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THE EQUIVALENCE OF DIFFERENT APPROACHES FOR GENERATING MULTISOLITON SOLUTIONS OF THE KPII EQUATION

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The unexpectedly rich structure of the multisoliton solutions of the KPII equation has previously been explored using different approaches ranging from the dressing method to twisting transformations and the τ-function formulation. All these approaches proved useful for displaying different properties of these solutions and the corresponding Jost solutions. The aim of our investigation is to establish explicit formulas relating all these approaches. We discuss some hidden invariance properties of these multisoliton solutions.

Keywords: KPII equation, Bäcklund transformation, tau function, soliton

1. Introduction

The Kadomtsev–Petviashvili (KP) equation in its version called KPII,

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = -3u_{x_2x_2}, \tag{1.1}$$

where $u = u(x,t)$, $x = (x_1, x_2)$, and the subscripts x_1, x_2, and t denote partial derivatives, is a (2+1)-dimensional generalization of the celebrated Korteweg–de Vries (KdV) equation. There are two nonequivalent versions of the KP equations corresponding to the sign choice in the right-hand side of (1.1) and to the opposite choice (the KPI equation). The KP equations were derived as a model of small-amplitude, long-wavelength, weakly two-dimensional waves in a weakly dispersive medium [1]. Since the beginning of the 1970s, they are known [2], [3] to be integrable and can be considered prototypical (2+1)-dimensional integrable equations.

The KPII equation is integrable via its association with the operator

$$\mathcal{L}(x, \partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u(x), \tag{1.2}$$

which defines the well-known heat conduction equation or the heat equation, for short. The spectral theory of operator (1.2) in the case of a real potential $u(x)$ rapidly decaying at spatial infinity was developed in [4]–[7]. But this case is not the most interesting, because the KPII equation was proposed in [1] to deal with a two-dimensional weak transverse perturbation of the one-soliton solution of the KdV equation. In fact, KPII admits a one-soliton solution of the form

$$u(x,t) = -\frac{(a-b)^2}{2}\operatorname{sch}^2\left[\frac{a-b}{2}x_1 + \frac{a^2-b^2}{2}x_2 - 2(a^3-b^3)t\right], \tag{1.3}$$

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where a and b are arbitrary real constants.

In the literature, different methods for constructing multisoliton solutions of KPII are known: the Hirota method in [8], the dressing method in [9] and [10], Darboux transformations in [11], and the Wronskian technique in [12]. In addition, inelastic and resonant scattering of solitons was described in [13]–[17]; the existence of such scattering essentially distinguishes KPII from KPI. The problem of finding the most general N-soliton solution and their interactions has recently attracted much attention. In [18], a sort of dressing method was used to "superimpose" N solitons on an arbitrary smooth decaying background and to obtain the corresponding Jost solutions. There, the two-soliton case was studied in detail, showing that solitons can interact inelastically and that they can be created and annihilated. It was shown in a series of papers [19]–[23] using a finite-dimensional version of the Sato theory for the KP hierarchy [24] that the general N-soliton solution can be written in terms of τ-functions. A detailed study of these solutions showed that they exhibit nontrivial spatial interaction patterns, resonances, and web structures. These results were surveyed in [25], and applications to shallow water waves were given in [25], [26]. In [27], solutions corresponding to N solitons "superimposed" on an arbitrary smooth decaying background and the corresponding Jost solutions were constructed using twisting transformations.

As regards a spectral theory of heat operator (1.2), the inverse scattering transform for a perturbed one-soliton potential was derived in [28]. In [29], the initial value problem for the KPII equation with data not decaying along a line was linearized with some additional constraints on the spectral data. But a theory also including N solitons must still be constructed. The extended resolvent approach was introduced in [30] in solving an analogous problem for the nonstationary Schrödinger operator associated with the KPI equation. Accordingly, to solve the spectral problem for the heat operator when the potential $u(x)$ describes N solitons, among different procedures used to derive these potentials, we must find just that one that can be used to build the corresponding extended resolvent. Therefore, our goal here is to explore these different procedures and their interrelations.

The paper is organized as follows. In Sec. 2, we formulate the basic concepts and sketch some basic topics of the standard scattering problem for operator (1.2) in the case of a decaying potential. In Sec. 3, we review the multisoliton potentials obtained in [18], solving a $\bar{\partial}$-problem given by a rational transformation of generic spectral data. In Sec. 4, we consider the multisoliton potentials obtained in [27] using twisting transformations, and we study their connection with the potentials given in Sec. 3. In Sec. 5, we derive some alternative, more symmetric representations of the purely solitonic potentials, and we show that they coincide with the representation obtained in [19]–[23] using the τ-function approach. In Sec. 6, we consider invariance properties of the multisoliton potentials under transformations of the soliton parameters. We also derive symmetric representations for the Jost solutions, showing that these solutions, similarly to the Baker–Akhiezer solutions, can be obtained by a Miwa shift [31] (also see [25]).

A preliminary version of this paper containing all the main results was published in the electronic archive [32]. Soon after that, the e-print [33] appeared, where some quite close results concerning the potential were presented and the classification of two-soliton potentials given in [18] was also reproduced.

2. Elements of scattering theory

The KPII equation can be expressed as the compatibility condition $[\mathcal{L},\mathcal{T}]=0$ of a Lax pair [2], [3], where $\mathcal{L}$ is the heat operator defined in (1.2) and $\mathcal{T}$ is given by

$$\mathcal{T}(x,\partial_x,\partial_t)=\partial_t+4\partial_{x_1}^3-6u\partial_{x_1}-3u_{x_1}-3\partial_{x_1}^{-1}u_{x_2}. \tag{2.1}$$

Because the heat operator is not self-dual, we should simultaneously consider its dual $\mathcal{L}^{\mathrm{d}}(x,\partial_x)=\partial_{x_2}+\partial_{x_1}^2-u(x)$ and introduce the Jost solution $\Phi(x,\mathbf{k})$ and the dual Jost solution $\Psi(x,\mathbf{k})$ satisfying the equations

$$\mathcal{L}(x,\partial_x)\Phi(x,\mathbf{k})=0,\qquad \mathcal{L}^{\mathrm{d}}(x,\partial_x)\Psi(x,\mathbf{k})=0, \tag{2.2}$$

where $\mathbf{k}$ is an arbitrary complex variable playing the role of a spectral parameter.

To normalize these Jost solutions, we introduce the two functions

$$\chi(x,\mathbf{k}) = e^{i\mathbf{k}x_1+\mathbf{k}^2x_2}\Phi(x,\mathbf{k}), \qquad \xi(x,\mathbf{k}) = e^{-i\mathbf{k}x_1-\mathbf{k}^2x_2}\Psi(x,\mathbf{k}) \tag{2.3}$$

satisfying the differential equations

$$\big(-\partial_{x_2} + \partial_{x_1}^2 - 2i\mathbf{k}\partial_{x_1} - u(x)\big)\chi(x,\mathbf{k}) = 0, \tag{2.4a}$$

$$\big(\partial_{x_2} + \partial_{x_1}^2 + 2i\mathbf{k}\partial_{x_1} - u(x)\big)\xi(x,\mathbf{k}) = 0. \tag{2.4b}$$

The normalization conditions at $\mathbf{k}$-infinity are then formulated as

$$\lim_{\mathbf{k}\to\infty}\chi(x,\mathbf{k}) = 1, \qquad \lim_{\mathbf{k}\to\infty}\xi(x,\mathbf{k}) = 1, \tag{2.5}$$

and these functions by (2.4) are hence related to the potential $u(x)$ of the heat equation by

$$u(x) = -2i\lim_{\mathbf{k}\to\infty}\mathbf{k}\partial_{x_1}\chi(x,\mathbf{k}) = 2i\lim_{\mathbf{k}\to\infty}\mathbf{k}\partial_{x_1}\xi(x,\mathbf{k}). \tag{2.6}$$

The realness of the potential $u(x)$, which we always assume here, is equivalent to the conjugation properties

$$\overline{\chi(x,\mathbf{k})} = \chi(x,-\bar{\mathbf{k}}), \qquad \overline{\xi(x,\mathbf{k})} = \xi(x,-\bar{\mathbf{k}}). \tag{2.7}$$

In the case of a potential $u(x) \equiv u_0(x)$ rapidly decaying at spatial infinity, according to [4]–[7], the main tools for building the spectral theory of operator (1.2), as in the one-dimensional case, are the integral equations whose solutions define the associated normalized Jost solutions $\chi_0(x,\mathbf{k})$ and $\xi_0(x,\mathbf{k})$, i.e.,

$$\chi_0(x,\mathbf{k}) = 1 + \int dx'\, G_0(x-x',\mathbf{k})u_0(x')\chi_0(x',\mathbf{k}), \tag{2.8a}$$

$$\xi_0(x,\mathbf{k}) = 1 + \int dx'\, G_0(x'-x,\mathbf{k})u_0(x')\xi_0(x',\mathbf{k}), \tag{2.8b}$$

where

$$G_0(x,\mathbf{k}) = -\frac{\operatorname{sgn} x_2}{2\pi}\int d\alpha\,\theta\big(\alpha(\alpha-2\mathrm{Re}\mathbf{k})x_2\big)e^{i\alpha x_1-\alpha(\alpha-2\mathbf{k})x_2} \tag{2.9}$$

is the Green's function of the bare heat operator, $(-\partial_{x_2}+\partial_{x_1}^2)G_0(x,\mathbf{k}) = \delta(x)$.

By (2.8), the functions χ_0 and ξ_0 have the asymptotic values on the x plane

$$\lim_{x\to\infty}\chi_0(x,\mathbf{k}) = 1, \qquad \lim_{x\to\infty}\xi_0(x,\mathbf{k}) = 1. \tag{2.10}$$

But if the potential $u(x)$ does not decay at spatial infinity, as is the case with soliton potentials, then integral equations (2.8) are ill defined, and we need to find a more general approach. A spectral theory of the KPII equation that also includes soliton solutions has been investigated using the resolvent approach. In that framework, it was possible to develop the inverse scattering transform for a solution describing one soliton on an arbitrary background [28] and to study the existence of the (extended) resolvent for some multisoliton solutions [27]. The general theory, to a certain extent, is still a work in progress. But we here only consider different approaches for constructing soliton potentials and the corresponding Jost solutions.

3. Multisoliton potentials in the dressing method

In [18], we used a sort of dressing method to construct a potential $u(x)$ describing N solitons superimposed on a general background potential $u_0(x)$. More precisely, we considered a potential $u(x)$ with spectral data obtained by a rational transformation of the spectral data of a generic background $u_0(x)$ (smooth, decaying at infinity), and we transformed the problem of finding this $u(x)$ into a $\bar{\partial}$-problem on the corresponding Jost solutions, which was solved by a dressing procedure. The rational transformation was chosen to depend on N pairs of distinct real parameters a_j and b_j, $j = 1, \dots, N$, and the Jost solutions $\Phi(x, \mathbf{k})$ and $\Psi(x, \mathbf{k})$ were required to have suitable analyticity properties, normalization like in (2.3) and (2.5), and conjugation property (2.7).

The transformed potential $u(x)$ is given by the formula

$$u(x) = u_0(x) - 2\partial_{x_1}^2 \det\bigl[C + \mathcal{F}(x)\bigr], \tag{3.1}$$

where C is a real $N\times N$ constant matrix and $\mathcal{F}(x)$ is an $N\times N$ matrix function of x with the elements

$$\mathcal{F}_{lj}(x) = \int_{\substack{(b_1-a_1)\infty<y_1<x_1,\\ y_2=x_2}} dy_1\, \Psi_0(y, ib_l)\Phi_0(y, ia_j), \quad l, j = 1, \dots, N, \tag{3.2}$$

where $\Phi_0(x, \mathbf{k})$ and $\Psi_0(x, \mathbf{k})$ are the Jost and dual Jost solutions of Eqs. (2.2) with the potential $u_0(x)$. We note that the matrix elements of $\mathcal{F}(x)$ are given in terms of values of the so-called Cauchy–Baker–Akhiezer function [34] at the points a_j and b_l, $j, l = 1, \dots, N$. We showed in [18] that the matrix C need not be regular or diagonal. We also mentioned that to obtain a real potential, it suffices to consider a more general situation with complex parameters a_j and b_j such that

$$a_j = \bar{a}_{\pi_a(j)}, \qquad b_j = \bar{b}_{\pi_b(j)}, \quad j = 1, \dots, N, \tag{3.3}$$

where the bar denotes the complex conjugate and π_a and π_b are some permutations of the indices (with a proper modification of the constraints on the constant matrix C). But this case is essentially more complicated to investigate, and we do not consider it here.

If the background u_0 is taken a posteriori to be identically zero, then the eigenfunctions in (3.2) are pure exponentials, i.e.,

$$\Phi_0(x, ia_j) = e^{A_j(x)}, \qquad \Psi_0(x, ib_l) = e^{-B_l(x)},$$

where we introduce

$$A_j(x) = a_j x_1 + a_j^2 x_2, \qquad B_l(x) = b_l x_1 + b_l^2 x_2, \quad j, l = 1, 2, \dots, N, \tag{3.4}$$

for future convenience. Then (3.2) becomes

$$\mathcal{F}_{lj}(x) = e^{-B_l(x)}\Lambda_{lj}e^{A_j(x)}, \tag{3.5}$$

where Λ is the Cauchy matrix,

$$\Lambda_{lj} = \frac{1}{a_j - b_l}. \tag{3.6}$$

We note that the time dependence was not specified in [18], but this dependence simply amounts to taking the time dependence of the original Jost solutions $\Phi_0(x, \mathbf{k})$ and $\Psi_0(x, \mathbf{k})$ fixed by the choice of second Lax operator (2.1) into account. In the case of $u_0(x, t) \equiv 0$, this means that we must use

$$A_j(x, t) = a_j x_1 + a_j^2 x_2 - 4a_j^3 t, \qquad B_l(x, t) = b_l x_1 + b_l^2 x_2 - 4b_l^3 t \tag{3.7}$$

instead of (3.4).

In [18], we addressed the problem of the regularity of the potentials and also their asymptotic behavior only in the case $N = 2$, i.e., for two-soliton potentials. In that particular case, we could formulate the conditions on the 2×2 matrix C guaranteeing that the potential $u(x)$ is regular, and we showed that at large distances in the x plane, the potential decays exponentially except along certain specific rays, where it has a one-dimensional solitonic behavior of form (1.3). The classification of the two-soliton potentials obtained in [18] was subsequently generalized to the case of N solitons in [23], [25].

4. Multisoliton potentials in the method of twisting transformations

4.1. Equivalence of the N-soliton potentials derived in [18] and in [27]. In this section, we consider pure soliton potentials of the heat equation obtained by twisting transformations (see [27] for the details of the construction). The transformation of a general background potential $u_0(x)$ (smooth, decaying at infinity) into a new potential $u(x)$ describing N solitons "superimposed" on the background $u_0(x)$ is parameterized by two sets of real parameters $\{a_1,\dots,a_{N_a}\}$ and $\{b_1,\dots,b_{N_b}\}$, which we assume all distinct, and an $N_a\times N_b$ real matrix c. Here N_a and N_b are not necessarily equal, but

$$N_a, N_b \geq 1, \tag{4.1}$$

and we set $N = \max\{N_a, N_b\}$. The expression for the pure N-soliton potential follows directly from the expressions derived in [27] by taking $u_0(x) \equiv 0$. Precisely, we have

$$u(x) = -2\partial_{x_1}^2 \log \tau_1(x), \tag{4.2}$$

where

$$\tau_1(x) = \det\bigl(E_{N_a} + c\mathcal{F}(x)\bigr) \equiv \det\bigl(E_{N_b} + \mathcal{F}(x)c\bigr) \tag{4.3}$$

and we introduce the $N_a\times N_a$ and $N_b\times N_b$ identity matrices E_{N_a} and E_{N_b}, the diagonal matrices

$$e^{A(x)} = \operatorname{diag}\bigl\{e^{A_j(x)}, j = 1,\dots,N_a\bigr\}, \qquad e^{-B(x)} = \operatorname{diag}\bigl\{e^{-B_l(x)}, l = 1,\dots,N_b\bigr\} \tag{4.4}$$

(see (3.4)), and the $N_b\times N_a$ matrix function $\mathcal{F}(x) = \bigl\|\mathcal{F}_{lj}(x)\bigr\|_{l=1,\dots,N_b}^{j=1,\dots,N_a}$ (cf. (3.5)), and hence

$$\mathcal{F}(x) = e^{-B(x)}\Lambda e^{A(x)}, \tag{4.5}$$

where Λ is a constant $N_b\times N_a$ matrix with elements given in (3.6).

We now show that the potentials given in (3.1) in terms of the $N\times N$ matrices C and $\mathcal{F}$ coincide with those obtained by the twisting transformations and expressed in terms of the matrices c and $\mathcal{F}$ in (4.3). This is obvious when $N_a = N_b$ and matrices C and c are nonsingular. It then suffices to set $c = C^{-1}$, and the determinants in (3.1) and (4.3) are then equal up to an inessential constant factor.

The general situation is a somewhat more complicated. To make the size of the matrices involved explicit, we here use the notation A_N for an $N\times N$ matrix A. For definiteness, we consider the case $N_a \leq N_b \equiv N$. Let c_N denote the $N\times N$ matrix constructed by adding $N_b - N_a$ zero rows to the matrix c. Taking into account that the product $\mathcal{F}c$ in the second equality in (4.3) is an $N\times N$ matrix, we see that $(\mathcal{F})_N c_N = \mathcal{F}c$, where

$$\mathcal{F}_N = (e^{-B})_N \Lambda_N (e^{A})_N \tag{4.6}$$

and where the parameters a_j, $j = N_a+1, N_a+2,\dots,N$ in $(e^A)_N$ and in Λ_N can be chosen arbitrarily under the condition that they differ from the parameters b_l. It is well known that the Cauchy matrix

Λ_N is invertible and that $\Lambda_N^{-1} = G_N \tilde{\Lambda}_N F_N$, where the matrix $\tilde{\Lambda}_N$ is obtained from the matrix Λ_N by interchanging a_j and b_j and where G_N and F_N are two invertible diagonal matrices. Therefore, for the second determinant in (4.3), we have

$$\det(E_{N_b} + \mathcal{F}c) = \det(E_N + \mathcal{F}_N c_N) = \\ = \det(e^{-B}\Lambda e^{A} GF)_N \det\bigl((e^{-A})_N \tilde{\Lambda}_N (e^{B})_N + G_N^{-1} c_N F_N^{-1}\bigr).$$

The factor $\det(e^{-B}\Lambda e^{A} GF)_N$ does not contribute to the potential, because of the derivative in (4.2). Moreover, by (3.5) and (4.6), the $N\times N$ matrix $e^{-A(x)}\tilde{\Lambda} e^{B(x)}$ coincides with the $N\times N$ matrix $\mathcal{F}(x)$ in (3.5) after the parameters a_j and b_j are interchanged. Therefore, relations (4.3) and (4.2) indeed generate the same potential if

$$c_N = G_N C F_N \tag{4.7}$$

and the parameters a_j and b_j are interchanged.

Remark 4.1. In the case of a nonzero background potential $u_0(x)$, we can prove that the two potentials obtained by the two approaches are equivalent up to an additional term $\partial_{x_1}^2 \log\det \mathcal{F}_N(x)$.

Remark 4.2. Here and hereafter, we omit the time dependence because it can be easily switched on by using (3.7) instead of (3.4).

4.2. Regularity conditions for the potential. In view of the discussion about the necessary and sufficient regularity conditions for the multisoliton solutions of the equations in the hierarchy of the KPII equation, in an equivalent formulation, we here recover sufficient conditions for the regularity of multisoliton potentials of the heat equation and then of multisoliton solutions of KPII, already given in [20], [25].

We again consider the second equality in (4.3). We have

$$\det(E_{N_b} + \mathcal{F}c) = \sum_{n=0}^{N_b} \sum_{1\le l_1<l_2<\cdots<l_n\le N_b} (\mathcal{F}c)\begin{pmatrix} l_1, l_2, \dots, l_n \\ l_1, l_2, \dots, l_n \end{pmatrix}. \tag{4.8}$$

Here and hereafter, $A\bigl(\begin{smallmatrix} l_1, l_2, \dots, l_n \\ l_1, l_2, \dots, l_n \end{smallmatrix}\bigr)$ denotes the minor of the matrix A obtained by selecting rows $l_1, l_2, \dots, l_n$ and columns $j_1, j_2, \dots, j_n$. The principal minor of the product $\mathcal{F}c$ in (4.8) can be written by the Binet–Cauchy formula as

$$(\mathcal{F}c)\begin{pmatrix} l_1, l_2, \dots, l_n \\ l_1, l_2, \dots, l_n \end{pmatrix} = \sum_{1\le j_1<j_2<\cdots<j_n\le N_a} \Biggl(\prod_{m=1}^{n} e^{-B_{l_m}(x)}\Biggr)\Biggl(\prod_{m=1}^{n} e^{A_{j_m}(x)}\Biggr)\times \\ \times \Lambda \begin{pmatrix} l_1, l_2, \dots, l_n \\ j_1, j_2, \dots, j_n \end{pmatrix} c \begin{pmatrix} j_1, j_2, \dots, j_n \\ l_1, l_2, \dots, l_n \end{pmatrix}. \tag{4.9}$$

Taking into account that $\mathcal{F}$ is an $N_b\times N_a$ matrix and c is an $N_a\times N_b$ matrix, we find that all minors of $\mathcal{F}c$ with $n > \min(N_a, N_b)$ are zero.

Recalling that the term corresponding to $n = 0$ in (4.8) is 1, i.e., greater than zero, we deduce that the sufficient condition for the regularity of the soliton potential is given by the characterization conditions on the real matrix c

$$\Lambda \begin{pmatrix} l_1, l_2, \dots, l_n \\ j_1, j_2, \dots, j_n \end{pmatrix} c \begin{pmatrix} j_1, j_2, \dots, j_n \\ l_1, l_2, \dots, l_n \end{pmatrix} \ge 0, \tag{4.10}$$

for any $1 \leq n \leq \min(N_a, N_b)$ and all minors, i.e., for any choice of $1 \leq j_1 < j_2 < \cdots < j_n \leq N_a$ and $1 \leq l_1 < l_2 < \cdots < l_n \leq N_b$. To obtain a nontrivial potential under the substitution of τ_1 in (4.2), we must then impose the condition that at least one inequality in (4.10) is strict.

Next, because any submatrix of a Cauchy matrix Λ is itself a Cauchy matrix, for an arbitrary minor of Λ, we have

$$\Lambda\begin{pmatrix} l_1, l_2, \ldots, l_p \\ j_1, j_2, \ldots, j_p \end{pmatrix} = \frac{\prod_{1\leq i<m\leq p}(a_{j_i} - a_{j_m})(b_{l_m} - b_{l_i})}{\prod_{i,m=1}^{p}(a_{j_i} - b_{j_m})}. \tag{4.11}$$

We then deduce that there are orders of the a and b parameters for which all minors of Λ are positive. For instance, if

$$a_1 < a_2 < \cdots < a_{N_a-1} < a_{N_a} < b_1 < b_2 < \cdots < b_{N_b-1} < b_{N_b}, \tag{4.12}$$

then regularity conditions (4.10) become

$$c\begin{pmatrix} l_1, l_2, \ldots, l_n \\ j_1, j_2, \ldots, j_n \end{pmatrix} \geq 0, \tag{4.13}$$

i.e., all minors of c must be nonnegative. Such matrices are called totally nonnegative matrices [35]. Obviously, we obtain the same result by using the first equality in (4.3), and the Binet–Cauchy formula allows directly verifying that the two determinants in (4.3) are equal.

5. Equivalence to the τ-function representation

5.1. Determinant representations for the potential and Jost solutions. In [27], we derived the transformed potential $u(x)$ (see (4.2) and (4.3)) and the Jost solutions of direct and dual problems (2.2) related to this potential. The corresponding Jost solutions normalized by (2.3) are expressed as

$$\chi(x,\mathbf{k}) = 1 + \sum_{j=1}^{N_a}\sum_{l,l'=1}^{N_b} e^{A_j(x)} c_{jl'} \big(E_{N_b} + \mathcal{F}(x)c\big)^{-1}_{l'l} \frac{e^{-B_l(x)}}{b_l + i\mathbf{k}} \equiv$$

$$\equiv 1 + \sum_{j,j'=1}^{N_a}\sum_{l=1}^{N_b} e^{A_j(x)} \big(E_{N_a} + c\mathcal{F}(x)\big)^{-1}_{jj'} \frac{c_{j'l}e^{-B_l(x)}}{b_l + i\mathbf{k}}, \tag{5.1a}$$

$$\xi(x,\mathbf{k}) = 1 - i\sum_{j=1}^{N_a}\sum_{l,l'=1}^{N_b} \frac{e^{A_j(x)}}{a_j + i\mathbf{k}} c_{jl} \big(E_{N_b} + \mathcal{F}(x)c\big)^{-1}_{ll'} e^{-B_l(x)} \equiv$$

$$\equiv 1 - \sum_{j,j'=1}^{N_a}\sum_{l=1}^{N_b} \frac{e^{A_j(x)}}{a_j + i\mathbf{k}} \big(E_{N_a} + c\mathcal{F}(x)\big)^{-1}_{jj'} c_{j'l} e^{-B_l(x)}. \tag{5.1b}$$

By analogy with (4.3), all relations are given in two equivalent forms. Although keeping only one form suffices, we continue to consider both to highlight the symmetry of the whole construction with respect to N_a and N_b, which play the role of topological charges.

To obtain a representation of (4.3) and (5.1) in terms of τ-functions, we must perform some simple algebraic operations. First, by the standard identity for the determinant of a bordered matrix, we can rewrite (5.1) as

$$\chi(x,\mathbf{k}) = \frac{\tau_{1,\chi}(x,\mathbf{k})}{\tau_1(x)}, \qquad \xi(x,\mathbf{k}) = \frac{\tau_{1,\xi}(x,\mathbf{k})}{\tau_1(x)}, \tag{5.2}$$

where $\tau_1(x)$ is given in (4.3) and

$$\tau_{1,\chi}(x,\mathbf{k}) = \det\begin{pmatrix} E_{N_b} + \mathcal{F}(x)c & \dfrac{-e^{-B_*(x)}}{b_* + i\mathbf{k}} \\ \sum_{j=1}^{N_a} e^{A_j(x)}c_{j*} & 1 \end{pmatrix} \equiv \tag{5.3a}$$

$$\equiv \det\begin{pmatrix} E_{N_a} + c\mathcal{F}(x) & -\sum_{l=1}^{N_b} \dfrac{c_{*l}e^{-B_l(x)}}{b_l + i\mathbf{k}} \\ e^{A_*(x)} & 1 \end{pmatrix}, \tag{5.3b}$$

$$\tau_{1,\xi}(x,\mathbf{k}) = \det\begin{pmatrix} E_{N_b} + \mathcal{F}(x)c & e^{-B_*(x)} \\ \sum_{j=1}^{N_a} \dfrac{e^{A_j(x)}}{a_j + i\mathbf{k}}c_{j*} & 1 \end{pmatrix} \equiv \tag{5.3c}$$

$$\equiv \det\begin{pmatrix} E_{N_a} + c\mathcal{F}(x) & \sum_{l=1}^{N_b} c_{*l}e^{-B_l(x)} \\ \dfrac{e^{A_*(x)}}{a_* + i\mathbf{k}} & 1 \end{pmatrix}. \tag{5.3d}$$

In the above formulas, the subscript $*$ in the row and column that border the matrices $E_{N_b} + \mathcal{F}c$ and $E_{N_a} + c\mathcal{F}$ denotes an index respectively ranging from 1 to N_b and from 1 to N_a. We note that the function τ_1 in (4.3) can also be obtained as a limit value, i.e.,

$$\tau_1(x) = \lim_{\mathbf{k}\to\infty} \tau_{1,\chi}(x,\mathbf{k}) = \lim_{\mathbf{k}\to\infty} \tau_{1,\xi}(x,\mathbf{k}). \tag{5.4}$$

Using (2.3) to return from (5.2) and (5.3) to the Jost solutions $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$, we can verify that they have respective poles at $\mathbf{k} = ib_l$, $l = 1,\dots,N_b$, and $\mathbf{k} = ia_j$, $j = 1,\dots,N_a$ with the residues

$$\Phi_{b_l}(x) = \operatorname{res}_{\mathbf{k}=ib_l} \Phi(x,\mathbf{k}), \qquad \Psi_{a_j}(x) = \operatorname{res}_{\mathbf{k}=ia_j} \Psi(x,\mathbf{k}), \tag{5.5}$$

which satisfy the relations

$$\Phi_{b_l}(x) = -i\sum_{j=1}^{N_a} \Phi(x, ia_j)c_{jl}, \qquad \Psi_{a_j}(x) = i\sum_{l=1}^{N_b} c_{jl}\Psi(x, ib_l). \tag{5.6}$$

We note that these equations, together with the analyticity requirement and normalization condition (2.10), allow reconstructing normalized Jost solutions (5.2) and (5.3) and then, via (2.6), the potential $u(x)$ in form (4.2), (4.3).

We next rewrite (5.3) as

$$\tau_{1,\chi}(x,\mathbf{k}) = \left(\prod_{l=1}^{N_b} \frac{e^{-B_l(x)}}{b_l + i\mathbf{k}}\right) \det\begin{pmatrix} (b_* + i\mathbf{k})\big(e^{B(x)} + \Lambda e^{A(x)}c\big) & -1_* \\ \sum_{j=1}^{N_a} e^{A_j(x)}c_{j*} & 1 \end{pmatrix} \equiv$$

$$\equiv \left(\prod_{j=1}^{N_a} e^{A_j(x)}\right) \det\begin{pmatrix} e^{-A(x)} + ce^{-B(x)}\Lambda & -\sum_{l=1}^{N_b} \dfrac{c_{*l}e^{-B_l(x)}}{b_l + i\mathbf{k}} \\ 1_* & 1 \end{pmatrix},$$

$$\tau_{1,\xi}(x,\mathbf{k}) = \left(\prod_{l=1}^{N_b} e^{-B_l(x)}\right) \det\begin{pmatrix} e^{B(x)} + \Lambda e^{A(x)}c & 1_* \\ \sum_{j=1}^{N_a} \dfrac{e^{A_j(x)}}{a_j + i\mathbf{k}}c_{j*} & 1 \end{pmatrix} \equiv$$

$$\equiv \left(\prod_{j=1}^{N_a} \frac{e^{A_j(x)}}{a_j + i\mathbf{k}}\right) \det\begin{pmatrix} \big(e^{-A(x)} + ce^{-B(x)}\Lambda\big)(a_* + i\mathbf{k}) & \sum_{l=1}^{N_b} c_{*l}e^{-B_l(x)} \\ 1_* & 1 \end{pmatrix}.$$

By elementary transformations of the matrices in the right-hand sides, we can reduce them to a form where rows and columns with all elements equal to1 or -1 are transformed into rows and columns with all elements equal to 0 except 1 in the last place.

To present the result of these transformations, we introduce

$$\tau_{2,\chi}(x,\mathbf{k}) = \left(\prod_{l=1}^{N_b} e^{B_l(x)}(b_l + i\mathbf{k})\right)\tau_{1,\chi}(x,\mathbf{k}), \tag{5.7a}$$

$$\tau'_{2,\chi}(x,\mathbf{k}) = \left(\prod_{j=1}^{N_a} \frac{e^{-A_j(x)}}{a_j + i\mathbf{k}}\right)\tau_{1,\chi}(x,\mathbf{k}), \tag{5.7b}$$

$$\tau_{2,\xi}(x,\mathbf{k}) = \left(\prod_{l=1}^{N_b} \frac{e^{B_l(x)}}{b_l + i\mathbf{k}}\right)\tau_{1,\xi}(x,\mathbf{k}), \tag{5.7c}$$

$$\tau'_{2,\xi}(x,\mathbf{k}) = \left(\prod_{l=1}^{N_a} e^{-A_j(x)}(a_j + i\mathbf{k})\right)\tau_{1,\xi}(x,\mathbf{k}). \tag{5.7d}$$

Then

$$\tau_{2,\chi}(x,\mathbf{k}) = \det\left(\delta_{ll'}(b_l + i\mathbf{k})e^{B_l(x)} + \sum_{j=1}^{N_a} \frac{a_j + i\mathbf{k}}{a_j - b_l} e^{A_j(x)} c_{jl'}\right)_{l,l'=1}^{N_b}, \tag{5.8a}$$

$$\tau'_{2,\chi}(x,\mathbf{k}) = \det\left(\delta_{jj'}\frac{e^{-A_j(x)}}{a_j + i\mathbf{k}} + \sum_{l=1}^{N_b} \frac{c_{jl}e^{-B_l(x)}}{(a_{j'} - b_l)(b_l + i\mathbf{k})}\right)_{j,j'=1}^{N_a}, \tag{5.8b}$$

$$\tau_{2,\xi}(x,\mathbf{k}) = \det\left(\delta_{ll'}\frac{e^{B_l(x)}}{b_l + i\mathbf{k}} + \sum_{j=1}^{N_a} \frac{e^{A_j(x)}c_{jl'}}{(a_j - b_l)(a_j + i\mathbf{k})}\right)_{l,l'=1}^{N_b}, \tag{5.8c}$$

$$\tau'_{2,\xi}(x,\mathbf{k}) = \det\left(\delta_{jj'}(a_j + i\mathbf{k})e^{-A_j(x)} + \sum_{l=1}^{N_b} \frac{c_{jl}(b_l + i\mathbf{k})}{a_{j'} - b_l} e^{-B_l(x)}\right)_{j,j'=1}^{N_a}. \tag{5.8d}$$

It is easy to see that the limit values

$$\tau_2(x) = \lim_{\mathbf{k}\to\infty} (i\mathbf{k})^{-N_b}\tau_{2,\chi}(x,\mathbf{k}) = \lim_{\mathbf{k}\to\infty} (i\mathbf{k})^{N_b}\tau_{2,\xi}(x,\mathbf{k}),$$

$$\tau'_2(x) = \lim_{\mathbf{k}\to\infty} (i\mathbf{k})^{N_a}\tau'_{2,\chi}(x,\mathbf{k}) = \lim_{\mathbf{k}\to\infty} (i\mathbf{k})^{-N_a}\tau'_{2,\xi}(x,\mathbf{k})$$

of both the τ-functions and τ'-functions with the indices χ and ξ coincide and, by (5.8), are equal to

$$\tau_2(x) = \det\left(\delta_{ll'}e^{B_l(x)} + \sum_{j=1}^{N_a} \frac{e^{A_j(x)}c_{jl'}}{a_j - b_l}\right)_{l,l'=1}^{N_b}, \tag{5.9a}$$

$$\tau'_2(x) = \det\left(\delta_{jj'}e^{-A_j(x)} + \sum_{l=1}^{N_b} \frac{c_{jl}e^{-B_l(x)}}{a_{j'} - b_l}\right)_{j,j'=1}^{N_a}. \tag{5.9b}$$

By (5.4),

$$\tau_2(x) = \left(\prod_{l=1}^{N_b} e^{B_l(x)}\right)\tau_1(x), \qquad \tau'_2(x) = \left(\prod_{j=1}^{N_a} e^{-A_j(x)}\right)\tau_1(x), \tag{5.10}$$

and these expressions are equivalent in the sense that they generate the same potential

$$u(x) = -2\partial_{x_1}^2 \log \tau_2(x) = -2\partial_{x_1}^2 \log \tau_2'(x). \tag{5.11}$$

For the functions $\chi(x,\mathbf{k})$ and $\xi(x,\mathbf{k})$ (see (2.3)), using (5.2) and (5.7), we obtain

$$\chi(x,\mathbf{k}) = \left(\prod_{l=1}^{N_b}(b_l+i\mathbf{k})^{-1}\right)\frac{\tau_{2,\chi}(x,\mathbf{k})}{\tau_2(x)} \equiv \left(\prod_{j=1}^{N_a}(a_j+i\mathbf{k})\right)\frac{\tau_{2,\chi}'(x,\mathbf{k})}{\tau_2'(x)}, \tag{5.12a}$$

$$\xi(x,\mathbf{k}) = \left(\prod_{l=1}^{N_b}(b_l+i\mathbf{k})\right)\frac{\tau_{2,\xi}(x,\mathbf{k})}{\tau_2(x)} \equiv \left(\prod_{j=1}^{N_a}(a_j+i\mathbf{k})^{-1}\right)\frac{\tau_{2,\xi}'(x,\mathbf{k})}{\tau_2'(x)}. \tag{5.12b}$$

We note that the functions in (5.8) can be obtained from the functions in (5.9) by substituting

$$\tau_2(x) \to \tau_{2,\chi}(x,\mathbf{k}) \quad \text{for } e^{A_j} \to e^{A_j}(a_j+i\mathbf{k}), \quad e^{B_l} \to e^{B_l}(b_l+i\mathbf{k}), \tag{5.13a}$$

$$\tau_2'(x) \to \tau_{2,\chi}'(x,\mathbf{k}) \quad \text{for } e^{A_j} \to e^{A_j}(a_j+i\mathbf{k}), \quad e^{B_l} \to e^{B_l}(b_l+i\mathbf{k}), \tag{5.13b}$$

$$\tau_2(x) \to \tau_{2,\xi}(x,\mathbf{k}) \quad \text{for } e^{A_j} \to \frac{e^{A_j}}{a_j+i\mathbf{k}}, \qquad e^{B_l} \to \frac{e^{B_l}}{b_l+i\mathbf{k}}, \tag{5.13c}$$

$$\tau_2'(x) \to \tau_{2,\xi}'(x,\mathbf{k}) \quad \text{for } e^{A_j} \to \frac{e^{A_j}}{a_j+i\mathbf{k}}, \qquad e^{B_l} \to \frac{e^{B_l}}{b_l+i\mathbf{k}}. \tag{5.13d}$$

These rules allow significantly shortening the list of formulas given below (also see Remark 6.2).

Remark 5.1. By (5.8), (5.9), and (5.12), the function χ is transformed into ξ and vice versa under the transformation $x \to -x$, $N_a \longleftrightarrow N_b$, $\{a_1,\dots,a_{N_a}\} \to \{b_1,\dots,b_{N_a}\}$, $\{b_1,\dots,b_{N_b}\} \to \{a_1,\dots,a_{N_b}\}$, and $c_{jl} \to -c_{lj}$.

5.2. Symmetric representations for the potential and comparison with the τ-function approach. Here, we prove that expression (5.11) for the potential is equivalent to the one obtained using the τ-function approach in the series of papers cited in the introduction and surveyed in [25].

We already mentioned that the double representations for the potential and the Jost solutions derived above highlight the symmetric role played by the parameters a_j and b_l. To demonstrate this fact explicitly, we introduce N_a+N_b (real) parameters

$$\{\kappa_1,\dots,\kappa_{N_a+N_b}\} = \{a_1,\dots,a_{N_a},b_1\dots,b_{N_b}\} \tag{5.14}$$

and by analogy with (3.4) set

$$K_n(x) = \kappa_n x_1 + \kappa_n^2 x_2, \quad n=1,\dots,N_a+N_b, \tag{5.15}$$

and hence $K_n(x) = A_n(x)$, $n=1,\dots,N_a$, and $K_n(x) = B_{n-N_a}(x)$, $n=N_a+1,\dots,N_a+N_b$.

According to (3.7), the time dependence is taken into account simply by adding a term $-4\kappa_n^3 t$ to the right-hand side of $K_n(x)$ in (5.15).

Let d denote an $N_a{\times}N_b$ real matrix with elements given in terms of the elements of the matrix c as

$$d_{jl} = \prod_{l'=1}^{N_b}(a_j-b_{l'})^{-1}c_{jl}\prod_{\substack{1\le l'\le N_b,\\ l'\ne l}}(b_l-b_{l'}), \quad j=1,\dots,N_a, \quad l=1,\dots,N_b. \tag{5.16}$$

We also introduce the constant $(N_a{+}N_b){\times}N_b$ and $N_a{\times}(N_a{+}N_b)$ matrices $\mathcal{D}$ and $\mathcal{D}'$ with the block structures

$$\mathcal{D} = \begin{pmatrix} d \\ E_{N_b} \end{pmatrix}, \qquad \mathcal{D}' = (E_{N_a}, -d), \tag{5.17}$$

and the constant diagonal real $(N_a{+}N_b){\times}(N_a{+}N_b)$ matrix

$$\gamma = \operatorname{diag}\Bigg\{ \prod_{n'=1,n'\neq n}^{N_a+N_b} (\kappa_n - \kappa_{n'})^{-1},\ n = 1,\ldots,N_a+N_b \Bigg\}. \tag{5.18}$$

We prove now that with one more rescaling of $\tau_2(x)$ and $\tau_2'(x)$,

$$\tau(x) = \Bigg(\prod_{1\le l<l'\le N_b} (b_{l'} - b_l) \Bigg) \tau_2(x), \tag{5.19a}$$

$$\tau'(x) = \Bigg(\prod_{1\le j<j'\le N_a} (a_j - a_{j'})^{-1} \Bigg) \Bigg(\prod_{j=1}^{N_a} \prod_{l=1}^{N_b} (a_j - b_l)^{-1} \Bigg) \tau_2'(x), \tag{5.19b}$$

we obtain

$$\tau(x) = \frac{\det(\mathcal{K} e^{K(x)} \mathcal{D})}{\prod_{1\le l<l'\le N_b} (b_l - b_{l'})}, \qquad \tau'(x) = \frac{\det(\mathcal{D}' e^{-K(x)} \gamma \mathcal{K}')}{\prod_{1\le j<j'\le N_a} (a_j - a_{j'})} \tag{5.20}$$

instead of (5.9), where we introduce the diagonal $(N_a{+}N_b){\times}(N_a{+}N_b)$ matrix

$$e^{K(x)} = \operatorname{diag}\{e^{K_n(x)},\ n = 1,\ldots,N_a+N_b\} \tag{5.21}$$

by analogy with (4.4) and the new constant $N_b{\times}(N_a{+}N_b)$ and $(N_a{+}N_b){\times}N_a$ matrices

$$\mathcal{K} = \|\mathcal{K}_{ln}\|, \qquad \mathcal{K}_{ln} = \prod_{\substack{1\le l'\le N_b,\\ l'\neq l}} (\kappa_n - b_{l'}), \quad l = 1,\ldots,N_b, \tag{5.22}$$

and

$$\mathcal{K}' = \|\mathcal{K}'_{nj}\|, \qquad \mathcal{K}'_{nj} = \prod_{\substack{1\le j\le N_b,\\ j'\neq j}} (a_{j'} - \kappa_n), \quad j = 1,\ldots,N_a; \tag{5.23}$$

where $n = 1,\ldots,N_a+N_b$. We note that they have a block structure because $\mathcal{K}_{l,N_a+l'}$ and $\mathcal{K}'_{j'j}$, $l,l' = 1,\ldots,N_b$, $j,j' = 1,\ldots,N_a$ are diagonal submatrices. We also emphasize that because $N_a, N_b \ge 1$, as stated in (4.1), the constant matrices $\mathcal{K}$, $\mathcal{K}'$, $\mathcal{D}$, and $\mathcal{D}'$ are not square and the determinants therefore cannot be decomposed into products of determinants, which would imply $u(x) \equiv 0$.

To prove (5.20), we note that

$$(\mathcal{K} e^{K(x)} \mathcal{D})_{l,l'=1}^{N_b} \equiv \Bigg(\sum_{n=1}^{N_a+N_b} \Bigg(\prod_{\substack{1\le l''\le N_b,\\ l''\neq l}} (\kappa_n - b_{l''}) \Bigg) e^{K_n(x)} \mathcal{D}_{nl'} \Bigg)_{l,l'=1}^{N_b} =$$

$$= \sum_{j=1}^{N_a} e^{A_j(x)} d_{jl'} \prod_{\substack{l''=1,\\ l''\neq l}}^{N_b} (a_j - b_{l''}) + \sum_{l'''=1}^{N_b} e^{B_{l'''}(x)} \delta_{l''',l'} \prod_{\substack{1\le l''\le N_b,\\ l''\neq l}} (b_{l'''} - b_{l''}),$$

where (5.17), (5.21), and (5.22) are used. All terms in the last sum except those with $l''' = l$ vanish. Therefore, taking (5.16) into account, we obtain

$$\det(\mathcal{K}e^{K(x)}\mathcal{D})_{l,l'=1}^{N_b} = (-1)^{N_b(N_b-1)/2}\Biggl(\prod_{1\le l<l'\le N_b}(b_{l'}-b_l)^2\Biggr)\times$$
$$\times\det\Biggl(\delta_{ll''}e^{B_l(x)} + \sum_{j=1}^{N_a}\frac{e^{A_j(x)}c_{jl'}}{a_j-b_l}\Biggr)_{l,l'=1}^{N_b}.$$

The determinant in the right-hand side coincides with the determinant in (5.9a), which proves the first equality in (5.20). The second equality reduces to (5.9b) analogously, where the explicit expression for the matrix γ in (5.18) should also be taken into account.

Considering appropriate linear operations on the rows and columns of the matrices in (5.20), we can obtain more symmetric expressions for $\tau(x)$ and $\tau'(x)$ involving only the parameters κ_n. We consider the first equality in (5.20). We subtract the last row of the matrix $\mathcal{K}$ from all the preceding rows. Then the lth ($l > 1$) row becomes $(b_l - b_{N_b})\prod_{l'=1,l'\neq l}^{N_b-1}(\kappa_n - b_{l'})$. We extract all factors $b_l - b_{N_b}$ for $l = 1,\ldots,N_b$ from the determinant, repeat the same procedure with the next-to-last row, and so on up to the second row. The first row then has all units, and the second row has the elements $\kappa_l - b_1$ for $l = 1,\ldots,N_a+N_b$. We can then easily shift $\kappa_l - b_1 \to \kappa_l$ using the first row. A similar transformation can be used to transform the general element of the subsequent rows into κ_n^j. Finally, instead of (5.20), we obtain

$$\tau(x) = \det(\mathcal{V}e^{K(x)}\mathcal{D}), \qquad \tau'(x) = \det(\mathcal{D}'e^{-K(x)}\gamma\mathcal{V}'), \tag{5.24}$$

where $\mathcal{V}$ and $\mathcal{V}'$ denote the "incomplete Vandermonde" matrices, i.e., the $N_b\times(N_a+N_b)$ and $(N_a+N_b)\times N_a$ matrices

$$\mathcal{V} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ \kappa_1^{N_b-1} & \cdots & \kappa_{N_a+N_b}^{N_b-1}\end{pmatrix}, \qquad \mathcal{V}' = \begin{pmatrix} 1 & \cdots & \kappa_1^{N_a-1} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & \kappa_{N_a+N_b}^{N_a-1}\end{pmatrix}. \tag{5.25}$$

Remark 5.2. In equalities (5.20), all objects except $K_n(x)$ are invariant under an overall shift of all parameters a_j and b_l (or, equivalently, all κ_n) by the same constant, while this invariance is not obvious in (5.25). In fact, following the same procedure used to transform (5.20) into (5.25), we can obtain matrices $\mathcal{V}$ and $\mathcal{V}'$ constructed from powers of $\kappa_n + z$ instead of κ_n, where z is totally arbitrary.

Expressions (5.11) for the potential are invariant under rescaling (5.19), and consequently

$$u(x) = -2\partial_{x_1}^2\log\tau(x) = -2\partial_{x_1}^2\log\tau'(x). \tag{5.26}$$

The fact that τ and τ' give the same potential also follows directly because by (5.10), (5.14), (5.15), and (5.19),

$$\tau(x) = (-1)^{N_aN_b+N_a(N_a-1)/2}\Biggl(\prod_{n=1}^{N_a+N_b}e^{K_n(x)}\Biggr)V(\kappa_1,\ldots,\kappa_{N_a+N_b})\tau'(x), \tag{5.27}$$

where V denotes the Vandermonde determinant

$$V(\kappa_1,\ldots,\kappa_{N_a+N_b}) = \det\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ \kappa_1^{N_a+N_b-1} & \cdots & \kappa_{N_a+N_b}^{N_a+N_b-1}\end{pmatrix} \equiv \prod_{1\le m<n\le N_a+N_b}(\kappa_n-\kappa_m). \tag{5.28}$$

We have thus shown that representations (4.2) (or (5.11)) for the potential are equivalent to those expressed in (5.24) as determinants of the product of three and four matrices. These representations coincide with the representations obtained using the τ-function approach in [20] and studied in detail in [21]–[23], [25]. We note that the special block form of the matrices $\mathcal{D}$ and $\mathcal{D}'$ in (5.17) is not preserved in general, when we relabel the parameters κ_n in determinants (5.24), for instance, to obtain $\kappa_1 < \kappa_2 < \cdots < \kappa_{N_a+N_b}$. This problem is studied in detail in Sec. 6.3.

5.3. Explicit representation for the τ-functions. To study the behavior of the potential and Jost solutions, which we plan to do in a forthcoming publication, it is convenient to derive explicit representations for the determinants involved (also see [19]–[23], [25]). These representations involve the maximal minors of the matrices $\mathcal{D}$ and $\mathcal{D}'$ (see (5.17)), for which we use the simplified expressions

$$\mathcal{D}(n_1,\dots,n_{N_b}) = \mathcal{D}\begin{pmatrix} n_1, & n_2, & \dots, & n_{N_b} \\ 1, & 2, & \dots, & N_b \end{pmatrix}, \tag{5.29}$$

$$\mathcal{D}'(n_1,\dots,n_{N_a}) = \mathcal{D}'\begin{pmatrix} 1, & 2, & \dots, & N_a \\ n_1, & n_2, & \dots, & n_{N_a} \end{pmatrix}, \tag{5.30}$$

i.e., the determinant of the $N_b \times N_b$ matrix consisting of $n_1,\dots,n_{N_b}$ rows of $\mathcal{D}$ and all its columns and the determinant of the $N_a \times N_a$ matrix consisting of all rows of $\mathcal{D}'$ and $n_1,\dots,n_{N_a}$ columns of this matrix. Then, using the Binet–Cauchy formula for the determinant of a product of matrices and notation (5.28), we can rewrite relations (5.24) in the form

$$\tau(x) = \sum_{1\le n_1<n_2<\dots<n_{N_b}\le N_a+N_b} f_{n_1,\dots,n_{N_b}} \prod_{l=1}^{N_b} e^{K_{n_l}(x)}, \tag{5.31a}$$

$$\tau'(x) = \sum_{1\le n_1<n_2<\dots<n_{N_a}\le N_a+N_b} f'_{n_1,\dots,n_{N_a}} \prod_{j=1}^{N_a} e^{-K_{n_j}(x)}, \tag{5.31b}$$

where

$$f_{n_1,n_2,\dots,n_{N_b}} = V(\kappa_{n_1},\dots,\kappa_{n_{N_b}})\mathcal{D}(n_1,\dots,n_{N_b}), \tag{5.32a}$$

$$f'_{n_1,n_2,\dots,n_{N_a}} = V(\kappa_{n_1},\dots,\kappa_{n_{N_a}})\mathcal{D}'(n_1,\dots,n_{N_a}) \prod_{j=1}^{N_a} \gamma_{n_j}. \tag{5.32b}$$

It follows from (5.27) that the coefficients f and f' in expansions (5.31) are related by the equation

$$f'_{n_1,\dots,n_{N_a}} = (-1)^{N_a N_b + N_a(N_a-1)/2} \frac{f_{\tilde{n}_1,\tilde{n}_2,\dots,\tilde{n}_{N_b}}}{V(\kappa_1,\dots,\kappa_{N_a+N_b})}, \tag{5.33}$$

where $\{n_1,\dots,n_{N_a}\}$ and $\{\tilde{n}_1,\tilde{n}_2,\dots,\tilde{n}_{N_b}\}$ are two disjoint ordered subsets of the set of numbers ranging from 1 to N_a+N_b.

We mention that in our construction, the two equivalent representations for the τ-functions in (5.27) are obtained as a consequence of the two equalities in (2.6), while they can of course be proved directly using (5.31) and (5.33) (see [23], where this property is called duality).

Remark 5.3. In agreement with (5.32) and (5.33), we have

$$\prod_{j=1}^{N_a} \gamma_{n_j} V(\kappa_{n_1}, \ldots, \kappa_{n_{N_a}}) = \det \pi \cdot (-1)^{N_a N_b + N_a(N_a-1)/2} \frac{V(\kappa_{\tilde{n}_1}, \ldots, \kappa_{\tilde{n}_{N_b}})}{V(\kappa_1, \ldots, \kappa_{N_a+N_b})} \tag{5.34}$$

and

$$\mathcal{D}'(n_1, \ldots, n_{N_a}) = \det \pi \cdot \mathcal{D}(\tilde{n}_1, \ldots, \tilde{n}_{N_b}), \tag{5.35}$$

where π is the matrix permuting $(\kappa_{n_1}, \ldots, \kappa_{n_{N_a}}, \kappa_{\tilde{n}_1}, \ldots, \kappa_{\tilde{n}_{N_b}})$ into $(\kappa_1, \ldots, \kappa_{N_a+N_b})$.

6. Jost solutions and invariance properties

6.1. Properties of the matrices $\mathcal{D}$ and $\mathcal{D}'$. The matrices $\mathcal{D}$ and $\mathcal{D}'$ introduced in (5.17) have rather interesting properties. As follows directly from the definition, they are orthogonal in the sense that

$$\mathcal{D}'\mathcal{D} = 0, \tag{6.1}$$

where the zero in the right-hand side is an $N_a \times N_b$ matrix. Moreover, because the matrices

$$\mathcal{D}^\dagger \mathcal{D} = E_{N_b} + d^\dagger d, \qquad \mathcal{D}' \mathcal{D}'^\dagger = E_{N_a} + dd^\dagger, \tag{6.2}$$

where $\dagger$ denotes Hermitian conjugation of matrices (in fact, transposition here), are invertible, the matrices (see [36])

$$(\mathcal{D})^{(-1)} = (E_{N_b} + d^\dagger d)^{-1} \mathcal{D}^\dagger, \tag{6.3}$$

$$(\mathcal{D}')^{(-1)} = \mathcal{D}'^\dagger (E_{N_a} + dd^\dagger)^{-1}, \tag{6.4}$$

are the respective left inverse of $\mathcal{D}$ and right inverse of $\mathcal{D}'$, i.e.,

$$(\mathcal{D})^{(-1)} \mathcal{D} = E_{N_b}, \qquad \mathcal{D}'(\mathcal{D}')^{(-1)} = E_{N_a}. \tag{6.5}$$

The products of these matrices in the opposite order give the real self-adjoint $(N_a+N_b) \times (N_a+N_b)$ matrices

$$P = \mathcal{D}(\mathcal{D})^{(-1)} = \mathcal{D}(E_{N_b} + d^\dagger d)^{-1} (\mathcal{D})^\dagger, \tag{6.6}$$

$$P' = (\mathcal{D}')^{(-1)} \mathcal{D}' = (\mathcal{D}')^\dagger (E_{N_a} + dd^\dagger)^{-1} \mathcal{D}', \tag{6.7}$$

which are orthogonal projectors, i.e.,

$$P^2 = P, \qquad (P')^2 = P', \qquad PP' = 0 = P'P, \tag{6.8}$$

and are complementary in the sense that

$$P + P' = E_{N_a+N_b}. \tag{6.9}$$

The orthogonality of the projectors follows from (6.1), and the last equality follows from obvious relations of the kind $(E_{N_b} + d^\dagger d)^{-1} d^\dagger = d^\dagger (E_{N_a} + dd^\dagger)^{-1}$.

6.2. Symmetric representations for the Jost solutions. To obtain a τ-representation for the Jost solutions, we use (5.12) and note that rescaling (5.19) does not modify the substitution rules given in (5.13). In terms of notation (5.14), (5.15), and (5.21), these rules are

$$\tau(x) \to \tau_\chi(x,\mathbf{k}) \quad \text{for } e^K \to e^K(\kappa + i\mathbf{k}), \tag{6.10a}$$

$$\tau'(x) \to \tau'_\chi(x,\mathbf{k}) \quad \text{for } e^K \to e^K(\kappa + i\mathbf{k}), \tag{6.10b}$$

$$\tau(x) \to \tau_\xi(x,\mathbf{k}) \quad \text{for } e^K \to \frac{e^K}{\kappa + i\mathbf{k}}, \tag{6.10c}$$

$$\tau'(x) \to \tau'_\xi(x,\mathbf{k}) \quad \text{for } e^K \to \frac{e^K}{\kappa + i\mathbf{k}}, \tag{6.10d}$$

where $\kappa + i\mathbf{k}$ denotes the diagonal $(N_a{+}N_b)\times(N_a{+}N_b)$ matrix

$$\kappa + i\mathbf{k} = \operatorname{diag}\{\kappa_1 + i\mathbf{k}, \ldots, \kappa_{N_a+N_b} + i\mathbf{k}\}, \tag{6.11}$$

and analogously for the matrix $(\kappa + i\mathbf{k})^{-1}$. Explicitly, these replacements give (see (5.24))

$$\tau_\chi(x,\mathbf{k}) = \det\bigl(\mathcal{V} e^{K(x)}(\kappa + i\mathbf{k})\mathcal{D}\bigr), \tag{6.12a}$$

$$\tau'_\chi(x,\mathbf{k}) = \det\bigl(\mathcal{D}' e^{-K(x)}(\kappa + i\mathbf{k})^{-1}\gamma\mathcal{V}'\bigr), \tag{6.12b}$$

$$\tau_\xi(x,\mathbf{k}) = \det\bigl(\mathcal{V} e^{K(x)}(\kappa + i\mathbf{k})^{-1}\mathcal{D}\bigr), \tag{6.12c}$$

$$\tau'_\xi(x,\mathbf{k}) = \det\bigl(\mathcal{D}' e^{-K(x)}(\kappa + i\mathbf{k})\gamma\mathcal{V}'\bigr). \tag{6.12d}$$

Hence, by (5.12), we have

$$\left(\prod_{l=1}^{N_b}(b_l + i\mathbf{k})\right)\chi(x,\mathbf{k}) = \frac{\tau_\chi(x,\mathbf{k})}{\tau(x)} \equiv \left(\prod_{n=1}^{N_a+N_b}(\kappa_n + i\mathbf{k})\right)\frac{\tau'_\chi(x,\mathbf{k})}{\tau'(x)}, \tag{6.13a}$$

$$\left(\prod_{l=1}^{N_b}(b_l + i\mathbf{k})^{-1}\right)\xi(x,\mathbf{k}) = \frac{\tau_\xi(x,\mathbf{k})}{\tau(x)} \equiv \left(\prod_{n=1}^{N_a+N_b}(\kappa_n + i\mathbf{k})^{-1}\right)\frac{\tau'_\xi(x,\mathbf{k})}{\tau'(x)}. \tag{6.13b}$$

The Jost solutions themselves are then given by (2.3), and to simplify their analyticity properties, it is convenient to renormalize them as

$$\Phi(x,\mathbf{k}) \to \frac{\Phi(x,\mathbf{k})}{\prod_{l=1}^{N_b}(b_l + i\mathbf{k})}, \qquad \Psi(x,\mathbf{k}) \to \left(\prod_{l=1}^{N_b}(b_l + i\mathbf{k})\right)\Psi(x,\mathbf{k}). \tag{6.14}$$

To preserve the relations given in (2.3), we then also normalize $\chi(x,\mathbf{k})$ and $\xi(x,\mathbf{k})$ accordingly such that instead of (6.13), we have

$$\chi(x,\mathbf{k}) = \frac{\tau_\chi(x,\mathbf{k})}{\tau(x)} \equiv \left(\prod_{n=1}^{N_a+N_b}(\kappa_n + i\mathbf{k})\right)\frac{\tau'_\chi(x,\mathbf{k})}{\tau'(x)}, \tag{6.15a}$$

$$\xi(x,\mathbf{k}) = \frac{\tau_\xi(x,\mathbf{k})}{\tau(x)} \equiv \left(\prod_{n=1}^{N_a+N_b}(\kappa_n + i\mathbf{k})^{-1}\right)\frac{\tau'_\xi(x,\mathbf{k})}{\tau'(x)}. \tag{6.15b}$$

Now, $\chi(x,\mathbf{k})$ is a polynomial in $\mathbf{k}$ of the order $\mathbf{k}^{N_b}$, and $\xi(x,\mathbf{k})$ is a meromorphic function of $\mathbf{k}$ that becomes a polynomial of the order $\mathbf{k}^{N_a}$ after multiplication by $\prod_{n=1}^{N_a+N_b}(\kappa_n+i\mathbf{k})$. In other words, $\Phi(x,\mathbf{k})$ is now an entire function of $\mathbf{k}$, and $\Psi(x,\mathbf{k})$ is meromorphic with poles at all points $\mathbf{k}=i\kappa_n$, $n=1,\dots,N_a+N_b$. Introducing the discrete values of $\Phi(x,\mathbf{k})$ at these points as an N_a+N_b row

$$\Phi(x,i\kappa)=\big\{\Phi(x,i\kappa_1),\dots,\Phi(x,i\kappa_{N_a+N_b})\big\}, \tag{6.16}$$

and the residues at these points

$$\Psi_{\kappa_n}(x)=\operatorname{res}_{\mathbf{k}=i\kappa_n}\Psi(x,\mathbf{k}) \tag{6.17}$$

as an N_a+N_b column

$$\Psi_\kappa(x)=\big\{\Psi_{\kappa_1}(x),\dots,\Psi_{\kappa_{N_a+N_b}}(x)\big\}^{\mathrm{T}}, \tag{6.18}$$

by (5.14), (5.16), and (5.17), we find that relations (5.6) take the more symmetric form

$$\Phi(x,i\kappa)\mathcal{D}=0, \qquad \mathcal{D}'\Psi_\kappa(x)=0. \tag{6.19}$$

We must mention that after renormalization (6.13), asymptotic conditions (2.5) become

$$\lim_{\mathbf{k}\to\infty}(i\mathbf{k})^{-N_b}\chi(x,\mathbf{k})=1, \qquad \lim_{\mathbf{k}\to\infty}(i\mathbf{k})^{N_b}\xi(x,\mathbf{k})=1, \tag{6.20}$$

and relations (2.6) become

$$u(x)=-2\lim_{\mathbf{k}\to\infty}(i\mathbf{k})^{-N_b+1}\partial_{x_1}\chi(x,\mathbf{k})=2\lim_{\mathbf{k}\to\infty}(i\mathbf{k})^{N_b+1}\partial_{x_1}\xi(x,\mathbf{k}). \tag{6.21}$$

Finally, we point out that if an infinite set of times is introduced or, more precisely, $K_n(x)$ in (5.31) and (6.12) is replaced with the formal series (see [31])

$$K_n(t_1,t_2,\dots)=\sum_{j=1}^{\infty}\kappa_n^j t_j, \qquad (t_1,t_2)=(x_1,x_2), \tag{6.22}$$

then the choice of t_j as an evolution parameter provides multisoliton solutions of the $(j-2)$th nonlinear evolution equation in the hierarchy related to the KPII equation and also the corresponding Jost solutions.

Remark 6.1. The multitime multisoliton solution of all equations of the KPII hierarchy is given by (5.31), where all exponentials are now independent. Thus, (4.10) is the necessary and sufficient condition for the regularity of the solution of an arbitrary equation of the hierarchy.

Remark 6.2. We showed in (6.10) that the Jost solutions can be obtained using transformations that are equivalent to the formal Miwa shift used to construct Baker–Akhiezer functions in terms of τ-functions. In fact, substitutions (6.10a) and (6.10b) can be obtained (up to an inessential factor) by considering the multitime τ-function obtained by replacing K_n with the infinite formal series in (6.22) and then shifting $t_j\to t_j-(i/\mathbf{k})^j(1/j)$. Similarly, substitutions (6.10c) and (6.10d) are obtained (again up to an inessential factor) by the shifts $t_j\to t_j+(i/\mathbf{k})^j(1/j)$. Then, nonformal Jost solutions of the heat equation with a potential being a solution of KPII are derived by choosing $t_3=-4t$ and setting all higher times equal to zero.

6.3. Invariance properties of the multisoliton potential. In some cases, it is useful to relabel the spectral parameters $\kappa_n \to \tilde{\kappa}_n$, for instance, such that the relabeled parameters $\tilde{\kappa}_n$ are ordered as

$$\tilde{\kappa}_1 < \tilde{\kappa}_2 < \cdots < \tilde{\kappa}_{\mathcal{N}}. \tag{6.23}$$

This permutation can be performed using an $(N_a{+}N_b)\times(N_a{+}N_b)$ matrix π such that

$$(\tilde{\kappa}_1, \dots, \tilde{\kappa}_{N_a+N_b}) = (\kappa_1, \dots, \kappa_{N_a+N_b})\pi. \tag{6.24}$$

This matrix is unitary,

$$\pi^\dagger = \pi^{-1}, \tag{6.25}$$

and all rows and columns have one element equal to 1 and all others equal to 0. It is convenient to write this matrix in a block form like

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}, \tag{6.26}$$

where π_{11} is an $N_a\times N_a$ matrix, π_{22} is an $N_b\times N_b$ matrix, π_{12} is an $N_a\times N_b$ matrix, and π_{21} is an $N_b\times N_a$ matrix.

Representations (5.24) then keep the same form under transformation (6.24) if the matrices $\mathcal{D}$ and $\mathcal{D}'$ are transformed as

$$\mathcal{D} \to \widetilde{\mathcal{D}} = \pi\mathcal{D} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \qquad \mathcal{D}' \to \widetilde{\mathcal{D}}' = \mathcal{D}'\pi^\dagger = (d_1', -d_2'), \tag{6.27}$$

where we have

$$d_1 = \pi_{11}d + \pi_{12}, \qquad d_2 = \pi_{21}d + \pi_{22}, \tag{6.28}$$

$$d_1' = \pi_{11}^\dagger - d\pi_{12}^\dagger, \qquad -d_2' = \pi_{21}^\dagger - d\pi_{22}^\dagger \tag{6.29}$$

by (5.17). While the sizes of blocks in (6.27) are the same as in (5.17), the block structure generally differs. Nevertheless, because the matrix π is unitary, relation (6.1) remains valid for the transformed matrices, which by (6.27) means that

$$d_1'd_1 = d_2'd_2. \tag{6.30}$$

Relations (6.19) are also preserved for the transformed quantities, and transformed projectors $\widetilde{P}$ and $\widetilde{P}'$ can be built.

We note that the special block structure of the matrices $\mathcal{D}$ and $\mathcal{D}'$ in (5.17) determines them uniquely for a given potential. We can relax this constraint without changing the potential by multiplying $\mathcal{D}$ by any nonsingular $N_b\times N_b$ matrix from the right and $\mathcal{D}'$ by any nonsingular $N_a\times N_a$ matrix from the left. In fact, the determinants of these matrices cancel in (5.26) and (6.13). Conversely, we can use this procedure to bring $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}}'$ back from block structure (6.27) to the special form (5.17). For this, we use matrices $(d_2)^{-1}$ and $(d_1')^{-1}$, if they exist, and perform the transformations

$$\widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}(d_2)^{-1} = \begin{pmatrix} d_1(d_2)^{-1} \\ E_{N_b} \end{pmatrix}, \qquad \widetilde{\mathcal{D}}' \to (d_1')^{-1}\widetilde{\mathcal{D}}' = \big(E_{N_a}, -(d_1')^{-1}d_2'\big). \tag{6.31}$$

We then note that by (6.30),

$$d_1(d_2)^{-1} = (d_1')^{-1}d_2' = \tilde{d}, \tag{6.32}$$

where $\tilde{d}$ is defined by this equality. Taking into account that both representations in (5.24) are equivalent, we deduce that the matrices d_2 and d'_1 are simultaneously invertible. We have thus proved that in the case of a permutation π such that the matrix d_2 (or d'_1) is nonsingular, permuting the parameters κ_n is equivalent to a transformation of the matrix d to $\tilde{d}$ or, by (5.16), to a corresponding transformation of the matrix c. In particular, both matrices d_2 and d'_1 are invertible, and we can use the above substitution when the matrix π has a diagonal block structure, i.e., when $\pi_{12} = \pi_{21} = 0$. We note that in this case, permuting the parameters κ_n does not mix the original parameters a_j with b_l (see (5.14)).

In general, permuting the parameters κ_n, for instance, reducing them to order (6.23), we lose block structure (5.17) and can only say that the τ-functions are given by (5.24) and (6.12), where both matrices $\mathcal{D}$ and $\mathcal{D}'$ have at least two nonzero maximal minors.

7. Concluding remarks

We have described relations between different representations known in the literature for the multisoliton potentials of the heat operator and derived forms of these representations that will allow studying the asymptotic behavior of the potentials themselves and the corresponding Jost solutions on the x plane in detail in a forthcoming publication. We also presented various formulations of the conditions guaranteeing that the potential is regular. Nevertheless, the essential problems of determining the necessary conditions for the regularity of the multisoliton solution of KPII remains open (cf. Sec. 6.1). In this context, we mention the selected role of the specific subclass of potentials defined by the strict inequalities in (4.10), which by (5.31a) is equivalent to requiring that all $f_{n_1,\dots,n_{N_b}}$ have the same sign (analogously, by (5.31b), that all $f'_{n_1,\dots,n_{N_a}}$ have the same sign). These conditions identify fully resonant soliton solutions (cf. [19], [20]). When such conditions are imposed, all maximal minors of the matrices $\mathcal{D}$ and $\mathcal{D}'$ are nonzero, as follows from (5.32); then, because of the invariance properties discussed above, we can always permute the parameters κ_n in any way (as in (6.23) for instance) and simultaneously deal with transformed matrices $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}}'$ with a special block structure like in (5.17).

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