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SPECTRAL THEORY OF THE NONSTATIONARY SCHRÖDINGER EQUATION WITH A TWO-DIMENSIONALLY PERTURBED ARBITRARY ONE-DIMENSIONAL POTENTIAL

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We consider the nonstationary Schrödinger equation with the potential being a perturbation of a generic one-dimensional potential by means of a decaying two-dimensional function in the framework of the extended resolvent approach. We give the corresponding modification of the Jost and advanced/retarded solutions and spectral data and present relations between them.

Keywords: inverse scattering transform, resolvent approach, Kadomtsev–Petviashvili equation

1. Introduction

The Kadomtsev–Petviashvili equation in its version called KPI

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = 3u_{x_2x_2} \tag{1.1}$$

is a (2+1)-dimensional generalization of the celebrated Korteweg–de Vries (KdV) equation [1], [2]. As a consequence, the KPI equation admits solutions that behave at spatial infinity like the solutions of the KdV equation. For instance, if $u_1(t, x_1)$ satisfies the KdV equation, then $u(t, x_1, x_2) = u_1(t, x_1 + \mu x_2 + 3\mu^2 t)$ solves the KPI equation for an arbitrary constant $\mu \in \mathbb{R}$. It is therefore natural to consider solutions of (1.1) that are not decaying in all directions at spatial infinity but have one-dimensional rays with behavior of the type of u_1. Even though the KPI equation has been known to be integrable for about three decades [2], its general theory is far from complete. Indeed, the Cauchy problem for the KPI equation with rapidly decaying initial data was resolved in [3] using the inverse scattering transform (IST) based on the spectral analysis of the nonstationary Schrödinger operator

$$\mathcal{L}(x, \partial_x) = i\partial_{x_2} + \partial_{x_1}^2 - u(x), \qquad x = (x_1, x_2), \tag{1.2}$$

which gives the associated linear problem for the KPI equation. But it is known that the standard approach to the spectral theory of operator (1.2), based on integral equations for the Jost solutions, fails for potentials with one-dimensional asymptotic behavior.

In [4], the method of the extended resolvent was suggested as a way to pursue a generalization of the IST that allows studying the spectral theory of operators with nontrivial asymptotic behavior at spatial infinity. Here, we consider the case where there is only one direction of nondecaying behavior, exactly,

$$u(x) = u_1(x_1) + u_2(x_1, x_2). \tag{1.3}$$

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Without loss of generality, we chose the constant $\mu = 0$ (the generic case is reconstructed via the Galileo invariance of (1.1)). The spectral theory for the simplest example of such potentials in (1.2) was developed in [5], where the Cauchy problem for the KPI equation was considered with initial data of type (1.3) with $u_1(x_1)$ being the value at time $t = 0$ of the one-soliton solution of the KdV equation and $u_2(x)$ being a smooth, real, rapidly decaying function on the plane (x_1, x_2). In [5], the direct problem was studied using a modified integral equation, and it was shown that the modified Jost solution, in addition to the standard jump across the real axis of the spectral parameter $\mathbf{k}$, has a jump across a segment of the imaginary axis of the complex $\mathbf{k}$ plane. But some essential properties of the Jost solutions and relations between spectral data were stated without proof in [5]. In [6], in the framework of the extended resolvent approach, the problem was completely solved for the case where u_1 is a pure one-soliton potential. Here, we present a generalization of these results to the case where $u_1(x_1)$ in (1.3) is an arbitrary smooth real one-dimensional potential rapidly decaying with respect to x_1. For brevity, some results are stated here without proof, which we will give in [7], where we also derive characterization equations for the spectral data and study their properties.

2. Case of a one-dimensional potential

2.1. Basic objects of the resolvent approach. In this section, we briefly review the basic elements of the extended resolvent approach (see [4] for further details). Let $A(x, x'; q)$ belong to the space $\mathcal{S}'$ of tempered distributions of the six real variables $x = (x_1, x_2)$, $x' = (x'_1, x'_2)$, and $q = (q_1, q_2)$. We call $A(q)$ an operator with the kernel $A(x, x'; q) \in \mathcal{S}'$. For generic operators $A(q)$ and $B(q)$ with the kernels $A(x, x'; q)$ and $B(x, x'; q)$, we introduce the standard composition law

$$(AB)(x, x'; q) = \int dx''\, A(x, x''; q)\, B(x'', x'; q) \tag{2.1}$$

if the integral exists in terms of distributions. An operator A can have an inverse in the sense of this composition, $AA^{-1} = I$ or $A^{-1}A = I$, where I is the unit operator in $\mathcal{S}'$, $I(x, x'; q) = \delta(x - x')$, where $\delta(x) = \delta(x_1)\delta(x_2)$ is the two-dimensional δ-function. A subclass essential for us in this space of operators is obtained as follows. Let $\mathcal{A} = \mathcal{A}(x, \partial_x)$ be a differential operator with the kernel $A(x, x') = \mathcal{A}(x, \partial_x)\delta(x - x')$. In what follows, we consider differential operators whose coefficients are smooth functions of x. We introduce an *extension* of differential operators, i.e., with any differential operator $\mathcal{A}$, we associate the operator $A(q)$ with the kernel

$$A(x, x'; q) \equiv \mathcal{A}(x, \partial_x + q)\delta(x - x') = e^{-q(x-x')} A(x, x'), \tag{2.2}$$

where $x, x', q \in \mathbb{R}^2$ and $qx = q_1x_1 + q_2x_2$. Such a kernel obviously belongs to the space $\mathcal{S}'(\mathbb{R}^6)$.

With any operator $A(q)$ with the kernel $A(x, x'; q)$, we associate its "hat kernel"

$$\hat{A}(x, x'; q) = e^{q(x-x')} A(x, x'; q). \tag{2.3}$$

If $A(q)$ is a differential operator then this procedure is inverse to (2.2), $\hat{A}(x, x'; q) = A(x, x')$, and for any (not necessary differential) operator $B(q)$, the relations

$$\begin{aligned} (\widehat{AB})(x, x'; q) &= \mathcal{A}(x, \partial_x)\widehat{B}(x, x'; q), \\ (\widehat{BA})(x, x'; q) &= \mathcal{A}^{\mathrm{d}}(x', \partial_{x'})\widehat{B}(x, x'; q) \end{aligned} \tag{2.4}$$

hold, where $\mathcal{A}^{\mathrm{d}}$ is the operator dual to $\mathcal{A}$. In what follows, we use the notation

$$\widehat{AB} = \overrightarrow{\mathcal{A}}\widehat{B}, \qquad \widehat{BA} = \widehat{B}\overleftarrow{\mathcal{A}} \tag{2.5}$$

for equalities of type (2.4).

The operator extension $L(q)$ of $\mathcal{L}(x, \partial_x)$ in (1.2) is given by

$$L(q) = L_0(q) - U, \tag{2.6}$$

where L_0 by (2.2) has the kernel

$$L_0(x, x'; q) = \bigl[i(\partial_{x_2} + q_2) + (\partial_{x_1} + q_1)^2\bigr]\delta(x - x') \tag{2.7}$$

and the kernel of operator U is

$$U(x, x'; q) = u(x)\delta(x - x'), \tag{2.8}$$

where $u(x)$ is always assumed to be real. The main object in our approach is the (extended) resolvent $M(q)$ of the operator $L(q)$, which is defined as the inverse of the operator L:

$$LM = ML = I. \tag{2.9}$$

For brevity, we do not here specify the additional conditions that guarantee the uniqueness of the extended resolvent as a solution of (2.9) (see, e.g., [6]). The hat version (cf. (2.3)) of the resolvent of the "bare" operator L_0 is given by

$$\widehat{M}_0(x, x'; q) = \frac{1}{2\pi i} \int_{\mathbf{k}_\Im = q_1} d\mathbf{k}_\Re \left[\theta(x_2 - x_2') - \theta(\ell_{2\Im}(\mathbf{k}) - q_2)\right] \Phi_0(x, \mathbf{k}) \Psi_0(x', \mathbf{k}), \tag{2.10}$$

where $\Phi_0(x, \mathbf{k}) = e^{-i\ell(\mathbf{k})x}$, $\Psi_0(x, \mathbf{k}) = e^{i\ell(\mathbf{k})x}$, and the two-component vector

$$\ell(\mathbf{k}) = (\mathbf{k}, \mathbf{k}^2), \qquad \mathbf{k} = \mathbf{k}_\Re + i\mathbf{k}_\Im \in \mathbb{C}. \tag{2.11}$$

We use the boldface font to indicate that the corresponding variables are complex. For any $\mathbf{k} \in \mathbb{C}$, the functions $\Phi_0(x, \mathbf{k})$ and $\Psi_0(x, \mathbf{k})$ solve nonstationary Schrödinger equation (1.2) and its dual in the case of a zero potential and have the conjugation property $\overline{\Phi_0(x, \mathbf{k})} = \Psi_0(x, \bar{\mathbf{k}})$. Using notation (2.5), we can rewrite Eqs. (2.9) for the case of a zero potential in the form $\overrightarrow{\mathcal{L}}_0 \widehat{M}_0(q) = \widehat{M}_0(q) \overleftarrow{\mathcal{L}}_0 = I$, which shows that $\widehat{M}_0(q)$ is a two-parameter set of Green's functions of (1.2) and its dual.

From (2.10), we directly deduce that

$$\frac{\partial \widehat{M}_0(q)}{\partial q_1} = \frac{i}{\pi} \int_{\mathbf{k}_\Im = q_1} d\mathbf{k}_\Re \, \bar{\mathbf{k}} \delta(\ell_{2\Im}(\mathbf{k}) - q_2) \, \Phi_0(\mathbf{k}) \otimes \Psi_0(\mathbf{k}), \tag{2.12}$$

$$\frac{\partial \widehat{M}_0(q)}{\partial q_2} = \frac{1}{2\pi i} \int_{\mathbf{k}_\Im = q_1} d\mathbf{k}_\Re \, \delta(\ell_{2\Im}(\mathbf{k}) - q_2) \, \Phi_0(\mathbf{k}) \otimes \Psi_0(\mathbf{k}) \tag{2.13}$$

for $q_1 \neq 0$, where $\ell_{2\Im}$ is the imaginary part of the second component of the vector introduced in (2.11) and the direct product is standardly defined as an operator with the kernel

$$(\Phi_0(\mathbf{k}) \otimes \Psi_0(\mathbf{k}))(x, x') = \Phi_0(x, \mathbf{k}) \Psi_0(x', \mathbf{k}).$$

In terms of M_0, the resolvent of the generic operator L in (2.6) can also be defined as the solution of the integral equations

$$M = M_0 + M_0 U M, \qquad M = M_0 + M U M_0. \tag{2.14}$$

Again referring to [4], we here mention that the main tool for investigating the properties of the resolvent is given by the identity

$$M' - M = -M'(L' - L)M, \tag{2.15}$$

where L' is another operator of type (1.2) and M' is its resolvent. This Hilbert identity (we call it thus in analogy with the standard spectral theory of operators) is used to investigate the continuity and differentiability properties of the resolvent with respect to q variables.

2.2. Resolvent approach in the case of a one-dimensional potential. In this section, we consider the embedding of the standard one-dimensional scattering transform (see [8]) in the theory of a two-dimensional differential operator with a one-dimensional potential. This provides a basic example of potential (1.3) with $u_2 \equiv 0$. We use the results of this construction in the following section, where we study a generic situation with $u_2 \neq 0$. We here consider the extended differential operator

$$L_1(q) = L_0(q) - U_1, \qquad U_1(x, x'; q) = u_1(x_1)\delta(x - x'), \tag{2.16}$$

where $L_0(q)$ is defined in (2.7). For the Jost solution of the nonstationary Schrödinger equation and its dual, we use the notation

$$\varphi(x, \mathbf{k}) = e^{-i\mathbf{k}^2 x_2}\Phi_1(x_1, \mathbf{k}) = e^{-i\mathbf{k}x_1 - i\mathbf{k}^2 x_2}\chi_1(x_1, \mathbf{k}), \tag{2.17}$$

$$\psi(x, \mathbf{k}) = e^{i\mathbf{k}^2 x_2}\Psi_1(x_1, \mathbf{k}) = e^{i\mathbf{k}x_1 + i\mathbf{k}^2 x_2}\xi_1(x_1, \mathbf{k}), \tag{2.18}$$

where $\Phi_1(x_1, \mathbf{k})$ and $\Psi_1(x_1, \mathbf{k})$ are the Jost solutions of the one-dimensional Sturm–Liouville operator, namely, χ_1 is defined by the integral equation

$$\chi_1(x_1, \mathbf{k}) = 1 + \int_{-\mathbf{k}_\Im\infty}^{x_1} dy_1\, \frac{e^{2i\mathbf{k}(x_1 - y_1)} - 1}{2i\mathbf{k}} u_1(y_1)\chi_1(y_1, \mathbf{k}), \tag{2.19}$$

and $\xi_1(x_1, \mathbf{k})$ is defined by a similar equation. The functions $\varphi(x, \mathbf{k})$ and $\psi(x, \mathbf{k})$, as well as their boundary values at the real axis, $\varphi^{\pm}(x, k) = \varphi(x, k \pm i0)$ and $\psi^{\pm}(x, k) = \psi(x, k \pm i0)$, have the conjugation properties

$$\overline{\varphi(x, \mathbf{k})} = \psi(x, \overline{\mathbf{k}}), \quad \mathbf{k} \in \mathbb{C}, \qquad \overline{\varphi^{\pm}(x, k)} = \psi^{\mp}(x, k), \quad k \in \mathbb{R}, \tag{2.20}$$

and satisfy orthogonality and completeness relations of the forms

$$\frac{1}{2\pi}\int dx_1\, \psi(x, \mathbf{k} + p)\varphi(x, \mathbf{k}) = \frac{\delta(p)}{t(\mathbf{k})}, \quad p \in \mathbb{R}, \quad \mathbf{k} \in \mathbb{C}, \tag{2.21}$$

$$\int dx_1\, \psi(x, i\varkappa_j)\varphi(x, \mathbf{k}) = \int dx_1\, \psi(x, \mathbf{k})\varphi(x, i\varkappa_j) = 0, \quad |\mathbf{k}_\Im| < \varkappa_j, \tag{2.22}$$

$$\int dx_1 \psi(x, i\varkappa_j)\varphi(x, i\varkappa_{j'}) = \frac{i\delta_{j,j'}}{t_j}, \tag{2.23}$$

$$\begin{aligned} &\frac{1}{2\pi}\int_{x_2' = x_2} d\mathbf{k}_\Re\, t(\mathbf{k})\varphi(x, \mathbf{k})\psi(x', \mathbf{k}) - \\ &\qquad - i\sum_{j=1}^{N} t_j\theta(\varkappa_j - |\mathbf{k}_\Im|)\varphi(x, i\varkappa_j)\psi(x', i\varkappa_j)\big|_{x_2' = x_2} = \delta(x_1 - x_1'), \end{aligned} \tag{2.24}$$

where $t(\mathbf{k})$ is the transmission coefficient of the one-dimensional problem with poles at the points $\pm i\varkappa_j$, $j = 1, \dots N$ (for definiteness, we assume $\varkappa_j > 0$ for all j). We let the residues of $t(\mathbf{k})$ in the upper half-plane be denoted by

$$t_j = \operatorname*{res}_{\mathbf{k}=i\varkappa_j} t(\mathbf{k}). \tag{2.25}$$

Another set of discrete spectral data for the one-dimensional problem is given by the coefficients b_j defined by one of the equalities

$$\Phi_1(x_1, i\varkappa_j) = b_j \Phi_1(x_1, -i\varkappa_j), \qquad \Psi_1(x_1, -i\varkappa_j) = b_j \Psi_1(x_1, i\varkappa_j). \tag{2.26}$$

These coefficients are real and nonzero, and $\operatorname{sgn}(it_j b_j) = -1$. By (2.20),

$$\varphi(x, i\varkappa_j) = b_j \varphi(x, -i\varkappa_j), \qquad \varphi(x, i\varkappa_j) = b_j \overline{\psi(x, i\varkappa_j)}. \tag{2.27}$$

It is easy to verify by (2.24) that the hat kernel (cf. (2.3)) of the resolvent $M_1(q)$ of the two-dimensional operator $L_1(q)$ is given in terms of the above notation by

$$\begin{aligned}\widehat{M}_1(x, x'; q) = \frac{1}{2\pi i} \int_{\mathbf{k}_\Im = q_1} d\mathbf{k}_\Re \left[\theta(x_2 - x_2') - \theta(2\mathbf{k}_\Re \mathbf{k}_\Im - q_2)\right] t(\mathbf{k}) \varphi(x, \mathbf{k}) \psi(x', \mathbf{k}) - \\ - \sum_j \theta(\varkappa_j^2 - q_1^2) t_j \left[\theta(x_2 - x_2') - \theta(-q_2)\right] \varphi(x, i\varkappa_j) \psi(x', i\varkappa_j),\end{aligned} \tag{2.28}$$

i.e., that in notation (2.5),

$$\overrightarrow{\mathcal{L}}_1 \widehat{M}_1(q) = \widehat{M}_1(q) \overleftarrow{\mathcal{L}}_1 = I, \tag{2.29}$$

which is equivalent to (2.9). Moreover, returning to the kernel $M_1(x, x'; q)$ and using properties of the one-dimensional Jost solutions and definitions (2.18), we find that $M_1(x, x'; q) \in \mathcal{S}'$.

The behavior of $\widehat{M}_1(x, x'; q)$ with respect to the q variables is determined by a combination of θ-functions and the pole behavior of $t(\mathbf{k})$. Introducing terms compensating these pole singularities in the integrand, we decompose the resolvent into the sum

$$\widehat{M}_1(q) = \widehat{M}_{1,\mathrm{reg}}(q) + \sum_j \Gamma_j(q) \, \varphi(i\varkappa_j) \otimes \psi(i\varkappa_j), \tag{2.30}$$

where $\widehat{M}_{1,\mathrm{reg}}(q)$ is a regular function of q and $\Gamma_j(q)$ are x-independent functions,

$$\Gamma_j(q) = \frac{t_j \operatorname{sgn} q_1}{2\pi i} \log \frac{q_2 + 2iq_1(q_1 - \varkappa_j)}{q_2 + 2iq_1(q_1 + \varkappa_j)}, \quad j = 1, \dots, N, \tag{2.31}$$

where the logarithm has a cut along the negative part of the real axis of its argument. This proves that the extended resolvent in the case of a one-dimensional potential has logarithmic singularities at the points $q = (\pm\varkappa_j, 0)$ and a cut along $q_2 = 0$. For the discontinuity of the resolvent at $q_2 = 0$, we obtain

$$\widehat{M}_1(q_1, +0) - \widehat{M}_1(q_1, -0) = -\sum_j t_j \theta(\varkappa_j - |q_1|) \, \varphi(i\varkappa_j) \otimes \psi(i\varkappa_j) \tag{2.32}$$

from (2.28). For all other values of q, resolvent (2.28) has derivatives with respect to q of form (2.12), (2.13). For instance,

$$\frac{\partial \widehat{M}_1(q)}{\partial q_2} = \frac{1}{2\pi i}\int_{\mathbf{k}_\Im = q_1} d\mathbf{k}_\Re\, \delta(\ell_{2\Im}(\mathbf{k}) - q_2)\, t(\mathbf{k})\varphi(\mathbf{k}) \otimes \psi(\mathbf{k}), \tag{2.33}$$

which must be considered in the distributional sense in the vicinity of the point $q_1 = 0$. The Green's function $\mathcal{G}_1(x, x', \mathbf{k})$ of Eq. (1.2) is introduced using a reduction,

$$\mathcal{G}_1(\mathbf{k}) = \widehat{M}_1(q)\big|_{q=\ell_\Im(\mathbf{k})} \tag{2.34}$$

(see (2.11)), the same as in the case of a decaying potential [6]. From (2.28), we then obtain the bilinear representation

$$\mathcal{G}_1(x, x', \mathbf{k}) = \frac{1}{2\pi i}\int d\alpha \left[\theta(x_2 - x_2') - \theta(\mathbf{k}_\Im(\alpha - \mathbf{k}_\Re))\right] t(\alpha + i\mathbf{k}_\Im)\varphi(x, \alpha + i\mathbf{k}_\Im)\psi(x', \alpha + i\mathbf{k}_\Im) -$$
$$- \sum_j t_j \theta(\varkappa_j - |\mathbf{k}_\Im|)\left[\theta(x_2 - x_2') - \theta(-\mathbf{k}_\Re \mathbf{k}_\Im)\right]\varphi(x, i\varkappa_j)\psi(x', i\varkappa_j), \tag{2.35}$$

which generalizes the Green's function used in [5] and [6] for a pure one-soliton potential to the case of a generic potential u_1. This Green's function has the conjugation property $\overline{\mathcal{G}_1(x, x', \mathbf{k})} = \mathcal{G}_1(x', x, \overline{\mathbf{k}})$ and the product $g_1(x, x', \mathbf{k}) = e^{i\ell(\mathbf{k})(x-x')}\mathcal{G}_1(x, x', \mathbf{k})$ decays as $\mathbf{k} \to \infty$. Taking into account that the resolvent satisfies (2.29), we find that the Green's function satisfies the integral equations

$$\mathcal{G}_1(\mathbf{k}) = \mathcal{G}_0(\mathbf{k}) + \mathcal{G}_0(\mathbf{k})U_1\mathcal{G}_1(\mathbf{k}), \qquad \mathcal{G}_1(\mathbf{k}) = \mathcal{G}_0(\mathbf{k}) + \mathcal{G}_1(\mathbf{k})U_1\mathcal{G}_0(\mathbf{k}), \tag{2.36}$$

it is analytic for $\mathbf{k}_\Re \mathbf{k}_\Im \neq 0$, and we have

$$\frac{\partial \mathcal{G}_1(\mathbf{k})}{\partial \mathbf{k}_\Im} = \frac{\operatorname{sgn} \mathbf{k}_\Im}{2\pi}\, t(\mathbf{k})\varphi(\mathbf{k}) \otimes \psi(\mathbf{k}) \tag{2.37}$$

in this region. Because of (2.34), decomposition (2.30) gives an analogous decomposition for the Green's function:

$$\mathcal{G}_1(\mathbf{k}) = \mathcal{G}_{1,\mathrm{reg}}(\mathbf{k}) + \sum_j \gamma_j(\mathbf{k})\varphi(i\varkappa_j) \otimes \psi(i\varkappa_j), \tag{2.38}$$

where by (2.31),

$$\gamma_j(\mathbf{k}) = \Gamma_j(\ell_\Im(\mathbf{k})) \equiv \frac{t_j \operatorname{sgn} \mathbf{k}_\Im}{2\pi i} \log \frac{\mathbf{k} - i\varkappa_j}{\mathbf{k} + i\varkappa_j}. \tag{2.39}$$

This proves that the Green's function, in addition to the standard discontinuity at the real axis, has logarithmic singularities at the points $\mathbf{k} = \pm i\varkappa_j$ and a discontinuity at the imaginary axis for $|\mathbf{k}_\Im| < \max_j \varkappa_j$, which generalizes the behavior discovered in [5] for the case of a one-soliton potential u_1.

In [4], we showed that in the case of a decaying potential, the Jost solutions themselves result as special truncations and reductions of the Green's function. But when applied to the above Jost solutions φ and ψ, these procedures, as expected, lead to divergent expressions. They can be regularized for $\mathbf{k}_\Im \neq 0$ by the limiting procedure

$$\varphi(x, \mathbf{k}) = \lim_{\varepsilon \to +0} \int dx' \left(\mathcal{G}_1(\mathbf{k})\overleftarrow{\mathcal{L}}_0\right)(x, x')e^{-i\ell(\mathbf{k})x' - i\varepsilon x_2'}, \tag{2.40}$$

$$\psi(x, \mathbf{k}) = \lim_{\varepsilon \to +0} \int dx'\, e^{i\ell(\mathbf{k})x' + i\varepsilon x_2'}\left(\overrightarrow{\mathcal{L}}_0\mathcal{G}_1(\mathbf{k})\right)(x', x), \tag{2.41}$$

which gives identities when $\mathcal{G}_1(\mathbf{k})$ given by (2.35) is substituted.

The transmission coefficient, Green's function, and Jost solutions are discontinuous at the real $\mathbf{k}$ axis. In what follows, we use the notation

$$
\begin{aligned}
t^{\pm}(k) &= \lim_{\mathbf{k}_\Im \to \pm 0} t(k + i\mathbf{k}_\Im), \quad k \in \mathbb{R}, \\
\varphi^{\pm}(k) &= \lim_{\mathbf{k}_\Im \to \pm 0} \varphi(k + i\mathbf{k}_\Im), \qquad \psi^{\pm}(k) = \lim_{\mathbf{k}_\Im \to \pm 0} \psi(k + i\mathbf{k}_\Im), \\
\mathcal{G}_1^{\pm}(k) &= \lim_{\mathbf{k}_\Im \to \pm 0} \mathcal{G}_1(k + i\mathbf{k}_\Im)
\end{aligned}
\tag{2.42}
$$

for their boundary values. By definition $t^{\pm}(k)$, $\varphi^{\pm}(k)$, and $\psi^{\pm}(k)$ are finite smooth functions of k, while the boundary value of the Green's function is discontinuous at $k = 0$, as follows from (2.35).

Because of (2.28), the resolvent $M_1(q)$ is discontinuous at $q = 0$ and $q_2 = 0$. We first consider the limit $q_2 \to 0$ when $q_1 \neq 0$. From (2.34), we obtain

$$
\lim_{q_2 \to \pm 0\, \mathbf{k}_\Im} \left(\widehat{M}_1(q)|_{q_1 = \mathbf{k}_\Im} \right) = \mathcal{G}_1(\pm 0 + i\mathbf{k}_\Im), \tag{2.43}
$$

and from (2.32), we have

$$
\mathcal{G}_1(+0 + i\mathbf{k}_\Im) - \mathcal{G}_1(-0 + i\mathbf{k}_\Im) = -\operatorname{sgn} \mathbf{k}_\Im \sum_j t_j \theta(\varkappa_j - |\mathbf{k}_\Im|) \varphi(i\varkappa_j) \otimes \psi(i\varkappa_j) \tag{2.44}
$$

for the discontinuity of the Green's function.

The discontinuity of the resolvent $M_1(q)$ with respect to q_2 at zero for $q_1 = 0$ leads to introducing advanced/retarded Green's functions as in the case of a decaying potential, i.e.,

$$
\mathcal{G}_{1,\pm}(x, x') = \lim_{q_2 \to \pm 0} \lim_{q_1 \to 0} \widehat{M}_1(x, x'; q), \tag{2.45}
$$

where the limit $q_1 \to 0$ is independent of the sign. By (2.28), we then obtain the bilinear representation for these Green's functions in terms of the Jost solutions on the real axis:

$$
\mathcal{G}_{1,\pm}(x, x') = \pm\theta(\pm(x_2 - x_2')) \left(\frac{1}{2\pi i} \int d\alpha\, t^{\sigma}(\alpha) \varphi^{\sigma}(x, \alpha) \psi^{\sigma}(x', \alpha) - \sum_j t_j \varphi(x, i\varkappa_j) \psi(x', i\varkappa_j) \right), \tag{2.46}
$$

where $\sigma = +, -$ and notation (2.42) is used for the boundary values of the Jost solutions and $t(\mathbf{k})$. It is clear that the advanced/retarded Green's functions are independent of the sign of σ in (2.46) and have the conjugation property $\overline{\mathcal{G}_{1,\pm}(x, x')} = \mathcal{G}_{1,\mp}(x', x)$. The advanced/retarded solutions of (1.2) are defined as in the case of a decaying potential, i.e.,

$$
\varphi_{\pm}(x, k) = \int dx' \left(\mathcal{G}_{1,\pm} \overleftarrow{\mathcal{L}}_0 \right)(x, x') e^{-i\ell(k)x'}, \tag{2.47}
$$

$$
\psi_{\pm}(x, k) = \int dx'\, e^{i\ell(k)x'} \left(\overrightarrow{\mathcal{L}}_0 \mathcal{G}_{1,\pm} \right)(x', x). \tag{2.48}
$$

Using notation (2.42) we obtain

$$
\begin{aligned}
\mathcal{G}_1^{\sigma}(k) - \mathcal{G}_{1,\pm} = \frac{\mp 1}{2\pi i} \int d\alpha\, \theta(\pm\sigma(\alpha - k)) t^{\sigma}(\alpha) \varphi^{\sigma}(\alpha) \otimes \psi^{\sigma}(\alpha) \pm \\
\pm \theta(\mp\sigma k) \sum_j t_j \varphi(i\varkappa_j) \otimes \psi(i\varkappa_j), \quad \sigma = +, -,
\end{aligned}
\tag{2.49}
$$

from (2.35) and (2.46).

The advanced/retarded solutions and the limiting values of the Jost solutions at the real axis are related to each other by

$$\varphi_{\pm}(k) = \int dp\, \varphi^{\sigma}(p)\, \overline{r_{\pm}^{-\sigma}(p,k)}, \qquad t^{\sigma}(k)\varphi^{\sigma}(k) = \int dp\, \varphi_{\pm}(p) r_{\pm}^{\sigma}(p,k), \tag{2.50}$$

where

$$r_{\pm}^{\sigma}(p,k) = \delta(k-p)\big[\theta(\pm\sigma k) + \theta(\mp\sigma k)t(\sigma k)\big] + \theta(\mp\sigma k)\delta(k+p)r^{\sigma}(k) \tag{2.51}$$

and $r^{\pm}(k)$ are the standard (left and right) reflection coefficients (see (2.53) below). The analogous relations for the dual solutions follow from (2.20) and the conjugation property,

$$\overline{\varphi_{\pm}(k)} = \psi_{\mp}(k). \tag{2.52}$$

In the standard theory of the Sturm–Liouville equation, the advanced/retarded solutions are not used, in contrast to the case of the nonstationary Schrödinger equation. We introduce them here for the convenience of future reference. The standard relations between boundary values of the Jost solutions on the real axis (see [8]) follow by substituting the first equation in (2.50) in the second one:

$$t^{\sigma}(k)\varphi^{\sigma}(k) = \int dp\, \varphi^{-\sigma}(p) f^{-\sigma}(p,k), \quad \sigma = +,-, \tag{2.53}$$

where we introduce the spectral data

$$f^{-\sigma}(k,k') = \int dk''\, \overline{r_{\pm}^{\sigma}(k'',k)} r_{\pm}^{\sigma}(k'',k') \equiv \delta(k-k') + \delta(k+k')r^{-\sigma}(k) \tag{2.54}$$

and (2.51) is used in the last equality.

3. Inverse scattering transform on a nontrivial background: Two-dimensional perturbation of the one-dimensional potential

3.1. Resolvent. We now study the spectral properties of the operator $\mathcal{L}$ given by (1.2) with the potential given by (1.3). Its extension $L(q)$ is defined in (2.6), and the operator inverse to $L(q)$, i.e., the resolvent $M(q)$, is defined by one of the integral equations

$$M(q) = M_1(q) + M_1(q)U_2M(q), \qquad M(q) = M_1(q) + M(q)U_2M_1(q), \tag{3.1}$$

where $U_2(x,x';q) = u_2(x)\delta(x-x')$. We assume that $u_2(x)$ is real, smooth, and rapidly decaying with respect to both variables (x_1,x_2). Moreover, we assume that it is "small" in the sense that the solutions $M(x,x';q)$ of both equations in (3.1) exist in $\mathcal{S}'(\mathbb{R}^6)$ and coincide. It then obviously follows from (3.1) that the resolvent satisfies (2.9). Its properties with respect to the q variables are inherited from the corresponding properties of the resolvent $M_1(q)$. For instance, $M(q)$ is a continuous function of q when $q \neq 0$ and $q_2 \neq 0$.

As already mentioned, to investigate the properties of the resolvent, we use Hilbert identity (2.15), which we write in the form

$$M(q') - M(q) = M(q')L_1(q')\big(M_1(q') - M_1(q)\big)L_1(q)M(q), \tag{3.2}$$

and for the derivatives of hat kernel (2.3) of the resolvent, we then obtain

$$\frac{\partial \widehat{M}(q)}{\partial q_j} = \widehat{M}(q) \overleftarrow{\mathcal{L}}_1 \frac{\partial \widehat{M}_1(q)}{\partial q_j} \overrightarrow{\mathcal{L}}_1 \widehat{M}(q), \quad j = 1, 2, \quad q_2 \neq 0.$$

Using (2.33) in the r.h.s., we obtain the equalities

$$\frac{\partial \widehat{M}(q)}{\partial q_1} = \frac{i}{\pi} \int_{\mathbf{k}_\Im = q_1} d\mathbf{k}_\Re \, \bar{\mathbf{k}} \delta(\ell_{2\Im}(\mathbf{k}) - q_2) \, t(\mathbf{k}) \Phi(\mathbf{k}) \otimes \Psi(\mathbf{k}), \tag{3.3}$$

$$\frac{\partial \widehat{M}(q)}{\partial q_2} = \frac{1}{2\pi i} \int_{\mathbf{k}_\Im = q_1} d\mathbf{k}_\Re \, \delta(\ell_{2\Im}(\mathbf{k}) - q_2) \, t(\mathbf{k}) \Phi(\mathbf{k}) \otimes \Psi(\mathbf{k}) \tag{3.4}$$

(see (2.11)), which hold for $q_2 \neq 0$, where we set

$$\Phi(x, \mathbf{k}) = \int dx' \, \big(\mathcal{L}_1^{\mathrm{d}}(x', \partial_{x'}) \mathcal{G}(x, x', \mathbf{k})\big) \varphi(x', \mathbf{k}), \tag{3.5}$$

$$\Psi(x', \mathbf{k}) = \int dx \, \psi(x, \mathbf{k}) \, \mathcal{L}_1(x, \partial_x) \, \mathcal{G}(x, x', \mathbf{k}), \tag{3.6}$$

$\mathcal{L}_1^{\mathrm{d}}$ is the operator dual to $\mathcal{L}_1$, and the Green's function $\mathcal{G}(x, x', \mathbf{k})$ is defined as a specific value of the resolvent itself (cf. (2.34)),

$$\mathcal{G}(x, x', \mathbf{k}) = \widehat{M}(x, x'; q)\big|_{q = \ell_\Im(\mathbf{k})} \equiv \widehat{M}(x, x'; \mathbf{k}_\Im, 2\mathbf{k}_\Im \mathbf{k}_\Re). \tag{3.7}$$

In what follows, we consider the function $\mathcal{G}(x, x', \mathbf{k})$ as the kernel of the operator $\mathcal{G}(\mathbf{k})$ and the functions $\Phi(x, \mathbf{k})$ and $\Psi(x', \mathbf{k})$ as the "vector" $\Phi(\mathbf{k})$ and "covector" $\Psi(\mathbf{k})$. For brevity, we write equations of type (3.5)–(3.7) omitting the dependence on x and x' as

$$\Phi(\mathbf{k}) = \mathcal{G}(\mathbf{k}) \overleftarrow{\mathcal{L}}_1 \varphi(\mathbf{k}), \qquad \Psi(\mathbf{k}) = \psi(\mathbf{k}) \overrightarrow{\mathcal{L}}_1 \mathcal{G}(\mathbf{k}), \tag{3.8}$$

$$\mathcal{G}(\mathbf{k}) = \widehat{M}(q)\big|_{q = \ell_\Im(\mathbf{k})}. \tag{3.9}$$

Taking $\mathcal{L} = \mathcal{L}_1 - U_2$ into account, we verify that $\mathcal{G}(\mathbf{k})$ satisfies the differential equations $\overrightarrow{\mathcal{L}} \mathcal{G}(\mathbf{k}) = \mathcal{G}(\mathbf{k}) \overleftarrow{\mathcal{L}} = I$, as follows from (2.9) and (2.4). By (3.8), we then have $\overrightarrow{\mathcal{L}} \Phi(\mathbf{k}) = 0 = \Psi(\mathbf{k}) \overleftarrow{\mathcal{L}}$, $\mathbf{k} \in \mathbb{C}$, i.e., the functions $\Phi(x, \mathbf{k})$ and $\Psi(x, \mathbf{k})$ introduced in (3.5) and (3.6) are the respective Jost solutions of operator (1.2) and its dual. We note that these definitions already do not need any regularization of type (2.40), (2.41).

The resolvent $M(q)$, as well as $M_1(q)$, is discontinuous at $q = 0$ and $q_2 = 0$. In the case where the limit $q_1 \to 0$ is performed first, we introduce the advanced/retarded Green's functions by analogy with (2.45) by the relations

$$\mathcal{G}_\pm(x, x') = \lim_{q_2 \to \pm 0} \lim_{q_1 \to 0} \widehat{M}(x, x'; q). \tag{3.10}$$

On the other hand, if $q_1 \neq 0$, then by (3.9), the limit values of the resolvent are given (cf. (2.43)) in terms of the limit values of the Green's function on the imaginary axis:

$$\lim_{q_2 \to \pm 0 \, \mathbf{k}_\Im} \big(\widehat{M}(q)\big|_{q_1 = \mathbf{k}_\Im}\big) = \mathcal{G}(\pm 0 + i\mathbf{k}_\Im). \tag{3.11}$$

3.2. Discontinuity of the Green's function at the imaginary axis. By (3.1) and definition (3.9), the Green's function satisfies the integral equations

$$\mathcal{G}(\mathbf{k}) = \mathcal{G}_1(\mathbf{k}) + \mathcal{G}_1(\mathbf{k})U_2\mathcal{G}(\mathbf{k}), \qquad \mathcal{G}(\mathbf{k}) = \mathcal{G}_1(\mathbf{k}) + \mathcal{G}(\mathbf{k})U_2\mathcal{G}_1(\mathbf{k}) \tag{3.12}$$

and has the conjugation property

$$\overline{\mathcal{G}(x, x', \mathbf{k})} = \mathcal{G}(x', x, \overline{\mathbf{k}}). \tag{3.13}$$

The analytic properties of the Green's function follow from the properties of $\mathcal{G}_1(\mathbf{k})$ and our assumption that the perturbation u_2 is "small." In particular, this means that $\mathcal{G}(\mathbf{k})$ is analytic in the region $\mathbf{k}_\Re\mathbf{k}_\Im \neq 0$, where it satisfies

$$\frac{\partial\mathcal{G}(\mathbf{k})}{\partial\mathbf{k}_\Re} = \frac{\operatorname{sgn}\mathbf{k}_\Im}{2\pi i}\,\Phi(\mathbf{k})\otimes\Psi(\mathbf{k}), \qquad \frac{\partial\mathcal{G}(\mathbf{k})}{\partial\mathbf{k}_\Im} = \frac{\operatorname{sgn}\mathbf{k}_\Im}{2\pi}\,\Phi(\mathbf{k})\otimes\Psi(\mathbf{k}) \tag{3.14}$$

by (3.3) and (3.4). The Green's function has the standard cut at $\mathbf{k}_\Im = 0$ and an additional cut at $\mathbf{k}_\Re = 0$ when $|\mathbf{k}_\Im| < \max_j \varkappa_j$.

To describe the discontinuity at the imaginary axis, we use Hilbert identity (3.2), using (2.43) and (3.11) to write it in terms of the Green's functions. Because of (2.44), we then obtain

$$\begin{aligned}\mathcal{G}(+0 + i\mathbf{k}_\Im) - \mathcal{G}(-0 + i\mathbf{k}_\Im) = {} & -\operatorname{sgn}\mathbf{k}_\Im \sum_j t_j\theta(\varkappa_j - |\mathbf{k}_\Im|)\times \\ & \times\big(\mathcal{G}(\pm0 + i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j)\big)\otimes\big(\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\mathcal{G}(\mp0 + i\mathbf{k}_\Im)\big),\end{aligned} \tag{3.15}$$

where the expressions in parentheses are understood in the sense of (3.8). Equality (3.15) suggests introducing the functions $\Phi_j(x, \mathbf{k}_\Im)$ and $\Psi_j(x, \mathbf{k}_\Im)$ by the relations

$$\Phi_j(\mathbf{k}_\Im) = \mathcal{G}(+0 + i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j), \qquad \Psi_j(\mathbf{k}_\Im) = \psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\mathcal{G}(+0 + i\mathbf{k}_\Im). \tag{3.16}$$

By (3.15) (bottom sign), we now derive the expression

$$\mathcal{G}(+0 + i\mathbf{k}_\Im) - \mathcal{G}(-0 + i\mathbf{k}_\Im) = -\sum_{l,m}\theta(\varkappa_l - |\mathbf{k}_\Im|)\Phi_l(\mathbf{k}_\Im)(A(\mathbf{k}_\Im)^{-1})_{lm}\otimes\Psi_m(\mathbf{k}_\Im) \tag{3.17}$$

for the discontinuity, where we introduce the matrix $A(\mathbf{k}_\Im)$ with the elements

$$A_{lm}(\mathbf{k}_\Im) = \frac{\delta_{lm}}{t_l\operatorname{sgn}\mathbf{k}_\Im} - \theta(\min\{\varkappa_l, \varkappa_m\} - |\mathbf{k}_\Im|)\big(\psi(i\varkappa_l)\overrightarrow{\mathcal{L}}_1\mathcal{G}(+0 + i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_m)\big), \tag{3.18}$$

where the "expectation" in parentheses denotes

$$\int dx\int dx'\,\psi(x, i\varkappa_l)\varphi(x', i\varkappa_m)\mathcal{L}_1(x, \partial_x)\mathcal{L}_1^{\mathrm{d}}(x', \partial'_x)\mathcal{G}(x, x', +0 + i\mathbf{k}_\Im)$$

(cf. notation (3.8)), i.e., a function of only $\mathbf{k}_\Im$.

Under the assumption that the integral equation for the Green's function has a unique solution, the matrix $A(\mathbf{k}_\Im)$ is nonsingular, and because of (2.27) and (3.13), it is easy to see that

$$A(\mathbf{k}_\Im)^\dagger = BA(-\mathbf{k}_\Im)B^{-1}, \tag{3.19}$$

where we introduce the diagonal matrix $B = \operatorname{diag}\{b_1, \ldots, b_N\}$. We note that B is Hermitian because the b_j in (2.26) are real.

3.3. Discontinuity of the Jost solutions at the imaginary axis. Because of (3.12), the Jost solutions $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$ introduced in (3.8) satisfy the integral equations

$$\Phi(\mathbf{k}) = \varphi(\mathbf{k}) + \mathcal{G}_1(\mathbf{k})U_2\Phi(\mathbf{k}), \qquad \Psi(\mathbf{k}) = \psi(\mathbf{k}) + \Psi(\mathbf{k})U_2\mathcal{G}_1(\mathbf{k}), \tag{3.20}$$

and by (3.8) and (3.13), we have

$$\overline{\Phi(x,\mathbf{k})} = \Psi(x,\overline{\mathbf{k}}). \tag{3.21}$$

Because of the properties of the Green's function $\mathcal{G}(\mathbf{k})$, the Jost solutions are analytic functions of $\mathbf{k} \in \mathbb{C}$ when $\mathbf{k}_\Re \mathbf{k}_\Im \neq 0$ and in the generic situation have the standard discontinuity at the real axis, $\mathbf{k}_\Im = 0$, and an additional discontinuity at the segment $\mathbf{k}_\Re = 0$, $|\mathbf{k}_\Im| \leq \max_j \varkappa_j$, of the imaginary axis. Taking into account that the functions $\varphi(\mathbf{k})$ and $\psi(\mathbf{k})$ are continuous at $\mathbf{k}_\Re = 0$ and applying the operation $\overleftarrow{\mathcal{L}}_1\varphi(i\mathbf{k}_\Im)$ to (3.15), from (3.8), we obtain

$$\Phi(x, +0 + i\mathbf{k}_\Im) - \Phi(x, -0 + i\mathbf{k}_\Im) = \sum_l \theta(\varkappa_l - |\mathbf{k}_\Im|)\Phi_l(x,\mathbf{k}_\Im)w_l(\mathbf{k}_\Im) \tag{3.22}$$

for this discontinuity, where we introduce the functions $w_l(\mathbf{k}_\Im)$ as

$$w_l(\mathbf{k}_\Im) \equiv t_l \operatorname{sgn} \mathbf{k}_\Im \big(\psi(i\varkappa_l)\overrightarrow{\mathcal{L}}_1\mathcal{G}(-0 + i\mathbf{k}_\Im)\overleftarrow{\mathcal{L}}_1\varphi(i\mathbf{k}_\Im)\big) \tag{3.23}$$

and the expression in parentheses is understood as in (3.18). The functions $w_l(\mathbf{k}_\Im)$ are the spectral data describing the discontinuity of the Jost solution at the imaginary axis. An analogous relation for the dual solution $\Psi(x,\mathbf{k})$ can be obtained from conjugation properties (3.21) and the conditions

$$\overline{\Phi_j(x,\mathbf{k}_\Im)} = b_j\Psi_j(x,-\mathbf{k}_\Im), \qquad \overline{\Psi_j(x,\mathbf{k}_\Im)} = \frac{\Phi_j(x,-\mathbf{k}_\Im)}{b_j}, \tag{3.24}$$

which in turn follow from (2.27) and (3.13). By (3.22), the discontinuity of the Jost solution at the imaginary axis is thus given in terms of the functions $\Phi_l(x,\mathbf{k}_\Im)$ and $\Psi_l(x,\mathbf{k}_\Im)$ introduced in (3.16). In the same way as for the Jost solutions, we prove that they satisfy the differential equations $\overrightarrow{\mathcal{L}}\Phi_l(\mathbf{k}_\Im) = \Psi_l(\mathbf{k}_\Im)\overleftarrow{\mathcal{L}} = 0$. These functions generalize the functions $\varphi(i\varkappa_l)$ and $\psi(i\varkappa_l)$ to the case $u_2 \neq 0$, while in the contrast to the latter ones, $\Phi_l(x,\mathbf{k}_\Im)$ and $\Psi_l(x,\mathbf{k}_\Im)$ are not equal to the values of the Jost solutions at some points and depend on $\mathbf{k}_\Im$ nontrivially. By (3.15), we need these functions only on the interval $|\mathbf{k}_\Im| < \varkappa_l$. In what follows, we call them the auxiliary Jost solutions. These solutions are discontinuous at $\mathbf{k}_\Im = 0$, as follows from the properties of the total Green's function.

3.4. Behavior of the Green's function, Jost solutions, and spectral data at the points $\pm i\varkappa_j$. In (2.38), it was shown that $\mathcal{G}_1(\mathbf{k})$ has logarithmic singularities at all points $\mathbf{k} = \pm i\varkappa_j$, $j = 1,\dots,N$. We consider how these singularities affect the properties of the total Green's function $\mathcal{G}(\mathbf{k})$ and the objects constructed using it. We choose some $\varkappa_j$ and let $\mathbf{k}$ belong to some neighborhood of a point $i\varkappa_j$ or $-i\varkappa_j$ that does not include other points $\pm i\varkappa_l$, $l \neq j$, or points on the real axis. We introduce

$$\mathcal{G}_{1,j}(\mathbf{k}) = \mathcal{G}_1(\mathbf{k}) - \gamma_j(\mathbf{k})\varphi(\mathbf{k}) \otimes \psi(\mathbf{k}), \tag{3.25}$$

which is finite in the vicinities of $\pm i\varkappa_j$ by (2.38) but can be discontinuous there in accordance with (2.44). This disadvantage of regularization (3.25), if compared with $\mathcal{G}_{1,\mathrm{reg}}(\mathbf{k})$ in (2.38), is compensated by the product $e^{i\ell(\mathbf{k})(x-x')}\mathcal{G}_{1,j}(x,x',\mathbf{k})$ being bounded on the x plane. We now define a new Green's function of the nonstationary Schrödinger equation with potential (1.3) via the integral equation $\mathcal{G}_j(\mathbf{k}) = \mathcal{G}_{1,j}(\mathbf{k}) + \mathcal{G}_{1,j}(\mathbf{k})U_2\mathcal{G}_j(\mathbf{k})$ or its dual. Then $\mathcal{G}_j(\mathbf{k})$ is a regularization of $\mathcal{G}(\mathbf{k})$ in neighborhoods of $\pm i\varkappa_j$ and is such that

$$\mathcal{G}(\mathbf{k}) = \mathcal{G}_j(\mathbf{k}) + \gamma_j(\mathbf{k})\,\widetilde{\Phi}_j(\mathbf{k}) \otimes \Psi(\mathbf{k}), \tag{3.26}$$

where we use notation (3.8) for the Jost solutions and analogously introduce

$$\widetilde{\Phi}_j(\mathbf{k}) = \mathcal{G}_j(\mathbf{k})\overleftarrow{\mathcal{L}}_1\varphi(\mathbf{k}), \qquad \widetilde{\Psi}_j(\mathbf{k}) = \psi(\mathbf{k})\overrightarrow{\mathcal{L}}_1\mathcal{G}_j(\mathbf{k}). \tag{3.27}$$

The functions $\widetilde{\Phi}_j(\mathbf{k})$ and $\widetilde{\Psi}_j(\mathbf{k})$ are also solutions of the nonstationary Schrödinger equation and its dual. These functions are bounded in vicinities of the points $\pm i\varkappa_j$ but can be discontinuous in these vicinities. The same properties hold for the functions

$$g_j(\mathbf{k}) = \bigl(\psi(\mathbf{k})\overrightarrow{\mathcal{L}}_1\mathcal{G}_j(\mathbf{k})\overleftarrow{\mathcal{L}}_1\varphi(\mathbf{k})\bigr), \tag{3.28}$$

where the expression in parentheses is understood as in (3.18). Applying $\psi(\mathbf{k})\overrightarrow{\mathcal{L}}_1$ to (3.26) from the left, we find from (3.8) and (3.27), (3.28) that $\Psi(\mathbf{k}) = \widetilde{\Psi}(\mathbf{k})(1-\gamma_j(\mathbf{k})g_j(\mathbf{k}))^{-1}$, and (3.26) can therefore be written as

$$\mathcal{G}(\mathbf{k}) = \mathcal{G}_j(\mathbf{k}) + \frac{\gamma_j(\mathbf{k})}{1-\gamma_j(\mathbf{k})g_j(\mathbf{k})}\widetilde{\Phi}_j(\mathbf{k}) \otimes \widetilde{\Psi}_j(\mathbf{k}). \tag{3.29}$$

We now see that the behavior of the Green's function and of the Jost solutions in vicinities of the points $\pm i\varkappa_j$ is determined by the behavior of the functions $g_j(\mathbf{k})$. If

$$g_j(\mathbf{k}) = o(1), \qquad \mathbf{k} \sim \pm i\varkappa_j, \tag{3.30}$$

then the Green's function has logarithmic singularities at the points $\pm i\varkappa_j$, and the Jost solutions are bounded at these points. On the other hand, if

$$g_j(\mathbf{k}) = O(1), \qquad \mathbf{k} \sim \pm i\varkappa_j \tag{3.31}$$

and the limit (also depending on how the limit is taken) is nonzero, then

$$\Phi(\mathbf{k}) = o(1), \qquad \Psi(\mathbf{k}) = o(1), \qquad \mathbf{k} \sim \pm i\varkappa_j, \tag{3.32}$$

and we have terms of the order $1/\log(|\mathbf{k} \mp i\varkappa_j|)$ in the r.h.s.

In what follows, we assume that condition (3.31) is satisfied for all $j = 1,\ldots,N$ (there is only one condition for any j because $\overline{g_j(\mathbf{k})} = g_j(\overline{\mathbf{k}})$ by (2.20) and (3.13)). In particular, under this condition,

$$\Phi_j(\pm\varkappa_j) = \Psi_j(\pm\varkappa_j) = 0, \quad j = 1,\ldots,N, \tag{3.33}$$

while the values $\Phi_m(\pm\varkappa_j)$ and $\Psi_m(\pm\varkappa_j)$ of the auxiliary solutions for m such that $\varkappa_m > \varkappa_l$ are finite and nonzero, and

$$\psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\Phi_m(\pm\varkappa_j) = 0, \qquad \Psi_m(\pm\varkappa_j)\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j) = 0. \tag{3.34}$$

Under condition (3.31), the matrix elements $A_{lm}(\mathbf{k}_\Im)$, as follows from (3.18), have finite limits when $\mathbf{k}_\Im = \pm\varkappa_j$. In particular, by (3.27) and (3.34), we have

$$A(\pm\varkappa_j)_{jm} = A(\pm\varkappa_j)_{mj} = \frac{\pm\delta_{jm}}{t_j}, \tag{3.35}$$

which coincides with the values of $A_{jm}(\mathbf{k}_\Im)$ and $A_{mj}(\mathbf{k}_\Im)$ when $\mathbf{k}_\Im > \min\{\varkappa_j, \varkappa_m\}$. Finally, by (3.23), we can prove that under condition (3.31),

$$w_l(\pm\varkappa_j)_{jm} = 0 \quad \text{for all } j \text{ and } l\text{: } \varkappa_j \le \varkappa_l. \tag{3.36}$$

The properties of the Jost solutions and spectral data so far derived allow reconstructing the auxiliary Jost solutions in terms of the boundary values of the Jost solutions. Indeed, integrating (3.14) at $\mathbf{k}_\Re = \pm 0$ and using notation (2.42) for the limit values of the total Green's function on the real axis, we obtain

$$\mathcal{G}(\pm 0 + i\mathbf{k}_\Im) = \mathcal{G}^\sigma(\pm 0) + \frac{\operatorname{sgn}\mathbf{k}_\Im}{2\pi}\int_0^{\mathbf{k}_\Im} d\alpha\, t(i\alpha)\Phi(\pm 0 + i\alpha)\otimes\Psi(\pm 0 + i\alpha), \tag{3.37}$$

where $\sigma = \operatorname{sgn}\mathbf{k}_\Im$. In the same way, by (3.16), (3.18), (3.23), and (3.33), we obtain

$$\Phi_j(\mathbf{k}_\Im) = \frac{\operatorname{sgn}\mathbf{k}_\Im}{2\pi}\int_{\varkappa_j \operatorname{sgn}\mathbf{k}_\Im}^{\mathbf{k}_\Im} d\alpha\, t(i\alpha)\Phi(+0 + i\alpha)\sum_l b_l\overline{w_l(-\alpha)}A_{lj}(\alpha) \tag{3.38}$$

in the region $|\mathbf{k}_\Im| < \varkappa_j$. This expression generalizes one-dimensional relations (2.27). From (3.14), (3.18), (3.23), and (3.35), we also obtain

$$\begin{aligned}(A^{-1}(\mathbf{k}_\Im))_{jm} = t_m \operatorname{sgn}\mathbf{k}_\Im\delta_{jm} + \theta(\min\{\varkappa_m, \varkappa_j\} - |\mathbf{k}_\Im|)\frac{\operatorname{sgn}\mathbf{k}_\Im}{2\pi}b_m \times \\ \times \int_{\min\{\varkappa_m,\varkappa_j\}\operatorname{sgn}\mathbf{k}_\Im}^{\mathbf{k}_\Im} d\alpha\, t(i\alpha)w_j(\alpha)\overline{w_m(-\alpha)},\end{aligned} \tag{3.39}$$

which gives an expression for the spectral data $A_{mj}(\mathbf{k}_\Im)$ in terms of $w_j(\mathbf{k}_\Im)$. It is straightforward to verify that the properties of the Jost solutions and spectral data described above are compatible with representations (3.38) and (3.39).

3.5. Bilinear representation for the resolvent and relations between the Green's functions. Let $q_1 \neq 0$. The derivative of $\widehat{M}(q)$ with respect to q_2 when $q_2 \neq 0$ is given by (3.4), and the discontinuity at $q_2 = 0$ follows from (3.11) and (3.17). These equalities allow reconstructing the hat kernel $\widehat{M}(x, x'; q)$ of the resolvent up to a term independent of q_2. This term is fixed by the condition that the kernel $M(x, x'; q)$ (defined by (2.3)) belongs to $\mathcal{S}'(\mathbb{R}^6)$. Omitting the details of this construction (see [7]), we here present the final expression:

$$\begin{aligned}\widehat{M}(x, x'; q) = \frac{1}{2\pi i}\int_{\mathbf{k}_\Im = q_1} d\mathbf{k}_\Re\left[\theta(x_2 - x_2') - \theta(2\mathbf{k}_\Re\mathbf{k}_\Im - q_2)\right]t(\mathbf{k})\Phi(x, \mathbf{k})\Psi(x', \mathbf{k}) - \\ - \operatorname{sgn} q_1\left[\theta(x_2 - x_2') - \theta(-q_2)\right]\sum_{l,m}\theta(\min\{\varkappa_l, \varkappa_m\} - |q_1|)\bigl(A(q_1)^{-1}\bigr)_{lm}\times \\ \times \Phi_l(x, q_1)\Psi_m(x', q_1),\end{aligned} \tag{3.40}$$

which gives a bilinear representation of the resolvent in terms of the Jost solutions. We note that because the kernel $M(x,x';q)$ is bounded in q and is continuous in q_1 for $q_2 \neq 0$, representation (3.40) holds for any q.

This bilinear representation for the resolvent leads via (3.7) to the bilinear representation for the Green's function:

$$\begin{aligned}\mathcal{G}(x,x',\mathbf{k}) = \int \frac{dk'}{2\pi i}\big[\theta(x_2-x_2')-\theta(\mathbf{k}_{\Im}k')\big]\,t(k'+\mathbf{k})\Phi(x,k'+\mathbf{k})\Psi(x',k'+\mathbf{k}) - \\ -\operatorname{sgn}\mathbf{k}_{\Im}\big[\theta(x_2-x_2')-\theta(-\mathbf{k}_{\Re}\mathbf{k}_{\Im})\big]\times \\ \times\sum_{l,m}\theta(\min\{\varkappa_l,\varkappa_m\}-|\mathbf{k}_{\Im}|)\big(A(\mathbf{k}_{\Im})^{-1}\big)_{lm}\Phi_l(x,\mathbf{k}_{\Im})\Psi_m(x',\mathbf{k}_{\Im}),\end{aligned} \tag{3.41}$$

which generalizes (2.35). Moreover, because of (3.10), bilinear representation (3.40) also gives a representation for the advanced/retarded Green's functions in terms of the Jost solutions:

$$\mathcal{G}_{\pm}(x,x') = \pm\theta(\pm(x_2-x_2'))\bigg(\frac{1}{2\pi i}\int dk\,t^{\sigma}(k)\Phi^{\sigma}(x,k)\Psi^{\sigma}(x',k)-\sigma\sum_{l,m}(A^{\sigma})^{-1}_{lm}\Phi^{\sigma}_l(x)\Psi^{\sigma}_m(x')\bigg), \tag{3.42}$$

where $\sigma = +,-$ and, by analogy with (2.42), we introduce the notation ($k \in \mathbb{R}$)

$$\Phi^{\pm}(x,k) = \lim_{\mathbf{k}_{\Im}\to\pm0}\Phi(x,k+i\mathbf{k}_{\Im}), \qquad \Psi^{\pm}(x) = \lim_{\mathbf{k}_{\Im}\to\pm0}\Psi(x,k+i\mathbf{k}_{\Im}), \tag{3.43}$$

$$\Phi^{\pm}_l(x) = \lim_{\mathbf{k}_{\Im}\to\pm0}\Phi_l(x,\mathbf{k}_{\Im}), \qquad \Psi^{\pm}_l(x) = \lim_{\mathbf{k}_{\Im}\to\pm0}\Psi_l(x,\mathbf{k}_{\Im}) \tag{3.44}$$

for the values of the Jost solutions on the real axis. Because of (3.21) and (3.24), these limit values have the conjugation properties

$$\overline{\Phi^{\pm}(x,k)} = \Psi^{\mp}(x,k), \qquad \overline{\Phi^{\pm}_l(x)} = b_l\Psi^{\mp}_l(x), \quad k\in\mathbb{R}, \quad l=1,\dots,N. \tag{3.45}$$

We mention without proof here that the l.h.s. of (3.42) is independent of the sign σ.

It is straightforward to verify that the functions $\mathcal{G}_{\pm}$ satisfy the differential equations $\overrightarrow{\mathcal{L}}\mathcal{G}_{\pm} = \mathcal{G}_{\pm}\overleftarrow{\mathcal{L}} = I$, the conjugation property

$$\overline{\mathcal{G}_{\pm}(x,x')} = \mathcal{G}_{\mp}(x',x), \tag{3.46}$$

the integral equations $\mathcal{G}_{\pm} = \mathcal{G}_{1,\pm} + \mathcal{G}_{1,\pm}U_2\mathcal{G}_{\pm}$, and their dual.

The standard and auxiliary advanced/retarded solutions are defined as (cf. (3.8) and (3.16))

$$\Phi_{\pm}(k) = \mathcal{G}_{\pm}\overleftarrow{\mathcal{L}}_1\varphi_{\pm}(k), \qquad \Phi_{\pm,j} = \mathcal{G}_{\pm}\overleftarrow{\mathcal{L}}_1\varphi(i\varkappa_j), \tag{3.47}$$

$$\Psi_{\pm}(k) = \psi_{\pm}(k)\overrightarrow{\mathcal{L}}_1\mathcal{G}_{\pm}, \qquad \Psi_{\pm,j} = \psi(i\varkappa_j)\overrightarrow{\mathcal{L}}_1\mathcal{G}_{\pm}. \tag{3.48}$$

These functions satisfy the nonstationary Schrödinger equation with the potential u and the conjugation properties

$$\overline{\Phi_{\pm}(x,k)} = \Psi_{\mp}(x,k), \qquad \overline{\Phi_{\pm,j}} = b_j\Psi_{\mp,j}. \tag{3.49}$$

The limit values of the Green's function on the real axis, as follows from (3.7), also give limits of $M(q)$ as $q \to 0$, which are different from (3.10). For these boundary values, we use the notation $\mathcal{G}^{\pm}(k)$ as in (2.42). Because the r.h.s. of equalities (3.41) and (3.42) are given in terms of the Jost solutions, taking the limit as $\mathbf{k} \to k + i\sigma 0$ in (3.41), we then obtain

$$\mathcal{G}^{\sigma}(k) - \mathcal{G}_{\pm} = \frac{\mp 1}{2\pi i} \int dk'\, \theta(\pm\sigma(k'-k))\, t^{\sigma}(k')\Phi^{\sigma}(k') \otimes \Psi^{\sigma}(k') \pm \\ \pm \sigma\theta(\mp\sigma k) \sum_{l,m} (A^{\sigma})^{-1}_{lm} \Phi^{\sigma}_{l} \otimes \Psi^{\sigma}_{m} \tag{3.50}$$

from (3.43) and (3.44). Another expression for this difference follows from Hilbert identity (3.2). For this, we use (3.7) (again in the limit as $\mathbf{k} \to k + i\sigma 0$) and relations (3.14), (2.49), and (2.50). Then

$$\mathcal{G}^{\sigma}(k) - \mathcal{G}_{\pm} = \frac{\mp 1}{2\pi i} \int d\alpha\, \theta(\pm\sigma(\alpha-k)) \int dp\, \Phi_{\pm}(p) r^{\sigma}_{\pm}(p,\alpha) \otimes \psi^{\sigma}(\alpha) \overrightarrow{\mathcal{L}}_{1} \mathcal{G}^{\sigma}(k) \pm \\ \pm \theta(\mp\sigma k) \sum_{j} t_j \Phi_{\pm,j} \otimes \psi(i\varkappa_j) \overrightarrow{\mathcal{L}}_{1} \mathcal{G}^{\sigma}(k), \tag{3.51}$$

where $r^{\sigma}_{\pm}$ is defined in (2.51).

3.6. Spectral data. The last equality gives the difference between the Green's functions in terms of the advanced/retarded solutions. It thus leads directly to the expression for the Jost solution in terms of the advanced/retarded solutions. Indeed, by definitions (3.8) and (3.16) ((3.43) and (3.44) in the limit case) and equalities (3.47) and (3.48), we obtain

$$t^{\sigma}(k)\Phi^{\sigma}(k) = \int dp\, \Phi_{\pm}(p) \mathcal{R}^{\sigma}_{\pm}(p,k) + \sum_{j} \Phi_{\pm,j} \mathcal{R}^{\sigma}_{\pm,j}(k), \quad \sigma = +,-, \tag{3.52}$$

$$\Phi^{\sigma}_{m} = \int dp\, \Phi_{\pm}(p) \widehat{\mathcal{R}}^{\sigma}_{\pm,m}(p) + \theta(\pm\sigma)\Phi_{\pm,m} \mp \theta(\mp\sigma) \sum_{j} t_j \Phi_{\pm,j} A^{\sigma}_{jm}, \tag{3.53}$$

where we use (3.18) in the limit as $\mathbf{k}_{\Im} \to \sigma 0$ in the last line, we introduce the spectral data $\mathcal{R}^{\sigma}_{\pm}$, $\mathcal{R}^{\sigma}_{\pm,j}$, and $\widehat{\mathcal{R}}^{\sigma}_{\pm,m}$ as

$$\mathcal{R}^{\sigma}_{\pm}(p,k) = \int d\alpha\, r^{\sigma}_{\pm}(p,\alpha) R^{\sigma}_{\pm}(\alpha,k), \tag{3.54}$$

where $R^{\sigma}_{\pm}$ is a triangular operator, $R^{\sigma}_{\pm}(p,k) = \delta(p-k) \mp \theta(\pm\sigma(p-k)) R^{\sigma}(p,k)$, with

$$R^{\sigma}(p,k) = \frac{t^{\sigma}(k)}{2\pi i} \big(\psi^{\sigma}(p) \overrightarrow{\mathcal{L}}_{1} \mathcal{G}^{\sigma}(k) \overleftarrow{\mathcal{L}}_{1} \varphi^{\sigma}(k)\big), \tag{3.55}$$

and the expression in parentheses is again understood as in (3.18). Next, $\mathcal{R}^{\sigma}_{\pm,j}(k) = \pm\theta(\mp\sigma k)\mathcal{R}^{\sigma}_{j}(k)$, where

$$\mathcal{R}^{\sigma}_{j}(k) = t_j t^{\sigma}(k) \big(\psi(i\varkappa_j) \overrightarrow{\mathcal{L}}_{1} \mathcal{G}^{\sigma}(k) \overleftarrow{\mathcal{L}}_{1} \varphi^{\sigma}(k)\big), \tag{3.56}$$

and

$$\widehat{\mathcal{R}}^{\sigma}_{\pm,m}(p) = \int d\alpha\, r^{\sigma}_{\pm}(p,\alpha) \widehat{R}^{\sigma}_{\pm,m}(\alpha), \tag{3.57}$$

where $\widehat{R}^{\sigma}_{\pm,m}(k) = \mp\theta(\pm\sigma k)\widehat{R}^{\sigma}_{m}(k)$, and

$$\widehat{R}^{\sigma}_{m}(k) = \frac{1}{2\pi i}\bigl(\psi^{\sigma}(k)\overrightarrow{\mathcal{L}}_1\mathcal{G}^{\sigma}(+0)\overleftarrow{\mathcal{L}}_1\varphi^{\sigma}(i\varkappa_m)\bigr). \tag{3.58}$$

The inverse relations that give the advanced/retarded solutions in terms of the Jost solutions follow in the same way, but by reducing (3.50). By the above definitions of the spectral data, we obtain

$$\Phi_{\pm}(k) = \int dk'\Phi^{\sigma}(k')\overline{\mathcal{R}^{-\sigma}_{\pm}(k,k')} - 2\pi i\sigma\sum_{l,m}\Phi^{\sigma}_{l}(A^{\sigma})^{-1}_{lm}\frac{1}{b_m}\overline{\widehat{\mathcal{R}}^{-\sigma}_{\pm,m}(k)}, \tag{3.59}$$

$$\Phi_{\pm,j} = \frac{-b_j}{2\pi i t_j}\int dk'\Phi^{\sigma}(k')\overline{\mathcal{R}^{-\sigma}_{\pm,j}(k')} + \theta(\pm\sigma)\Phi^{\sigma}_{j} \mp \frac{\theta(\mp\sigma)}{t_j}\sum_{l}\Phi^{\sigma}_{l}(A^{\sigma})^{-1}_{lj}. \tag{3.60}$$

Finally, substituting these equalities in (3.52) and (3.53), we obtain the relations between the limit values of the Jost solutions on the real axis:

$$t^{\sigma}(k)\Phi^{\sigma}(k) = \int dp\,\Phi^{-\sigma}(p)\mathcal{F}^{-\sigma}(p,k) + 2\pi i\sigma\sum_{l,m}\Phi^{-\sigma}_{l}(A^{-\sigma})^{-1}_{lm}\frac{1}{b_m}\mathcal{F}^{-\sigma}_{m}(k), \tag{3.61}$$

$$\Phi^{\sigma}_{j} = \int dp\,\Phi^{-\sigma}(p)\widehat{\mathcal{F}}^{-\sigma}_{j}(p) + 2\pi i\sigma\sum_{l,m}\Phi^{-\sigma}_{l}(A^{-\sigma})^{-1}_{lm}\frac{1}{b_m}\mathcal{F}^{-\sigma}_{mj}, \tag{3.62}$$

which generalize (2.53). By analogy with (2.54), we here introduce the alternative spectral data by the equalities

$$\mathcal{F}^{-\sigma}(k,k') = \int dk''\,\overline{\mathcal{R}^{\sigma}_{\pm}(k'',k)}\mathcal{R}^{\sigma}_{\pm}(k'',k') - \sum_{j}\frac{b_j}{2\pi i t_j}\overline{\mathcal{R}^{\sigma}_{\pm,j}(k)}\mathcal{R}^{\sigma}_{\pm,j}(k'), \tag{3.63}$$

$$\mathcal{F}^{-\sigma}_{l}(k') = -\frac{\theta(\pm\sigma)b_l}{2\pi i t_l}\mathcal{R}^{\sigma}_{\pm,l}(k) \pm \frac{\theta(\mp\sigma)}{2\pi i}\sum_{m=1}^{N}b_l A^{-\sigma}_{lm}\mathcal{R}^{\sigma}_{\pm,m}(k) + \int dk''\,\overline{\widehat{\mathcal{R}}^{\sigma}_{\pm,l}(k'')}\mathcal{R}^{\sigma}_{\pm}(k'',k'), \tag{3.64}$$

$$\widehat{\mathcal{F}}^{-\sigma}_{j}(k) = -\frac{\theta(\pm\sigma)b_j}{2\pi i t_j}\overline{\mathcal{R}^{\sigma}_{\pm,j}(k)} \pm \sum_{l}\frac{b_l\theta(\mp\sigma)}{2\pi i}A^{\sigma}_{lj}\overline{\mathcal{R}^{\sigma}_{\pm,l}(k)} + \int dk'\overline{\mathcal{R}^{\sigma}_{\pm}(k',k)}\,\widehat{\mathcal{R}}^{\sigma}_{\pm,j}(k'), \tag{3.65}$$

$$\mathcal{F}^{-\sigma}_{lj} = -\frac{\theta(\pm\sigma)b_l}{2\pi i t_l}\delta_{lj} + \frac{\theta(\mp\sigma)}{2\pi i}\sum_{m=0}^{N}b_l A^{-\sigma}_{lm}t_m A^{\sigma}_{mj} + \int dk'\,\overline{\widehat{\mathcal{R}}^{\sigma}_{\pm,l}(k')}\widehat{\mathcal{R}}^{\sigma}_{\pm,j}(k'). \tag{3.66}$$

4. Conclusion

The existence of bilinear representation (3.40) is a main advantage of the resolvent approach. This relation gives the extended resolvent in terms of the Jost solutions. These solutions themselves, like any other relevant solutions of the scattering problem under consideration and its dual, are given as specific reductions of the corresponding Green's functions. The Green's functions in turn are values of the resolvent at some specific points. This property provides a simple and regular way to derive relations between different kinds of solutions of the linear problem as well as between the spectral data. The properties of both sets of the spectral data ((3.54)–(3.58) and (3.63)–(3.66)) deserve special study. The same holds for the characterization equations for them, which follow from the combination of (3.52), (3.53) and (3.61), (3.62) and also from the r.h.s. of (3.63)–(3.66) being independent of the signs $\pm$. These relations, as well as formulation of the inverse problem based on the above results, will be given in [7].

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