

As in the decaying-potential case, we can introduce the alternative spectral data

$$\widetilde{F}^{\sigma} = (\widetilde{R}^{-\sigma}_{\pm})^{\dagger}\widetilde{R}^{-\sigma}_{\mp}. \tag{2.100}$$

Relations (2.90) and (2.91) are modified and become

$$|\tilde{\nu}^{\sigma}\rangle\widetilde{T}^{\sigma} = |\tilde{\nu}^{-\sigma}\rangle\widetilde{F}^{-\sigma}, \qquad \widetilde{T}^{\sigma}\langle\tilde{\omega}^{\sigma}| = (\widetilde{F}^{\sigma})^{\dagger}\langle\tilde{\omega}^{-\sigma}|, \tag{2.101}$$

$$\widetilde{F}^{\sigma} = \widetilde{T}^{\sigma}(\widetilde{F}^{-\sigma})^{-1}\widetilde{T}^{-\sigma}, \tag{2.102}$$

while the self-adjointness property of $\widetilde{F}^{\sigma}$ is preserved.

Moreover, we can express the new spectral data in terms of the different elements composing the final potential $\tilde{u}$ by

$$\widetilde{R}^{+}_{\pm} = w_{\pm}R^{+}_{\pm}\widetilde{T}^{+}, \qquad \widetilde{R}^{-}_{\pm} = w_{\pm}R^{-}_{\pm}, \tag{2.103}$$

where

$$w_{\pm}(p;\mathbf{q}) = \delta(p)\left(1 - \theta\big(\pm(\mathbf{q}_{1\Re} - \mu)\big)\frac{2i\kappa}{\mathbf{q}_{1\Re} - \mu + i\kappa}\right) \tag{2.104}$$

and $R^{\sigma}_{\pm}$ are the spectral data of $u(x)$ that satisfy characterization equations (2.85)–(2.87). The parameter κ is chosen to be greater than zero, and this explains the asymmetry in the formulas (2.103). An additional real constant must also be introduced for the soliton, by analogy with the normalization constants in the one-dimensional case.

3. Superimposition of a soliton on a decaying background

We already mentioned that analyzing the properties of the solutions of integral equations (2.18) and (2.65), (2.66) defining the resolvent and the Jost solutions for potentials of type (1.3) is particularly difficult. For experience in a sufficiently general case, we consider the potential describing a soliton on a background obtained by computing the Bäcklund transformation of an arbitrary potential vanishing at large distances in the x plane.

Because we are interested in the corresponding Jost solution, it is natural to use the Darboux procedure that simultaneously furnishes the Bäcklund transformation of the potential and the transformation of the solution of Eq. (1.1). However, the Darboux transformation of the Jost solution is not the Jost solution of the transformed potential in general, nor does it necessarily have a good behavior at large x. Therefore, the formal algebraic scheme of [24] involving the combined use (called binary) of solutions of the spectral problem and its dual must be used with an accurate analysis enabling us to verify at each step that we are working with proper mathematical objects. We find it convenient to follow an approach closer to the one in [25], where the Darboux transformations of the continuous spectrum of a decaying potential were studied. Thus, we also obtain the explicit formula for the resolvent. The study of its analytic properties is essential in the attempt to generalize the result to perturbed one-soliton solutions.

3.1. Binary Bäcklund transformations

We work in the space $\mathcal{S}'_{p,\mathbf{q}}$, where products and commutators are defined by composition law (2.3). In particular, $A^{-1}(p;\mathbf{q})$ means the inversion of $A(p;\mathbf{q})$ *according to this composition law*, not $1/A(p;\mathbf{q})$. If $\mathcal{A}(x)$ is a distribution for which $\log \mathcal{A}(x)$ is a well-defined distribution, we also introduce the LOG function of its image $A(p;\mathbf{q}) \equiv A(p)$ in the space $\mathcal{S}'_{p,\mathbf{q}}$ (that is, of its Fourier transform):

$$(\mathrm{LOG}\, A)(p) = \frac{1}{(2\pi)^2} \int dx\, e^{ipx} \log \mathcal{A}(x). \tag{3.1}$$

The notation LOG is admissible because it has the characteristic property $\mathrm{LOG}\, A_1 + \mathrm{LOG}\, A_2 = \mathrm{LOG}(A_1 A_2)$. Furthermore, if A has an inverse, then

$$[Q_1, \mathrm{LOG}\, A] = [Q_1, A]A^{-1}. \tag{3.2}$$

Let the potential $v(p)$ in the nonstationary Schrödinger operator L introduced in Eqs. (2.10) and (2.11) be regular, that is, the Fourier transform of a potential $u(x)$ rapidly decaying at large distances in the x plane. A new potential $v'(p)$ and the corresponding nonstationary Schrödinger operator L' can be generated using a gauge transformation B,

$$L'B = BL. \tag{3.3}$$

Particularly simple gauges of the form

$$B = Q_1 + \alpha, \qquad \alpha(p;\mathbf{q}) \equiv \alpha(p), \tag{3.4}$$

are usually called elementary Bäcklund gauges [30]. Any other such can be obtained by repeatedly composing such gauges and their inverses. Without loss of generality (for details, see [25]), we can rewrite α as

$$\alpha = -[Q_1, \mathrm{LOG}\, \beta] - \lambda I, \tag{3.5}$$

where $\beta(p;\mathbf{q}) \equiv \beta(p)$ and the spectral parameter λ is chosen in the upper half-plane

$$\lambda = \mu + i\kappa, \qquad \kappa > 0. \tag{3.6}$$

Inserting (3.4) into (3.3), we obtain the new potential

$$v' = v + 2\big[Q_1, [Q_1, \mathrm{LOG}\, \beta]\big] \tag{3.7}$$

and the equation satisfied by β

$$L\beta + 2[Q_1, \beta](Q_1 - \lambda I) = \beta L_0. \tag{3.8}$$

Comparing this equation with differential equation (2.54), we can see that β can be chosen equal to the value of the Jost solution of the original potential at $\mathbf{q}_1 = \lambda$,

$$\beta\delta(Q_1 - \lambda I) = |\nu\rangle\delta(Q_1 - \lambda I), \tag{3.9}$$

where we define

$$\big(\delta(Q_1 - \lambda I)\big)(p;\mathbf{q}) = \delta(p)\delta(\mathbf{q}_1 - \lambda) \tag{3.10}$$

such that

$$\beta(p) = |\nu\rangle(p;\lambda). \tag{3.11}$$

Then, Bäcklund gauge (3.4) is

$$B = \beta(Q_1 - \lambda I)\beta^{-1}. \tag{3.12}$$

As was shown in [24], the condition that both the potentials u and u' are real (or, equivalently, that both v and v' are self-adjoint) drastically reduces the class of potentials to which the elementary Bäcklund transformation can be applied, in fact, to only one-dimensional potentials (see [25]). To obtain a truly two-dimensional self-adjoint potential, we must (as suggested in [24]) consider an elementary Bäcklund transformation and its inverse with different parameters and compose them. The resulting Bäcklund transformation is called binary.

Thus, let $\tilde{L}$ be another L-operator with potential $\tilde{v}$, and by analogy with (3.3), let

$$L'\tilde{B} = \tilde{B}\tilde{L}, \tag{3.13}$$

where L' is the same as above and $\tilde{B}$ is of type (3.12), i.e.,

$$\tilde{B} = Q_1 - [Q_1, \operatorname{LOG}\tilde{\beta}] - \tilde{\lambda} I \tag{3.14}$$

with another value $\tilde{\lambda}$ of the spectral parameter. Then $\tilde{\beta}$, by analogy with (3.8), satisfies

$$\tilde{L}\tilde{\beta} + 2[Q_1, \tilde{\beta}](Q_1 - \tilde{\lambda} I) = \tilde{\beta} L_0, \tag{3.15}$$

and

$$v' = \tilde{v} + 2\big[Q_1, [Q_1, \operatorname{LOG}\tilde{\beta}]\big]. \tag{3.16}$$

We show in the following that $\tilde{\beta}$ neither is a value of the Jost solution at $\mathbf{q}_1 = \tilde{\lambda}$ (in contrast to the solution β of (3.8)) nor has an inverse in the space $\mathcal{S}'_{\mathbf{p},\mathbf{q}}$. This means, in particular, that formula (3.2) for $[Q_1, \operatorname{LOG}\tilde{\beta}]$ cannot be used.

We can eliminate the intermediate potential v' using (3.7) and directly write

$$\tilde{v} = v - 2\big[Q_1, \big[Q_1, \operatorname{LOG}(\tilde{\beta}\beta^{-1})\big]\big]. \tag{3.17}$$

We now impose the condition that both the potentials v and $\tilde{v}$ are self-adjoint,

$$v^\dagger = v, \qquad \tilde{v}^\dagger = \tilde{v}. \tag{3.18}$$

As in [25], it is easy to show that because of (3.17), this requirement is equivalent, without loss of generality, to

$$(\tilde{\beta}\beta^{-1})^\dagger = \tilde{\beta}\beta^{-1}, \tag{3.19}$$

$$\tilde{\lambda} = \bar{\lambda} = \mu - i\kappa. \tag{3.20}$$

As we show in what follows, this choice guarantees that the resulting binary Bäcklund transformation adds a soliton to v.

Let U denote the Darboux transformation that generates this binary Bäcklund transformation from v to $\tilde{v}$, i.e.,

$$U\tilde{L} = LU. \tag{3.21}$$

Then, by (3.3) and (3.13),

$$U = B^{-1}\tilde{B}, \tag{3.22}$$

and because of (3.12) and (3.14), the gauge U is given by

$$U = I - \beta(Q_1 - \lambda I)^{-1} A \beta^{-1}, \tag{3.23}$$

where

$$A = \left[Q_1, \mathrm{LOG}(\tilde\beta\beta^{-1})\right] - 2i\kappa I, \qquad A^\dagger = -A. \tag{3.24}$$

We now must find a differential equation (in the space $\mathcal{S}'_{p,\mathbf{q}}$) for $\tilde\beta\beta^{-1}$ in terms of β. For this, we insert $\tilde v$ from (3.17) into (3.15) and then use (3.8) to eliminate v in the resulting equation. We obtain

$$\left[Q_2 - 2\mu Q_1, \mathrm{LOG}(\tilde\beta\beta^{-1})\right] + [Q_1, A] = A^2 + 2i\kappa A + 2A[Q_1, \mathrm{LOG}\,\beta]. \tag{3.25}$$

Adding this equation to its Hermitian conjugate, we obtain

$$\left[Q_1, (\tilde\beta\beta^\dagger)^{-1}A\right] = 0. \tag{3.26}$$

Therefore, there exists a c commuting with Q_1 such that

$$A = ic\tilde\beta\beta^\dagger.$$

Because of self-adjointness condition (3.19), this c must be Hermitian. Because of (3.17) and (3.24), both $\tilde v$ and A are independent of the choice of c, and we choose $c = -I$, i.e.,

$$A = -i\tilde\beta\beta^\dagger. \tag{3.27}$$

Using this equality to replace A in (3.24), we obtain a Riccati equation for $\tilde\beta\beta^{-1}$,

$$[Q_1, \tilde\beta\beta^{-1}] = 2i\kappa\tilde\beta\beta^{-1} - i(\tilde\beta\beta^{-1})^2\beta\beta^\dagger. \tag{3.28}$$

The solution of this equation can be written in the form

$$\tilde\beta = \beta s(I + sf)^{-1}, \tag{3.29}$$

where

$$s(p;\mathbf{q}) \equiv s(p) = \frac{ie^{ip_1x_0}\delta(p_2 - 2\mu p_1)}{2\sinh\frac{\pi(p_1+i0)}{2\kappa}} = \frac{\kappa}{2\pi^2}\int\frac{dx\, e^{ipx}}{1 + e^{-2\kappa(x_1+2\mu x_2-x_0)}}, \tag{3.30}$$

x_0 is a constant of integration, and

$$f(p;\mathbf{q}) = i\frac{(\beta^\dagger\beta)(p) - \delta(p)}{p_1 + 2i\kappa}. \tag{3.31}$$

We note that s and consequently $\tilde\beta$ have no inverse in the space $\mathcal{S}'_{p,\mathbf{q}}$. From the definition of s and f, it directly follows that

$$[Q_1, s] = -is(s - 2\kappa I), \tag{3.32}$$

$$[Q_1, f] = i(\beta^\dagger\beta - I - 2\kappa f), \tag{3.33}$$

$$s^\dagger = s, \qquad f^\dagger = f. \tag{3.34}$$

Therefore, because of (3.29),

$$\tilde\beta^\dagger = \beta^\dagger s(I + sf)^{-1}, \tag{3.35}$$

and self-adjointness requirement (3.19) is satisfied.

We must now verify that the solution $\tilde\beta$ found in (3.29) satisfies (3.25), as only the real part of (3.25) has been used so far. Inserting (3.26) into (3.25), we obtain

$$\left[Q_2 - 2\mu Q_1, \mathrm{LOG}(\tilde\beta\beta^{-1})\right] = -A\left[Q_1, \mathrm{LOG}(\beta^\dagger\beta^{-1})\right]$$

and then, using (3.27),

$$\left[Q_2 - 2\mu Q_1, \mathrm{LOG}(\tilde{\beta}\beta^{-1})\right] = -i\tilde{\beta}\beta^{-1}\left([Q_1,\beta]\beta^\dagger - [Q_1,\beta^\dagger]\beta\right). \tag{3.36}$$

This equation is easily verified if we notice that because of definitions (3.30) and (3.31),

$$[Q_2 - 2\mu Q_1, s] = 0, \tag{3.37}$$

$$[Q_2 - 2\mu Q_1, f] = i\left([Q_1,\beta]\beta^\dagger - [Q_1,\beta^\dagger]\beta\right). \tag{3.38}$$

From (3.17), using (3.29), (3.32), and (3.33), we finally obtain

$$\tilde{v} = v - 2\left[Q_1, \left[Q_1, \mathrm{LOG}\left(s(I+sf)^{-1}\right)\right]\right] = v + 2i\left[Q_1, s\beta^\dagger\beta(I+sf)^{-1}\right] \tag{3.39}$$

for the new potential, which can also be written as

$$\tilde{v} = v + 2i[Q_1, \beta^\dagger\tilde{\beta}] \tag{3.40}$$

because of (3.29). It follows from (3.29) and (3.35) that $\beta^\dagger\tilde{\beta} = \tilde{\beta}^\dagger\beta$ in accordance with the self-adjointness property of $\tilde{v}$. It is also easy to verify that the potential

$$\tilde{u}(x) = \int dp\, e^{-ipx}\tilde{v}(p) \tag{3.41}$$

(the inversion of (2.12)) has no singularities, because in the x space, the Fourier transform of $s(I+sf)^{-1}$ is strictly positive.

On the other hand, because of (3.30), $s(p)$ is proportional to $\delta(p_2 - 2\mu p_1)$, and $\delta(p)$ is the leading singularity of $(I+sf)^{-1}(p)$. Consequently, $\tilde{v}(p)$ has a term proportional to $\delta(p_2 - 2\mu p_1)$. Therefore, the potential $\tilde{u}(x)$ is indeed a special case of (1.3). The behavior of this potential at large distances is given in detail in Sec. 3.7.

Because of the self-adjointness of v and $\tilde{v}$, it follows by analogy with the first equality in (2.19) that $\tilde{L}$ is also self-adjoint; consequently, from (3.21), we have

$$\tilde{L}U^\dagger = U^\dagger L. \tag{3.42}$$

3.2. Resolvent

3.2.1. Orthogonal decomposition of $\tilde{L}$ and $\widetilde{M}$

Our aim is to explicitly construct the extended resolvent $\widetilde{M}$ of the potential $\tilde{v}$ considered in the previous section. In this case, we can avoid solving integral equations (2.18) and can directly use definition (2.13) of $\widetilde{M}$ as the inverse of $\tilde{L}$,

$$\tilde{L}\widetilde{M} = I, \qquad \widetilde{M}\tilde{L} = I. \tag{3.43}$$

Using the properties of the gauge operator U that performs the binary Darboux transformation, we first show that the operators $\tilde{L}$ and $\widetilde{M}$ are naturally decomposed into the sum of two orthogonal terms. Using (3.20), (3.24), (3.27), and (3.19), we obtain the following explicit formulas for the operator U and its Hermitian conjugate from (3.23):

$$U = I + i\beta(Q_1 - \lambda I)^{-1}\tilde{\beta}^\dagger = I + i\beta(Q_1 - \lambda I)^{-1}s(I+sf)^{-1}\beta^\dagger, \tag{3.44}$$

$$U^\dagger = I - i\tilde{\beta}(Q_1 - \bar{\lambda} I)^{-1}\beta^\dagger = I - i\beta s(I+sf)^{-1}(Q_1 - \bar{\lambda} I)^{-1}\beta^\dagger. \tag{3.45}$$

Using (3.33) and (3.19), it is easy to verify that the operator U is partially isometric but not unitary. More precisely, $U^\dagger$ is the right inverse of U,

$$UU^\dagger = I. \tag{3.46}$$

Furthermore,

$$\Pi = U^\dagger U \tag{3.47}$$

is a nontrivial orthogonal projection operator,

$$\Pi^\dagger = \Pi, \qquad \Pi^2 = \Pi, \tag{3.48}$$

which plays a crucial role in the following. Using (3.33), we obtain

$$\Pi - I = \beta(I + sf)^{-1}(\Pi_1 - I)(I + sf)^{-1}\beta^\dagger, \tag{3.49}$$

where

$$\Pi_1 = I + \frac{i}{2\kappa}\left[s(Q_1 - \bar{\lambda}I)^{-1}(s - 2\kappa I) - (s - 2\kappa I)(Q_1 - \lambda I)^{-1}s\right]. \tag{3.50}$$

Multiplying (3.21) by $U^\dagger$ from the left and (3.42) by U from the right, we find that because of (3.47), this projection commutes with $\tilde{L}$ and

$$\Pi\tilde{L} = \tilde{L}\Pi = U^\dagger LU. \tag{3.51}$$

Thus, we obtain the decomposition

$$\tilde{L} = U^\dagger LU + \tilde{L}(I - \Pi). \tag{3.52}$$

Since

$$U(I - \Pi) = (I - \Pi)U^\dagger = 0 \tag{3.53}$$

because of (3.46) and (3.47), the product of the two terms in the right-hand side of (3.52) is equal to zero, and the decomposition is orthogonal.

It now follows from (3.43) that Π commutes with $\widetilde{M}$, and because of (2.13), we can also write the orthogonal decomposition for $\widetilde{M}$

$$\widetilde{M} = U^\dagger MU + \widetilde{M}(I - \Pi), \tag{3.54}$$

where M and U in the first term together with the arbitrary background v are considered given data. The second term in (3.54),

$$\widehat{M} = \widetilde{M}(I - \Pi), \tag{3.55}$$

is uniquely fixed by the requirements that it be orthogonal to Π, i.e.,

$$\widehat{M}\Pi = \Pi\widehat{M} = 0, \tag{3.56}$$

and that it satisfy

$$\widehat{M}\tilde{L}(I - \Pi) = \tilde{L}(I - \Pi)\widehat{M} = (I - \Pi). \tag{3.57}$$

Therefore, we use these two equations to compute $\widehat{M}$ below. We start construction of $\widetilde{M}$ with the special case where the original potential is equal to zero, i.e., $v \equiv 0$, and $\tilde{v}$ is therefore a pure one-soliton potential.

3.2.2. One-soliton case

The quantities obtained in the special case $v \equiv 0$ are distinguished by the subscript 1. For instance, we change the notation

$$\tilde{v} \longrightarrow v_1$$

and

$$\tilde{L} = L_0 - \tilde{v} \longrightarrow L_1 = L_0 - v_1. \tag{3.58}$$

We obtain the quantities in the pure one-soliton case by simply setting $\beta = I$ and $f = 0$ in the previous formulas. We thus obtain

$$U_1 = I + i(Q_1 - \lambda I)^{-1}s, \qquad U_1^\dagger = I - is(Q_1 - \bar{\lambda} I)^{-1}, \tag{3.59}$$

$$U_1 U_1^\dagger = I, \qquad U_1^\dagger U_1 = \Pi_1, \tag{3.60}$$

$$\Pi_1^\dagger = \Pi_1, \qquad \Pi_1^2 = \Pi_1, \tag{3.61}$$

where the orthogonal projection operator Π_1 is given by (3.50).

Orthogonal decomposition (3.52) is now

$$L_1 = U_1^\dagger L_0 U_1 + L_1(I - \Pi_1). \tag{3.62}$$

According to the general discussion in Sec. 3.2.1, we must find an expression for $L_1(I - \Pi_1)$ for which (3.57) can be easily solved. We consider the difference $L_1 - U_1^\dagger L_0 U_1$. Taking into account that $v_1 = 2i[Q_1, s]$ follows from (3.40), we obtain

$$\begin{aligned} L_1 - U_1^\dagger L_0 U_1 = &- 2i[Q_1, s] + is(Q_1 - \bar{\lambda} I)^{-1} L_0 - iL_0(Q_1 - \lambda I)^{-1}s - \\ &- s(Q_1 - \bar{\lambda} I)^{-1} L_0 (Q_1 - \lambda I)^{-1} s. \end{aligned} \tag{3.63}$$

Inserting L_0 rewritten in the form

$$L_0 = (Q_2 - 2\mu Q_1 + |\lambda|^2 I) - (Q_1 - \bar{\lambda} I)(Q_1 - \lambda I) \tag{3.64}$$

in the right-hand side of (3.63), using (3.37) and (3.32), and recalling the definition of Π_1 in (3.50), we obtain an explicit orthogonal decomposition for the pure one-soliton operator:

$$L_1 = U_1^\dagger L_0 U_1 + (I - \Pi_1)\big(Q_2 - 2\mu Q_1 + |\lambda|^2 I\big). \tag{3.65}$$

Because of (3.37) and (3.50),

$$[\Pi_1, Q_2 - 2\mu Q_1] = 0, \tag{3.66}$$

and it is therefore easy to obtain the orthogonal decomposition for the resolvent of the pure one-soliton potential

$$M_1 = U_1^\dagger M_0 U_1 + (I - \Pi_1)\big(Q_2 - 2\mu Q_1 + |\lambda|^2 I\big)^{-1}. \tag{3.67}$$

For studying the analytic properties of $M_1(p; \mathbf{q})$ with respect to $\mathbf{q}$, it is convenient to explicitly compute both terms in the right-hand side of (3.67). Inserting s from (3.30) into definition (3.50) of Π_1, we obtain

$$\begin{aligned} (I - \Pi_1)(p; \mathbf{q}) = &\frac{\delta(p_2 - 2\mu p_1)}{8i\kappa} e^{ip_1 x_0} \times \\ &\times \int dp_1' \Bigg[\frac{1}{(p_1' - i\kappa + \mathbf{q}_1 - \mu) \cosh \frac{\pi(p_1 - (p_1' - i(\kappa - 0)))}{2\kappa} \cosh \frac{\pi(p_1' - i(\kappa - 0))}{2\kappa}} - \\ &- \frac{1}{(p_1' + i\kappa + \mathbf{q}_1 - \mu) \cosh \frac{\pi(p_1 - (p_1' + i(\kappa - 0)))}{2\kappa} \cosh \frac{\pi(p_1' + i(\kappa - 0))}{2\kappa}} \Bigg]. \end{aligned}$$

which is simply the Cauchy formula applied to the contour of the strip $|\mathbf{q}_{1\Im}| \le \kappa$ in the complex $\mathbf{q}_1$ plane. We thus obtain

$$(I - \Pi_1)(p;\mathbf{q}) = \theta(\kappa - |\mathbf{q}_{1\Im}|)\frac{\pi\delta(p_2 - 2\mu p_1)e^{ip_1x_0}}{4\kappa\cosh\frac{\pi(p_1+\mathbf{q}_1-\mu)}{2\kappa}\cosh\frac{\pi(\mathbf{q}_1-\mu)}{2\kappa}} \tag{3.68}$$

and conclude that $(I - \Pi_1)(p;\mathbf{q})$ is independent of $\mathbf{q}_2$, is analytic with respect to $\mathbf{q}_1$ in the strip $|\mathbf{q}_{1\Im}| \le \kappa$, and is zero outside it. In addition, it is a product of two analytic functions that respectively depend on $p_1 + \mathbf{q}_1$ and on $\mathbf{q}_1$.

For explicitly computing the first term of the orthogonal decomposition of M_1 in (3.67), $U_1^\dagger M_0 U_1$, we need the integral

$$\int dp' \, \frac{s(p-p')s(p')}{\mathbf{w}+p_1'} = \frac{2\kappa}{\mathbf{w}}s(p) + is(p)\left[\psi\left(\frac{\mathbf{w}+p_1}{2i\kappa}\right) - \psi\left(\frac{\mathbf{w}}{2i\kappa}\right)\right] + \\ + \frac{i\pi e^{ip_1x_0}\theta(-\mathbf{w}_\Im)\delta(p_2-2\mu p_1)}{2\sinh\frac{\pi(p_1+\mathbf{w})}{2\kappa}\sinh\frac{\pi\mathbf{w}}{2\kappa}}, \tag{3.69}$$

(where $\psi(z)$ is the logarithmic derivative of the Γ-function). Paying special attention to the fact that it is a distribution, we can obtain $s(p)$ from a recursion relation derived by considering the limit as $R \to \infty$ of the integral $\int_{C_R} dp_1' \, \frac{s(p-p')s(p')}{\mathbf{w}+p_1'}$ along the rectangle C_R in the complex p_1' plane with the width $2R$ and height 2κ and with the lower border centered on the real axis. Because of (3.30), this integral depends on p_2 only via the multiplier $\delta(p_2 - 2\mu p_1)$.

The correctness of the result can be verified by comparing the analytic properties of the left- and right-hand sides of (3.69). In particular, the simple poles of $\psi(z)$ at $z = 0, -1, \ldots, -n, \ldots$ are exactly cancelled by the poles of the last term.

Because $U_1(p;\mathbf{q})$ and $U_1^\dagger(p;\mathbf{q})$ contain the factor $\delta(p_2 - 2\mu p_1)$, the first term in (3.67) can be rewritten as

$$U_1^\dagger M_0 U_1 = M_a + M_b + M_c + M_d, \tag{3.70}$$

where

$$M_{a,\ldots,d}(p;\mathbf{q}) = M_{1a,\ldots,1d}(p_1;\mathbf{q})\delta(p_2 - 2\mu p_1) \tag{3.71}$$

and

$$M_a = M_0 + i\big(Q_2 - 2\mu Q_1 + |\lambda|^2 I\big)^{-1}\big[s, (Q_1 - \bar{\lambda}I)M_0\big], \tag{3.72}$$

$$M_{1b}(p;\mathbf{q}) = \frac{is(p)}{2\zeta(\mathbf{q}_2 - 2\mu\mathbf{q}_1 + |\lambda|^2)}\left[\psi\left(\frac{\mathbf{q}_1+p_1-\mu+\zeta}{2i\kappa}\right) - \right. \\ \left. - \psi\left(\frac{\mathbf{q}_1+p_1-\mu-\zeta}{2i\kappa}\right) - \psi\left(\frac{\mathbf{q}_1-\mu+\zeta}{2i\kappa}\right) + \psi\left(\frac{\mathbf{q}_1-\mu-\zeta}{2i\kappa}\right)\right], \tag{3.73}$$

$$M_{1c}(p;\mathbf{q}) = \frac{i\pi e^{ip_1x_0}}{4\zeta(\mathbf{q}_2 - 2\mu\mathbf{q}_1 + |\lambda|^2)}\times \\ \times\left[\frac{\theta(-\mathbf{q}_{1\Im} - \zeta_\Im)}{\sinh\frac{\pi(\mathbf{q}_1+p_1-\mu+\zeta)}{2\kappa}\sinh\frac{\pi(\mathbf{q}_1-\mu+\zeta)}{2\kappa}} - \frac{\theta(-\mathbf{q}_{1\Im} + \zeta_\Im)}{\sinh\frac{\pi(\mathbf{q}_1+p_1-\mu-\zeta)}{2\kappa}\sinh\frac{\pi(\mathbf{q}_1-\mu-\zeta)}{2\kappa}}\right], \tag{3.74}$$

$$M_{1d}(p;\mathbf{q}) = \frac{\pi e^{ip_1x_0}\theta(\kappa - |\mathbf{q}_{1\Im}|)}{4\kappa(\mathbf{q}_2 - 2\mu\mathbf{q}_1 + |\lambda|^2)\sinh\frac{\pi(\mathbf{q}_1+p_1-\lambda)}{2\kappa}\sinh\frac{\pi(\mathbf{q}_1-\lambda)}{2\kappa}}, \tag{3.75}$$

with

$$\zeta = \sqrt{\mathbf{q}_2 - 2\mu\mathbf{q}_1 + \mu^2}. \tag{3.76}$$

We note that (3.72)-(3.75) do not dependent on of the sign of this square root.

We now know all terms in the right-hand side of (3.67). Moreover, we can see that because of (3.68), the last term is exactly canceled by the term M_d of $U_1^\dagger M_0 U_1$ in (3.70), and we obtain the explicit formula for the one-soliton resolvent

$$M_1 = M_a + M_b + M_c. \tag{3.77}$$

Because the above-mentioned terms canceled out, the resolvent has no discontinuity on the lines $\mathbf{q}_{1\Im} = \pm\kappa$. We also note that the poles of the ψ's in M_b are exactly canceled by the poles of M_c.

Because of (3.71), $M_1(p;\mathbf{q})$ is proportional to $\delta(p_2 - 2\mu p_1)$. This factor shows (cf. (2.62)) the one-dimensional character of the one-soliton resolvent, and $M_1(p;\mathbf{q})$ can be considered the embedding of a one-dimensional resolvent in two dimensions. Indeed, the image of the stationary Schrödinger operator $L = \partial_{x_1}^2 - u(x_1) + z$, where $u(x_1)$ is a one-dimensional potential and z is a (complex) spectral parameter, is the element of $\mathcal{S}'_{p,\mathbf{q}}$ with the form

$$L_z(p_1;\mathbf{q}_1) = \delta(p_1)(z - \mathbf{q}_1^2) - v(p_1), \tag{3.78}$$

where $v(p_1)$ is the Fourier transform of $u(x_1)$. The one-dimensional resolvent $M_z(p_1;\mathbf{q}_1)$ is defined (by analogy with (2.13)) by the equation $(L_z M_z)(p_1;\mathbf{q}_1) = \delta(p_1)$. In the case considered here, i.e., where $u(x_1)$ is a pure one-soliton potential, it is easy to verify that the corresponding resolvent is given by

$$M_z(p_1;\mathbf{q}_1) = (M_{1a} + M_{1b} + M_{1c})(p_1;\mathbf{q}_1 + \mu, z + 2\mu\mathbf{q}_1 + \mu^2), \tag{3.79}$$

which can be explicitly expressed using (3.72)-(3.74). In Secs. 3.4 and 4, we describe the properties of M_1 and M_z in detail.

3.2.3. One soliton on a background

We now consider a potential $\tilde{v}$ obtained by superimposing a soliton on an arbitrary regular potential v using a binary Bäcklund transformation.

To obtain the resolvent, we follow a procedure analogous to that used in the previous section. We first compute an explicit formula for the second term in (3.52), the orthogonal decomposition of $\tilde{L}$. Because of (3.40), (3.44), and (3.45), we have

$$\begin{aligned}\tilde{L} - U^\dagger L U = &- 2i[Q_1, \beta^\dagger\tilde{\beta}] + i\tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}\beta^\dagger L - iL\beta(Q_1 - \lambda I)^{-1}\tilde{\beta}^\dagger - \\ &- \tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}\beta^\dagger L\beta(Q_1 - \lambda I)^{-1}\tilde{\beta}^\dagger.\end{aligned}$$

Substituting $L\beta$ and $\beta^\dagger L$ from (3.8) and its Hermitian conjugate, we obtain

$$\begin{aligned}\tilde{L} - U^\dagger L U = &- 2i\beta^\dagger[Q_1, \tilde{\beta}] + 2i\tilde{\beta}^\dagger[Q_1, \beta] + i\tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}L_0\beta^\dagger - i\beta L_0(Q_1 - \lambda I)^{-1}\tilde{\beta}^\dagger - \\ &- \tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}\beta^\dagger\beta L_0(Q_1 - \lambda I)^{-1}\tilde{\beta}^\dagger + 2\tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}\beta^\dagger[Q_1, \beta]\tilde{\beta}^\dagger.\end{aligned} \tag{3.80}$$

Multiplying this equation by $(I + sf)\beta^{-1}$ from the left and by $(I + sf)(\beta^\dagger)^{-1}$ from the right and using (3.33) and (3.38), we obtain

$$\begin{aligned}(I + sf)\beta^{-1}(\tilde{L} - U^\dagger L U)(I + sf)(\beta^\dagger)^{-1} = &- 2i[Q_1, s] + is(Q_1 - \bar{\lambda}I)^{-1}L_0 - \\ &- iL_0(Q_1 - \lambda I)^{-1}s - s(Q_1 - \bar{\lambda}I)^{-1}L_0(Q_1 - \lambda I)^{-1}s.\end{aligned}$$

Because of (3.63), the right-hand side is exactly $L_1 - U_1^\dagger L_0 U_1$. Using (3.65), we finally derive the orthogonal decomposition for $\tilde{L}$ in the form

$$\tilde{L} = U^\dagger L U + \beta(I+sf)^{-1}(I-\Pi_1)\big(Q_2 - 2\mu Q_1 + |\lambda|^2 I\big)\beta^\dagger(I+sf)^{-1}. \tag{3.81}$$

It is now easy to prove that the orthogonal decomposition of the resolvent is given by

$$\widetilde{M} = U^\dagger M U + \widehat{M}, \tag{3.82}$$

where

$$\widehat{M} = \beta(I+sf)^{-1}(I-\Pi_1)\big(Q_2 - 2\mu Q_1 + |\lambda|^2 I\big)^{-1}\beta^\dagger(I+sf)^{-1}. \tag{3.83}$$

In accordance with the requirements at the end of Sec. 3.2.1, we must verify that $\widehat{M}$ satisfies (3.56) and (3.57). First, using (3.49) and (3.66), we write

$$\widehat{M} = \beta(I+sf)^{-1}\big(Q_2 - 2\mu Q_1 + |\lambda|^2 I\big)^{-1}\beta^{-1}(I+sf)(I-\Pi) \tag{3.84}$$

and

$$\tilde{L}(I-\Pi) = (I-\Pi)(\beta^\dagger)^{-1}(I+sf)\big(Q_2 - 2\mu Q_1 + |\lambda|^2 I\big)\beta^\dagger(I+sf)^{-1}.$$

Then, using (3.48), (3.49), and (3.66), we obtain $\widehat{M}\Pi = 0$ and $\widehat{M}\tilde{L}(I-\Pi) = (I-\Pi)$. We can verify analogously that $\Pi\widehat{M} = 0$ and $\tilde{L}(I-\Pi)\widehat{M} = (I-\Pi)$.

3.3. Jost solutions

The potential $\tilde{v}$ of the previous sections was in fact obtained via a binary Darboux transformation, i.e., using the gauge transformation U of L. Therefore, we expect that the corresponding Jost solutions $|\widetilde{\nu}\rangle$ and $\langle\widetilde{\omega}|$ are related to $U^\dagger|\nu\rangle$ and $\langle\omega|U$ respectively. Indeed, because of (3.42),

$$\tilde{L}U^\dagger|\nu\rangle = U^\dagger L|\nu\rangle = U^\dagger|\nu\rangle L_0. \tag{3.85}$$

That is, $U^\dagger|\nu\rangle$ satisfies the differential equation for the Jost solution, and $\langle\omega|U$ similarly satisfies the dual equation.

To find the exact relation, we must compute the integral equation satisfied by $U^\dagger|\nu\rangle$ and compare it with (2.65). Because of (3.85), we obtain $\tilde{v}U^\dagger|\nu\rangle = (L_0 - \tilde{L})U^\dagger|\nu\rangle = \big[L_0, U^\dagger|\nu\rangle\big]$ and, after inserting (3.45),

$$\tilde{v}U^\dagger|\nu\rangle = v|\nu\rangle - i\big[L_0, \tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}\beta^\dagger|\nu\rangle\big].$$

Because $[L_0, A](p;\mathbf{q}) = \big(p_2 - p_1(p_1 + 2\mathbf{q}_1)\big)A(p;\mathbf{q})$ for any A, the last term of integral equation (2.65) with $|\widetilde{\nu}\rangle$ replaced by $U^\dagger|\nu\rangle$ can be rewritten in the form

$$\frac{1}{p_1 + i0\mathbf{q}_{1\Im}}\left(\frac{p_1\big(\tilde{v}U^\dagger|\nu\rangle\big)(p;\mathbf{q})}{p_2 - p_1(p_1+2\mathbf{q}_1)}\right) = \frac{\big(v|\nu\rangle\big)(p;\mathbf{q})}{p_2 - p_1(p_1+2\mathbf{q}_1)} - i\frac{\big(p_1\big(\tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}\beta^\dagger|\nu\rangle\big)(p;\mathbf{q})\big)}{p_1 + i0\mathbf{q}_{1\Im}}.$$

As we know, the Jost solution corresponding to the regular potential v satisfies integral equation (2.30), and we can therefore substitute $\big(|\nu\rangle - I\big)(p;\mathbf{q})$ for the first term. To compute the second term, we must analyze the leading singularity of the distribution $\big(\tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}\beta^\dagger|\nu\rangle\big)(p;\mathbf{q})$ at $p_1 = 0$. The leading singularity of β and $|\nu\rangle$ is I, and because of (3.29), the leading singularity of $\tilde{\beta}$ is the same as the singularity of s, i.e., of the form $(p_1 + i0)^{-1}$. Therefore, because the product of two distributions of the form $(p_1 + i0)^{-1}$ is well defined, we can write

$$\frac{\left(p_1\left(\tilde{\beta}\left(Q_1 - \bar{\lambda}I\right)^{-1}\beta^\dagger|\nu\rangle\right)(p;\mathbf{q})\right)}{p_1 + i0\mathbf{q}_{1\Im}} = \left(\tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}\beta^\dagger|\nu\rangle\right)(p;\mathbf{q}) + \\ + 2\pi i\theta(-\mathbf{q}_\Im)\delta(p_1)\left(p_1\left(\tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}\beta^\dagger|\nu\rangle\right)(p;\mathbf{q})\right) = \\ = \left(\tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}\beta^\dagger|\nu\rangle\right)(p;\mathbf{q}) + 2\pi i\theta(-\mathbf{q}_\Im)\delta(p_1)\left(\frac{p_1 s(p;\mathbf{q})}{\mathbf{q}_1 - \bar{\lambda}}\right).$$

Thus, because of (3.30) and (3.45), we obtain

$$U^\dagger|\nu\rangle(p;\mathbf{q}) - \frac{1}{p_1 + i0\mathbf{q}_{1\Im}}\left(\frac{p_1\left(\bar{v}U^\dagger|\nu\rangle\right)(p;\mathbf{q})}{p_2 - p_1(p_1 + 2\mathbf{q}_1)}\right) = \delta(p)\left(\theta(\mathbf{q}_{1\Im}) + \theta(-\mathbf{q}_{1\Im})\frac{\mathbf{q}_1 - \lambda}{\mathbf{q}_1 - \bar{\lambda}}\right),$$

which gives the inhomogeneous term of the integral equation for $U^\dagger|\nu\rangle$. Therefore, to construct the Jost solution obeying (2.65), the $\mathbf{q}$-dependent multiplier must be compensated, and we conclude that

$$|\tilde{\nu}\rangle = U^\dagger|\nu\rangle\left(\theta(Q_{1\Im}) + \theta(-Q_{1\Im})\frac{Q_1 - \bar{\lambda}I}{Q_1 - \lambda I}\right), \tag{3.86}$$

where

$$\theta(Q_{1\Im})(p;\mathbf{q}) = \theta(\mathbf{q}_{1\Im})\delta(p). \tag{3.87}$$

This construction also proves the existence of the Jost solution.

We obtain

$$\langle\tilde{\omega}| = \left(\theta(-Q_{1\Im}) + \theta(Q_{1\Im})\frac{Q_1 - \lambda I}{Q_1 - \bar{\lambda}I}\right)\langle\omega|U \tag{3.88}$$

similarly. Taking into account the explicit formulas for U and $U^\dagger$ in (3.44) and (3.45) and the analytic properties of $|\nu\rangle$ and $\langle\omega|$, we can verify that $|\tilde{\nu}\rangle$ and $\langle\tilde{\omega}|$ are piecewise analytic in the complex $\mathbf{q}_1$ plane with a cut on the real axis, in spite of the involved properties of the resolvent described in Sec. 3.4.

Comparing

$$\langle\omega|UU^\dagger|\nu\rangle = I, \tag{3.89}$$

which is a direct consequence of (3.46) and (2.58), with (2.72) we deduce that

$$\widetilde{T} = \theta(Q_{1\Im})\frac{Q_1 - \bar{\lambda}I}{Q_1 - \lambda I} + \theta(-Q_{1\Im})\frac{Q_1 - \lambda I}{Q_1 - \bar{\lambda}I} \tag{3.90}$$

explicitly proving (2.74), where now

$$\tilde{t}(\mathbf{q}) = \frac{\mathbf{q}_1 - \mu + i\kappa\,\mathrm{sgn}\,\mathbf{q}_{1\Im}}{\mathbf{q}_1 - \mu - i\kappa\,\mathrm{sgn}\,\mathbf{q}_{1\Im}}. \tag{3.91}$$

It is easy to see that the transmission coefficient $\widetilde{T}$ is self-adjoint,

$$\widetilde{T}^\dagger = \widetilde{T}. \tag{3.92}$$

For simplicity, let

$$\lambda_\pm = \mu \pm i\kappa. \tag{3.93}$$

such that $\lambda_+ = \lambda$ (cf. (3.6)) and $\lambda_- = \bar{\lambda}$. Then the values of the transmission coefficient on the two sides $\mathbf{q}_{1\Re} \pm i0$ of the real axis in the complex $\mathbf{q}_1$ plane are equal to

$$\widetilde{T}^{\pm} = \frac{Q_{1\Re} - \lambda_{\mp} I}{Q_{1\Re} - \lambda_{\pm} I} \tag{3.94}$$

and satisfy

$$\widetilde{T}^{+}\widetilde{T}^{-} = I, \qquad (\widetilde{T}^{+})^{\dagger} = \widetilde{T}^{-}. \tag{3.95}$$

Because of (3.91), $\tilde{t}(\mathbf{q})$ has poles at $\mathbf{q}_1 = \lambda_{\pm}$, and the residua of $\tilde{t}$ at these points are

$$\tilde{t}_{\pm} = \pm 2i\kappa. \tag{3.96}$$

It is clear that $|\tilde{\nu}\rangle\widetilde{T}$ and $\widetilde{T}\langle\widetilde{\omega}|$ also satisfy differential equation (3.85) and its dual, because $\widetilde{T}$ and L_0 commute. These are the solutions that were defined in (2.75) and (2.76). From (2.74),

$$\big(|\tilde{\nu}\rangle\widetilde{T}\big)(p;\mathbf{q}) = \tilde{t}(\mathbf{q})|\tilde{\nu}\rangle(p;\mathbf{q}) \qquad \text{and} \qquad \big(\widetilde{T}\langle\tilde{\omega}|\big)(p;\mathbf{q}) = \tilde{t}(\mathbf{q}+p)\langle\tilde{\omega}|(p;\mathbf{q}).$$

The first function therefore has poles at $\mathbf{q}_1 = \lambda_{\pm}$, and the second one at $\mathbf{q}_1 = \lambda_{\pm} - p_1$. The corresponding residua (up to multiples of $\pm 2i\kappa$) are

$$|\tilde{\nu}_{1,\pm}\rangle(p) = |\tilde{\nu}\rangle(p;\lambda_{\pm}), \qquad \langle\widetilde{\omega}_{1,\pm}|(p) = \langle\widetilde{\omega}|(p;\lambda_{\pm} - p_1). \tag{3.97}$$

For brevity, we call them the discrete Jost solutions.

From (3.86), (3.45), and (3.11), we obtain

$$|\tilde{\nu}_{1,+}\rangle(p) = \big(U^{\dagger}|\nu\rangle\big)(p;\mathbf{q})\big|_{\mathbf{q}_1=\lambda_+} = \big(\beta - i\tilde{\beta}(Q_1 - \bar{\lambda}I)^{-1}\beta^{\dagger}\beta\big)(p;\lambda_+) \tag{3.98}$$

and, inserting (3.33) and using (3.29),

$$|\tilde{\nu}_{1,+}\rangle = -\frac{\beta}{2\kappa}(s - 2\kappa I)(I + sf)^{-1}. \tag{3.99}$$

From (3.88) and (3.44), we obtain

$$\langle\widetilde{\omega}_{1,+}|(p) = \left(\frac{Q_1 - \lambda I}{Q_1 - \bar{\lambda} I}\langle\omega|\big[I + i\beta(Q_1 - \lambda I)^{-1}s(I + sf)^{-1}\beta^{\dagger}\big]\right)(p;\mathbf{q})\Big|_{\mathbf{q}_1=\lambda_+ - p_1}.$$

Recalling from (3.11) that β is a Jost solution, we can see that

$$\langle\omega|\beta = I + \textit{ less singular terms},$$

and we therefore obtain

$$\langle\widetilde{\omega}_{1,+}| = \frac{1}{2\kappa}s(I + sf)^{-1}\beta^{\dagger}.$$

We use

$$|\tilde{\nu}_{1,\pm}\rangle^{\dagger} = \langle\widetilde{\omega}_{1,\mp}| \tag{3.100}$$

to obtain the values of the Jost solutions at λ_-. In particular,

$$|\tilde{\nu}_{1,-}\rangle = \frac{1}{2\kappa}s(I + sf)^{-1}\beta. \tag{3.101}$$

By analogy with (2.27) and (2.29), we determine the Jost solutions in the x space as

$$\widetilde{\Phi}(x,\mathbf{k}) = \int dp\, e^{-i(p+\ell(\mathbf{k}))x} |\tilde{\nu}\rangle(p;\mathbf{k}), \tag{3.102}$$

$$\widetilde{\Psi}(x,\mathbf{k}) = \int dp\, e^{-i(p-\ell(\mathbf{k}))x} \langle\tilde{\omega}|(p;\mathbf{k}-p_1), \tag{3.103}$$

where ℓ is defined in (2.38). These functions solve (1.1) (and, correspondingly, its dual) for the transformed potential $\tilde{u}$ (see (3.41), where $\tilde{v}(p)$ is given in (3.39)). Letting $\widetilde{\Phi}_{1,\pm}(x)$ and $\widetilde{\Psi}_{1,\pm}(x)$ denote the values of $\widetilde{\Phi}(x,\mathbf{k})$ and $\widetilde{\Psi}(x,\mathbf{k})$ at $\mathbf{k} = \mu \pm i\kappa$, we obtain

$$\widetilde{\Phi}_{1,\pm}(x) = \int dp\, e^{-i(p+\ell(\lambda_\pm))x} |\tilde{\nu}_{1,\pm}\rangle(p), \tag{3.104}$$

$$\widetilde{\Psi}_{1,\pm}(x) = \int dp\, e^{-i(p-\ell(\lambda_\pm))x} \langle\tilde{\omega}_{1,\pm}|(p) \tag{3.105}$$

from (3.97). Then, considering (3.99) and (3.101) and taking (3.30) into account, we obtain

$$\widetilde{\Phi}_{1,+}(x) = e^{2\kappa x_0}\widetilde{\Phi}_{1,-}(x), \qquad \widetilde{\Psi}_{1,+}(x) = e^{-2\kappa x_0}\widetilde{\Psi}_{1,-}(x), \tag{3.106}$$

where the second equality is obtained analogously.

Finally, we consider the asymptotic behavior of the Jost solutions at $\mathbf{q}_1 \to \infty$. Because of (3.45),

$$U^\dagger(p;\mathbf{q}) = \delta(p) - \frac{i}{\mathbf{q}_1}(\tilde{\beta}\beta^\dagger)(p) + o(\mathbf{q}_1^{-1}), \qquad \mathbf{q}_1 \to \infty,$$

and taking (2.42) and (2.43) into account and expanding (3.86), we obtain the asymptotic conditions

$$|\tilde{\nu}\rangle(p,\mathbf{q}) = \delta(p) + O(1/\mathbf{q}_1), \tag{3.107}$$

$$p_1|\tilde{\nu}\rangle(p,\mathbf{q}) = -\frac{1}{2\mathbf{q}_1}\tilde{v}(p) + O(1/\mathbf{q}_1^2), \tag{3.108}$$

where (3.40) is also taken into account for (3.108).

3.4. Bilinear representation of the resolvent and its properties

Orthogonal decomposition (3.82) enables us to derive a bilinear representation of the resolvent $\widetilde{M}$ for a nondecaying potential $\tilde{u}$ of type (1.3) obtained by means of the Bäcklund transformation (see (3.39)). The form of this bilinear representation is modified in comparison with bilinear representation (2.57) in the decaying-potential (regular) case. Indeed, inserting (2.57) into the first term of decomposition (3.82) and using (3.86), (3.88), (3.90), and the commutativity of $\widetilde{T}$ with M_0, we obtain

$$\widetilde{M} = |\tilde{\nu}\rangle\widetilde{T}M_0\langle\tilde{\omega}| + \widehat{M}. \tag{3.109}$$

Then, inserting (3.50) into formula (3.83) for $\widehat{M}$ and using (3.90), (3.99), and (3.101), we also obtain the bilinear representation of $\widehat{M}$:

$$\widehat{M} = -\sum_{\pm} |\tilde{\nu}_{1,\pm}\rangle \frac{\pm 2i\kappa}{(Q_1 - \lambda_\pm I)(Q_2 - 2\mu Q_1 + |\lambda|^2 I)} \langle\tilde{\omega}_{1,\pm}|. \tag{3.110}$$

These formulas are coherent with the general formulas obtained in [22] for the bilinear representation of the resolvent for a nondecaying potential $\tilde{u}(x)$ under some additional assumptions. They prove that the general approach developed in [22] is not void, because we explicitly found a sufficiently general example with all the required properties, at least for $N = 1$.

From the expression for $\widehat{M}$ in (3.110), we deduce that $\widehat{M}(p; \mathbf{q}_1, \mathbf{q}_2 + 2\mu\mathbf{q}_1)$ is analytic in $\mathbf{q}_1$ inside the strip $|\mathbf{q}_{1\Im}| \leq \kappa$, zero outside it (see (3.68) and (3.84)), and analytic in $\mathbf{q}_2$ with a cut at $\mathbf{q}_{2\Im} = 0$.

The condition that $\widehat{M}$ vanishes outside the strip $|\mathbf{q}_{1\Im}| \leq \kappa$ is equivalent to the condition for the proportionality of the discrete Jost solutions in (3.106). To prove this, we note that because of (3.110), $\widehat{M}(p; \mathbf{q}_1, \mathbf{q}_2 + 2\mu\mathbf{q}_1)$ decays for $\mathbf{q}_2 \to \infty$. Therefore, because of its analytic property with respect to $\mathbf{q}_2$, it is zero outside the strip iff the discontinuity of $\widehat{M}(p; \mathbf{q}_1, \mathbf{q}_2 + 2\mu\mathbf{q}_1)$ at $\mathbf{q}_{2\Im} = 0$ is zero outside the strip. By (3.109), this discontinuity of $\widehat{M}$ is equal to the discontinuity of $\widetilde{M}$ itself. In summary, $\widehat{M}(p; \mathbf{q})$ is zero outside the strip $|\mathbf{q}_{1\Im}| \leq \kappa$ iff the discontinuity

$$\Delta\widetilde{M}(p; \mathbf{q}) = \widetilde{M}(p; \mathbf{q}_1, \mathbf{q}_{2\Re} + 2i\mu\mathbf{q}_{1\Im} + i0) - \widetilde{M}(p; \mathbf{q}_1, \mathbf{q}_{2\Re} + 2i\mu\mathbf{q}_{1\Im} - i0) \tag{3.111}$$

is zero outside this strip.

On the other hand, from (3.110), we have

$$\Delta\widetilde{M} = -4\pi\kappa \sum_{\pm} \pm|\tilde{\nu}_{1,\pm}\rangle \frac{\delta(Q_{2\Re} - 2\mu Q_{1\Re} + |\lambda|^2 I)}{Q_1 - \lambda_\pm I} \langle\tilde{\omega}_{1,\pm}|. \tag{3.112}$$

Taking its Fourier transform and recalling (3.104) and (3.105), we obtain

$$\frac{1}{(2\pi)^2} \int dp \int d\mathbf{q}_\Re \, e^{-i(p+\mathbf{q}_\Re)x + i\mathbf{q}_\Re y} \Delta\widetilde{M}(p; \mathbf{q}) =$$

$$= 2i\kappa e^{-\mathbf{q}_{1\Im}(x_1 - y_1 + 2\mu x_2 - 2\mu y_2)} \sum_{\pm} \pm\theta\big((x_1 - y_1 + 2\mu x_2 - 2\mu y_2)(\mathbf{q}_{1\Im} \mp \kappa)\big) \operatorname{sgn}(\mathbf{q}_{1\Im} \mp \kappa) \widetilde{\Phi}_{1,\pm}(x) \widetilde{\Psi}_{1,\pm}(y),$$

which is zero outside the strip iff

$$\widetilde{\Phi}_{1,+}(x)\widetilde{\Psi}_{1,+}(y) = \widetilde{\Phi}_{1,-}(x)\widetilde{\Psi}_{1,-}(y), \tag{3.113}$$

that is, iff

$$\widetilde{\Phi}_{1,+}(x) = b\widetilde{\Phi}_{1,-}(x), \qquad \widetilde{\Psi}_{1,+}(x) = b^{-1}\widetilde{\Psi}_{1,-}(x), \tag{3.114}$$

where b is a real (by (3.100)) nonzero constant. These equalities coincide with (3.106), and the exact value of b given there follows from the substitution of (3.99) and (3.101) in (3.104).

Because of this relation, the Fourier transform of $\Delta\widetilde{M}(p; \mathbf{q})$ can be written as a direct product of two functions, i.e.,

$$\frac{1}{(2\pi)^2} \int dp \int d\mathbf{q}_\Re \, e^{-i(p+\mathbf{q}_\Re)x + i\mathbf{q}_\Re y} \Delta\widetilde{M}(p; \mathbf{q}) =$$

$$= -2i\kappa\theta(\kappa - |\mathbf{q}_{1\Im}|) e^{-\mathbf{q}_{1\Im}(x_1 - y_1 + 2\mu x_2 - 2\mu y_2)} \widetilde{\Phi}_{1,+}(x) \widetilde{\Psi}_{1,+}(y). \tag{3.115}$$

In the right-hand side of (3.109), the discontinuity of $\widehat{M}$ at $\mathbf{q}_{1\Im} = \pm\kappa$,

$$\widehat{M}|_{Q_{1\Im} = \pm(\kappa+0)I} - \widehat{M}|_{Q_{1\Im} = \pm(\kappa-0)I} = -4\pi\kappa|\tilde{\nu}_{1,\pm}\rangle \frac{\delta(Q_{1\Re} - \mu I)}{Q_2 - \lambda_\pm^2 I} \langle\tilde{\omega}_{1,\pm}|$$

(which can be directly obtained from (3.110)), is compensated by the discontinuity of the first term originating from the poles of $\widetilde{T}$. In fact, taking into account (2.17), (2.74), (3.91), (3.97), and (3.93), we obtain

$$|\tilde{\nu}\rangle\widetilde{T}M_0\langle\tilde{\omega}|\big|_{Q_{1\Im}=\pm(\kappa+0)I} - |\tilde{\nu}\rangle\widetilde{T}M_0\langle\tilde{\omega}|\big|_{Q_{1\Im}=\pm(\kappa-0)I} = 4\pi\kappa|\tilde{\nu}_{1,\pm}\rangle\frac{\delta(Q_{1\Re}-\mu I)}{Q_2-\lambda_{\pm}^2 I}\langle\tilde{\omega}_{1,\pm}|.$$

Therefore, for any $\mathbf{q}_2$, the resolvent $\widetilde{M}(p;\mathbf{q})$ has no discontinuities at the borders of the strip, as was explicitly demonstrated for the pure one-soliton case in (3.77).

On the other hand, the discontinuity of $\widehat{M}$ at $Q_{2\Im} = 2\mu Q_{1\Im}$ is not compensated by the first term in (3.109), and its presence in the resolvent $\widetilde{M}$ is a characteristic manifestation of the soliton content of the potential $\tilde{u}(x)$.

The behavior of $\widetilde{M}(p;\mathbf{q})$ at the end points $\mathbf{q}_{1\Im} = \pm\kappa$ of this cut needs special study. In the limit

$$\mathbf{q}_{1\Im} \to \pm\kappa, \qquad \mathbf{q}_{2\Im} \to \pm 2\mu\kappa, \tag{3.116}$$

$\widetilde{M}(p;\mathbf{q})$ has logarithmic singularities.

To show this, we consider the first term in Eq. (3.109). Although the factors $|\tilde{\nu}\rangle$ and $\langle\tilde{\omega}|$ are regular in this limit, as was mentioned in Sec. 3.3, because of (2.74), (3.91), and (2.17), $\widetilde{T}$ and M_0 behave as

$$\frac{\mathbf{q}_1-\mu+i\kappa\operatorname{sgn}\mathbf{q}_{1\Im}}{\mathbf{q}_1-\mu-i\kappa\operatorname{sgn}\mathbf{q}_{1\Im}} \to \frac{\mathbf{q}_{1\Re}-\mu\pm 2i\kappa}{\mathbf{q}_{1\Re}-\mu\pm i0\big(|\mathbf{q}_{1\Im}|-\kappa\big)}, \tag{3.117}$$

$$\frac{1}{\mathbf{q}_2-\mathbf{q}_1^2} \to \frac{1}{\mathbf{q}_{2\Re}-\mathbf{q}_{1\Re}^2+\kappa^2\mp 2i\kappa(\mathbf{q}_{1\Re}-\mu)}, \tag{3.118}$$

and product of the limiting values is not defined. To separate the singularity, we note that

$$\frac{1}{\mathbf{q}_2-\mathbf{q}_1^2} + i\pi\operatorname{sgn}(\mathbf{q}_{2\Im}-2\mathbf{q}_{1\Re}\mathbf{q}_{1\Im})\delta(\mathbf{q}_{2\Re}-\mathbf{q}_{1\Re}^2+\kappa^2) \to$$

$$\to \frac{1}{\mathbf{q}_{2\Re}-\mathbf{q}_{1\Re}^2+\kappa^2\mp 2i\kappa(\mathbf{q}_{1\Re}-\mu)} \mp i\pi\operatorname{sgn}(\mathbf{q}_{1\Re}-\mu)\delta(\mathbf{q}_{2\Re}-\mathbf{q}_{1\Re}^2+\kappa^2) \tag{3.119}$$

in the limit (3.116). This distribution can be multiplied by the distribution in the right-hand side of (3.117), because for $\mathbf{q}_{1\Re}\to\mu$, the right-hand side of (3.119) is equal to $(\mathbf{q}_{2\Re}-\mathbf{q}_{1\Re}^2+\kappa^2)^{-1}$ in the sense of the principal value.

We now write the product we are interested in as

$$\frac{\mathbf{q}_1-\mu+i\kappa\operatorname{sgn}\mathbf{q}_{1\Im}}{\mathbf{q}_1-\mu-i\kappa\operatorname{sgn}\mathbf{q}_{1\Im}}\frac{1}{\mathbf{q}_2-\mathbf{q}_1^2} =$$

$$= \frac{\mathbf{q}_1-\mu+i\kappa\operatorname{sgn}\mathbf{q}_{1\Im}}{\mathbf{q}_1-\mu-i\kappa\operatorname{sgn}\mathbf{q}_{1\Im}}\left(\frac{1}{\mathbf{q}_2-\mathbf{q}_1^2}+i\pi\operatorname{sgn}(\mathbf{q}_{2\Im}-2\mathbf{q}_{1\Re}\mathbf{q}_{1\Im})\delta(\mathbf{q}_{2\Re}-\mathbf{q}_{1\Re}^2+\kappa^2)\right) -$$

$$- i\pi\frac{\mathbf{q}_1-\mu+i\kappa\operatorname{sgn}\mathbf{q}_{1\Im}}{\mathbf{q}_1-\mu-i\kappa\operatorname{sgn}\mathbf{q}_{1\Im}}\operatorname{sgn}(\mathbf{q}_{2\Im}-2\mathbf{q}_{1\Re}\mathbf{q}_{1\Im})\delta(\mathbf{q}_{2\Re}-\mathbf{q}_{1\Re}^2+\kappa^2), \tag{3.120}$$

such that only the last term is singular in the limit. Using the spectral parameter $\mathbf{k}$ introduced in (2.41) and recalling that the sign of $\mathbf{q}_{1\Im}$ is equal to $\pm$ in the limit (3.116), this term can be rewritten as

$$\frac{\pm i\pi\operatorname{sgn}(\mathbf{q}_{1\Re}-\mathbf{k}_{\Re})}{\mathbf{q}_{1\Re}-\mathbf{k}_{\Re}+\mathbf{k}-\lambda_{\pm}}\delta(\mathbf{q}_{2\Re}-\mathbf{q}_{1\Re}^2+\kappa^2)(\mathbf{q}_{1\Re}-\mu\pm 2i\kappa). \tag{3.121}$$

We must study it in the limit

$$\mathbf{k} \to \lambda_{\pm}, \tag{3.122}$$

which is the equivalent formulation of limit (3.116) in terms of $\mathbf{k}$ because of (2.41) and (3.93). It is easy to prove that in the sense of the Schwartz distributions in p,

$$\frac{\operatorname{sgn} p}{p + \mathbf{z}} = -\delta(p)\big(\log \mathbf{z} + \log(-\mathbf{z})\big) + \frac{1}{|p|} + o(1), \qquad \mathbf{z} \to 0, \tag{3.123}$$

for complex $\mathbf{z}$, $\mathbf{z}_{\Im} \neq 0$, where $\log \mathbf{z}$ is the principal branch of the logarithmic function and $1/|p|$ is the standard distribution (see [31]) defined by

$$\left(\frac{1}{|p|}, \phi\right) = \int \frac{dp}{|p|}\big[\phi(p) - \theta(1 - |p|)\phi(0)\big] \tag{3.124}$$

with $\phi(p)$ being an arbitrary test function.

Now, from (3.120) in the limit (3.122), taking (3.117), (3.121), and (3.123) into account, we obtain

$$\begin{aligned}
\frac{\mathbf{q}_1 - \mu + i\kappa \operatorname{sgn} \mathbf{q}_{1\Im}}{\mathbf{q}_1 - \mu - i\kappa \operatorname{sgn} \mathbf{q}_{1\Im}} \frac{1}{\mathbf{q}_2 - \mathbf{q}_1^2} \to{}& \frac{\mathbf{q}_{1\Re} - \mu \pm 2i\kappa}{\mathbf{q}_{1\Re} - \mu} \times \\
&\times \left(\frac{1}{\mathbf{q}_{2\Re} - \mathbf{q}_{1\Re}^2 + \kappa^2 \mp 2i\kappa(\mathbf{q}_{1\Re} - \mu)} \mp i\pi \operatorname{sgn}(\mathbf{q}_{1\Re} - \mu)\delta(\mathbf{q}_{2\Re} - \mathbf{q}_{1\Re}^2 + \kappa^2) \right) \pm \\
&\pm i\pi \frac{\mathbf{q}_{1\Re} - \mu \pm 2i\kappa}{|\mathbf{q}_{1\Re} - \mu|} \delta(\mathbf{q}_{2\Re} - \mathbf{q}_{1\Re}^2 + \kappa^2) + \\
&+ 2\pi\kappa\delta(\mathbf{q}_{1\Re} - \mu)\delta(\mathbf{q}_{2\Re} - \mathbf{q}_{1\Re}^2 + \kappa^2)\big(\log(\mathbf{k} - \lambda_{\pm}) + \log(\lambda_{\pm} - \mathbf{k})\big) + \\
&+ \frac{2\pi\kappa \operatorname{sgn}\big(|\mathbf{q}_{1\Im}| - \kappa\big)}{\mathbf{q}_{2\Re} - \mu^2 + \kappa^2} \delta(\mathbf{q}_{1\Re} - \mu) + o(1),
\end{aligned} \tag{3.125}$$

where $(\mathbf{q}_{1\Re} - \mu)^{-1}$ and $(\mathbf{q}_{2\Re} - \mu^2 + \kappa^2)^{-1}$ are principal values, $|\mathbf{q}_{1\Re} - \mu|^{-1}$ is defined in (3.124), and $o(1)$ denotes distributions vanishing in the limit.

Substituting (3.125) in $|\tilde{\nu}\rangle \widetilde{T} M_0 \langle\widetilde{\omega}|$, we obtain

$$\begin{aligned}
|\tilde{\nu}\rangle \widetilde{T} M_0 \langle\widetilde{\omega}| \to{}& 2\pi\kappa\big(\log(\mathbf{k} - \lambda_{\pm}) + \log(\lambda_{\pm} - \mathbf{k})\big)|\tilde{\nu}_{1,\pm}\rangle \delta(Q_{1\Re} - \mu I)\delta\big(Q_{2\Re} - (\lambda^2)_{\Re} I\big)\langle\widetilde{\omega}_{1,\pm}| + \\
&+ 2\pi\kappa \operatorname{sgn}\big(|\mathbf{k}_{\Im}| - \kappa\big)|\tilde{\nu}_{1,\pm}\rangle \frac{\delta(Q_{1\Re} - \mu I)}{Q_{2\Re} - (\lambda^2)_{\Re} I} \langle\widetilde{\omega}_{1,\pm}| + O(1),
\end{aligned} \tag{3.126}$$

where all the terms that are finite in limit (3.122) and do not depend on the sign of $|\mathbf{k}_{\Im}| - \kappa$ are included in $O(1)$. Terms depending on this sign, as was shown above, must be compensated by $\widehat{M}$.

Indeed, by (3.110), we have

$$\begin{aligned}
\widehat{M} \to{}& \mp 2i\kappa|\tilde{\nu}_{1,\pm}\rangle \frac{1}{Q_{1\Re} - \mu I + i0(Q_{1\Im} \mp \kappa I)} \frac{1}{Q_{2\Re} - 2\mu Q_{1\Re} + |\lambda|^2 I \pm i0(\mathbf{k}_{\Re} - \mu) I} \langle\widetilde{\omega}_{1,\pm}| \pm \\
&\pm 2i\kappa|\tilde{\nu}_{1,\mp}\rangle \frac{1}{(Q_{1\Re} - \mu I \pm 2i\kappa I)} \frac{1}{\big(Q_{2\Re} - 2\mu Q_{1\Re} + |\lambda|^2 I \pm i0(\mathbf{k}_{\Re} - \mu) I\big)} \langle\widetilde{\omega}_{1,\mp}|
\end{aligned} \tag{3.127}$$

in limit (3.122). On the other hand, the requirement that $\widehat{M}$ vanish outside the strip $|\mathbf{q}_{1\Im}| \leq \kappa$ gives

$$\mp 2i\kappa|\tilde{\nu}_{1,\pm}\rangle \frac{1}{(Q_{1\Re} - \mu I \pm i0)(Q_{2\Re} - 2\mu Q_{1\Re} + |\lambda|^2 I \pm i0(\mathbf{k}_\Re - \mu)I)} \langle\widetilde{\omega}_{1,\pm}| \pm$$

$$\pm 2i\kappa|\tilde{\nu}_{1,\mp}\rangle \frac{1}{(Q_{1\Re} - \mu I \pm 2i\kappa I)(Q_{2\Re} - 2\mu Q_{1\Re} + |\lambda|^2 I \pm i0(\mathbf{k}_\Re - \mu)I)} \langle\widetilde{\omega}_{1,\mp}| = 0.$$

Subtracting this equality from (3.127), we obtain

$$\widehat{M} = 4\pi\kappa\theta(\kappa \mp \mathbf{k}_\Im)|\tilde{\nu}_{1,\pm}\rangle \frac{\delta(Q_{1\Re} - \mu I)}{Q_{2\Re} - (\lambda^2)_\Re I \pm i0(\mathbf{k}_\Re - \mu)I} \langle\widetilde{\omega}_{1,\pm}| + o(1) \tag{3.128}$$

in limit (3.122).

Summing (3.126) and (3.128), we finally obtain the behavior of $\widetilde{M}$ in limit (3.122) from (3.109),

$$\widetilde{M} = 4\pi\kappa \log(\mp i(\mathbf{k} - \lambda_\pm))|\tilde{\nu}_{1,\pm}\rangle\delta(Q_{1\Re} - \mu I)\delta(Q_{2\Re} - (\lambda^2)_\Re I)\langle\widetilde{\omega}_{1,\pm}| + O(1), \tag{3.129}$$

where $O(1)$ denotes distributions that have well-defined limiting values under (3.122).

Therefore, $\widetilde{M}(p;\mathbf{q})$ has logarithmic singularities at the end points of the cut $\mathbf{k}_\Re = \mu$, $|\mathbf{k}_\Im| \leq \kappa$ (because of (2.41) at $\mathbf{q}_{2\Im} = 2\mu\mathbf{q}_{1\Im}$, $|\mathbf{q}_{1\Im}| = \kappa$). We note that the first term in the right-hand side of (3.109) gives logarithmic singularities with two infinite cuts at $\mathbf{q}_{1\Im} = \pm\kappa$. The second term, $\widehat{M}$, cancels these two cuts and replaces them with a finite cut in the complex $\mathbf{k}$ plane connecting the two points $\mathbf{k} = \lambda_\pm$.

By analogy with (3.115), the asymptotic behavior in (3.129) has a simple form in the x space because of condition (3.113). We have

$$\frac{1}{(2\pi)^2}\int dp \int d\mathbf{q}_\Re\, e^{-i(p+\mathbf{q}_\Re)x + i\mathbf{q}_\Re y}\widetilde{M}(p;\mathbf{q}) =$$

$$= \frac{\kappa}{\pi}\log(\mp i(\mathbf{k} - \lambda_\pm))e^{\mp\kappa(x_1 - y_1 + 2\mu x_2 - 2\mu y_2)}\widetilde{\Phi}_{1,+}(x)\widetilde{\Psi}_{1,+}(y) + O(1). \tag{3.130}$$

Discontinuity (3.115) in the neighborhoods of the end points follows immediately from this asymptotic expansion.

Finally, we note that because of the special structure of $\widetilde{M}$, the reduction procedure described in Sec. 2.3 for obtaining the Jost solutions $|\tilde{\nu}\rangle$ and $\langle\widetilde{\omega}|$ cancels these additional singularities, as is explicitly shown in Sec. 3.3. The relevance of these singularities to any extension of the theory to more general potentials of form (1.3) is examined in Sec. 4.

3.5. Spectral data

In this section, we derive the spectral data corresponding to the potential $\tilde{v}$ (see (3.39)). We represent these spectral data in terms of the spectral data of the original potential v and the Bäcklund transformation parameter λ. A study of their properties demonstrates that the spectral data satisfy the conditions formulated in Sec. 2.5.

We start with the spectral data $\widetilde{F}^\pm$ introduced in (2.101). Applying limiting procedure (2.77) to (3.86) and taking (3.90) and (3.93) into account, we obtain

$$|\tilde{\nu}^\sigma\rangle\widetilde{T}^\sigma = U^\dagger|\nu^\sigma\rangle(\widetilde{T}^\sigma\theta(\sigma) + \theta(-\sigma)), \qquad Q_{1\Im} = 0, \qquad \sigma = \pm, \tag{3.131}$$

where $\theta(+) = 1$, $\theta(-) = 0$, and the continuity of U at the real axis following from (3.44) is used. Now, we obtain

$$|\tilde{\nu}^\sigma\rangle\widetilde{T}^\sigma = U^\dagger|\nu^{-\sigma}\rangle F^{-\sigma}(\widetilde{T}^\sigma\theta(\sigma) + \theta(-\sigma))$$

from (2.90). Changing the sign of σ in (3.131) and substituting $U^\dagger|\nu^{-\sigma}\rangle$, we obtain the first equality in (2.101) with

$$\widetilde{F}^{+} = F^{+}, \qquad \widetilde{F}^{-} = \widetilde{T}^{-}F^{-}\widetilde{T}^{+}. \tag{3.132}$$

Characterization property (2.102) and the self-adjointness of $\widetilde{F}^{\pm}$ follow from the corresponding properties of the spectral data $F^{\pm}$ (see (2.91)). The second equality in (2.101) follows via the Hermitian conjugation of the first one.

It is well known, as was mentioned following (2.91), that for a solvable inverse problem, even in the decaying-potential case, the spectral data F^{σ} must be decomposable into the product of triangular operators $R^{\sigma}_{\pm}$ with a trivial (unity) "diagonal" part. The construction of the triangular operators $\widetilde{R}^{\sigma}_{\pm}$ that gives decomposition (2.100) of $\widetilde{F}^{\sigma}$ is more involved than construction of $\widetilde{F}^{\sigma}$ themselves. Moreover, the operators have a nontrivial "diagonal" part and an additional singularity in the "off-diagonal" part.

We must return to definition (2.94). Using the integral equation for the Jost solution $|\tilde{\nu}\rangle$ in (2.65) and the formula for $|\tilde{\nu}\rangle$ derived in (3.86), we obtain

$$\begin{aligned}\big(\widetilde{R}^{\sigma}_{\pm}(\widetilde{T}^{\sigma})^{-1}\big)(p;\mathbf{q}) = \delta(p) + \Bigg[\frac{1}{p_1 + i0\sigma}\left(\frac{\big(p_1\mathcal{P}(p;\mathbf{q}_{1\Re})\big(U^\dagger|\nu^{\sigma}\rangle\big)(p;\mathbf{q})\big)}{\mathcal{P}(p;\mathbf{q}_{1\Re}) - i0\sigma p_1}\right) - \\ - \frac{\big(\mathcal{P}(p;\mathbf{q}_{1\Re})\big(U^\dagger|\nu^{\sigma}\rangle\big)(p;\mathbf{q})\big)}{\mathcal{P}(p;\mathbf{q}_{1\Re}) \pm i0}\Bigg]\big(\theta(\sigma) + \theta(-\sigma)\tilde{t}^{-\sigma}(\mathbf{q})\big),\end{aligned} \tag{3.133}$$

where $\mathcal{P}$ is defined in (2.32). We can express $U^\dagger|\nu^{\sigma}\rangle$ as a sum of some distributions singular at $p_1 = 0$ and at $\mathcal{P}(p;\mathbf{q}_{1\Re}) = 0$ and a regular term (denoted by Reg):

$$U^\dagger|\nu^{\sigma}\rangle = |\nu^{\sigma}\rangle - is(Q_{1\Re} - \bar{\lambda}I)^{-1} - is(Q_{1\Re} - \bar{\lambda}I)^{-1}\big(|\nu^{\sigma}\rangle - I\big) + \text{Reg}\,. \tag{3.134}$$

The regular term does not contribute to (3.133). Therefore, because the first term (once multiplied by $\mathcal{P}(p;\mathbf{q}_{\Re})$) and the third term are regular at $p_1 = 0$, we insert (3.134) into (3.133) and obtain

$$\widetilde{R}^{\sigma}_{\pm}(\widetilde{T}^{\sigma})^{-1} = I + (A^{\sigma}_{\pm} + B^{\sigma}_{\pm} + C^{\sigma}_{\pm})\big(\theta(\sigma)I + \theta(-\sigma)\widetilde{T}^{-\sigma}\big),$$

where

$$A^{\sigma}_{\pm} = \left[\frac{\big(\mathcal{P}(p;\mathbf{q}_{1\Re})|\nu^{\sigma}\rangle(p;\mathbf{q})\big)}{\mathcal{P}(p;\mathbf{q}_{1\Re}) - i0\sigma p_1} - \frac{\big(\mathcal{P}(p;\mathbf{q}_{1\Re})|\nu^{\sigma}\rangle(p;\mathbf{q})\big)}{\mathcal{P}(p;\mathbf{q}_{1\Re}) \pm i0}\right],$$

$$\begin{aligned}B^{\sigma}_{\pm} = \Bigg[\frac{1}{p_1 + i0\sigma}\left(\frac{\big(p_1\mathcal{P}(p;\mathbf{q}_{1\Re})\big(-is(Q_{1\Re} - \bar{\lambda}I)^{-1}\big)(p;\mathbf{q})\big)}{\mathcal{P}(p;\mathbf{q}_{1\Re}) - i0\sigma p_1}\right) - \\ - \frac{\big(\mathcal{P}(p;\mathbf{q}_{1\Re})\big(-is(Q_{1\Re} - \bar{\lambda}I)^{-1}\big)(p;\mathbf{q})\big)}{\mathcal{P}(p;\mathbf{q}_{1\Re}) \pm i0}\Bigg],\end{aligned}$$

$$\begin{aligned}C^{\sigma}_{\pm} = \Bigg[\frac{\big(\mathcal{P}(p;\mathbf{q}_{1\Re})\big(-is(Q_{1\Re} - \bar{\lambda}I)^{-1}\big(|\nu^{\sigma}\rangle - I\big)\big)(p;\mathbf{q})\big)}{\mathcal{P}(p;\mathbf{q}_{1\Re}) - i0\sigma p_1} - \\ - \frac{\big(\mathcal{P}(p;\mathbf{q}_{1\Re})\big(-is(Q_{1\Re} - \bar{\lambda}I)^{-1}\big(|\nu^{\sigma}\rangle - I\big)\big)(p;\mathbf{q})\big)}{\mathcal{P}(p;\mathbf{q}_{1\Re}) \pm i0}\Bigg].\end{aligned}$$

These expressions can be essentially simplified. Because of (2.82) and (2.92), we have

$$A^\sigma_\pm(p;\mathbf{q}) = r^\sigma_\pm(p;\mathbf{q}).$$

Because it follows from (3.30) that

$$\big(-is(Q_{1\Re} - \bar\lambda I)^{-1}\big)(p;\mathbf{q}) = \frac{\kappa}{\pi}\frac{\delta(p_2 - 2\mu p_1)}{p_1 + i0}\frac{1}{\mathbf{q}_{1\Re} - \lambda_-} + \mathrm{Reg}\,, \tag{3.135}$$

we obtain

$$B^\sigma_\pm(p;\mathbf{q}) = -2i\sigma\kappa\frac{\theta\big(\pm\sigma(\mathbf{q}_{1\Re} - \mu)\big)}{\mathbf{q}_{1\Re} - \lambda_-}\delta(p).$$

The computation of $C^\sigma_\pm$ is more involved. We first write

$$C^\sigma_\pm(p;\mathbf{q}) = \pm 2i\pi\theta(\pm\sigma p_1)\delta\big(\mathcal{P}(p;\mathbf{q}_{1\Re})\big)\big(\mathcal{P}(p;\mathbf{q}_{1\Re})\big(-is(Q_{1\Re} - \bar\lambda I)^{-1}(|\nu^\sigma\rangle - I)\big)(p;\mathbf{q})\big).$$

We must therefore evaluate the singular part of $-is(Q_1 - \bar\lambda I)^{-1}\big(|\nu^\sigma\rangle - I\big)$ at $\mathcal{P}(p;\mathbf{q}_{1\Re}) = 0$. Because of (3.135), we obtain

$$\begin{aligned}\big(-is(Q_1 - \bar\lambda I)^{-1}(|\nu^\sigma\rangle - I)\big)(p;\mathbf{q}) =& \frac{\kappa}{\pi}\int dp_1'\,\frac{\big(v|\nu^\sigma\big)(p_1', p_2 - 2\mu(p_1 - p_1');\mathbf{q})}{(p_1 - p_1' + i0)(\mathbf{q}_{1\Re} + p_1' - \lambda_-)}\times\\ &\times\Bigg[\frac{1}{\mathcal{P}(p;\mathbf{q}_{1\Re}) + (p_1 - p_1' + i0)(p_1 + p_1' + 2\mathbf{q}_{1\Re} - 2\mu - i0)} +\\ &+ 2\pi i\theta\big(\sigma p_1'(p_1' + \mathbf{q}_{1\Re} - \mu)\big)\,\mathrm{sgn}(p_1' + \mathbf{q}_{1\Re} - \mu)\times\\ &\times\delta\big(\mathcal{P}(p;\mathbf{q}_{1\Re}) + (p_1 - p_1')(p_1 + p_1' + 2\mathbf{q}_{1\Re} - 2\mu)\big)\Bigg] + \mathrm{Reg}\,.\end{aligned}$$

Because the first term is regular at $\mathcal{P}(p;\mathbf{q}_{1\Re}) = 0$, we obtain

$$C^\sigma_\pm(p;\mathbf{q}) = -2i\kappa\frac{\theta\big(\sigma p_1(p_1 + \mathbf{q}_{1\Re} - \mu)\big)}{\mathbf{q}_{1\Re} + p_1 - \lambda_-}r^\sigma_\pm(p;\mathbf{q}).$$

Therefore,

$$\begin{aligned}\widetilde{R}^\sigma_\pm(p;\mathbf{q}) = \delta(p)&\left[1 + 2i\sigma\kappa\frac{\theta\big(\mp\sigma(\mathbf{q}_{1\Re} - \mu)\big)}{\mathbf{q}_{1\Re} - \lambda_\sigma}\right] +\\ &+\left[1 - 2i\kappa\frac{\theta\big(\pm(p_1 + \mathbf{q}_{1\Re} - \mu)\big)}{\mathbf{q}_{1\Re} + p_1 - \lambda_-}\right]r^\sigma_\pm(p;\mathbf{q})\big(\theta(-\sigma) + \theta(\sigma)\tilde{t}^\sigma(\mathbf{q})\big),\end{aligned}$$

where the triangular character of $r^\sigma_\pm(p;\mathbf{q})$ is taken into account. Using (2.82), (3.94), and the first equality in (3.95), we can rewrite the spectral data in a more compact form as

$$\widetilde{R}^+_\pm = w_\pm R^+_\pm\widetilde{T}^+, \qquad \widetilde{R}^-_\pm = w_\pm R^-_\pm, \tag{3.136}$$

where

$$w_\pm(p;\mathbf{q}) \equiv w_\pm(\mathbf{q}_{1\Re})\delta(p) = \left[\theta\big(\mp(\mathbf{q}_{1\Re} - \mu)\big) + \theta\big(\pm(\mathbf{q}_{1\Re} - \mu)\big)\frac{\mathbf{q}_{1\Re} - \lambda_+}{\mathbf{q}_{1\Re} - \lambda_-}\right]\delta(p). \tag{3.137}$$

We note that by (2.1),

$$w_\pm^\dagger w_\pm = I \tag{3.138}$$

and by (3.94),

$$\widetilde{T}^+ = (w_\mp w_\pm)^\dagger, \qquad \widetilde{T}^- = w_\mp w_\pm. \tag{3.139}$$

It is easy to verify that any $\widetilde{R}_\pm^\sigma$ of form (3.136) with spectral data $R_\pm^\sigma$ satisfying characterization equations (2.85)–(2.87) and $w_\pm$ satisfying (3.138) satisfies characterization equations (2.96)–(2.98). Moreover, it is easy to verify via (2.89) and (3.138) that (2.100) is satisfied for $\widetilde{F}^\sigma$ obtained in (3.132). To demonstrate the triangularity of the operators $\widetilde{R}_\pm^\sigma$, we substitute (2.82) in (3.136) and obtain

$$\widetilde{R}_\pm^\sigma = \tilde{d}_\pm^\sigma + \tilde{r}_\pm^\sigma, \tag{3.140}$$

where $\tilde{d}_\pm^\sigma$ and $\tilde{r}_\pm^\sigma$ are structurally independent distributions,

$$\tilde{d}_\pm^+ = w_\pm \widetilde{T}^+, \qquad \tilde{d}_\pm^- = w_\pm, \tag{3.141}$$

and

$$\tilde{r}_\pm^+ = w_\pm r_\pm^+ \widetilde{T}^+, \qquad \tilde{r}_\pm^- = w_\pm r_\pm^-. \tag{3.142}$$

Because of (3.90) and (3.137), $\tilde{d}_\pm^\sigma(p;\mathbf{q})$, $\sigma = +,-$, is proportional to $\delta(p)$. Because of (2.83), (3.90), and (3.137), $\tilde{r}_\pm^\sigma(p;\mathbf{q})$ is proportional to $\theta(\pm\sigma p_1)\delta\big(\mathcal{P}(p;\mathbf{q}_{1\Re})\big)$ and is singular at $p_1+\mathbf{q}_{1\Re} = \mu$. Therefore, the spectral data of the continuous spectrum under the Bäcklund transformation have a nontrivial "diagonal" term $\tilde{d}_\pm^\sigma(p;\mathbf{q})$ and an additional discontinuity in the "off-diagonal" term $\tilde{r}_\pm^\sigma(p;\mathbf{q})$.

3.6. Inverse problem

We proved that the Jost solution $|\tilde{\nu}\rangle$ constructed using the Darboux transformation is analytic in the upper and lower half-planes of the spectral parameter $\mathbf{q}_1$, that it obeys asymptotic condition (3.107) and that its limiting values $|\tilde{\nu}^\pm\rangle$ on the two sides of the real axis are related by the first equality of (2.101), for example, for $\sigma = -$,

$$|\tilde{\nu}^-\rangle\widetilde{T}^- = |\tilde{\nu}^+\rangle\widetilde{F}^+, \tag{3.143}$$

where (see (3.94) and (3.132)) $\widetilde{T}^-$ and $\widetilde{F}^+$ are given in terms of the soliton parameter λ and the spectral data F^+ of the original (decaying) potential. In addition, we showed that the values of the Jost solution at $\mathbf{q}_1 = \lambda_\pm$, Eqs. (3.97), are not independent because their gauged Fourier transforms (3.104) are related by (3.106).

Now, we prove that $|\tilde{\nu}\rangle(p;\mathbf{q})$ in (3.86) is the unique solution of the nonlocal Riemann–Hilbert problem (3.143) under conditions (3.107) and (3.106). The proof is based on the assumption that F^σ in (3.132) are spectral data that, via the first equality of (2.90) and the asymptotic condition (2.42), define a uniquely solvable nonlocal Riemann–Hilbert problem for a decaying potential.

First, we note that under limiting procedure (2.77), the scalar product of Jost solutions (2.55) and completeness relation (2.58) become

$$\langle\omega^\pm|\nu^\pm\rangle = I, \qquad |\nu^\pm\rangle\langle\omega^\pm| = I. \tag{3.144}$$

Therefore, because of (2.90), we have $F^+ = \langle\omega^+|\nu^-\rangle$, and because of (3.132), we can rewrite (3.143) as $|\tilde{\nu}^-\rangle\widetilde{T}^- = |\tilde{\nu}^+\rangle\langle\omega^+|\nu^-\rangle$ or, using completeness, as

$$|\tilde{\nu}^+\rangle\langle\omega^+| = |\tilde{\nu}^-\rangle\widetilde{T}^-\langle\omega^-|. \tag{3.145}$$

The left-hand side of (3.145) can be considered the limiting value on the upper side of the real axis of the function $\big(|\tilde{\nu}\rangle\langle\omega|\big)(p;\mathbf{q})\big)$, which is analytic in the upper half plane, and the right-hand side is the limiting value on the lower side of the real axis of the function $\big(|\tilde{\nu}\rangle\widetilde{T}\langle\omega|\big)(p;\mathbf{q})$, which is analytic in the lower half plane with a cut at $\mathbf{q}_{1\Im} = -\kappa$ because of the special analytic properties of $\widetilde{T}$ (see (3.90)). Taking into account the definition of $\beta(p)$ in (3.9) (more exactly, of its Hermitian conjugate) and the definition of $|\tilde{\nu}_{1,-}\rangle$ in (3.97) and using (3.90), we find that the above discontinuity of $|\widetilde{\nu}\rangle\widetilde{T}\langle\omega|$ coincides with the discontinuity of $-2i\kappa|\tilde{\nu}_{1,-}\rangle(Q_1-\bar{\lambda}I)^{-1}\beta^{\dagger}$. Therefore, rewriting (3.145) as

$$|\tilde{\nu}^{+}\rangle\langle\omega^{+}| + |\tilde{\nu}_{1,-}\rangle\frac{2i\kappa}{Q_{1\Re}-\bar{\lambda}I}\beta^{\dagger} = |\tilde{\nu}^{-}\rangle\widetilde{T}^{-}\langle\omega^{-}| + |\tilde{\nu}_{1,-}\rangle\frac{2i\kappa}{Q_{1\Re}-\bar{\lambda}I}\beta^{\dagger}, \tag{3.146}$$

we obtain an equality between the boundary values of two functions analytic in the corresponding half-planes, which, because of (2.42), (3.90), and condition (3.107), tend to I as $\mathbf{q}_1 \to \infty$. Therefore, we have

$$|\tilde{\nu}\rangle\big[\theta(Q_{1\Im}) + \widetilde{T}\theta(-Q_{1\Im})\big]\langle\omega| + |\tilde{\nu}_{1,-}\rangle\frac{2i\kappa}{Q_1-\bar{\lambda}I}\beta^{\dagger} = I,$$

where notation (3.87) is used. Because of (2.55) and (3.90), we finally obtain

$$|\tilde{\nu}\rangle = \left[I - |\tilde{\nu}_{1,-}\rangle\frac{2i\kappa}{Q_1-\bar{\lambda}I}\beta^{\dagger}\right]|\nu\rangle\left[\theta(Q_{1\Im}) + \theta(-Q_{1\Im})\frac{Q_1-\bar{\lambda}I}{Q_1-\lambda I}\right]. \tag{3.147}$$

In the right-hand side of (3.147), all terms except $|\tilde{\nu}_{1,-}\rangle$ are given in terms of the known objects $|\nu\rangle$ and λ. Therefore, to solve the inverse problem, we must also determine $|\tilde{\nu}_{1,-}\rangle$ in terms of these objects. If we use definition (3.97) with the minus sign and evaluate (3.147) at $\mathbf{q}_1 = \lambda_-$, we obtain an identity. We therefore need the first condition in (3.106) to build $|\tilde{\nu}_{1,-}\rangle$. Using (3.97) with the plus sign and (3.11), we obtain

$$|\tilde{\nu}_{1,+}\rangle(p) = \beta(p) - 2i\kappa\int dp'\,|\tilde{\nu}_{1,-}\rangle(p-p')\frac{(\beta^{\dagger}\beta)(p')}{p'_1+2i\kappa}$$

from (3.147). Taking definition (3.31) into account, we obtain

$$|\tilde{\nu}_{1,+}\rangle = \beta - |\tilde{\nu}_{1,-}\rangle(I+2\kappa f). \tag{3.148}$$

We now introduce

$$S(x) = \frac{2\kappa}{1+e^{-2\kappa(x_1+2\mu x_2-x_0)}}, \tag{3.149}$$

$$F(x) = i\int dp\,e^{-ipx}\frac{(\beta^{\dagger}\beta-I)(p)}{p_1+2i\kappa}, \tag{3.150}$$

i.e., the Fourier transforms of s and f (see (3.30) and (3.31)). Using (3.104), we rewrite (3.148) in the form

$$\widetilde{\Phi}_{1,+}(x) = \Phi(x,\lambda_+) - e^{2\kappa(x_1+2\mu x_2)}\widetilde{\Phi}_{1,-}(x)\big[1+2\kappa F(x)\big].$$

Using (3.106) to replace $\widetilde{\Phi}_{1,+}(x)$, we obtain

$$\widetilde{\Phi}_{1,-}(x) = \frac{e^{-2\kappa(x_1+2\mu x_2)}}{2\kappa}\Phi(x,\lambda_+)\frac{S(x)}{1+S(x)F(x)}. \tag{3.151}$$

Inverting gauged Fourier transform (3.104) and using (3.149) and (3.150), we can verify that (3.151) is the Fourier transform of (3.101). In turn, (3.147) is exactly (3.86) with $U^\dagger$ replaced with (3.45). This proves the uniqueness of the solution of the nonlocal Riemann–Hilbert problem formulated in this section.

3.7. Behavior of the potential at large distances

To consider the behavior of the potential $\tilde{u}(x)$ at large distances, we use (3.1) and (3.39) to rewrite (3.41) as

$$\tilde{u}(x) = u(x) - 2\partial_{x_1}^2 \log\left(\frac{1}{S(x)} + F(x)\right), \tag{3.152}$$

where $S(x)$ and $F(x)$ are defined in (3.149) and (3.150). We choose a direction on the x plane,

$$\xi' \equiv x_1 + 2\mu' x_2 = \text{const}, \tag{3.153}$$

where μ' is a real parameter. The special direction with $\mu' = \mu$ is denoted by ξ. Because of (3.149), we have

$$\tilde{u}(x) = u(x) - 2\partial_{\xi'}^2 \log\left(1 + e^{-2\kappa(\xi'-x_0)-4\kappa(\mu-\mu')x_2} + 2\kappa F(\xi' - 2\mu' x_2, x_2)\right) \tag{3.154}$$

instead of (3.152). The asymptotic behavior at large distances along the direction $\xi' = \text{const}$ is then obtained by evaluating $F(\xi' - 2\mu' x_2, x_2)$ at large x_2. We obtain the equality $(2\pi)^2\beta(p) = \int dx\, e^{ipx}\chi(x,\lambda)$ from definitions (2.27) and (3.11). Inserting it into (3.150), we obtain

$$F(\xi' - 2\mu' x_2, x_2) = \int_{-\infty}^{\xi'} dy_1\, e^{2\kappa(y_1-\xi')}\left[\left|\chi(y_1 - 2\mu' x_2, x_2, \lambda)\right|^2 - 1\right], \tag{3.155}$$

and its asymptote can therefore be obtained from that of $\chi(y_1 - 2\mu' x_2, x_2, \lambda)$. To obtain the leading term, we must know the asymptotic behavior of $\chi(y_1 - 2\mu' x_2, x_2, \lambda)$ at large x_2 up to and including the order x_2^{-3}. Using integral equation (2.33) and taking (2.36) into account, we obtain

$$\chi(y_1 - 2\mu' x_2, x_2, \lambda) = 1 + \frac{d_{10}}{x_2} + \frac{d_{20} + d_{21}y_1}{x_2^2} + \frac{d_{30} + d_{31}y_1 + d_{32}y_1^2}{x_2^3} + O(x_2^{-4}), \tag{3.156}$$

where

$$d_{10} = \frac{\pi\gamma(0,\lambda)}{\lambda - \mu'}, \qquad d_{21} = -\frac{\pi\gamma(0,\lambda)}{2(\lambda-\mu')^2}, \qquad d_{32} = \frac{\pi\gamma(0,\lambda)}{4(\lambda-\mu')^3} \tag{3.157}$$

with

$$\gamma(0;\lambda) = \frac{1}{(2\pi)^2}\int dx\, u(x)\chi(x,\lambda). \tag{3.158}$$

The evaluation of the other constant coefficients d_{ij} is rather involved, but, in fact, we only need d_{10} and d_{32}.

We can now formulate the asymptotic behavior of $\tilde{u}(x)$ with its dependence on the direction. In the direction $\xi = \text{const}$, that is, for $\mu' = \mu$ in (3.153), we have a pure one-soliton behavior as expected, corrected, however, by a term $O(x_2^{-1})$. Precisely, we obtain

$$\tilde{u}(\xi - 2\mu x_2, x_2) - u(\xi - 2\mu x_2, x_2) = -\frac{2\kappa^2}{\cosh^2\kappa(\xi - x_0)}\left[1 - \frac{d_{10} + \bar{d}_{10}}{x_2}\tanh\kappa(\xi - x_0)\right] + O(x_2^{-2}), \tag{3.159}$$

where $\xi \equiv x_1 + 2\mu x_2 = \text{const}$, $x_2 \to \infty$.

In a direction $\xi' = \text{const}$ with $\mu' \neq \mu$, for $(\mu - \mu')x_2 \to -\infty$, the behavior of $\tilde{u}(x)$ with respect to that of the background potential $u(x)$ is corrected by an additional term which is exponentially decreasing,

$$\tilde{u}(\xi' - 2\mu' x_2, x_2) - u(\xi' - 2\mu' x_2, x_2) = -8\kappa^2 e^{2\kappa(\xi' - x_0) + 4\kappa(\mu - \mu')x_2}(1 + O(x_2^{-1})), \tag{3.160}$$

whereas for $(\mu - \mu')x_2 \to +\infty$, the correction decreases algebraically,

$$\tilde{u}(\xi' - 2\mu' x_2, x_2) - u(\xi' - 2\mu' x_2, x_2) = -4\frac{d_{32} + \bar{d}_{32}}{x_2^3} + O(x_2^{-4}). \tag{3.161}$$

We conclude that the direction $\xi = \text{const}$ of the wave soliton emerging from the background at large distances divides the x plane into two regions with different asymptotic behavior. In the first region (to the left of the soliton), the original potential $u(x)$ is modified at large distances by an exponentially decreasing term, and in the second region (to the right), by a term decreasing as x_2^{-3}. If we consider the case $\kappa < 0$, the behavior in the two regions is interchanged.

We stress that this asymptotic behavior does not depend in any way on the smoothness or the decay properties of the background potential $u(x)$. This background potential can be as small in any norm as we want, but nevertheless the asymptotic behavior is modified as shown above. Moreover, because the coefficients d_{ij} depend on the direction $\xi' = \text{const}$ via (3.157), the only way to cancel the leading term in the above asymptote is to impose the condition

$$\gamma(0; \lambda) = 0. \tag{3.162}$$

It is easy to show that we are then left with terms of the next orders; to cancel all rational terms, it is necessary to impose an infinite set of constraints on the background potential $u(x)$. On the other hand, to cancel the x_2^{-1} correction to the pure one-soliton asymptote at large distances, i.e., the leading term in (3.159) due to (3.157), it is necessary and sufficient to impose the condition

$$\gamma_{\Im}(0; \lambda) = 0. \tag{3.163}$$

The function $\gamma(0; \lambda)$ is known to be the generator of the integrals of motion for the KPI evolution (1.2) and can be expressed in terms of the spectral data using the dispersion relation (see formula (224) in [10])

$$\gamma(0; \lambda) = v(0) - \frac{1}{8\pi^2}\int \frac{d\lambda'}{\lambda' - \lambda}\sum_{\sigma=\pm}\int dp_2'' \left((R_+^\sigma - R_-^\sigma)(R_+^{-\sigma} - R_-^{-\sigma})^\dagger\right)(0, p_2''; \lambda'). \tag{3.164}$$

The constraints on the spectral data obtained by inserting (3.164) into either (3.162) or (3.163) are consequently compatible with the KPI time evolution.

4. Dressing the one-soliton potential

In the preceding sections, we studied the spectral properties of the nonstationary Schrödinger equation (1.1) with a special potential describing a soliton on an arbitrary background. This potential was obtained using a Bäcklund transformation that gives a nonlinear superimposition of a pure one-soliton potential on an arbitrary decaying potential. In this case, we explicitly proved that in spite of the presence of logarithmic singularities of the resolvent, the Jost solutions are still piecewise analytic with a cut along the real axis of the spectral parameter, that the spectral data satisfy characterization equations (2.96)–(2.98), and, furthermore, that the general scheme in Sec. 2 is indeed applicable to nondecaying potentials.

The potential obtained by means of such a Bäcklund transformation, however, is not the most general perturbation of a soliton potential. The general case can be studied using a special modification of the dressing procedure. Such a modification was suggested in [26], where the KPII equation was considered.

Its main advantage is that it is developed in the framework of the resolvent approach and is therefore independent of the special choice of the Lax operator L under consideration. Of course, we cannot write explicit formulas using this method as when using Bäcklund transformations, but we can get specific information about the analytic properties of the Jost solutions and then about the structural properties of the spectral data.

Let the resolvent M' and consequently the Jost solution $|\nu'\rangle$ and the spectral data of a potential u' be known, and let a potential u be given by

$$u = u' + u_2, \tag{4.1}$$

where $u_2(x_1, x_2)$ is a decaying smooth real function of two variables considered as a perturbation of u'. We can then consider Hilbert identity (2.20), where L' and M' are now the spectral operator and the resolvent of u'. For the resolvent M of u, we obtain the integral equations

$$M = M' + M'v_2M, \qquad M = M' + Mv_2M', \tag{4.2}$$

where v_2 is Fourier transform (2.12) of u_2. For the Jost solution $|\nu\rangle$, using reduction procedure (2.63), we then obtain the integral equation

$$|\nu\rangle(p;\mathbf{q}) = |\nu'\rangle(p;\mathbf{q}) + \int dp'\, G'(p,p',\mathbf{q}_1)\big(v_2|\nu\rangle\big)(p';\mathbf{q}), \tag{4.3}$$

where the Green's function G', as in (2.39), is given in terms of the resolvent M' as

$$G'(p,p',\mathbf{q}_1) = M'\big(p-p';\ell(\mathbf{q}_1)+p'\big) \tag{4.4}$$

with ℓ defined in (2.38). Equations (4.2) and (4.3) generalize the original integral equations (2.18) and (2.30) for the resolvent and the Jost solution.

The analytic properties of M and $|\nu\rangle$ with respect to the complex variable $\mathbf{q}$ can be determined from the analytic properties of M' using the same methods used with integral equations (2.18) and (2.30). Of course, we do not obtain new information if both u' and u_2 are decaying smooth potentials. The case where u' is not decaying and, correspondingly, v' is not regular is more interesting. In particular, we are interested in the case $u' = \tilde{u}$ where $\tilde{u}$ is the potential constructed in Sec. 3. Correspondingly, $M' = \widetilde{M}$, $G' = \widetilde{G}$, and $|\nu'\rangle = |\widetilde{\nu}\rangle$. Because of (4.2) and (4.3), the resolvent M and the Jost solution $|\nu\rangle$ of u inherit the properties of the resolvent $\widetilde{M}$, which we derived above. In particular, $M(p;\mathbf{q})$ has an additional cut at $\mathbf{q}_{2\Im} = 2\mu\mathbf{q}_{1\Im}$, $|\mathbf{q}_{1\Im}| < \kappa$.

For studying the additional cut of the Jost solution, we transform (4.3) using (2.27) and obtain

$$\chi(x,\mathbf{k}) = \tilde{\chi}(x,\mathbf{k}) + \int dx'\, \widetilde{G}(x,x',\mathbf{k})u_2(x')\chi(x',\mathbf{k}), \tag{4.5}$$

where the Green's function, in correspondence with (2.28) and (2.39), is

$$\begin{aligned}\widetilde{G}(x,x',\mathbf{k}) &= \frac{1}{(2\pi)^2}\int dp \int dp'\, e^{-ipx+ip'x'}\widetilde{M}\big(p-p';p'+\ell(\mathbf{k})\big) = \\ &= \frac{1}{(2\pi)^2}e^{i\ell_{\Re}(\mathbf{k})(x-x')}\int dp\int d\mathbf{q}_{\Re}\, e^{-i(p+\mathbf{q}_{\Re})x+i\mathbf{q}_{\Re}x'}\widetilde{M}\big(p;\mathbf{q}_{\Re}+i\ell_{\Im}(\mathbf{k})\big).\end{aligned} \tag{4.6}$$

Taking (3.113), (3.130), and (2.29) into account, we find that this Green's function has a cut for $\mathbf{k}_{\Re} = \mu$, $\kappa > \mathbf{k}_{\Im} > -\kappa$, and has the logarithmic singularities

$$\begin{aligned}\widetilde{G}(x,x',\mathbf{k}) &= \frac{\kappa}{\pi}\log\big(\mp i(\mathbf{k}-\lambda_\pm)\big)e^{i\ell(\lambda_\pm)(x-x')}\widetilde{\Phi}_{1,+}(x)\widetilde{\Psi}_{1,+}(x') + O(1) = \\ &= \frac{\kappa}{\pi}\log\big(\mp i(\mathbf{k}-\lambda_\pm)\big)\tilde{\chi}_{1,\pm}(x)\tilde{\xi}_{1,\pm}(x') + O(1), \qquad \mathbf{k}\to\lambda_\pm.\end{aligned} \tag{4.7}$$

at the end points $\lambda_\pm$. The cut is inherited from the discontinuity of $\widetilde{M}(p;\mathbf{q})$ at $\mathbf{q}_{2\Im} = 2\mu\mathbf{q}_{1\Im}$ derived in Sec. 3.5. Indeed, introducing

$$\Delta\widetilde{G}(x,x',\mathbf{k}) = \widetilde{G}(x,x',\mu+0+i\mathbf{k}_\Im) - \widetilde{G}(x,x',\mu-0+i\mathbf{k}_\Im) \tag{4.8}$$

and using (2.41), (3.111), and (4.6), we obtain

$$\begin{aligned}\Delta\widetilde{G}(x,x',\mathbf{k}) =& \frac{\operatorname{sgn}\mathbf{k}_\Im}{(2\pi)^2} e^{i\ell_\Re(\mu+i\mathbf{k}_\Im)(x-x')}\times \\ &\times \int dp \int d\mathbf{q}_\Re\, e^{-i(p+\mathbf{q}_\Re)x+i\mathbf{q}_\Re x'} \Delta\widetilde{M}(p;\mathbf{q}_\Re + i\ell_\Im(\mu+i\mathbf{k}_\Im))\end{aligned} \tag{4.9}$$

and then use (3.115) to obtain

$$\begin{aligned}\Delta\widetilde{G}(x,x',\mathbf{k}) &= -2i\kappa \operatorname{sgn}\mathbf{k}_\Im \theta(\kappa - |\mathbf{k}_\Im|) e^{i\ell(\mu+i\mathbf{k}_\Im)(x-x')}\widetilde{\Phi}_{1,+}(x)\widetilde{\Psi}_{1,+}(x') = \\ &= -2i\kappa \operatorname{sgn}\mathbf{k}_\Im \theta(\kappa - |\mathbf{k}_\Im|) e^{i[\ell(\mu+i\mathbf{k}_\Im)-\ell(\lambda_\pm)](x-x')}\widetilde{\chi}_{1,\pm}(x)\widetilde{\xi}_{1,\pm}(x').\end{aligned} \tag{4.10}$$

In the special case where $\tilde{u} = u_1$ (the pure one-soliton potential as in Sec. 3.2.2), the presence of this additional cut in the spectral theory of a perturbed soliton potential was discovered and studied (using more traditional methods) some years ago [27].

From (4.5), the Jost solution has an additional discontinuity for $\mathbf{k}_\Re = \mu$ and $\kappa > \mathbf{k}_\Im > -\kappa$. Specifically, using (4.8), we obtain

$$\chi(x,\mu+0+i\mathbf{k}_\Im) = \tilde{\chi}(x,\mu+i\mathbf{k}_\Im) + \int dx'\,\widetilde{G}(x,x',\mu+0+i\mathbf{k}_\Im)u_2(x')\chi(x',\mu+0+i\mathbf{k}_\Im), \tag{4.11}$$

$$\begin{aligned}\chi(x,\mu-0+i\mathbf{k}_\Im) =& \tilde{\chi}(x,\mu+i\mathbf{k}_\Im) + \int dx'\,\widetilde{G}(x,x',\mu+0+i\mathbf{k}_\Im)u_2(x')\chi(x',\mu-0+i\mathbf{k}_\Im) - \\ &- \int dx'\,\Delta\widetilde{G}(x,x',\mathbf{k})u_2(x')\chi(x',\mu-0+i\mathbf{k}_\Im).\end{aligned} \tag{4.12}$$

Therefore, we expect that in the case with an arbitrary perturbation u_2, it is necessary to introduce additional spectral data and that the theory must consequently be substantially modified in comparison with the traditional case. The solution of this problem is deferred.

Inserting (4.10) into (4.12), we find that this additional cut is absent iff

$$\int dx\,\widetilde{\Psi}_{1,+}(x)u_2(x)\Phi(x,\mu+i\mathbf{k}_\Im) = 0, \qquad \kappa \geq \mathbf{k}_\Im \geq -\kappa, \tag{4.13}$$

where Φ is constructed using χ in (2.29). Because Φ is the Jost solution of u, this is an implicit condition on the perturbation $u_2(x)$.

The existence of logarithmic singularities of the Green's function and Jost solutions seems to contradict the known properties of these objects in the one-dimensional theory. In fact, there is no contradiction because these additional singularities are a special consequence of the embedding of the one-dimensional objects in two spatial dimensions; they disappear upon returning to one dimension. To show this, we consider the special case where $\tilde{u}$ is the pure one-soliton potential u_1, and choosing $\mu = 0$ for simplicity, we have $u_1 = u_1(x_1)$. In this case, the resolvent is given by (2.62), where $M_{\mathbf{z}}(p_1;\mathbf{q}_1)$ is the one-dimensional resolvent of u_1. From (4.6), we then obtain

$$G_1(x,x',\mathbf{k}) = \frac{1}{(2\pi)^2}\int dp_1 \int dp_1' \int dp_2\, e^{-ip_1x_1+ip_1'x_1'-ip_2(x_2-x_2')} M_{p_2+\mathbf{k}^2}(p_1-p_1';p_1'+\mathbf{k}). \tag{4.14}$$

We emphasize that this is the two-dimensional Green's function of the one-dimensional potential $u_1(x_1)$ that satisfies the equation

$$(i\partial_{x_2} + \partial_{x_1}^2 - 2i\mathbf{k}\partial_{x_1} - u_1(x_1))G_1(x, x', \mathbf{k}) = \delta(x - x'), \tag{4.15}$$

where $\delta(x)$ is the two-dimensional δ-function and that to construct the one-dimensional Green's function $g_1(x_1, x_1, \mathbf{k})$ that satisfies the equation

$$(\partial_{x_1}^2 - 2i\mathbf{k}\partial_{x_1} - u_1(x_1))g_1(x_1, x_1', \mathbf{k}) = \delta(x_1 - x_1'), \tag{4.16}$$

it is necessary to integrate G_1 with respect to x_2. In discussing (2.69), it was shown that this integral does not exist, not even for the case of zero potential. Therefore, by analogy with definitions (2.63) and (2.64) of the Jost solutions, we must use a limiting procedure that has

$$g_1(x_1, x_1', \mathbf{k}) = \lim_{\epsilon \to +0} \int dx_2\, e^{i\epsilon(x_2 - x_2')} G_1(x, x', \mathbf{k}) \tag{4.17}$$

in the x space. When $\tilde{u} = 0$, it is easy to verify that this regularization is just the x version of the one used to construct the integral equations of the Jost solutions in (2.67) and (2.68). We now define Δg_1 by applying the same transformation to ΔG_1. Using (4.10), we then obtain

$$\frac{1}{4\pi i\kappa}\Delta g_1(x, x', \mathbf{k}) = -\operatorname{sgn}\mathbf{k}_\Im e^{(\pm\kappa - \mathbf{k}_\Im)(x_1 - x_1')}\tilde{\chi}_{1,\pm}(x_1)\tilde{\xi}_{1,\pm}(x_1')\theta(\kappa - |\mathbf{k}_\Im|)\delta(\varepsilon + \kappa^2 - \mathbf{k}_\Im^2), \tag{4.18}$$

which is identically zero because the product of θ- and δ-functions is zero. Therefore, as must be the case, the one-dimensional Green's function has neither an additional cut in the complex domain nor any logarithmic singularities.

A.K.P. thanks his colleagues in the Dipartimento di Fisica dell'Università di Lecce (Italy) for the kind hospitality and fruitful discussions.

This work is supported in part by M.U.R.S.T and Sezione INFN, Lecce, by the Russian Foundation for Basic Research (Grant No. 96-01-00344), and by INTAS (Grant No. 93-166-ext).

REFERENCES

1. V. S. Dryuma, *JETP Letters*, **19**, 381 (1974).
2. V. E. Zakharov and A. B. Shabat, *Funct. Anal. Appl.*, **8**, 226 (1974).
3. B. B. Kadomtsev and V. I. Petviashvili, *Sov. Phys. Doklady*, **192**, 539 (1970).
4. M. J. Ablowitz and P. A. Clarkson, "Solitons, nonlinear evolution equations and inverse scattering," in: *Lecture Notes Series*, Vol. 49, Cambridge Univ., Cambridge (1991).
5. B. G. Konopelchenko, *Solitons in Multidimensions*, World Scientific, Singapore (1993).
6. V. E. Zakharov and S. V. Manakov, *Sov. Sci. Rev. A*, **1**, 133 (1979); S. V. Manakov, *Physica D*, **3**, 420 (1981).
7. A. S. Fokas and M. J. Ablowitz, *Stud. Appl. Math.*, **69**, 211 (1983).
8. M. J. Ablowitz and J. Villarroel, *Stud. Appl. Math.*, **85**, 195 (1991).
9. M. Boiti, F. Pempinelli, and A. Pogrebkov, *Inverse Problems*, **10**, 505 (1994).
10. M. Boiti, F. Pempinelli, and A. Pogrebkov, *J. Math. Phys.*, **35**, 4683 (1994).
11. M. Boiti, J. Léon, and F. Pempinelli, *Phys. Lett. A*, **141**, 96 (1989).
12. M. Boiti, F. Pempinelli, A. K. Pogrebkov, and M. C. Polivanov, *Inverse Problems*, **8**, 331 (1992).
13. S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its, and V. B. Matveev, *Phys. Lett. A*, **63**, 205 (1977).
14. B. A. Dubrovin, T. M. Malanyuk, I. M. Krichever, and V. G. Makhankov, *Sov. J. Part. Nucl.*, **19**, 252 (1988).
15. A. S. Fokas and V. E. Zakharov, *J. Nonlinear Sci.*, **2**, 109 (1992).
16. J. M. Ghidaglia and J. C. Saut, *Nonlinearity*, **3**, 475 (1990).
17. M. Boiti, J. Léon, L. Martina, and F. Pempinelli, *Phys. Lett. A*, **132**, 432 (1988).
18. M. Boiti, L. Martina, and F. Pempinelli, *Chaos, Solitons, and Fractals*, **5**, 2377 (1995).

19. M. Boiti, F. Pempinelli, A. K. Pogrebkov, and M. C. Polivanov, "Resolvent approach for the nonstationary Schrödinger equation. Standard case of rapidly decreasing potential," in: *Nonlinear Evolution Equations and Dynamical Systems* (M. Boiti, L. Martina, and F. Pempinelli, eds.), World Scientific, Singapore (1992), p. 97.
20. M. Boiti, F. Pempinelli, A. K. Pogrebkov, and M. C. Polivanov, *Theor. Math. Phys.*, **93**, 1200 (1992).
21. M. Boiti, F. Pempinelli, and A. Pogrebkov, *Theor. Math. Phys.*, **99**, 511 (1994).
22. M. Boiti, F. Pempinelli, and A. Pogrebkov, "Spectral theory of solitons on a generic background for the KPI equation," in: *Nonlinear Physics. Theory and Experiment* (E. Alfinito, M. Boiti, L. Martina, and F. Pempinelli, eds.), World Scientific, Singapore (1996), p. 37.
23. M. Boiti, F. Pempinelli, and A. Pogrebkov, *Inverse Problems*, **13**, L7 (1997).
24. V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin (1991).
25. M. Boiti, F. Pempinelli, A. K. Pogrebkov, and M. C. Polivanov, *Inverse Problems*, **7**, 43 (1991).
26. M. Boiti, F. Pempinelli, and A. Pogrebkov, *Physica D*, **87**, 123 (1995).
27. A. S. Fokas and A. K. Pogrebkov, unpublished.
28. V. D. Lipovsky, *Funct. Anal. Appl.*, **20**, 282 (1986).
29. Xin Zhou, *Commun. Math. Phys.*, **128**, 551 (1990).
30. M. Boiti, B. Konopelchenko, and F. Pempinelli, *Inverse Problems*, **1**, 33 (1985).
31. I. M. Gelfand and G. E. Shilov, *Generalized Functions*, Vol. 1, *Properties and Operations*, Acad. Press, New York (1964).