Theoretical and Mathematical Physics, **159**(3): 721–733 (2009)

BUILDING AN EXTENDED RESOLVENT OF THE HEAT OPERATOR VIA TWISTING TRANSFORMATIONS

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We introduce twisting transformations for the heat operator. By the simultaneous use of these transformations, N solitons are superimposed à la Darboux on a generic smooth potential decaying at infinity, and the corresponding Jost solutions are generated. We also use these twisting operators to study the existence of the related extended resolvent. We study the existence and uniqueness of the extended resolvent in detail in the case of N solitons with N "incoming" rays and one "outgoing" ray.

Keywords: Darboux transformation, multidimensional soliton, annihilator

1. Introduction

The Kadomtsev–Petviashvili equation in the version called KPII,

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = -3u_{x_2x_2}, \tag{1.1}$$

is a (2+1)-dimensional generalization of the celebrated Korteweg–de Vries (KdV) equation. As a consequence, the KPII equation admits solutions that behave at space infinity like the solutions of the KdV equation. For instance, if $u_1(t,x_1)$ satisfies the KdV equation, then $u(t,x_1,x_2) = u_1(t,x_1+\mu x_2-3\mu^2 t)$ solves the KPII equation with an arbitrary constant $\mu \in \mathbb{R}$. It is therefore important to consider solutions of (1.1) that decay at space infinity in all directions with the exception of a finite number of one-dimensional rays with a behavior of the type of u_1. Although the KPII equation has been known for more than three decades to be integrable [1], [2], its general theory (involving such nondecaying solutions) is far from complete.

The Cauchy problem for the KPII equation with rapidly decaying initial data was solved in [3], [4] using the inverse scattering method based on the spectral analysis of the heat operator

$$\mathcal{L}(x,\partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u(x), \qquad x = (x_1, x_2), \tag{1.2}$$

that gives the associated linear problem for the KPII equation. The standard approach to the spectral theory of operator (1.2) is based on integral equations for the Jost solution $\Phi(x,\mathbf{k})$, where $\mathbf{k} \in \mathbb{C}$ denotes the spectral parameter, or for the Jost solution $\Psi(x,\mathbf{k})$ of the dual operator $\mathcal{L}^{\mathrm{d}}$. But it is known that these integral equations are ill-defined in the case of potentials with a one-dimensional asymptotic behavior. To overcome these difficulties, a resolvent approach was developed in [5]–[14].

In the resolvent approach, a space of operators $A(q)$ with kernels $A(x,x';q)$ belonging to the space of tempered distributions in the variables $x, x', q \in \mathbb{R}^2$ is introduced. Ordinary differential operators $\mathcal{L}(x,\partial_x)$

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Prepared from an English manuscript submitted by the authors; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 159, No. 3, pp. 364–378, June, 2009.

0040-5779/09/1593-0721 .

are embedded in this space as operators with the kernels

$$L(x,x';q) \equiv \mathcal{L}(x,\partial_x+q)\delta(x-x'). \tag{1.3}$$

These operators are called extended operators because they also depend on the parameter q. A generalization of the resolvent of a differential operator, called an extended resolvent, is introduced in this space, which allows considering the spectral theory of operators with a nontrivial asymptotic behavior at space infinity.

In [11]–[14], we considered the nonstationary Schrödinger and heat operators with potentials with only one direction of nondecaying behavior. The first step in solving the problem was to embed a pure one-dimensional spectral theory in the two-dimensional theory, building the two-dimensional extended resolvent for an operator with the potential $u(x) \equiv u_1(x_1)$. The second step was to dress this resolvent with an arbitrary bidimensional perturbation of the potential u_1. Finally, we obtained all the mathematical entities in the inverse scattering transform theory from this dressed resolvent by a reduction procedure, which allowed formulating the direct and inverse problems. We also note that the standard spectral theory for the heat operator in the case of potentials nondecaying in one space direction was developed under some implicit special conditions on the potential in [15].

Here, we consider a substantially more complicated problem, the case of a potential u nondecaying along multiple nonparallel rays. There is now no analogy with the one-dimensional case, and the whole theory must be constructed directly without embedding one-dimensional entities in two dimensions. We must therefore directly consider true bidimensional potentials as was previously done in [16], [17] for the nonstationary Schrödinger operator. In fact, we use the same procedure successfully used in that case. More precisely, to obtain the potential corresponding to N solitons "superimposed" on a generic smooth decaying potential and the related Jost solutions, we construct these entities directly using twisting transformations instead of recursively using the (binary) Darboux transformation. We thus recover potentials and Jost solutions that were obtained in [11] using a recursive procedure and also some alternative representations of the purely solitonic potentials obtained in [18]–[22] using the tau functions. We note that the heat operator is not self-dual and singular behaviors of the left and right twisting operators are hence not necessarily correlated. This explains why the structure of the N soliton solution is much richer for KPII than for KPI. In particular, the N soliton solution can have a different number of "incoming" and "outgoing" rays (in the sense of $x_2 \to -\infty$ or $x_2 \to +\infty$).

This paper is organized as follows. In Sec. 2, we briefly review some aspects and basic ideas of the extended resolvent approach using the example of operator (1.2) with a smooth rapidly decaying potential $u(x)$ (see [5]–[9] for further details). In Sec. 3, we introduce the twisting operators, i.e., operators that "twist" the operator L, an extension in the sense of (1.3) of the operator $\mathcal{L}$ in (1.2), into an operator L' of the same kind with a potential u' describing N solitons "superimposed" à la Darboux on the background potential u. In Sec. 4, under the assumption that the resolvent $M'(q)$ exists, we use these twisting operators to derive an explicit expression for its kernel $M'(x,x';q)$. The problem of proving the existence of the resolvent $M'(q)$ thus reduces to the problem of finding the region in the q plane where this kernel is a tempered distribution. This problem turns out to be more laborious than in the case of the nonstationary Schrödinger operator. Therefore, we first consider purely N-soliton potentials in Sec. 5 and then the special simple subclass of N-soliton potentials with N incoming rays and only one outgoing ray in Sec. 6. We prove that for $N>1$, the resolvent $M'(q)$ exists only in the region of the q plane outside a polygon with $N+1$ sides. More specifically, we prove that there exists a value $\mathbf{k}_0$ of the spectral parameter $\mathbf{k}$ for which the dual Jost solution $\Psi'(x,\mathbf{k})$ is such that $\Psi'(x,\mathbf{k}_0)e^{q_1x_1+q_2x_2}$ is exponentially decaying on the x plane for any value of q belonging to this polygon, and we consequently conclude that the twisted operator $L'(q)$ inside the polygon has a left annihilator and cannot have right inverse. Nevertheless, the Green's function of $\mathcal{L}'$

exists and can be uniquely derived via a reduction from $M'(q)$. In a forthcoming publication, we plan to consider the generic case of an arbitrary number of incoming and outgoing rays and to elucidate the role of these annihilators in the spectral theory of such potentials.

2. Background theory

We introduce the space of *extended* operators $A(q)$, i.e., operators with a kernel $A(x,x';q)$ belonging to the space $\mathcal{S}'$ of tempered distributions in the six real variables $x=(x_1,x_2)$, $x'=(x'_1,x'_2)$, and $q=(q_1,q_2)$. For two extended operators $A(q)$ and $B(q)$ with the kernels $A(x,x';q)$ and $B(x,x';q)$, we introduce the composition law

$$(AB)(x,x';q)=\int dx''\, A(x,x'';q)B(x'',x';q) \tag{2.1}$$

provided that the integral exists in terms of distributions. An operator A can have an inverse A^{-1} in the sense of this composition: $AA^{-1}=I$ and $A^{-1}A=I$, where I is the unity operator, i.e., the operator with the kernel $I(x,x';q)=\delta(x_1-x'_1)\delta(x_2-x'_2)$. A specific subclass of extended operators is given by the extensions $L(q)$ of differential operators as defined in (1.3), where $\mathcal{L}(x,\partial_x)$ denotes a differential operator whose coefficients are smooth functions of x. With any operator $A(q)$ with the kernel $A(x,x';q)$, we associate its "hat-kernel"

$$\hat{A}(x,x';q)=e^{q(x-x')}A(x,x';q), \tag{2.2}$$

where $qx=q_1x_1+q_2x_2$. For a differential operator $L(q)$, this procedure is the inverse of the extension introduced in (1.3), i.e., $\hat{L}(x,x';q)=L(x,x')$. But we note that because the kernels of the operators A are only subject to the requirement that they belong to the space of tempered distributions, the hat-kernel $\hat{A}$ in general depends on q and is not necessarily bounded. We also note that the operator $A^*(q)$ conjugate to an operator $A(q)$ is defined by the kernel

$$A^*(x,x';q)=\overline{A(x,x';q)}, \tag{2.3}$$

and a self-conjugate operator is therefore an operator with a real kernel.

It is convenient to introduce the representation of the operator $A(q)$ in the p space defined by the Fourier transformation,

$$A(p;\mathbf{q})=\frac{1}{(2\pi)^2}\int dx\int dx'\, e^{i(p+\mathbf{q}_\Re)x-i\mathbf{q}_\Re x'}A(x,x';\mathbf{q}_\Im), \tag{2.4}$$

where $p=(p_1,p_2)\in\mathbb{R}^2$ and we introduce a two-dimensional complex vector

$$\mathbf{q}=\mathbf{q}_\Re+i\mathbf{q}_\Im,\qquad q\equiv\mathbf{q}_\Im,\quad \mathbf{q}_\Re,\mathbf{q}_\Im\in\mathbb{R}^2. \tag{2.5}$$

Composition (2.1) in the p space becomes a sort of shifted convolution

$$(AB)(p;\mathbf{q})=\int dp'\, A(p-p';\mathbf{q}+p')B(p';\mathbf{q}). \tag{2.6}$$

With the defined notation, the extension of heat operator (1.2) becomes

$$L=L_0-u,\qquad u(x,x';q)=u(x)\delta(x-x'), \tag{2.7}$$

where L_0 is the extension in the sense of (1.3) of the differential part $\mathcal{L}_0$ of heat operator (1.2) and by (2.4) has the kernel in the p space given by

$$L_0(p;\mathbf{q})=(i\mathbf{q}_2-\mathbf{q}_1^2)\delta(p). \tag{2.8}$$

The main object in our approach is the extended resolvent (or resolvent, for short) $M(q)$ of the operator $L(q)$, i.e., the operator inverse to L in the sense of composition (2.1) (or (2.6)):

$$LM = ML = I. \tag{2.9}$$

It can be defined as the solution of the integral equations

$$M = M_0 + M_0 uM, \qquad M = M_0 + uMM_0, \tag{2.10}$$

where M_0 is the resolvent of the zero-potential (bare) operator L_0. By virtue of (2.8), we have $M_0(p;\mathbf{q}) = \delta(p)/(i\mathbf{q}_2 - \mathbf{q}_1^2)$ for the kernel of this operator in the p space. In the case of a rapidly decaying potential $u(x)$, the existence and uniqueness of the solution of the equations in (2.10) can be proved in analogy with [4]. We note that for a real potential $u(x)$, both L and M are self-conjugate operators in the sense of definition (2.3).

By (2.6) and the explicit form of $M_0(p;\mathbf{q})$ given above, integral equations (2.10) written in the p space show that the kernel $M(p;\mathbf{q})$ is singular for $\mathbf{q}_2 = -i\mathbf{q}_1^2$ and for $\mathbf{q}_2 + p_2 = -i(\mathbf{q}_1 + p_1)^2$. Therefore, it is natural to introduce truncated and reduced values of the resolvent,

$$\begin{aligned} \nu(p;\mathbf{q}) &= (ML_0)(p;\mathbf{q})\big|_{\mathbf{q}_2=-i\mathbf{q}_1^2}, \\ \omega(p;\mathbf{q}) &= (L_0M)(p;\mathbf{q})\big|_{\mathbf{q}_2=-i(\mathbf{q}_1+p_1)^2-p_2}. \end{aligned} \tag{2.11}$$

It can be shown that the operator $L(q)$ and its resolvent $M(q)$ admit the bilinear representations in terms of ν and ω

$$L = \nu L_0 \omega, \qquad M = \nu M_0 \omega. \tag{2.12}$$

Correspondingly, we call the operators ν and ω dressing operators because they "dress" the bare operators L_0 and M_0. We note that $\nu(p;\mathbf{q})$ and $\omega(p;\mathbf{q})$ have the asymptotic behavior

$$\lim_{\mathbf{q}_1\to\infty} \nu(p;\mathbf{q}) = \delta(p), \qquad \lim_{\mathbf{q}_1\to\infty} \omega(p;\mathbf{q}) = \delta(p)$$

and are independent of $\mathbf{q}_2$, which we make clear by writing $\nu(p;\mathbf{q}_1)$ and $\omega(p;\mathbf{q}_1)$, if necessary. The dressing operators ν and ω satisfy the equations

$$L\nu = \nu L_0, \qquad \omega L = L_0 \omega \tag{2.13}$$

and are mutually inverse,

$$\omega\nu = I, \qquad \nu\omega = I. \tag{2.14}$$

To determine the Jost solutions using the dressing operators, we introduce the Fourier transforms

$$\chi(x,\mathbf{q}_1) = \int dp\, e^{-ipx}\nu(p;\mathbf{q}_1), \qquad \xi(x,\mathbf{q}_1) = \int dp\, e^{-ipx}\omega(p;\mathbf{q}_1 - p_1). \tag{2.15}$$

Then the Jost and the dual Jost solutions can be respectively defined as

$$\Phi(x,\mathbf{k}) = e^{-i\mathbf{k}x_1 - \mathbf{k}^2 x_2}\chi(x,\mathbf{k}) \qquad \Psi(x,\mathbf{k}) = e^{i\mathbf{k}x_1 + \mathbf{k}^2 x_2}\xi(x,\mathbf{k}), \tag{2.16}$$

where we let $\mathbf{k}$ denote $\mathbf{q}_1$ in conformity with the standard notation for the spectral parameter. By (2.13), these solutions satisfy the heat equation and its dual,

$$\bigl(-\partial_{x_2} + \partial_{x_1}^2 - u(x)\bigr)\Phi(x,\mathbf{k}) = 0, \qquad \bigl(\partial_{x_2} + \partial_{x_1}^2 - u(x)\bigr)\Psi(x,\mathbf{k}) = 0. \tag{2.17}$$

3. Darboux transformations via twisting operators ζ and η

To build a two-dimensional potential describing N solitons superimposed on a generic smooth background, we bypass the recursive procedure used in [11] and directly construct the final entities with the operator formulation introduced in the preceding section using the twisting operators. We consider a transformation from the operator L in (2.7) to a new operator of the same form

$$L' = L_0 - u', \qquad u'(x, x'; q) = u'(x)\delta(x - x'), \tag{3.1}$$

given in terms of a pair of operators ζ and η according to the formulas

$$L'\zeta = \zeta L, \qquad \eta L' = L\eta, \tag{3.2}$$

"twisting" L into L'. We take the potential $u(x)$ in L as a real, smooth, rapidly decaying function of x and seek self-conjugate operators ζ and η such that the new potential $u'(x)$ is also real and smooth. In addition, we require that η be the left inverse of ζ, i.e., that the condition

$$\eta\zeta = I \tag{3.3}$$

be satisfied and hence

$$L = \eta L'\zeta. \tag{3.4}$$

To obtain a new potential $u'(x)$ not decaying along some directions of the plane, the two operators L and L' must be related by a transformation more general than a similarity transformation. We therefore seek twisting operators with a product $\zeta\eta$ not equal to I. Setting

$$\zeta\eta = I - P, \tag{3.5}$$

we find that P is an orthogonal self-conjugate projector because $P^2 = P$ as a consequence of (3.3) and $P^* = P$ because ζ and η are self-conjugate. The twisting operators ζ and η generate a new potential u' via (3.2) and also the new dressing operators ν' and ω'. In fact, taking (2.13) into account, we obtain $L'\zeta\nu = \zeta L\nu = \zeta\nu L_0$ and $\omega\eta L' = \omega L\eta = L_0\omega\eta$ by (3.2). Therefore, the operators ν' and ω' defined by

$$\nu' = \zeta\nu, \qquad \omega' = \omega\eta \tag{3.6}$$

satisfy the equations

$$L'\nu' = \nu' L_0, \qquad \omega' L' = L_0\omega', \tag{3.7}$$

analogous to (2.13) for the dressing operators ν and ω.

We note that because of (3.3), the scalar product of these dressing operators is equal to I just as for the original operators in (2.14):

$$\omega'\nu' = I. \tag{3.8}$$

But in contrast to ν and ω, these operators do not satisfy a completeness relation, because by (2.14) and unlike it, it follows from (3.5) that

$$\nu'\omega' + P = I. \tag{3.9}$$

To obtain a Darboux transformation, we must specify the analyticity properties of the kernels $\zeta(p; \mathbf{q})$ and $\eta(p; \mathbf{q})$ of the twisting operators with respect to the variables $\mathbf{q}$. Because $\zeta = \nu'\omega$ and $\eta = \nu\omega'$ by (2.14) and (3.6), the singularities of these kernels are related to the properties of $\nu'(p; \mathbf{q})$ and $\omega'(p; \mathbf{q})$. We impose the following conditions:

1. The kernels $\nu'(p;\mathbf{q})$ and $\omega'(p;\mathbf{q})$ are independent of $\mathbf{q}_2$ and have the asymptotic behavior

$$\lim_{\mathbf{q}_1\to\infty}\nu'(p;\mathbf{q}_1)=\delta(p),\qquad \lim_{\mathbf{q}_1\to\infty}\omega'(p;\mathbf{q}_1)=\delta(p).$$

2. The kernels $\nu'(p;\mathbf{q}_1)$ and $\omega'(p;\mathbf{q}_1)$ respectively have right and left simple poles with respect to the variable $\mathbf{q}_1$, i.e., there exist the nontrivial limits

$$\nu'_{b_l}(p)=\lim_{\mathbf{q}_1\to ib_l}(\mathbf{q}_1-ib_l)\nu'(p;\mathbf{q}_1), \tag{3.10}$$

$$\omega'_{a_j}(p)=\lim_{\mathbf{q}_1\to -p_1+ia_j}(\mathbf{q}_1+p_1-ia_j)\omega'(p;\mathbf{q}_1), \tag{3.11}$$

where $a_1,\dots,a_{N_a}$ and $b_1,\dots,b_{N_b}$ are N_a+N_b parameters, chosen all different and real to guarantee the realness of the transformed potential.

3. The kernels $\zeta(p;\mathbf{q})$ and $\eta(p;\mathbf{q})$ have no departures from analyticity with respect to $\mathbf{q}_1$ except the discontinuities at $\mathbf{q}_{1\Im}=b_l$ and $\mathbf{q}_{1\Im}=a_j$ following from condition 2.

The kernels of the operators ζ and η are then given by

$$\zeta(p;\mathbf{q}_1)=\delta(p)+\sum_{l=1}^{N_b}\int dp'\,\frac{\nu'_{b_l}(p-p')\omega(p';ib_l-p'_1)}{\mathbf{q}_1+p'_1-ib_l}, \tag{3.12}$$

$$\eta(p;\mathbf{q}_1)=\delta(p)+\sum_{j=1}^{N_a}\int dp'\,\frac{\nu(p-p';ia_j)\omega'_{a_j}(p')}{\mathbf{q}_1+p'_1-ia_j}, \tag{3.13}$$

where we use notation (3.10) and (3.11). We now let $\chi'(x,\mathbf{k})$ and $\xi'(p;\mathbf{k})$ denote the functions defined in terms of the dressing operators ν' and ω' by analogy with (2.15) and let $\chi'_{b_l}(x)$ and $\xi'_{a_j}(x)$ be their residues (cf. (3.10) and (3.11)). It is then easy to show that condition (3.3) is equivalent to the set of equations

$$\chi'_{b_l}(x)=-i\sum_{j=1}^{N_a}\chi(x,ia_j)m_{jl}(x),\quad l=1,\dots,N_b, \tag{3.14}$$

$$\xi'_{a_j}(x)=i\sum_{l=1}^{N_b}m_{jl}(x)\xi(x,ib_l),\qquad j=1,\dots,N_a, \tag{3.15}$$

where $m(x)$ is the $N_a\times N_b$ matrix with the elements

$$m_{jl}(x)=\int_{x_1}^{(a_j-b_l)\infty}dy_1\,e^{(a_j-b_l)(x_1-y_1)}\xi'_{a_j}(y)\chi'_{b_l}(y)\big|_{y_2=x_2}, \tag{3.16}$$

which are well defined for bounded χ'_{b_l} and ξ'_{a_j} and satisfy

$$\partial_{x_1}m_{jl}(x)=(a_j-b_l)m_{jl}(x)-\xi'_{a_j}(x)\chi'_{b_l}(x). \tag{3.17}$$

In addition, we obtain the transformed potential

$$u'(x)=u(x)-2\partial_{x_1}\sum_{j=1}^{N_a}\sum_{l=1}^{N_b}\xi(x,ib_l)\chi(x,ia_j)m_{jl}(x) \tag{3.18}$$

from (3.1) and (3.2). To define ζ and η, because of (3.12) and (3.13), we must specify ν'_{b_l} and ω'_{a_j}, i.e., $\chi'_{b_l}(x)$ and $\xi'_{a_j}(x)$. By (3.14) and (3.15), this means that we must define the matrix $m(x)$. Substituting equalities (3.14) and (3.15) in (3.17) and using (2.16), we write the equation for the matrix $\widehat{m}(x)$ with the elements $\widehat{m}_{jl}(x) = e^{(b_l-a_j)(x_1+(b_l+a_j)x_2)}m_{jl}(x)$,

$$\partial_{x_1}\widehat{m}_{jl}(x) = -\sum_{j'=1}^{N_a}\sum_{l'=1}^{N_b}\widehat{m}_{jl'}(x)\Psi(x, ib_{l'})\Phi(x, ia_{j'})\widehat{m}_{j'l}(x). \tag{3.19}$$

Omitting details, we present its solution in two equivalent forms:

$$\widehat{m}(x) = \big(E_{N_a} + c\mathcal{F}(x)\big)^{-1}c = c\big(E_{N_b} + \mathcal{F}(x)c\big)^{-1}, \tag{3.20}$$

where c is an arbitrary constant real $N_a\times N_b$ matrix, E_{N_a} and E_{N_b} are the respective $N_a\times N_a$ and $N_b\times N_b$ unit matrices, and $\mathcal{F}(x)$ is an $N_b\times N_a$ matrix with the elements $\mathcal{F}_{lj}(x) = \mathcal{F}(x, ib_l, ia_j)$, where

$$\mathcal{F}(x, \mathbf{k}, \mathbf{k}') = \int_{(\mathbf{k}_\Im - \mathbf{k}'_\Im)\infty}^{x_1} dx'_1\, \Psi(x', \mathbf{k})\Phi(x', \mathbf{k}')\big|_{x'_2=x_2}. \tag{3.21}$$

Potential (3.18) can now be also written in two forms:

$$u'(x) = u(x) - 2\partial_{x_1}^2 \log\det(E_{N_b} + \mathcal{F}c) = u(x) - 2\partial_{x_1}^2 \log\det(E_{N_a} + c\mathcal{F}). \tag{3.22}$$

Substituting the matrix m found above in (3.14) and (3.15) and then the resulting equations in (3.12) and (3.13), we derive the explicit formulas for the dressing operators ζ and η. Finally, using equation (2.16) and its analogue for the transformed (primed) Jost and dual Jost solutions, we obtain

$$\begin{aligned}\Phi'(x, \mathbf{k}) &= \Phi(x, \mathbf{k}) - \Phi(x, ia)\big(E_{N_a} + c\mathcal{F}(x)\big)^{-1}c\mathcal{F}(x, ib, \mathbf{k}) = \\ &= \Phi(x, \mathbf{k}) - \Phi(x, ia)c\big(E_{N_b} + \mathcal{F}(x)c\big)^{-1}\mathcal{F}(x, ib, \mathbf{k}),\end{aligned} \tag{3.23}$$

$$\begin{aligned}\Psi'(x, \mathbf{k}) &= \Psi(x, \mathbf{k}) - \mathcal{F}(x, \mathbf{k}, ia)\big(E_{N_a} + c\mathcal{F}(x)\big)^{-1}c\Psi(x, ib) = \\ &= \Psi(x, \mathbf{k}) - \mathcal{F}(x, \mathbf{k}, ia)c\big(E_{N_b} + \mathcal{F}(x)c\big)^{-1}\Psi(x, ib)\end{aligned} \tag{3.24}$$

from (3.6), where $j = 1, \dots, N_a$, $l = 1, \dots, N_b$, and

$$\Phi(x, ia) = \operatorname{diag}\{\Phi(x, ia_j)\}, \qquad \Psi(x, ib) = \operatorname{diag}\{\Psi(x, ib_l)\},$$

$$\mathcal{F}(x, \mathbf{k}, ia) = \operatorname{diag}\{\mathcal{F}(x, \mathbf{k}, ia_j)\}, \qquad \mathcal{F}(x, ib, \mathbf{k}) = \operatorname{diag}\{\mathcal{F}(x, ib_l, \mathbf{k})\}$$

and where either of the two forms (3.23) and (3.24) can be used. It is easy to see that both $\Phi'(x, \mathbf{k})$ and $\Psi'(x, \mathbf{k})$ respectively have poles at $\mathbf{k} = ib_l$ and $\mathbf{k} = ia_j$, and from (3.23) and (3.24), we find that the residues of these functions are given in terms of their values at the dual points by the relations

$$\Phi'_{b_l}(x) = -i\sum_{j=1}^{N_a}\Phi'(x, ia_j)c_{jl}, \qquad \Psi'_{a_j}(x) = i\sum_{l=1}^{N_b}c_{jl}\Psi'(x, ib_l), \tag{3.25}$$

as expected. It can be shown that the potential $u'(x)$ in (3.22) and the Jost solutions in (3.23) and (3.24) coincide with those in [11], including the case $N_a \neq N_b$ considered here, which is recovered by choosing some rows or columns in the constant matrix C introduced in [11] to be all zeros. We discuss the regularity conditions for the potential given by (3.22) in Sec. 5.

4. Resolvent

After the transformed operator L' is obtained, we can use the operators ζ and η to investigate its spectral properties and the existence and uniqueness of the corresponding resolvent M'. Multiplying the respective first and second equations in (3.2) from the right by η and from the left by ζ and recalling definition (3.5) of P, we obtain the intertwining relation

$$L' = \zeta L \eta + L_{\Delta}, \tag{4.1}$$

where we introduce

$$L_{\Delta} = L'P = PL'. \tag{4.2}$$

We therefore seek a resolvent in the form

$$M' = \zeta M \eta + M_{\Delta}, \tag{4.3}$$

where the product $\zeta M \eta$ is determined by the above construction. Indeed, we conclude from (2.12) and (2.14) that

$$\zeta M \eta = \nu' M_0 \omega', \tag{4.4}$$

and we then obtain the bilinear expression for the x-space kernel of this product in terms of the transformed Jost solutions

$$\begin{aligned}(\zeta M \eta)(x, x'; q) = {} & -\operatorname{sgn}(x_2 - x_2') \frac{e^{-q(x-x')}}{2\pi} \int dp_1\, \theta\big((q_2 + p_1^2 - q_1^2)(x_2 - x_2')\big) \times \\ & \times \Phi'(x; p_1 + iq_1) \Psi'(x'; p_1 + iq_1),\end{aligned} \tag{4.5}$$

where $\theta(\,\cdot\,)$ is the Heaviside step function.

The second term in (4.3), M_{Δ}, is to be determined. We conclude from (3.2), (2.9), and (3.5) that M' in (4.3) is the right or left inverse of L' iff M_{Δ} is a solution of the respective first or second of the operator equations

$$L' M_{\Delta} = P, \qquad M_{\Delta} L' = P. \tag{4.6}$$

Below, we consider the solvability of these equations in the case of pure soliton potentials.

To complete the discussion of the generic case, we note that an explicit form of the operator P defined in (3.5) can be derived by substituting ζ and η given by (3.12) and (3.13) in this equality. Again using (2.16), we obtain the expression for the hat-kernel of this operator:

$$\widehat{P}(x, x'; q) = i\delta(x_2 - x_2') \sum_{n=1}^{\mathcal{N}} \theta(q_1 - \alpha_n) \operatorname*{res}_{\mathbf{k}=i\alpha_n} \Phi'(x, \mathbf{k}) \Psi'(x', \mathbf{k}), \tag{4.7}$$

where we introduce the set of parameters

$$\{\alpha_1, \ldots, \alpha_{\mathcal{N}}\} = \{a_1, \ldots, a_{N_a}, b_1, \ldots, b_{N_b}\}, \qquad \mathcal{N} = N_a + N_b, \tag{4.8}$$

to make the symmetry properties explicit with respect to the parameters a_j and b_l used above. Kernel (4.7) is nonzero only in the interval of q_1 between the minimum and maximum values of the α_m. Indeed, this is obvious from (4.7) for q_1 below the interval and follows from the equality

$$\sum_{n=1}^{\mathcal{N}} \operatorname*{res}_{\mathbf{k}=i\alpha_n} \Phi'(x, \mathbf{k}) \Psi'(x', \mathbf{k}) = 0, \tag{4.9}$$

which is in turn a consequence of (3.25), for values of q_1 above the interval. According to the discussion above, we now expect that $\widehat{M}_\Delta$ has a structure similar to $\widehat{P}(x,x';q)$. It is clear that the kernel

$$\widehat{M}_\Delta(x,x';q) = \mp i\theta\big(\pm(x_2-x_2')\big)\sum_{n=1}^{\mathcal{N}}\theta(q_1-\alpha_n)\operatorname*{res}_{\mathbf{k}=i\alpha_n}\Phi'(x,\mathbf{k})\Psi'(x',\mathbf{k}) \tag{4.10}$$

satisfies the equations $\mathcal{L}'_x\widehat{M}_\Delta(x,x';q) = P(x,x';q)$ and $\mathcal{L}'^{\mathrm{d}}_{x'}\widehat{M}_\Delta(x,x';q) = P(x,x';q)$ for any sign. But it is easy to see that the kernel $M_\Delta(x,x';q)$ constructed using (2.2) in the general case increases at space infinity and cannot be the kernel of an extended operator as defined in Sec. 2. Below, we investigate this problem in detail with a specific example.

5. Pure soliton potential and Jost solutions

In the case $u(x)\equiv 0$, the transformed potential $u'(x)$ is the general purely N-soliton potential, where $N=\max\{N_a,N_b\}$. This potential and the corresponding Jost solutions can be easily obtained from the general expression found in Sec. 3. The pure soliton potential is given by either of the formulas in (3.22), where the matrix $\mathcal{F}(x)$ now has the elements

$$\mathcal{F}_{lj}(x) = \frac{e^{(a_j-b_l)(x_1+(a_j+b_l)x_2)}}{a_j-b_l}. \tag{5.1}$$

Expanding the determinants in the right-hand side of (3.22) and using the Binet–Cauchy formula, we find that for the potential to be regular, it suffices that the real matrix c satisfies the characterization requirements (equivalent to those in [20])

$$\Lambda\begin{pmatrix} l_1,l_2,\ldots,l_n \\ j_1,j_2,\ldots,j_n\end{pmatrix} c\begin{pmatrix} j_1,j_2,\ldots,j_n \\ l_1,l_2,\ldots,l_n\end{pmatrix}\geq 0, \qquad \Lambda_{lj}=\frac{1}{a_j-b_l}, \tag{5.2}$$

for any $1\leq n\leq\min\{N_a,N_b\}$ and all minors, i.e., any choice of $1\leq l_1<l_2<\cdots<l_n\leq N_b$ and $1\leq j_1<j_2<\cdots j_n\leq N_a$. We here use the standard notation

$$A\begin{pmatrix} j_1,j_2,\ldots,j_n \\ l_1,l_2,\ldots,l_n\end{pmatrix} = \det\begin{Vmatrix} a_{j_1l_1} & a_{j_1l_2} & \cdots & a_{j_1l_n} \\ a_{j_2l_1} & a_{j_2l_2} & \cdots & a_{j_2l_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j_nl_1} & a_{j_nl_2} & \cdots & a_{j_nl_n}\end{Vmatrix} \tag{5.3}$$

for the minors of the matrix $A=\|a_{jl}\|$. Formulas (3.23) and (3.24), after rather cumbersome calculations omitted here, can be more simply expressed as ratios of determinants. As in (3.22), we can alternatively, but equivalently, use determinants of $N_a\times N_a$ or $N_b\times N_b$ matrices. Below, we use the latter choice and symmetric notation (4.8). In these terms, we obtain

$$\chi'(x,\mathbf{k}) = \left(\prod_{l=1}^{N_b}(b_l+i\mathbf{k})^{-1}\right)\frac{\tau_\chi(x,\mathbf{k})}{\tau(x)}, \tag{5.4}$$

$$\xi'(x',\mathbf{k}) = \left(\prod_{l=1}^{N_b}(b_l+i\mathbf{k})\right)\frac{\tau_\xi(x',\mathbf{k})}{\tau(x')}, \tag{5.5}$$

for the functions χ' and ξ' related to the Jost solutions by (2.16), where the tau functions are the determinants

$$\begin{aligned}
\tau_\chi(x,\mathbf{k}) &= \det\bigl(\mathfrak{A}e^{\mathcal{A}(x)}(\alpha+i\mathbf{k})\mathcal{D}\bigr),\\
\tau_\xi(x',\mathbf{k}) &= \det\bigl(\mathfrak{A}e^{\mathcal{A}(x')}(\alpha+i\mathbf{k})^{-1}\mathcal{D}\bigr),\\
\tau(x) &= \det\bigl(\mathfrak{A}e^{\mathcal{A}(x)}\mathcal{D}\bigr),
\end{aligned}\tag{5.6}$$

i.e., determinants of $N_b\times N_b$ matrices obtained as products of square and rectangular matrices. More precisely,

$$\begin{aligned}
&\mathfrak{A}_{ln} = \alpha_n^{N_b-l}, && \alpha+i\mathbf{k} = \operatorname{diag}\{\alpha_n+i\mathbf{k}\},\\
&e^{\mathcal{A}(x)} = \operatorname{diag}\bigl\{e^{\mathcal{A}_n(x)}\bigr\}, && \mathcal{A}_n(x) = \alpha_n x_1 + \alpha_n^2 x_2,\\
&\mathcal{D} = \begin{pmatrix} d\\ E_{N_b}\end{pmatrix}, && d_{jl} = c_{jl}\frac{\prod_{l'=1,l'\neq l}^{N_b}(b_l-b_{l'})}{\prod_{l'=1}^{N_b}(a_j-b_{l'})},
\end{aligned}\tag{5.7}$$

where $l=1,2,\dots,N_b$, $j=1,2,\dots,N_a$, and $n=1,2,\dots,\mathcal{N}$. The potential can be expressed as $u'(x) = -2\partial_{x_1}^2\log\tau(x)$, and taking (5.6) into account, we recover the expression obtained in [19], [20], [22] using the tau functions.

6. The N-soliton potentials in the case $N_b = 1$

We restrict ourself to the special case of the N-soliton potential with $N_b = 1$ and arbitrary $N_a = N$. In addition, for simplicity, we choose $\alpha_1 < \alpha_2 < \cdots < \alpha_{\mathcal{N}}$ without loss of generality. In this situation, the potential $u'(x)$ has N_a incoming rays and one outgoing ray on the x plane, as shown schematically in Fig. 1. In this case, tau functions (5.6) take the simple form

$$\begin{aligned}
&\tau_\chi(x,\mathbf{k}) = \sum_{m=1}^{\mathcal{N}} f_m e^{\mathcal{A}(x)}(\alpha_m+i\mathbf{k}), \qquad \tau_\xi(x,\mathbf{k}) = \sum_{m=1}^{\mathcal{N}}\frac{f_m e^{\mathcal{A}(x)}}{\alpha_m+i\mathbf{k}},\\
&\tau(x) = \sum_{m=1}^{\mathcal{N}} f_m e^{\mathcal{A}(x)},
\end{aligned}\tag{6.1}$$

where the coefficients f_m are obtained from the elements d_{m1} of matrix d in (5.7) by a permutation that takes the chosen ordering $\alpha_1 < \alpha_2 < \cdots < \alpha_{\mathcal{N}}$ into account. Condition (5.2) here means that all $f_m > 0$.

We want to show that the extended operator $L'(q)$ with such a potential, as stated in the introduction, has a left self-conjugate annihilator $K(q)$ for q belonging to a domain of the q plane. More precisely, we consider the domain inside the polygon inscribed in the parabola $q_2 = q_1^2$ with vertices at the points $q_1 = \alpha_m$, $m = 1,\dots,N$, whose characteristic function is given by

$$\begin{aligned}
\kappa(q) = &\sum_{m=1}^{\mathcal{N}-1}[\theta(q_1-\alpha_{m+1}) - \theta(q_1-\alpha_m)]\times\\
&\times\bigl[\theta\bigl(q_2-(\alpha_m+\alpha_n)q_1+\alpha_m\alpha_n\bigr) - \theta\bigl(q_2-(\alpha_1+\alpha_{\mathcal{N}})q_1+\alpha_1\alpha_{\mathcal{N}}\bigr)\bigr].
\end{aligned}\tag{6.2}$$

We note that this polygon can be considered dual to the ray structure on the x plane (see Fig. 1 for the case $\mathcal{N} = 4$).

The main observation needed for obtaining the annihilator $K(q)$ is that the function

$$\psi(x;q) = \frac{\kappa(q)e^{qx}}{\tau(x)}\tag{6.3}$$

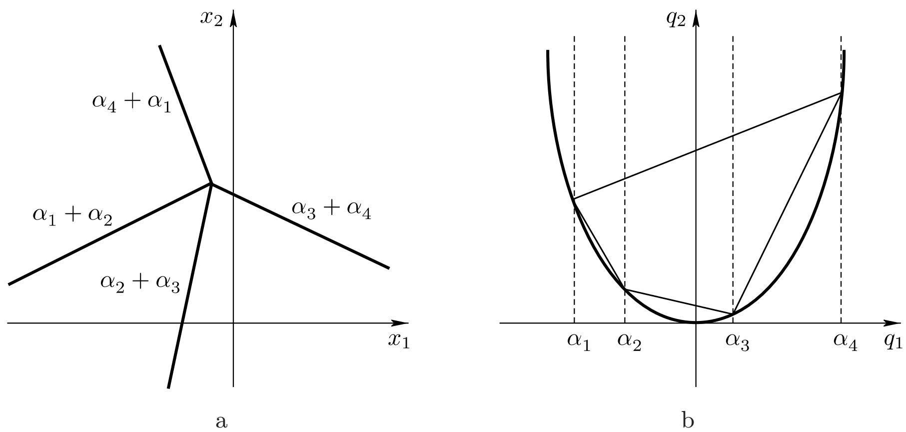

Fig. 1. The rays and polygon for $\mathcal{N} = 4$.

is bounded in the x plane. More precisely, as can be proved after a detailed study, it decays exponentially for x going to infinity in any direction on the plane for q inside the polygon and has directions of nondecaying (but bounded) behavior for q on the borders of the polygon.

By (2.16), (5.4), and (6.1), the function $1/\tau$ is proportional to the value of the dual Jost solution $\Psi'(x, \mathbf{k})$ at $\mathbf{k} = ib_1$. We then have

$$\mathcal{L}'^{\mathrm{d}}_{x'}\hat{\psi}(x'; q) = 0 \tag{6.4}$$

and consequently

$$KL' = 0 \tag{6.5}$$

for any operator K with the kernel

$$K(x, x'; q) = \varphi(x; q)\psi(x'; q), \tag{6.6}$$

where $\varphi(x; q)$ is any arbitrary self-conjugate function bounded in x and identically zero outside the polygon introduced above.

It can be easily verified that $KP = K$. If, in addition, we choose $\varphi(x; q)$ in the form $\varphi(x; q) = (P\gamma)(x; q)$, where $\gamma(x; q)$ is a self-conjugate function bounded in x and identically zero outside the polygon and satisfying the equality

$$\int dx\, \gamma(x; q)\psi(x, q) = \kappa(q), \tag{6.7}$$

then we also have

$$P\varphi = \varphi, \qquad \int dx\, \varphi(x; q)\psi(x, q) = \kappa(q). \tag{6.8}$$

It is then easy to verify that K is a self-conjugate projector commuting with the projector P, i.e., we have

$$K^* = K, \qquad K^2 = K, \qquad PK = KP = K. \tag{6.9}$$

The existence of this annihilator proves that the operator L' cannot have a right inverse for q belonging to the polygon defined by characteristic function (6.2). To prove that the resolvent $M'(q)$ on the contrary exists for q outside the polygon, we return to the hat-kernel $\widehat{M}_\Delta(x, x'; q)$ defined in (4.10) and study the boundedness properties of $M_\Delta(x,x';q) = e^{-q(x-x')}\widehat{M}_\Delta(x,x';q)$. Using (2.16), (5.4), (5.5), and (6.1), we write this function explicitly as

$$M_\Delta(x,x';q) = \frac{\pm e^{-q(x-x')}\theta\bigl(\pm(x_2-x_2')\bigr)}{\tau(x)\tau(x')} \times \\ \times \sum_{m,n=1}^{\mathcal{N}} f_m f_n \theta(q_1-\alpha_m)(\alpha_m-\alpha_n)e^{\mathcal{A}_m(x)+\mathcal{A}_n(x)}. \tag{6.10}$$

By (4.9), this function is identically equal to zero outside the strip $\alpha_1 < q_1 < \alpha_{N+1}$ on the q plane. The internal part of the strip is divided by the polygon. We can then prove that with the upper or lower sign for q respectively above or below the polygon in the strip, the kernel $M_\Delta(x,x';q)$ is a bounded function of x and q and defines the kernel of an extended operator $M_\Delta(q)$ according to the definition in Sec. 2. We conclude that $M_\Delta(q)$ for q outside the polygon satisfies Eqs. (4.6), and the resolvent $M'(q)$ therefore exists in this region and is given by (4.3).

We note that the total Green's function $G(x,x',\mathbf{k})$ of the operator $\mathcal{L}'$ can be defined (see [11], [12]) as the value of the kernel $M'(x,x';q)$ at $q_1 = \mathbf{k}_\Im$ and $q_2 = \mathbf{k}_\Im^2 - \mathbf{k}_\Re^2$ for a complex spectral parameter $\mathbf{k} = \mathbf{k}_\Re + i\mathbf{k}_\Im$. These values of q lie outside the parabola $q_2 = q_1^2$ and therefore outside the polygon and touch it only at the vertices. The Green's function then exists for any $\mathbf{k}$ but it is singular at the points $\mathbf{k} = i\alpha_m$ corresponding to the vertices of the polygon. These are the only singularities of the Green's function because the discontinuities of the first term in (4.3) at $q_1 = \alpha_m$ (see (4.5)) are compensated (outside the polygon) by the discontinuities of the second term, as follows from (4.10).

Finally, we note that the case where $N_a = 1$ and N_b is arbitrary can be handled analogously with the operator $L'(q)$ having a right instead of left annihilator for q inside the polygon and that the polygon in the case $N_a = N_b = 1$ reduces to the segment of the line $q_2 = (\alpha_1+\alpha_2)q_1 - \alpha_1\alpha_2$ with the endpoints $q_1 = \alpha_1$ and $q_1 = \alpha_2$. In this case, we recover the results obtained in [11].

Acknowledgments. One of the authors (A. K. P.) thanks the Department of Physics of the University of Salento (Lecce) for the kind hospitality.

This work is supported in part by the Russian Foundation for Basic Research (Grant No. 08-01-00501), the joint RFBR–CE (Grant No. 06-01-92057), the joint NWO–RFBR (Grant No. 047.011.2004.059), the Program for Supporting Leading Scientific Schools (Grant No. NSh-795.2008.1), the Russian Academy of Sciences program "Mathematical Methods of Nonlinear Dynamics," the INFN, and the Consortium E.I.N.S.T.E.I.N.

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