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Nonlinear Schrödinger Equations

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Historical Background

Ginzburg–Landau Equations

Nonlinear Schrödinger (NLS) equations have become one of the most important nonlinear systems studied in mathematics and physics. Actually, one can find the essence of NLS equations in the early work of Ginzburg and Landau (1950) and Ginzburg (1956) in their study of the macroscopic theory of superconductivity, and also of Ginzburg and Pitaevskii (1958), who subsequently investigated the theory of superfluidity.

By minimizing the free energy of a superconductor near the superconducting transition, Ginzburg and Landau arrived at what are now called the Ginzburg–Landau equations:

$$\frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}A\right)^2\psi + \alpha\psi + \beta|\psi|^2\psi = 0 \qquad [1]$$

$$J = -\frac{ie\hbar}{mc}[\psi^*\nabla\psi - \psi\nabla\psi^*] - \frac{e^2}{mc}|\psi|^2 A \qquad [2]$$

where α, β are phenomenological parameters, A the electromagnetic vector potential, and ψ^* denotes complex conjugate of ψ. The first equation determines the field ψ based on the applied magnetic field. The second equation provides the superconducting current J.

The equation describing the behavior of superfluid helium near the transition point in the stationary case derived in Ginzburg and Pitaevskii (1958) is completely analogous to eqn [1] in the phenomenological theory of superconductivity.

Equation [1] contains all the ingredients of the NLS equations which are discussed below. However, it was not until the 1960s that the wide physical importance of NLS equation became evident. The next section discusses how the NLS equation historically first appeared in the context of nonlinear optics.

Nonlinear Optics: Self-Focusing of Optical Beams in Nonlinear Media

In the mid-1960s, Chiao *et al.* (1964) and Talanov (1964) investigated the conditions under which an electromagnetic beam can produce its own dielectric waveguide and propagate without spreading. This is a reflection of the phenomenon of self-focusing. In fact, self-focusing of optical beams may occur in materials whose dielectric constant increases with field intensity. In the general situation, a beam of uniform intensity in a dielectric broadens due to diffraction. However, the refractive index of many physically important materials (the so-called Kerr materials, such as silica) depends on the field intensity as follows:

$$n = n_0 + n_2|E|^2 + \cdots$$

If the term $n_2|E|^2$ is large enough, the critical angle for total internal reflection at the beam's boundary can be greater than the angular divergence due to diffraction; thus, spreading does not occur as a result of diffraction. As a consequence, a beam above a certain critical power level is trapped and does not spread.

In a remarkable contribution, Kelley (1965) observed, using computational methods (years before computational methods became easy to implement and, consequently, so popular) that when the self-focusing effect due to the increase in the nonlinear index is not compensated by diffraction, there is a buildup in intensity of part of the beam as a function of the distance in the direction of propagation. Consequently, the intensity of the self-focused regions tended to become "anomalously large," that is, a singularity appeared to develop.

Consider as starting equation the electromagnetic wave equation in the presence of nonlinearities derived earlier by Chiao *et al.* (1964):

$$\nabla^2 E - \frac{\epsilon_0}{c^2}\partial_t^2\, E - \frac{\epsilon_2}{c^2}\partial_t^2\,(E^2 E) = 0 \qquad [3]$$

where $\epsilon_2|E|^2 \ll 1$. One assumes a linearly polarized wave of frequency ω, propagating along the z-axis, so that

$$E = \tfrac{1}{2}(\mathcal{E}e^{i(kz-\omega t)} + \text{c.c.})\hat{e}$$

where c.c. denotes complex conjugation, $k = \epsilon_0^{1/2}\omega/c$, the factor $\exp(ikz - \omega t)$ represents the propagating part, that is, the "carrier," of the wave, and $\mathcal{E}$ is the slowly varying part. Substituting the above expression for E into eqn [3], neglecting the third-harmonic term and the term $\partial_z^2\mathcal{E}$ from $\nabla^2 E$ (assuming it to be small), yields

$$2ik\partial_z\mathcal{E} + \left(\partial_x^2 + \partial_y^2\right)\mathcal{E} + \frac{3}{4}k^2\frac{\epsilon_2}{\epsilon_0}|\mathcal{E}|^2\mathcal{E} = 0 \qquad [4]$$

or, with a suitable rescaling of the dependent and independent variables $(\mathcal{E} \to \psi/((3/4)k^2\epsilon_2/\epsilon_0)^{1/2}$, $z \to 2kz)$,

$$i\partial_z\psi + \nabla_\perp^2\psi + 2|\psi|^2\psi = 0 \qquad [5]$$

which is the NLS equation in standard nondimensional form.

It should be remarked here that the name NLS equation for equations of the form of [5] is natural due to the formal analogy with the Schrödinger equation in quantum mechanics:

$$i\partial_t\psi + \nabla^2\psi + V\psi = 0 \qquad [6]$$

If one sets $V = 2|\psi|^2$ in eqn [6], the result is the NLS equation. In the context of quantum mechanics, a nonlinear potential arises in the "mean-field" description of interacting particles.

Modifications of [6] also arise as mean-field descriptions of Bose–Einstein condensates which is of keen interest in physics (see Pethick and Smith (2002) and references therein). The normalized equation is

$$i\partial_t\psi - \nabla^2\psi + \left(V(x,y) + 2|\psi|^2\right)\psi = 0 \qquad [7]$$

where V is an external potential. This is generally referred to as the Gross–Pitaevskii equation.

Talanov (1965) (see also Zakharov *et al.* (1971)) investigated the behavior of stationary light beams in a self-focusing nonlinear medium and found that for a purely cubic nonlinearity, "collapse" of the beam can take place. The proof that there is a singularity in eqn [5] is remarkably straightforward. This is discussed in the section "Wave collapse." In order to avoid wave collapse, other physical effects (e.g., saturable nonlinearity or dissipation) are required.

Universal Character of the NLS Equation

It turns out that almost any dispersive, energy-preserving system gives rise, in an appropriate limit, to the NLS equation. For instance, one can derive the NLS from other physically significant equations such as the Klein–Gordon equation

$$u_{tt} - u_{xx} + u + ku^3 = 0$$

and the Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0$$

Actually, the NLS equation provides a "canonical" description for the envelope dynamics of a quasi-monochromatic plane wave (the carrier wave) propagating in a weakly nonlinear dispersive medium when dissipative processes are negligible.

Indeed, consider a scalar nonlinear wave equation written symbolically as

$$L(\partial_t, \nabla)u + G(u) = 0$$

where L is a linear differential operator with constant coefficients and G a nonlinear function of u and its derivatives. For a real, small-amplitude solution of magnitude $\epsilon \ll 1$, the nonlinear effects can first be neglected, and the equation admits approximate monochromatic wave solutions

$$u = \epsilon\psi e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} + \text{c.c.} \qquad [8]$$

with small amplitude $\epsilon|\psi|$. Substituting [8] into the linear equation, one can find that the frequency ω and the wave vector $\boldsymbol{k}$ are related by the dispersion relation

$$L(-i\omega, i\boldsymbol{k}) = 0$$

Let

$$\omega = \omega(\boldsymbol{k})$$

be one of the solutions of the previous equation. Suppose one is interested in a solution ψ which is not constant, but slowly varying in space and time. This has the interpretation of $\boldsymbol{k}$ having a "sideband" wave vector and ω a "sideband" frequency. More precisely, restricting discussion, for simplicity, to the $(1+1)$-dimensional case, the slowly varying amplitude assumption corresponds to letting

$$\psi(x,t) = \psi(X,T) = \psi_0 e^{i(Kx - \Omega t)}$$

where $X = \epsilon x$ and $T = \epsilon t$. Note that $K = \epsilon k$ and $\Omega = \epsilon\omega$ are sometimes referred to as the sideband wave number and frequency, respectively, because they correspond to a deviation from the central wave number k and central frequency ω. Looking at these deviations from the point of view of operators, whereby $\omega \to i\partial_t$, $k \to -i\partial_x$ and $\Omega \to i\partial_T$, $K \to -i\partial_X$, one has

$$\omega_{\text{tot}} \sim \omega + \epsilon\Omega = \omega + i\epsilon\partial_T$$
$$k_{\text{tot}} \sim k + \epsilon K = k - i\epsilon\partial_X$$

Then $\omega(k)$ can be expanded in a Taylor series around the central wave number as

$$\omega(k - i\epsilon\partial_X) \sim \omega(k) - i\epsilon\omega'\partial_X - \epsilon^2\frac{\omega''}{2}\partial_X^2 + \cdots$$

Therefore,

$$\omega_{\text{tot}}(k)\psi \sim [\omega(k) + i\epsilon\partial_T]\psi$$
$$\sim \left(\omega(k) - i\epsilon\omega'\partial_X - \epsilon^2\frac{\omega''}{2}\partial_X^2\right)\psi$$

which shows that, to the leading order,

$$\mathrm{i}\epsilon\left(\frac{\partial\psi}{\partial T}+\omega'\frac{\partial\psi}{\partial X}\right)+\epsilon^2\frac{\omega''}{2}\frac{\partial^2\psi}{\partial X^2}=0 \qquad [9]$$

In the moving frame $\xi = X - \omega'(k)T$, $\tau = \epsilon T \equiv \epsilon^2 t$, eqn [9] transforms to

$$\epsilon^2\left(\mathrm{i}\psi_\tau+\frac{\omega''}{2}\psi_{\xi\xi}\right)=0$$

which is the linear Schrödinger equation with the canonical $\omega''(k)/2$ coefficient. On the other hand, if one considers rather general conservative nonlinear wave problems with leading quadratic or cubic nonlinearity, asymptotic analysis (e.g., multiple scale analysis which yields the so-called Stokes–Poincaré frequency shift) shows that a wave solution of the form

$$u(x,t)=\epsilon\psi(\tau)\mathrm{e}^{\mathrm{i}(kx-\omega t)}+\text{c.c.}$$

with $\tau = \epsilon^2 t$ has $\psi(\tau)$ satisfying

$$\mathrm{i}\frac{\partial\psi}{\partial\tau}+n|\psi|^2\psi=0 \qquad [10]$$

where the constant coefficient n depends on the particular equation under study. It should be remarked here that cubic nonlinearity yields an $O(\epsilon^3)$ contribution, which is balanced by a slow timescale of order ϵ^2. Putting the linear and nonlinear effects together (i.e., eqns [9] and [10]) implies that an NLS equation of the form

$$\mathrm{i}\frac{\partial\psi}{\partial t}+\frac{\omega''}{2}\frac{\partial^2\psi}{\partial\xi^2}+n|\psi|^2\psi=0$$

naturally arises. The NLS equation is viewed as a "universal" equation as it generically governs the slowly varying envelope of a monochromatic wave train (see also Benney and Newell (1969)).

Physical Applications

The nonlinear propagation of wave packets is governed by NLS-type systems in several different branches of scientific and technological applications, beyond what has been mentioned earlier. Some of these applications are discussed below.

NLS equation in Water Waves

The NLS equation in the context of small-amplitude water waves was derived by Zakharov (1968) (infinite depth) and Benney and Roskes (1969) (finite depth). The procedure for deriving the NLS equation from the Euler–Bernoulli equations of fluid dynamics in one horizontal direction will now be discussed, under the assumption of small-amplitude waves and deep water. The interested reader can also find the details of the derivation in Ablowitz and Clarkson (2006). The relevant equations are

$$\phi_{xx}+\phi_{zz}=0, \quad -\infty<z<\epsilon\eta(x,t) \qquad [11]$$

$$\phi_z=0, \quad z\to-\infty \qquad [12]$$

$$\phi_t+\frac{\epsilon}{2}\left(\phi_x^2+\phi_z^2\right)+g\eta=0, \quad z=\epsilon\eta \qquad [13]$$

$$\eta_t+\epsilon\eta_x\phi_x=\phi_z, \quad z=\epsilon\eta \qquad [14]$$

where ϕ is the velocity potential of an ideal (i.e., incompressible, irrotational, and inviscid) fluid, $\eta(x,t)$ is the free surface of the fluid, which is to be found, in addition to $\phi(x,z;t)$.

Equation [11] expresses the ideal nature of the fluid; the condition [12] expresses the requirement that there is no vertical flow at infinity; and eqn [13] is the Bernoulli equation of energy conservation. Finally, eqn [14] is a kinematic condition stating that no flow occurs transverse to the free surface.

At the free boundary, for small amplitudes, one can expand $\phi=\phi(t,x,\epsilon\eta)$ for $\epsilon \ll 1$ as

$$\phi=\phi(t,x,0)+\epsilon\eta\phi_z(t,x,0)+\frac{(\epsilon\eta)^2}{2}\phi_{zz}(t,x,0)+\cdots$$

and similarly for the derivatives. Second, one introduces slow temporal and spatial scales (one expects the slowly varying envelope of the wave to depend on slow variables $X=\epsilon x$, $Z=\epsilon z$, $T=\epsilon t$). Finally, because of the quadratic nonlinearity one expects second harmonics to be generated; hence,

$$\phi=\left(A\mathrm{e}^{\mathrm{i}\Theta+|k|z}+\text{c.c.}\right)+\epsilon\left(A_2\mathrm{e}^{2\mathrm{i}\Theta+2|k|z}+\text{c.c.}+\bar{\phi}\right)$$
$$\eta=\left(B\mathrm{e}^{\mathrm{i}\Theta}+\text{c.c.}\right)+\epsilon\left(B_2\mathrm{e}^{2\mathrm{i}\Theta}+\text{c.c.}+\bar{\eta}\right)$$

where A, A_2, $\bar{\phi}$ depend on X, Z, T and B, B_2, $\bar{\eta}$ depend on X, T ($\bar{\phi}$ and $\bar{\eta}$ are mean contributions, which are real) and $\Theta = kx - \omega t$ with the dispersion relation $\omega^2 = g|k|$. Substituting this ansatz into the equations, one obtains from the order-ϵ^2 terms

$$2\mathrm{i}\omega A_\tau-\left(\frac{v_g^2}{2\omega}A_{\xi\xi}+\frac{2k^4}{\omega}|A|^2A\right)=0 \qquad [15]$$

where $v_g=\omega'(k)=g/2\omega$ is the group velocity and the new variables $\tau=\epsilon T$, $\xi = X - v_g T$.

Equation [15] is the typical formulation of the (1 + 1)-dimensional NLS equation found in water wave theory for large depth.

In the section "NLS in nonlinear optics," a special solution to (a rescaled version of) eqn [15], namely a soliton solution, is discussed in the context of nonlinear optics. It should be remarked here that the coefficients of both terms $A_{\xi\xi}$ and $|A|^2A$ have the same sign. This is necessary for a decaying soliton solution to exist (see, e.g., Lighthill (1965)).

NLS in Nonlinear Optics

The NLS equation also describes self-compression and self-modulation of electromagnetic wave packets in weakly nonlinear media. Hasegawa and Tappert (1973a, b) first derived the NLS equation in the context of fiber optics. Light-wave propagation in a fiber is mainly affected by: (1) group velocity dispersion (GVD), that is, the frequency dependence of the group velocity originating from the refractive index of the fiber and (2) fiber nonlinearity (the so-called Kerr effect), originating from the dependence of the refractive index on the intensity of the optical pulse. In the presence of GVD and Kerr nonlinearity, the refractive index is expressed as

$$n(\omega, E) = n_0(\omega) + n_2|E|^2 \qquad [16]$$

where ω and E represent the frequency and electric field of the light wave, respectively, $n_0(\omega)$ is the frequency-dependent linear refractive index, and the constant n_2, referred to as the Kerr coefficient, is "small" but can have significant impact since the nonlinear effects accumulate over long distances. Normally, the electric field is modulated into a slowly varying amplitude of a carrier wave:

$$E(z,t) = \mathcal{E}(z,t)\mathrm{e}^{\mathrm{i}(k_0 z - \omega_0 t)} + \text{c.c.} \qquad [17]$$

where z denotes the distance along the fiber, t the time, $k_0 = k_0(\omega_0)$ the wave number, ω_0 the frequency, and $\mathcal{E}(z,t)$ the envelope of the electromagnetic field.

A Taylor series expansion of the dispersion relation (see also the section "Universal character of the NLS equation")

$$k(\omega, E) = \frac{\omega}{c}(n_0(\omega) + n_2|E|^2)$$

around the carrier frequency $\omega = \omega_0$ yields

$$\begin{aligned} k - k_0 = {} & k'(\omega_0)(\omega - \omega_0) + \frac{k''(\omega_0)}{2}(\omega - \omega_0)^2 \\ & + \frac{\omega_0 n_2}{c}|E|^2 \end{aligned} \qquad [18]$$

where the prime represents derivative with respect to ω and $k_0 = k(\omega_0)$. Replacing $k - k_0$ and $\omega - \omega_0$ by their Fourier operator equivalents, $\mathrm{i}\partial_z$ and $\mathrm{i}\partial_t$ resp., using $k - k_0 = (\omega/c)n_0(\omega)$ and letting eqn [18] operate on $\mathcal{E}$ yields

$$\mathrm{i}\left(\frac{\partial \mathcal{E}}{\partial z} + k_0'(\omega_0)\frac{\partial \mathcal{E}}{\partial t}\right) - \frac{k_0''(\omega_0)}{2}\frac{\partial^2 \mathcal{E}}{\partial t^2} + \nu|\mathcal{E}|^2\mathcal{E} = 0 \quad [19]$$

where $\nu = \omega_0 n_2 / cA_{\text{eff}}$, with A_{eff} being the effective cross-section area of the fiber (the factor $1/A_{\text{eff}}$ comes from a more detailed derivation which takes into account the finite size of the fiber; the factor $1/A_{\text{eff}}$ is needed in order to account for the variation of field intensity in the cross section of the fiber). Note that $k_0'(\omega_0) = 1/v_g$, where v_g represents the group velocity of the wave train. Introducing dimensionless variables $t' = t_{\text{ret}}/t_*, z' = z/z_*, q = \mathcal{E}/\sqrt{P_*}$ yields the NLS equation

$$\mathrm{i}\frac{\partial q}{\partial z'} + \frac{\operatorname{sgn}(-k_0''(\omega_0))}{2}\frac{\partial^2 q}{\partial t'^2} + |q|^2 q = 0 \qquad [20]$$

where t_*, P_* are the characteristic time and power, respectively, and $t_{\text{ret}} = t - k_0'(\omega_0)z = t - z/v_g$, $z_* = 1/\nu P_*$, with the constraint that the "nonlinear length" is balanced by the linear dispersion time, that is, $t_* = (z_*| - k''(\omega_0)|)^{1/2}$.

There are two cases of physical interest depending on the sign of k_0''. The so-called focusing case occurs when $k_0'' < 0$; this is called "anomalous" dispersion. The defocusing case obtains when the dispersion is "normal": $k_0'' > 0$.

Now write eqn [20] in the form

$$\mathrm{i}q_t + q_{xx} \pm 2|q|^2 q = 0 \qquad [21]$$

with $\pm$ corresponding to the focusing (+) and defocusing (−) case, respectively. The focusing NLS equation admits special solutions called "bright" solitons (solutions that are traveling localized "humps"). A pure one-soliton solution in the focusing (+) case has the form

$$q(x,t) = \eta\,\operatorname{sech}[\eta(x + 2\xi t - x_0)]\,\mathrm{e}^{-\mathrm{i}\Theta} \qquad [22]$$

where $\Theta = \xi x + (\xi^2 - \eta^2)t + \Theta_0$. The parameters ξ and η are such that $\lambda = \xi/2 + \mathrm{i}\eta/2$ is an eigenvalue from the inverse scattering transform analysis.

The defocusing (−) NLS equation does not admit solitons that decay at infinity. However, it does admit soliton solutions which have a nontrivial background intensity (called "dark" and "gray" solitons). A dark-soliton solution has the form

$$q(x,t) = \eta\,\tanh(\eta x)\,\mathrm{e}^{-2\mathrm{i}\eta^2 t} \qquad [23]$$

Note that $q \to \pm\eta$ as $x \to \pm\infty$. A gray-soliton solution is

$$q(x,t) = \eta\left[1 - B^2\operatorname{sech}^2(\eta B(x - x_0))\right]^{1/2}\mathrm{e}^{\mathrm{i}\phi(x,t)} \quad [24]$$

with

$$\phi(x,t) = -\eta^2(2-B^2)t + \eta\sqrt{1-B^2}x + \tan^{-1}\left(\frac{B\tanh(\eta Bx)}{\sqrt{1-B^2}}\right) + \phi_0$$

and $|B| < 1$. Note that as $B \to 1^-$, the gray soliton becomes a dark soliton, taking $\phi_0 = -\pi/2$.

Recall that the solutions [23] and [24] can be allowed to travel uniformly by making a Galilean transformation, that is, taking into account that if $q_1(x,t)$ is a solution of [21], then so is

$$q_2(x,t) = q_1(x - vt, t)\,e^{i(kx-\omega t)}$$

with $k = -v$ and $\omega = -k^2/2$.

It should also be remarked that Ablowitz *et al.* (1997) have shown that, in quadratically nonlinear optical materials, more complicated NLS-type equations arise. These equations are analogous to the finite-depth multidimensional nonlocal NLS-type systems derived in the context of water waves by Benney and Roskes (1967) and later by Davey and Stewartson (1974).

Optical Communications

Hasegawa and Tappert (1973) first suggested using solitons as the "bit" format for transmission of information in optical fiber systems. Motivated by this, in 1980, scientists at Bell Laboratories observed solitons (described by the NLS equation) in optical fibers (Mollenauer *et al.* 1980). The development of optical amplifiers (erbium-doped amplifiers) in the mid-1980s provided a mechanism to compensate fiber loss, and this permitted the transmission of information entirely optically over long distances. With damping and amplification included (see, e.g., Hasegawa and Kodama (1995)), the NLS equation [20] takes the form

$$i\frac{\partial q}{\partial z} + \frac{\mathrm{sgn}(-k_0''(\omega_0))}{2}\frac{\partial^2 q}{\partial t^2} + g(z)|q|^2 q = 0 \qquad [25]$$

where $g(z) = a_0^2 \exp(-2\Gamma z/z_a)$, $0 < z < z_a$, and periodically extended thereafter, and a_0^2 is determined by

$$< g > = \frac{1}{z_a}\int_0^{z_a} g(z/z_a)\mathrm{d}z = 1$$

with $z_a = l_a/z_*$, l_a being the amplifier length. Remarkably, asymptotic analysis ($z_a \ll 1$) shows that, to leading order, $q(z,t)$ still satisfies the NLS equation [20].

Amplifiers, however, introduce small amounts of noise to the system, which causes the temporal position of the soliton to fluctuate (cf. Gordon and Haus (1986)) and thus limits the distance signals can be reliably transmitted to. Soliton control mechanisms were introduced in the early 1990s in order to deal with these difficulties (cf. Mecozzi *et al.* (1991) and Kodama and Hasegawa (1992)).

By the mid-1990s, the development of all optical transmission systems began to take great advantage of wavelength-division-multiplexing (WDM), that is, the simultaneous transmission of multiple signals in different frequency (or equivalently wavelength) "channels" (Hasegawa 2000). However, it was found that a serious problem affected WDM systems. Namely, the interactions of solitons traveling at different velocities cause resonant amplifier-induced instabilities in adjacent frequency channels (four-wave mixing (Mamyshev and Mollenauer 1996, Ablowitz *et al.* 1996)). In order to avoid these instabilities, researchers developed and analyzed dispersion-managed (DM) transmission systems (cf. Hasegawa (2000)). In a DM transmission system, the fiber is composed of alternating sections of positive (normal) and negative (anomalous) dispersion fibers. The (dimensionless) NLS equation that governs this phenomenon is

$$i\frac{\partial q}{\partial z} + \frac{d(z)}{2}\frac{\partial^2 q}{\partial t^2} + g(z)|q|^2 q = 0 \qquad [26]$$

where $d(z)$ is usually taken to be a periodic, large, rapidly varying function of the form $d(z) = \delta_a + \Delta(z)$, with $|\Delta(z)| \gg 1$ and having zero average in the period z_a (generally the same as that of the amplifier). In fact, asymptotic analysis of [26] yields a nonlocal NLS-type equation (Gabitov and Turitsyn 1996, Ablowitz and Biondini 1998). It has also been shown that eqn [26] admits various types of optical pulses, such as DM solitons (Ablowitz and Biondini 1998), and quasilinear modes (Ablowitz *et al.* 2001).

NLS Equation in Other Settings

Many other interesting applications of the NLS equations exist in such different areas of physics as magnetic spin waves (see, e.g., the work by Zvezdin and Popkov (1983) and also by Kalinikos *et al.* (1997)), plasma physics (cf. the work by Zakharov (1972) on collapse of Langmuir waves), other areas of fluid dynamics, etc. (the interested reader can find an overview in the monograph by Ablowitz (1981)).

Mathematical Framework

Mathematically, the NLS equation had attained broad significance since it is integrable via inverse-scattering transform (IST), admits multisoliton solutions, has an infinite number of conserved quantities, and possesses many other interesting properties. Some of these are discussed below.

The Inverse-Scattering Transform

The IST method allows one to linearize a large class of nonlinear evolution equations and can be considered as a nonlinear version of the Fourier transform. An essential prerequisite of IST method is the association of the nonlinear evolution equation with a pair of linear problems (Lax pair), a linear eigenvalue problem, and a second associated linear problem, such that the given equation results as a compatibility condition between them. A key research breakthrough on NLS systems appeared in 1972, in the papers of Zakharov and Shabat (1972, 1973), who first analyzed the scalar NLS equation in the form

$$\mathrm{i}q_t = q_{xx} \pm 2|q|^2 q \qquad [27]$$

($\pm$ correspond to the focusing/defocusing case, respectively) and found the associated Lax pair

$$v_x = \begin{pmatrix} -\mathrm{i}k & q \\ \mp q^* & \mathrm{i}k \end{pmatrix} v \qquad [28]$$

$$v_t = \begin{pmatrix} 2\mathrm{i}k^2 \mp \mathrm{i}|q|^2 & -2kq - \mathrm{i}q_x \\ \pm 2kq^* \mp \mathrm{i}q_x^* & -2\mathrm{i}k^2 \pm \mathrm{i}|q|^2 \end{pmatrix} v \qquad [29]$$

where $v(x,t)$ is a two-component vector. The compatibility of [28] and [29] yields eqn [27], assuming that the eigenvalue parameter k is constant in time (so that [27] is often said to be isospectral).

The solution of the initial-value problem of a nonlinear evolution equation by IST proceeds in three steps, as follows:

1. *the forward problem* – the transformation of the initial data from the original "physical" variables to the transformed "scattering" variables;
2. *time dependence* – the evolution of the transformed data according to simple, explicitly solvable evolution equations; and
3. *the inverse problem* – the recovery of the evolved solution in the original variables from the evolved solution in the transformed variables.

The implementation of steps 1–3 described above is more concretely carried out as follows. The initial (Cauchy) datum $q(x,0)$ for eqn [27] is mapped into scattering data $S(k,0)$ (comprising, in general, discrete eigenvalues and associated normalization constants, and reflection coefficients) by means of eqn [28]. The data $S(k,0)$ are evolved via eqn [29] to get $S(k,t)$ at an arbitrary time $t>0$. Finally, by employing the methods of inverse scattering, eqn [28] allows one to reconstruct the evolved solution $q(x,t)$ from $S(k,t)$.

One can easily note the "formal" resemblance to the well-known method of Fourier transform for linear differential equations.

There is considerable literature on the subject and the interested reader is encouraged to consult, for instance, some of the following references: Ablowitz and Segur (1981), Calogero and Degasperis (1982), Novikov *et al.* (1984), Ablowitz and Clarkson (1991), Ablowitz *et al.* (2004).

Linear Stability Analysis

Consider a special solution of eqn [27] in the focusing (+sign) case: $q = a\exp(-2\mathrm{i}a^2 t)$. If this solution is perturbed as

$$q(x,t) = a\mathrm{e}^{2\mathrm{i}a^2 t}(1+\epsilon(x,t))$$

where $|\epsilon| \ll 1$, it is found that ϵ satisfies the condition

$$\mathrm{i}\epsilon_t = \epsilon_{xx} + 2a^2(\epsilon + \epsilon^*)$$

On the periodic spatial domain $0 < x < L$, ϵ has the Fourier expansion

$$\epsilon(x,t) = \sum_{-\infty}^{\infty} \hat{\epsilon}_n(t)\mathrm{e}^{\mathrm{i}\mu_n x}$$

where

$$\mu_n = \frac{2\pi n}{L} \qquad [30]$$

Assuming a solution of the form

$$\begin{pmatrix} \hat{\epsilon}_n \\ \hat{\epsilon}^*_{-n} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mathrm{e}^{\mathrm{i}\sigma_n t}$$

one finds that σ_n satisfies

$$\sigma_n^2 = \mu_n^2(\mu_n^2 - 4a^2) \qquad [31]$$

It then turns out that when $aL/\pi < n$ the system is unstable. Note that there are only a finite number of unstable modes (i.e., for fixed a, L, sufficiently high mode numbers n will not satisfy the above inequality). In the context of water waves, this corresponds to the famous experimental and theoretical result by Benjamin and Feir that the Stoke's water wave is unstable. Later, Benney and Roskes (1969) showed that all periodic wave solutions of the generalized nonlocal NLS equation resulting from water waves in $(2+1)$-dimensions are unstable. Also, in $(2+1)$-dimensions soliton solutions are unstable to weak transverse modulations.

Wave Collapse

The equation

$$i\psi_t + \Delta\psi + |\psi|^2\psi = 0, \qquad x \equiv (x, y) \in \mathbb{R}^2 \qquad [32]$$

has the following conserved quantities:

$$P = \int |\psi|^2 \, dx$$
$$M = \int \psi \nabla \psi \, dx$$
$$H = \int \left[|\nabla\psi|^2 - \frac{1}{2}|\psi|^4 \right] dx$$

that is, mass (power), momentum, and energy (Hamiltonian) are conserved. Remarkably, Talanov (1965) showed that eqn [32] satisfies the following equation:

$$\frac{\partial^2 V}{\partial t^2} = 8H \qquad [33]$$

where

$$V = \int (x^2 + y^2)|\psi|^2 \, dx \, dy$$

Equation [33] is also known as the "virial" theorem. Hence, it follows that

$$V = 4Ht^2 + c_1 t + c_2$$

and if $H < 0$ initially, then a singularity in eqn [32] results since V must be positive. Actually, one can further show (see, e.g., C Sulem and P L Sulem (1999), and references therein) that there exists a time t^* such that

$$\int |\nabla\psi|^2 \, dx$$

becomes infinite as $t \to t^*$, which in turn implies that ψ also becomes infinite as $t \to t^*$ (blowup in finite time).

Note also that for the more general equation

$$i\psi_t + \Delta_d\psi + |\psi|^{2\sigma}\psi = 0, \qquad x \in \mathbb{R}^d$$

where Δ_d is the d-dimensional Laplacian, one has the following types of solutions:

- *Supercritical* ($\sigma d > 2$): the solution blows up.
- *Critical* ($\sigma d = 2$): blowup can occur or global solution can exist.
- *Subcritical* ($\sigma d < 2$): global solutions exist.

Vector NLS Systems

In many applications vector NLS (VNLS) systems are the key governing equations. Physically, the VNLS arise under conditions similar to those described by NLS with the additional proviso that there are multiple wave trains moving nearly with the same group velocities (Roskes 1976). Importantly, VNLS also models systems where the field has more than one component. For example, in optical fibers and waveguides, the propagating electric field has two components transverse to the direction of propagation. The nondimensional system

$$iq_z^{(1)} = q_{xx}^{(1)} + 2\left(|q^{(1)}|^2 + |q^{(2)}|^2\right)q^{(1)} \qquad [34a]$$

$$iq_z^{(2)} = q_{xx}^{(2)} + 2\left(|q^{(1)}|^2 + |q^{(2)}|^2\right)q^{(2)} \qquad [34b]$$

is an asymptotic model which governs the propagation of the electric field in a waveguide, where z is the normalized distance along the waveguide and x a transversal spatial coordinate. It was first examined by Manakov (1974) (see also Anastassiou *et al.* (1999) and Soljačić *et al.* (2003)). Subsequently, this system was derived as a key model for light-wave propagation in optical fibers. More precisely, in optical fibers with constant birefringence (i.e., constant phase and group velocities as a function of distance) Menyuk (1987) has shown that the two polarization components of the electromagnetic field $\mathcal{E} = (u, v)^{\mathrm{T}}$ which are orthogonal to the direction of propagation, z, along the fiber asymptotically satisfy the following nondimensional equations (assuming anomalous dispersion):

$$i(u_z + \delta u_t) + \tfrac{1}{2}u_{tt} + (|u|^2 + \alpha|v|^2)u = 0 \qquad [35a]$$

$$i(v_z - \delta v_t) + \tfrac{1}{2}v_{tt} + (\alpha|u|^2 + |v|^2)u = 0 \qquad [35b]$$

where δ represents the group velocity "mismatch" between the u, v components of the electromagnetic field, α is a constant that depends on the polarization properties of the fiber, z the distance along the fiber, and t a retarded temporal frame. In deriving eqn [35], it is assumed that the electromagnetic field is slowly varying (as in the scalar problem); certain nonlinear (four-wave mixing) terms are neglected in the derivation of eqn [35], because the light wave is rapidly varying due to large, but constant, linear birefringence. In this context, birefringence means that the phase and group velocities of the electromagnetic wave in each polarization component are different. In a communications environment, due to the distances involved (hundreds to thousands of kilometers), the polarization properties evolve rapidly and randomly as the light wave evolves along the propagation distance, z. Not only does the birefringence evolve, but it does so randomly, and on a scale much faster than the distances required for communication transmission (birefringence polarization changes on a scale of 10–100 m). In this case, the relevant nonlinear equation is eqn [35] above, but with $\delta = 0$ and $\alpha = 1$. Indeed, this is the integrable VNLS equation first derived by Manakov (1974).

It should be remarked that the VNLS equation [34] and its generalization to an arbitrary number of components,

$$\mathrm{i}q_t = q_{xx} \pm 2\|q\|^2 q \qquad [36]$$

where q is an N-component vector and $\|\cdot\|$ is the Euclidean norm, are integrable by the IST. One has to suitably extend the analysis discussed earlier in this article (cf. e.g., Ablowitz *et al.* (2004)).

Discrete NLS Systems

Both the NLS and the VNLS equations discussed above admit integrable discretizations which, besides being used as the basis for constructing numerical schemes for the continuous counterparts, also have physical applications as discrete systems.

A natural discretization of NLS [27] is the following:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}q_n = \frac{1}{h^2}(q_{n+1} - 2q_n + q_{n-1}) \pm |q_n|^2(q_{n+1} + q_{n-1}) \qquad [37]$$

which is referred to as the integrable discrete NLS (IDNLS). It is an $O(h^2)$ finite-difference approximation of [27] which is integrable via the IST and has soliton solutions on the infinite lattice (Ablowitz and Ladik 1975, 1976). Note that if the nonlinear term in [37] is changed to $2|q_n|^2 q_n$, the equation, which is often called the discrete NLS (DNLS) equation, is apparently no longer integrable. It should be remarked that the (apparently nonintegrable) DNLS equation arises in many important physical contexts.

Correspondingly, one can consider the discretization of VNLS given by the following system:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}q_n = \frac{1}{h^2}(q_{n+1} - 2q_n + q_{n-1}) \pm \|q_n\|^2(q_{n+1} + q_{n-1}) \qquad [38]$$

where q_n is an N-component vector. Equation [38] for $q_n = q(nh)$ in the limit $h \to 0, nh = x$ gives VNLS [36]. The discrete vector NLS system [38] is also integrable (Ablowitz *et al.* 1999, Tsuchida *et al.* 1999). The interested reader can find further details in Ablowitz *et al.* (2004).

See also: Boundary-Value Problems for Integrable Equations; Dynamical Systems in Mathematical Physics: An Illustration from Water Waves; Evolution Equations: Linear and Nonlinear; Ginzburg–Landau Equation; Integrable Systems and Discrete Geometry; Integrable Systems: Overview; Partial Differential Equations: Some Examples; Riemann–Hilbert Methods in Integrable Systems; Schrödinger Operators.

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Non-Newtonian Fluids

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Introduction

The flow of a fluid, liquid or gas, is described by three conservation laws, the conserved physical quantities being the mass, the linear momentum, and the energy, and by constitutive equations. The constitutive equations are specific to each fluid, and link deformations to stresses.

A fluid is said to be Newtonian if it satisfies the simplest constitutive equation, which gives the stress tensor σ as a linear function of the rate of deformation tensor $\boldsymbol{D} = (1/2)(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})$, namely

$$\sigma = (\lambda \operatorname{tr} \boldsymbol{D} - p)\boldsymbol{I} + 2\eta \boldsymbol{D} \qquad [1]$$

where $\boldsymbol{u}$ is the fluid velocity, p is the hydrostatic pressure ($p \geq 0$), and λ and η are the Lamé viscosity coefficients of the fluid, satisfying $\eta \geq 0$ and $\lambda + 2\eta/3 \geq 0$. The superscript T designates the transpose operation, the abbreviation "tr" the trace operator of a tensor, and $\boldsymbol{I}$ the unit tensor. Water and glycerin are examples of Newtonian liquids.