Inverse scattering transform for the multi-component nonlinear Schrödinger equation with non-zero boundary conditions

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Abstract

The Inverse Scattering Transform (IST) for the defocusing vector nonlinear Schrödinger equation (NLS), with an arbitrary number of components and non-vanishing boundary conditions at space infinities, is formulated by adapting and generalizing the approach used by Beals, Deift and Tomei in the development of the IST for the N-wave interaction equations. Specifically, a complete set of sectionally meromorphic eigenfunctions is obtained from a family of analytic forms that are constructed for this purpose. As in the scalar and two-component defocusing NLS, the direct and inverse problems are formulated on a two-sheeted, genus-zero Riemann surface, which is then transformed into the complex plane by means of an appropriate uniformization variable. The inverse problem is formulated as a matrix Riemann-Hilbert problem with prescribed poles, jumps and symmetry conditions. In contrast to traditional formulations of the IST, the analytic forms and eigenfunctions are first defined for complex values of the scattering parameter, and extended to the continuous spectrum *a posteriori*.

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1 Introduction

Nonlinear Schrödinger (NLS) systems are prototypical dispersive nonlinear partial differential equations derived in many areas of physics (such as water waves, nonlinear optics, soft-condensed matter physics, plasma physics, etc) and analyzed mathematically for over forty years. The importance of NLS-type equations lies in their universal character, as generically speaking, most weakly nonlinear, dispersive, energy-preserving systems give rise, in an appropriate limit, to the NLS equation. Specifically, the NLS equation provides a "canonical" description for the envelope dynamics of a quasi-monochromatic plane wave propagating in a weakly nonlinear dispersive medium when dissipation can be neglected.

There are two inequivalent versions of the scalar NLS equation, depending on the dispersion regime, normal (defocusing) and anomalous (focusing). The focusing NLS equation admits the usual, bell-shaped solitons, while the defocusing NLS only admits soliton solutions with nontrivial boundary conditions. These solitons with non-zero boundary conditions are the so-called dark/gray solitons that appear as localized dips of intensity on a non-zero background field. The same is true for the vector (coupled) nonlinear Schrödinger (VNLS) equation. However, in the vector case the soliton zoology is richer. The focusing VNLS equation has vector bright soliton solutions, which, unlike scalar solitons, interact in a nontrivial way and may exhibit a polarization shift (i.e., an energy exchange among the components, cf. [18, 2]). The defocusing VNLS solitons include dark-dark soliton solutions, which have dark solitonic behavior in all components, as well as dark-bright soliton solutions, which have (at least) one dark and one (or more than one) bright components. Such solutions where first obtained by direct methods [12, 15, 17, 19], and spectrally characterized, in the 2-component case, in [16]. Dark-bright or dark-dark solitons in 2-component VNLS do not exhibit any polarization shift, but the situation might be different if at least one dark and more than one bright channel are present. This is one of the motivations for the present study of multicomponent defocusing NLS systems.

While the Inverse Scattering Transform (IST) as a method to solve the initial value problem for the scalar NLS equation was developed many years ago, both with vanishing and nonvanishing boundary conditions (BCs), the basic formulation of IST has not been fully developed for the VNLS equation:

$$i\mathbf{q}_t = \mathbf{q}_{xx} - 2\sigma\|\mathbf{q}\|^2\mathbf{q}\,, \tag{1.1}$$

where $\mathbf{q} = \mathbf{q}(x,t)$ is an N-component vector and $\|\cdot\|$ is the standard Euclidean norm. The focusing case ($\sigma = -1$) with vanishing BCs in two components was dealt with by Manakov in 1974 [13]. The IST for the VNLS with non-zero boundary conditions (NZBC) has been an open problem for over thirty years, and only the 2-component case was recently solved in [16] (partial results were obtained in [9]). The goal of this work is to present the IST on the full line ($-\infty < x < \infty$) for the N-component defocusing VNLS (1.1) with NZBC as $|x| \to \infty$: that is, $\mathbf{q}_\pm = \lim_{x\to\pm\infty}\mathbf{q} \neq 0$.

As is well-known, (1.1) admits a Lax pair for either dispersion regime, with dimension $N+1$. Already in the scalar case $N = 1$, the IST for the NLS equation with NZBC is complicated by the fact that the spectral parameter in the scattering problem is an element of a two-sheeted Riemann surface (instead of the

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complex plane, as it is customary in the case of zero BCs). In the scalar case, however, one still has two complete sets of analytic scattering eigenfunctions, and the direct and inverse problems can be carried out in more or less standard fashion, as shown in the early work by Zakharov and Shabat [21] (see also Faddeev and Takhtajan [8] for a more detailed treatment). When $N > 1$, however, $2(N-1)$ out of the $2(N+1)$ Jost eigenfunctions (defined as usual via Volterra integral equations) are not analytic, and one must somehow find a way to complete the eigenfunction basis. For the case $N = 2$, this last task was accomplished in [16] by generalizing the approach suggested by Kaup [10] for the three-wave interaction problem, and completing the basis of eigenfunctions with cross products of appropriate adjoint eigenfunctions. The major drawback of this approach, however, is that, at least in its present formulation, it is restricted to the case $N = 2$.

An alternative approach, used in [3, 4] for the N-wave interactions, makes use of Fredholm integral equations for the eigenfunctions. This approach, however, cannot be generalized "as is" to VNLS with NZBC, because the BCs $\mathbf{q}_\pm$ for the potential are in general different from each other. As a result, even though a bounded Green's function can be constructed, for instance, by asymmetric contour deformation in the plane of the scattering parameter, the convergence of an integral in x with either $\mathbf{q}-\mathbf{q}_-$ or $\mathbf{q}-\mathbf{q}_+$ will be assured only at one end. Therefore, to write down meaningful Fredholm integral equations, one should then first replace the given potential with one decaying smoothly at both ends, as suggested by Kawata and Inoue [11]. This process, however, introduces an "energy-dependent" potential, i.e., a potential with a complicated (though explicit) dependence on the scattering parameter, and it is not clear how to establish the analytic properties of eigenfunctions and scattering data with such a potential. Moreover, when $N \geq 3$, the eigenvalue associated with the nonanalytic scattering eigenfunctions becomes a multiple eigenvalue, with multiplicity $N-2$, in contrast to the case of the N-wave interaction, where all eigenvalues are assumed to be distinct.

The approach presented in this paper consists in generalizing to the VNLS with NZBC the methods developed by Beals, Deift and Tomei in [5] for general scattering and inverse scattering on the line with decaying potentials. Broadly speaking, the approach we propose is consistent with the usual development of IST. Namely, for the direct problem: (i) Find complete sets of sectionally meromorphic eigenfunctions for the scattering operator that are characterized by their asymptotic behavior. (ii) Identify a minimal set of data that describes the relations among these eigenfunctions, and which therefore defines the scattering data. For the inverse problem: reconstruct the scattering operator (and in particular the potential) from its scattering data. On the other hand, specific features of this approach, including those related to our extension of Beals, Deift and Tomei's work, are:

(a) A fundamentally different approach to direct and inverse scattering. Typically, eigenfunctions and scattering data are defined for values of the scattering parameter in the continuous spectrum (e.g., the real axis in the case of NLS with zero BCs), and are then extended to the complex plane. The approach used here will be exactly the opposite: the eigenfunctions are first defined away from the continuous spectrum, and the appropriate limits as the scattering parameter approaches the continuous spectrum are then evaluated.

(b) The use of forms (tensors constructed by wedge products of columns of the matrix eigenfunctions), which simplifies the analysis of the direct problem by reducing it to the study of Volterra equations.

(c) Departure from L^2-theory: as already pointed out and exploited in [16], bounded eigenfunctions are insufficient to characterize the discrete spectrum when the order of the scattering operator exceeds two.

The outline of this paper is the following. In Section 2 we state the problem and we introduce most of our notation. In Section 3: we define two fundamental tensor families associated with the scattering problem; we prove that they are analytic functions of the scattering parameter (Theorem 3.1) as well as point-wise decomposable (Lemma 3.2); we define the boundary data corresponding to the fundamental tensors (Theorems 3.2 and 3.3), we reconstruct two fundamental matrices of meromorphic eigenfunctions (Lemma 3.3, Theorems 3.4 and 3.5 and Corollaries 3.1–3.3); and we define a minimal set of scattering data (Theorems 3.6, 3.7 and 3.8). Finally, we describe the asymptotic behavior of the fundamental eigenfunctions with respect to the scattering parameter, and we discuss the symmetries of the scattering data. The inverse problem is formulated and formally linearized in Section 4, and the time evolution of the eigenfunctions and scattering data is derived in Section 5. In Section 6 we compare the results obtained in [16] for the direct problem in the 2-component case with the construction via fundamental tensors developed here. Section 7 offers some final remarks. Throughout, the body of the paper contains the logical steps of the method. All proofs are deferred to Appendix A, while Appendix B contains the derivation of the asymptotic behavior of the eigenfunctions via WKB expansions.

2 Scattering problem and preliminary considerations

2.1 Boundary conditions, eigenvalues and asymptotic eigenvectors

The Lax pair for the N-component defocusing VNLS equation [that is, Eq. (1.1) with $\sigma = 1$] is

$$v_x = \mathbf{L}v\,, \tag{2.1a}$$
$$v_t = \mathbf{T}v\,, \tag{2.1b}$$

with

$$\mathbf{L} = ik\mathbf{J} + \mathbf{Q} = \begin{pmatrix} -ik & \mathbf{q}^T \\ \mathbf{r} & ik\mathbf{I}_N \end{pmatrix}, \tag{2.2a}$$
$$\mathbf{T} = -2ik^2\mathbf{J} - i\mathbf{J}\mathbf{Q}^2 - 2k\mathbf{Q} - i\mathbf{J}\mathbf{Q}_x = \begin{pmatrix} 2ik^2 + i\mathbf{q}^T\mathbf{r} & -2k\mathbf{q}^T - i\mathbf{q}_x^T \\ -2k\mathbf{r} + i\mathbf{r}_x & -2ik^2\mathbf{I}_N - i\mathbf{r}\mathbf{q}^T \end{pmatrix}, \tag{2.2b}$$

where the superscripts x and t denote partial differentiation, $v = v(x,t,k)$ is an $N{+}1$-component vector, $\mathbf{I}_N$ is the $N \times N$ identity matrix,

$$\mathbf{J} = \operatorname{diag}\big(-1, \underbrace{1,\ldots,1}_{N}\big)\,, \qquad \mathbf{Q} = \begin{pmatrix} 0 & \mathbf{q}^T \\ \mathbf{r} & \mathbf{0}_N \end{pmatrix}, \tag{2.3a}$$
$$\mathbf{q}^T = (q_1,\ldots,q_N)\,, \qquad \mathbf{r} = \mathbf{q}^*\,, \tag{2.3b}$$

the asterisk denotes the complex conjugate and the superscript T denotes matrix transpose. The compatibility of the system of equations (2.1) [i.e., the equality of the mixed derivatives of the $(N+1)$-component vector v with respect to x and t], together with the constraints of constant k and $\mathbf{r} = \mathbf{q}^*$, is equivalent to requirement that $\mathbf{q}(x,t)$ satisfies (1.1) with $\sigma = 1$. As usual, (2.1a) is referred to as the scattering problem.

We consider potentials $\mathbf{q}(x,t)$ with non-zero boundary conditions at space infinity:

$$\lim_{x\to\pm\infty} \mathbf{Q}(x,t) = \mathbf{Q}_\pm(t) \equiv \begin{pmatrix} 0 & \mathbf{q}_\pm^T(t) \\ \mathbf{r}_\pm(t) & \mathbf{0}_N \end{pmatrix}, \tag{2.4}$$

with $\|\mathbf{q}_+\| = \|\mathbf{q}_-\| = q_0 \in \mathbb{R}^+$. Specifically, we restrict our attention to potentials in which the asymptotic phase difference is the same in all components, i.e., solutions such that

$$\mathbf{q}_\pm = \mathbf{q}_0\, e^{i\theta^\pm}\,, \tag{2.5}$$

with $\theta^{\pm} \in \mathbb{R}$. While this constraint significantly simplifies the analysis, it does not exclude multicomponent configurations with both vanishing and non-vanishing boundary conditions (such as for dark-bright solitons). Moreover, we assume that for all $t \geq 0$ the potentials are such that $(\mathbf{q}(\cdot,t) - \mathbf{q}_{-}(t)) \in L^1(-\infty, c)$ and $(\mathbf{q}(\cdot,t) - \mathbf{q}_{+}(t)) \in L^1(c,\infty)$ for all $c \in \mathbb{R}$, where the L^1 functional classes are defined as usual by:

$$L^1(a,b) = \left\{ \mathbf{f} : (a,b) \to \mathbb{C}^n \ : \quad \int_a^b \|\mathbf{f}(x)\| dx < \infty \right\}. \tag{2.6}$$

To deal efficiently with the above non-vanishing potentials as $x \to \pm\infty$, it is useful to introduce the asymptotic Lax operators

$$\mathbf{L}_{\pm} = ik\mathbf{J} + \mathbf{Q}_{\pm} \tag{2.7}$$

and write the scattering problem in (2.1a) in the form

$$v_x = \mathbf{L}_{\pm} v + (\mathbf{Q} - \mathbf{Q}_{\pm}) v \,, \tag{2.8}$$

where $\mathbf{L}_{\pm}$ is independent of x and $(\mathbf{Q} - \mathbf{Q}_{\pm}) \to 0$ sufficiently rapidly as $x \to \pm\infty$ for all $t \geq 0$. The eigenvalues of the asymptotic scattering problems $v_x = \mathbf{L}_{\pm} v$ are the elements of the diagonal matrix $i\mathbf{\Lambda}$ where

$$\mathbf{\Lambda}(k) = \mathrm{diag}(-\lambda, \underbrace{k, \ldots, k}_{N-1}, \lambda) \,, \tag{2.9}$$

and λ is a solution of

$$\lambda^2 = k^2 - q_0^2 \,. \tag{2.10}$$

The corresponding asymptotic eigenvectors can be chosen to be the columns of the respective $(N{+}1) \times (N{+}1)$ matrix

$$\mathbf{E}_{\pm}(k) = \begin{pmatrix} k+\lambda & \mathbf{0}_{1\times(N-1)} & k-\lambda \\ i\mathbf{r}_{\pm} & \mathbf{R}_0^{\perp} & i\mathbf{r}_{\pm} \end{pmatrix} \,, \tag{2.11}$$

where $\mathbf{R}_0^{\perp}$ is an $N \times (N{-}1)$ matrix each of whose $N{-}1$ columns is an N-component vector orthogonal to $\mathbf{r}_{\pm}$, i.e., according to (2.5):

$$\mathbf{r}_0^{\dagger} \mathbf{R}_0^{\perp} = \mathbf{0}_{1\times(N-1)} \,, \tag{2.12}$$

where the dagger signifies conjugate transpose. Then, by construction, it is

$$\mathbf{L}_{\pm} \mathbf{E}_{\pm} = \mathbf{E}_{\pm} \, i\mathbf{\Lambda} \,. \tag{2.13}$$

The condition (2.12) does not uniquely determine the matrix $\mathbf{R}_0^{\perp}$. It will be convenient to also require that the columns of $\mathbf{R}_0^{\perp}$ be mutually orthogonal, and each of norm q_0. That is, we take $\mathbf{R}_0^{\perp}$ to be such that $(\mathbf{R}_0^{\perp})^{\dagger} \mathbf{R}_0^{\perp} = q_0^2 \mathbf{I}_{N-1}$. This ensures that all the corresponding columns of $\mathbf{E}_{\pm}$ are orthogonal to its first and last columns. Note, however, that the first and last columns of $\mathbf{E}_{\pm}$ are *not* orthogonal to each other, which is a consequence of the fact that the matrix $\mathbf{L}_{\pm}$ is not normal. More in general, the scattering matrix $\mathbf{L}$ in (2.2a) is only self-adjoint for $k \in \mathbb{R}$, and, as a result, eigenvectors corresponding to distinct eigenvalues are not necessarily orthogonal to each other for $k \notin \mathbb{R}$. Note, however, that any two solutions $v(x,t,k)$ and $w(x,t,k)$ of the scattering problem are $\mathbf{J}$-orthogonal, namely:

$$\frac{\partial}{\partial x} \left(w^{\dagger}(x,t,k^*) \, \mathbf{J} \, v(x,t,k) \right) = 0 \,. \tag{2.14}$$

Therefore, if the two eigenfunctions are $\mathbf{J}$-orthogonal either as $x \to -\infty$ or as $x \to \infty$, their $\mathbf{J}$-orthogonality is preserved for all $x \in \mathbb{R}$.

2.2 Complexification

As in the scalar case [8, 21] and the 2-component case [16], the continuum spectrum of the scattering operator consists of all values of k such that the eigenvalue $\lambda(k) \in \mathbb{R}$, i.e., all $k \in \mathbb{R}$ with $|k| > q_0$. On the other hand, equation (2.10) does not uniquely define λ as a function of the complex variable k. To deal with the resulting loss of analyticity and recover the single-valuedness of λ in terms of k, it is therefore necessary to take k to be an element of a two-sheeted Riemann surface $\hat{\mathbb{C}}$. As usual, this Riemann surface is defined by "gluing" two copies of the complex k-plane, each containing a branch cut that connects the two branch points $\pm q_0$ through the point at infinity. We refer to the two sheets and to the branch cut respectively as $\mathbb{C}_{\mathrm{I}}$, $\mathbb{C}_{\mathrm{II}}$ and

$$\Sigma = (-\infty + i0, -q_0 + i0] \cup [-q_0 - i0, -\infty - i0) \cup (\infty + i0, q_0] \cup [q_0 - i0, \infty - i0)\,. \tag{2.15}$$

This choice for the cut results in the relations:

$$\begin{aligned} &\operatorname{Im}\lambda(k) > 0 \quad \text{and} \quad \operatorname{Im}(\lambda(k) \pm k) > 0 \qquad \forall k \in \mathbb{C}_{\mathrm{I}}\,,\\ &\operatorname{Im}\lambda(k) < 0 \quad \text{and} \quad \operatorname{Im}(\lambda(k) \pm k) < 0 \qquad \forall k \in \mathbb{C}_{\mathrm{II}}\,. \end{aligned}$$

(See [16] for further details.) Note that the construction of this Riemann surface is also necessary in the development of the IST for both scalar NLS [8, 21] and 2-component VNLS [16].

Following [5], on each sheet of the Riemann surface we order the eigenvalues and the corresponding eigenvectors by the decay rate of the corresponding solution of the scattering problem as $x \to -\infty$. More precisely, the ordering of the eigenvalues and eigenvectors implied by (2.9) and (2.11) provides maximal decay when $k \in \mathbb{C}_{\mathrm{I}}$, in the sense that for $k \notin \Sigma$ and $n = 1, \ldots, N+1$, the eigenfunction associated with the eigenvalue λ_n decays at least as fast as the one associated to λ_{n+1} as $x \to -\infty$. For $k \in \mathbb{C}_{\mathrm{II}}$, it is necessary to switch the first and last eigenvalue and the first and last eigenvector to achieve the desired maximal decay. We then define the eigenvalue matrix on the entire Riemann surface as:

$$\mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N, \lambda_{N+1})\,, \tag{2.16}$$

where $\lambda_1, \ldots, \lambda_{N+1}$ are given by (2.9) for $k \in \mathbb{C}_{\mathrm{I}}$ [namely, $\lambda_1 = -\lambda$ and $\lambda_{N+1} = \lambda$], while $\lambda_1 = \lambda$ and $\lambda_{N+1} = -\lambda$ for $k \in \mathbb{C}_{\mathrm{II}}$, and $\lambda_2 = \cdots = \lambda_N = k$ for all $k \in \hat{\mathbb{C}}$ regardless of the sheet. Correspondingly, the associated matrices of eigenvectors $\mathbf{E}_\pm(k)$ are defined by (2.11) for $k \in \mathbb{C}_{\mathrm{I}}$, and by

$$\mathbf{E}_\pm(k) = \begin{pmatrix} k-\lambda & \mathbf{0}_{1\times(N-1)} & k+\lambda \\ i\mathbf{r}_\pm & i\mathbf{R}_0^\perp & i\mathbf{r}_\pm \end{pmatrix}, \qquad k \in \mathbb{C}_{\mathrm{II}}\,. \tag{2.17}$$

The $(N+1) \times (N+1)$ matrix that switches the order of the eigenvectors and eigenvalues is simply

$$\boldsymbol{\pi} = (e_{N+1}, e_2, \ldots, e_N, e_1) = \begin{pmatrix} 0 & \mathbf{0}_{1\times(N-1)} & 1 \\ \mathbf{0}_{(N-1)\times 1} & \mathbf{I}_{N-1} & \mathbf{0}_{(N-1)\times 1} \\ 1 & \mathbf{0}_{1\times(N-1)} & 0 \end{pmatrix}, \tag{2.18}$$

where $e_1, \ldots, e_{N+1}$ are the vectors of the canonical basis of $\mathbb{C}^{N+1}$. That is, $\mathbf{\Lambda}^{\mathrm{I}} = \mathbf{\Lambda}^{\mathrm{II}}\,\boldsymbol{\pi}$ and $\mathbf{E}_\pm^{\mathrm{I}} = \mathbf{E}_\pm^{\mathrm{II}}\,\boldsymbol{\pi}$, where the superscripts I and II denote the values of the corresponding matrices on $\mathbb{C}_{\mathrm{I}}$ and $\mathbb{C}_{\mathrm{II}}$. Note that $\boldsymbol{\pi}$ is symmetric and an involution; i.e., $\boldsymbol{\pi}^{-1} = \boldsymbol{\pi}^T = \boldsymbol{\pi}$.

Denoting the columns of $\mathbf{E}_\pm$ by $e_1^\pm, \ldots, e_{N+1}^\pm$, the constraint (2.5) on the boundary conditions implies the following relations among the asymptotic eigenvectors $\mathbf{E}_-$ and $\mathbf{E}_+$:

$$e_n^+ \equiv e_n^-\,, \qquad \forall n = 2, \ldots, N \tag{2.19a}$$

while

$$\left(e_1^- \; e_{N+1}^-\right) = \left(e_1^+ \; e_{N+1}^+\right) \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}, \tag{2.19b}$$

or equivalently

$$\left(e_1^+ \; e_{N+1}^+\right) = e^{-i\Delta\theta} \left(e_1^- \; e_{N+1}^-\right) \begin{pmatrix} \eta_{22} & -\eta_{12} \\ -\eta_{21} & \eta_{11} \end{pmatrix}, \tag{2.19c}$$

with

$$\eta_{11}(k) = \frac{1}{2\lambda_1}\left[\lambda_1 - k + (\lambda_1 + k)e^{i\Delta\theta}\right], \qquad \eta_{12}(k) = \frac{\lambda_1 + k}{2\lambda_1}\left(e^{i\Delta\theta} - 1\right), \tag{2.20a}$$

$$\eta_{22}(k) = \frac{1}{2\lambda_1}\left[\lambda_1 + k + (\lambda_1 - k)e^{i\Delta\theta}\right], \qquad \eta_{21}(k) = \frac{\lambda_1 - k}{2\lambda_1}\left(e^{i\Delta\theta} - 1\right), \tag{2.20b}$$

and where $\Delta\theta = \theta^+ - \theta^-$ denotes the asymptotic phase difference in the potential.

2.3 Uniformization coordinate

Following [8, 16], in order to deal effectively with the Riemann surface we define a map from $\hat{\mathbb{C}}$ to the complex plane via the variable z (global uniformizing parameter):

$$z = k + \lambda(k), \tag{2.21a}$$

with inverse mapping

$$k = \tfrac{1}{2}(z + q_0^2/z), \qquad \lambda = z - k = \tfrac{1}{2}\left(z - q_0^2/z\right). \tag{2.21b}$$

With this mapping:

(i) The branch cut Σ on the two sheets of the Riemann surface is mapped onto the real z-axis.

(ii) The sheet $\mathbb{C}_{\mathrm{I}}$ is mapped onto the upper half of the complex z-plane, while $\mathbb{C}_{\mathrm{II}}$ is mapped to the lower half plane.

(iii) A half-neighborhood of $k = \infty$ on either sheet is mapped onto a half-neighborhood of either $z = \infty$ or $z = 0$, depending on the sign of $\operatorname{Im} k$.

(iv) The transformation $k - i0 \to k + i0$ for $k \in \Sigma$ (which changes the value of any function to its value on the opposite edge of the cut) is equivalent to the transformation $z \to q_0^2/z$ on the real z-axis.

(v) The segments $[-q_0, q_0]$ in $\mathbb{C}_{\mathrm{I}}$ and $\mathbb{C}_{\mathrm{II}}$ are mapped respectively onto the upper and lower half of the circle C_0 of radius q_0 centered at the origin.

In terms of the uniformization variable z, the matrix of asymptotic eigenvectors $\mathbf{E}_\pm$ is

$$\mathbf{E}_\pm(z) = \begin{pmatrix} z & \mathbf{0}_{1\times(N-1)} & q_0^2/z \\ i\mathbf{r}_\pm & i\mathbf{R}_0^\perp & i\mathbf{r}_\pm \end{pmatrix}, \qquad \operatorname{Im} z > 0, \tag{2.22a}$$

$$\mathbf{E}_\pm(z) = \begin{pmatrix} q_0^2/z & \mathbf{0}_{1\times(N-1)} & z \\ i\mathbf{r}_\pm & i\mathbf{R}_0^\perp & i\mathbf{r}_\pm \end{pmatrix}, \qquad \operatorname{Im} z < 0. \tag{2.22b}$$

Moreover, (2.20) gives

$$\eta_{11}(z) = \begin{cases} \dfrac{z^2 - q_0^2 e^{i\Delta\theta}}{z^2 - q_0^2} & \operatorname{Im} z > 0, \\[2ex] \dfrac{z^2 e^{i\Delta\theta} - q_0^2}{z^2 - q_0^2} & \operatorname{Im} z < 0, \end{cases} \tag{2.23}$$

with similar expression for the other coefficients. The uniformization coordinate simplifies the description of the asymptotic behavior of the eigenfunctions with respect to the scattering parameter (cf. Section 3.5), and it is crucial in our formulation of the inverse problem (cf. Section 4).

3 Direct problem

3.1 Fundamental matrix solutions

As usual, in the direct and inverse problems the temporal variable t is kept fixed. Consequently, we will systematically omit the time dependence of eigenfunctions and scattering data. As mentioned earlier, our approach for the direct problem will be to define complete sets of scattering eigenfunctions *off the cut*, i.e., for $k \in \hat{\mathbb{C}} \setminus \Sigma$, and then to consider the limits of the appropriate quantities as $k \to \Sigma$ from either sheet. More specifically, the problem of determining complete sets of analytic/meromorphic eigenfunctions is formulated as follows.

For a given $k \in \hat{\mathbb{C}} \setminus \Sigma$, we seek to determine a matrix solution $\Phi(x,k)$ of the scattering problem

$$\Phi_x = \mathbf{L}_- \Phi + (\mathbf{Q} - \mathbf{Q}_-)\Phi \,, \tag{3.1a}$$

with the asymptotic properties

$$\lim_{x \to -\infty} \Phi(x,k) e^{-i\boldsymbol{\Lambda} x} = \mathbf{E}_- \,, \tag{3.1b}$$

$$\limsup_{x \to +\infty} \left\| \Phi(x,k) e^{-i\boldsymbol{\Lambda} x} \right\| < \infty \,, \tag{3.1c}$$

where the asymptotic Lax operator is given by (2.7) and the asymptotic boundary condition (3.1b) is specified by (2.11) for $k \in \mathbb{C}_{\mathrm{I}}$ and by (2.17) for $k \in \mathbb{C}_{\mathrm{II}}$. Similarly, we seek to determine a matrix solution $\tilde{\Phi}(x,k)$ of

$$\tilde{\Phi}_x = \mathbf{L}_+ \tilde{\Phi} + (\mathbf{Q} - \mathbf{Q}_+)\tilde{\Phi} \,, \tag{3.2a}$$

with the asymptotic properties

$$\lim_{x \to +\infty} \tilde{\Phi}(x,k)\, e^{-i\boldsymbol{\Lambda} x} = \mathbf{E}_+ \,, \tag{3.2b}$$

$$\limsup_{x \to -\infty} \left\| \tilde{\Phi}(x,k)\, e^{-i\boldsymbol{\Lambda} x} \right\| < \infty \,. \tag{3.2c}$$

As usual, it is convenient to rewrite these problems without the asymptotic exponentials. Hence, we define

$$\Phi(x,k) = \boldsymbol{\mu}(x,k)\, e^{i\boldsymbol{\Lambda} x}, \tag{3.3a}$$

$$\tilde{\Phi}(x,k) = \tilde{\boldsymbol{\mu}}(x,k)\, e^{i\boldsymbol{\Lambda} x} \,. \tag{3.3b}$$

With these substitutions, the problems (3.1) and (3.2) can be stated as follows. Given $k \in \hat{\mathbb{C}} \setminus \Sigma$, we want to determine matrix functions $\boldsymbol{\mu}(x,k)$ and $\tilde{\boldsymbol{\mu}}(x,k)$ such that

$$\partial_x \boldsymbol{\mu} = \mathbf{L}_- \boldsymbol{\mu} - i\boldsymbol{\mu}\, \boldsymbol{\Lambda} + (\mathbf{Q} - \mathbf{Q}_-)\, \boldsymbol{\mu} \,, \tag{3.4a}$$

$$\lim_{x \to -\infty} \boldsymbol{\mu}(x,k) = \mathbf{E}_- \,, \tag{3.4b}$$

$$\boldsymbol{\mu}(x,k) \text{ is bounded for all } x\,, \tag{3.4c}$$

while

$$\partial_x \tilde{\boldsymbol{\mu}} = \mathbf{L}_+ \tilde{\boldsymbol{\mu}} - i\tilde{\boldsymbol{\mu}}\,\boldsymbol{\Lambda} + (\mathbf{Q} - \mathbf{Q}_+)\tilde{\boldsymbol{\mu}}\,, \tag{3.5a}$$

$$\lim_{x\to\infty} \tilde{\boldsymbol{\mu}}(x,k) = \mathbf{E}_+\,, \tag{3.5b}$$

$$\tilde{\boldsymbol{\mu}}(x,k) \text{ is bounded for all } x\,. \tag{3.5c}$$

We then give the following definition:

Definition 3.1 *A fundamental matrix solution (or simply fundamental matrix) for the operator* $\mathbf{L}$ *and the point* $k \in \hat{\mathbb{C}} \setminus \Sigma$ *is a solution* $\boldsymbol{\mu}(x,k)$ *of* (3.4) *or a solution* $\tilde{\boldsymbol{\mu}}(x,k)$ *of* (3.5).

We emphasize that, even though (3.4a) and (3.5a) are both an $(N{+}1)$-order linear system of differential equations, the respective sets of $(N{+}1)$ boundary conditions (3.4b) and (3.5b) are *not*, in general, sufficient to uniquely determine a solution. For example, a term proportional to the first column of $\boldsymbol{\mu}$, which corresponds to the solution of (2.1a) with maximal decay as $x \to -\infty$, can be added to any of the other columns without affecting the boundary conditions (3.4b). (This situation is often expressed by referring to such contributions as "subdominant" terms.) In general, however, these solutions will grow without bound as $x \to \infty$. Hence, the additional requirement of boundedness (3.4c) is imposed in order to remove the degeneracy and uniquely determine a solution. Similar considerations hold for (3.5). The following lemma asserts the uniqueness of such solutions:

Lemma 3.1 *For* $k \in \hat{\mathbb{C}} \setminus \Sigma$*, each of the problems* (3.4) *and* (3.5) *has at most one solution.*

Because $\partial_x[\det \boldsymbol{\mu}] = \partial_x[\det \tilde{\boldsymbol{\mu}}] = 0$ and $\det \mathbf{E}_\pm \neq 0$, the columns of the fundamental matrices are linearly independent for all $x \in \mathbb{R}$. Therefore, consistent with our terminology, any solution of (3.4a) or (3.5a) (which are, in fact, equivalent) can be written in terms of either the fundamental matrix $\boldsymbol{\mu}$ or of $\tilde{\boldsymbol{\mu}}$. A similar statement applies to the solutions of (3.1a) and (3.2a).

As with the eigenvector matrices, we use subscripts to denote the columns of the fundamental matrix solutions. That is, we write $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{N+1})$ and $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1 \ldots, \tilde{\mu}_{N+1})$, where μ_n and $\tilde{\mu}_n$ denote the n-th columns of $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$, respectively. For $k \in \hat{\mathbb{C}} \setminus \Sigma$, the corresponding vectors satisfy the following differential equations:

$$\partial_x \mu_n = [\mathbf{L}_- - i\lambda_n \mathbf{I} + (\mathbf{Q} - \mathbf{Q}_-)]\,\mu_n\,, \tag{3.6a}$$

$$\partial_x \tilde{\mu}_n = [\mathbf{L}_+ - i\lambda_n \mathbf{I} + (\mathbf{Q} - \mathbf{Q}_+)]\,\tilde{\mu}_n\,, \tag{3.6b}$$

for all $n = 1, \ldots, N+1$. From these equations, one could formulate Volterra integral equations whose solutions satisfy the original differential equations as well as the boundary conditions corresponding to either (3.4b) or (3.5b). Unfortunately, with the exception of the solutions with maximal asymptotic decay as either $x \to -\infty$ or $x \to +\infty$ (i.e., μ_1 and $\tilde{\mu}_{N+1}$), the resulting Volterra integral equations cannot, in general, be shown to admit solutions when $k \notin \Sigma$. Hence, to construct a complete set of analytic/meromorphic eigenfunctions, we are required to use a different approach.

3.2 Fundamental tensors

Let us begin by recalling some well-known results in tensor algebra [14]. The elements of the exterior algebra

$$\bigwedge(\mathbb{C}^{N+1}) = \bigoplus_{n=1}^{N+1} \bigwedge^n(\mathbb{C}^{N+1})$$

are n-forms (with $n = 1, \ldots, N+1$) constructed from the vector space $\mathbb{C}^{N+1}$ via the wedge product. Linear operators on $\mathbb{C}^{N+1}$ can be extended to act on the elements of the algebra. Specifically, a linear transformation $A : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ defines uniquely two linear maps from $\bigwedge(\mathbb{C}^{N+1})$ onto itself:

(i) a map $A^{(n)}$ such that, for all $u_1, \ldots, u_n \in \mathbb{C}^{N+1}$,

$$A^{(n)}\,(u_1 \wedge \cdots \wedge u_n) = \sum_{j=1}^{n} u_1 \wedge \cdots \wedge u_{j-1} \wedge Au_j \wedge u_{j+1} \wedge \cdots \wedge u_n\,; \tag{3.7}$$

(ii) a map A such that, for all $u_1, \ldots, u_n \in \mathbb{C}^{N+1}$,

$$A\,(u_1 \wedge \cdots \wedge u_n) = Au_1 \wedge \cdots \wedge Au_n\,, \tag{3.8}$$

where, with a slight abuse of notation, we used the same symbol to denote both the original operator and its extension to the tensor space.

With these definitions, one can extend linear systems of differential equations such as (3.1) and (3.2) to the tensor algebra $\bigwedge(\mathbb{C}^{N+1})$. We next show that such extended differential equations can be shown to admit unique solutions.

Given the columns of the fundamental matrices $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$, we can define the (totally antisymmetric) tensors:

$$f_n = \mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n\,, \tag{3.9a}$$
$$g_n = \tilde{\mu}_n \wedge \tilde{\mu}_{n+1} \wedge \cdots \wedge \tilde{\mu}_{N+1}\,, \tag{3.9b}$$

for all $n = 1, \ldots, N+1$. In the absence of an existence proof for the individual (vector) solutions (which is our object), the above definition is only formal. Nonetheless, via this construction, the differential equations (3.6) imply that the above tensors (if they exist) satisfy the *extended* differential equations

$$\partial_x f_n = \Big[\mathbf{L}_-^{(n)} - i(\lambda_1 + \cdots + \lambda_n)\mathbf{I}\Big]\, f_n + \Big[\mathbf{Q}^{(n)} - \mathbf{Q}_-^{(n)}\Big]\, f_n\,, \tag{3.10a}$$
$$\partial_x g_n = \Big[\mathbf{L}_+^{(N-n+2)} - i(\lambda_n + \cdots + \lambda_{N+1})\mathbf{I}\Big]\, g_n + \Big[\mathbf{Q}^{(N-n+2)} - \mathbf{Q}_+^{(N-n+2)}\Big]\, g_n\,, \tag{3.10b}$$

respectively, where $\mathbf{L}_\pm^{(n)}$, $\boldsymbol{\Lambda}^{(n)}$, $\mathbf{Q}^{(n)}$ and $\mathbf{Q}_\pm^{(n)}$ denote the n-th order extensions (3.7) of $\mathbf{L}_\pm$, $\boldsymbol{\Lambda}$, $\mathbf{Q}$ and $\mathbf{Q}_\pm$ to $\bigwedge^n(\mathbb{C}^{N+1})$. Similarly, the boundary conditions (3.4b) and (3.5b) imply that

$$\lim_{x\to-\infty} f_n(x,k) = e_1^- \wedge \cdots \wedge e_n^-\,, \tag{3.11a}$$
$$\lim_{x\to\infty} g_n(x,k) = e_n^+ \wedge \cdots \wedge e_{N+1}^+\,, \tag{3.11b}$$

respectively, where (as above) $e_j^\pm$ denotes the j-th column of the matrix $\mathbf{E}_\pm$. We now reverse the logic and use the tensor differential equations (3.10) and boundary conditions (3.11) as a *definition* of f_n and g_n, namely:

Definition 3.2 *A fundamental tensor family for the operator* $\mathbf{L}$ *and a point* $k \in \hat{\mathbb{C}} \setminus \Sigma$ *is a set of solutions* $\{f_n\,, g_n\}_{n=1,\ldots,N+1}$ *to* (3.10) *and* (3.11).

Unlike the vectors defined by equations (3.6) (or the equivalent integral equations), the elements of the fundamental tensor family *are* analytic functions of k, as described by the following theorem:

Theorem 3.1 *(Fundamental tensors) For each $k \in \hat{\mathbb{C}} \setminus \Sigma$ there exists a unique fundamental family of tensors for* $\mathbf{L}$. *On each of the sheets $\mathbb{C}_{\mathrm{I}} \setminus \Sigma$ and $\mathbb{C}_{\mathrm{II}} \setminus \Sigma$, the elements of the family are* analytic *functions of k. Moreover, the elements of the family extend smoothly to Σ from each sheet, and these extensions also satisfy the boundary conditions* (3.11).

As a "stand in" for the boundary conditions for the fundamental matrix solutions, the two parts of the fundamental family (namely, the f_n and the g_n) are defined by boundary conditions at opposite limits of x. Thus, equations that relate the members of the two parts of the family will provide a kind of spectral data (dependent on k, but independent of x) for the scattering potential. The following theorem defines and describes such data.

Theorem 3.2 *(Spectral data) There exist scalar functions $\Delta_1(k), \dots, \Delta_N(k)$, analytic on $\hat{\mathbb{C}} \setminus \Sigma$, with smooth extensions to $\Sigma \setminus \{\pm q_0\}$ from each sheet and such that, for all $n = 1, \dots, N$,*

$$f_n(x,k) \wedge g_{n+1}(x,k) = \Delta_n(k)\gamma_n(k)\, e_1 \wedge \cdots \wedge e_{N+1}\,, \tag{3.12}$$

where

$$\gamma_n(k) = \det\left(e_1^-, \dots, e_n^-, e_{n+1}^+, \dots, e_{N+1}^+\right) = 2i\lambda_1 q_0^N \eta_{11} e^{-i\theta_+}\,. \tag{3.13}$$

Moreover, for all $k \in \hat{\mathbb{C}}$ and all $x \in \mathbb{R}$,

$$f_{N+1}(x,k) \equiv e_1^- \wedge \cdots \wedge e_{N+1}^- = e^{i\Delta\theta}\, e_1^+ \wedge \cdots \wedge e_{N+1}^+\,, \tag{3.14a}$$

$$g_1(x,k) \equiv e_1^+ \wedge \cdots \wedge e_{N+1}^+ = e^{-i\Delta\theta}\, e_1^- \wedge \cdots \wedge e_{N+1}^-\,. \tag{3.14b}$$

Considering f_{N+1} and g_1 to be the extensions of the left-hand side of (3.12) to $n = N+1$ and $n = 0$, respectively, we define, consistently with the above theorem:

$$\Delta_0(k) = 1\,, \qquad \Delta_{N+1}(k) = 1\,, \tag{3.15a}$$

$$\gamma_0 = \det \mathbf{E}_+ = 2i\lambda_1 \det(\mathbf{R}_0^\perp, \mathbf{r}_+) = 2i\lambda_1 q_0^N\, e^{-i\theta_+}\,, \tag{3.15b}$$

$$\gamma_{N+1} = \det \mathbf{E}_- = 2i\lambda_1 \det(\mathbf{R}_0^\perp, \mathbf{r}_-) = 2i\lambda_1 q_0^N\, e^{-i\theta_-}. \tag{3.15c}$$

Also, we will denote the smooth extensions of the functions $\Delta_j(k)$ to $\Sigma \setminus \{\pm q_0\}$ as $\Delta_j^\pm(k)$, depending on whether the limit is taken from $\mathbb{C}_{\mathrm{I}}$ or $\mathbb{C}_{\mathrm{II}}$, respectively.

Note that (3.12) defines the $\Delta_n(k)$'s only for those values of k for which $\gamma_n(k) \neq 0$. Therefore, in principle, it would be necessary to exclude the points of $\hat{\mathbb{C}} \setminus \Sigma$ where $\eta_{11}(k) = 0$, i.e., the points $k = q_0 \cos(\Delta\theta/2)$ on each sheet. These points, however, will not play any role in what follows, as the Δ_n will be multiplied by either γ_n or η_{11}, or they will appear in ratios such that the behavior at these points cancels out.

On the other hand, the behavior at the branch points, $k = \pm q_0$, warrants some attention. In the scalar case, for instance, Faddeev and Takhajan [8] showed that, while the eigenfunctions are continuous also at the branch points, the scattering coefficients generically have simple poles at $k = \pm q_0$. (When the residues are zero, such that the poles are absent, the branch points are called virtual levels.) Here, as in [16], we assume that all scattering data are also continuous at the branch points.

Next we determine the asymptotic behavior of the fundamental tensors in the limit where x goes to the opposite infinity. This behavior is considerably more complex than in the scalar and 2-component cases, due to the fact that ik is an eigenvalue of the scattering problem with multiplicity $N-2$. As a result, the boundary conditions satisfied by the fundamental tensors contain a summation of terms in the subspace of the eigenvectors associated with the repeated eigenvalue, as described in the following theorem:

Theorem 3.3 *(Boundary data) For all $k \in \hat{\mathbb{C}} \setminus \Sigma$ it is*

$$\lim_{x\to+\infty} f_1(x,k) = \delta_\emptyset\, e_1^+\,, \tag{3.16a}$$

$$\lim_{x\to+\infty} f_n(x,k) = \sum_{2\le j_2<j_3<\cdots<j_n\le N} \delta_{j_2,\ldots,j_n}\, e_1^+ \wedge e_{j_2}^+ \wedge \cdots \wedge e_{j_n}^+\,, \qquad n = 2,\ldots,N\,, \tag{3.16b}$$

and

$$\lim_{x\to-\infty} g_{N+1}(x,k) = \tilde{\delta}_\emptyset\, e_{N+1}^-\,, \tag{3.17a}$$

$$\lim_{x\to-\infty} g_n(x,k) = \sum_{2\le j_n<j_{n+1}<\cdots<j_N\le N} \tilde{\delta}_{j_n,\ldots,j_N}\, e_{j_n}^- \wedge \cdots \wedge e_{j_N}^- \wedge e_{N+1}^-\,, \qquad n = 2,\ldots,N\,, \tag{3.17b}$$

where the functions $\delta_{j_2,\ldots,j_n}(k)$ and $\tilde{\delta}_{j_n,\ldots,j_N}(k)$ are all analytic on $\hat{\mathbb{C}} \setminus \Sigma$ with smooth extensions to $\Sigma \setminus \{\pm q_0\}$ from each sheet. Finally, for all $n = 1,\ldots,N-1$ it is

$$\tilde{\delta}_{n+1,\ldots,N}(k) = e^{-i\Delta\theta}\eta_{11}\Delta_n\,, \tag{3.18a}$$

and for all $n = 2,\ldots,N$ it is

$$\delta_{2,\ldots,n}(k) = \eta_{11}\Delta_n\,. \tag{3.18b}$$

Equations (3.18a) and (3.18b) also hold for $n = N$ and $n = 1$, respectively, in which case the function on the left-hand side reduce respectively to $\tilde{\delta}_\emptyset$ and to $\delta_\emptyset$. On the other hand, we emphasize that the asymptotic behaviors (3.16) and (3.17) do *not*, in general, hold on Σ, despite the fact that both the tensors f_n and g_n and the functions $\delta_{j_2,\ldots,j_n}$ and $\tilde{\delta}_{j_n,\ldots,j_N}$ can be extended smoothly onto the cut from each sheet.

3.3 Reconstruction of the fundamental matrices

The next step is to reconstruct fundamental matrices [i.e., solutions of (3.4) and (3.5)] from the fundamental tensors. To do so, we exploit the fact that these tensors are point-wise decomposable in order to extract vector-valued function "factors" from the wedge-products that define these tensors.

Lemma 3.2 *For all $k \in \hat{\mathbb{C}} \setminus \Sigma$, and for all $x \in \mathbb{R}$, there exist two sets of smooth functions $v_1(x,k),\ldots,$ $v_{N+1}(x,k)$ and $w_1(x,k),\ldots,w_{N+1}(x,k)$ such that, for all $n = 1,\ldots,N+1$,*

$$f_n(x,k) = v_1(x,k) \wedge \cdots \wedge v_n(x,k)\,, \qquad g_n(x,k) = w_n \wedge \cdots \wedge w_{N+1}\,. \tag{3.19}$$

Moreover, these functions have smooth extensions to Σ from each sheet.

As an aside, note that, as a result of Lemma 3.2, the fundamental tensors f_n and g_n are also decomposable asymptotically, i.e., as $|x| \to \infty$. Therefore, the boundary data introduced in Theorem 3.3 satisfy Plücker relations (e.g., cf. [20]): for all $2 \le j_2 < \cdots < j_{n-1} \le N$ and all $2 \le i_2 < \cdots < i_{n+1} \le N$, it is

$$\sum_{s=2}^{n+1} (-1)^s \delta_{j_2,\cdots,j_{n-1},i_s} \delta_{i_2,\cdots,i_{s-1},i_{s+1},i_{n+1}} = 0\,, \tag{3.20}$$

where the indices are rearranged in increasing order, if necessary, taking into account the signature of the corresponding permutation (i.e., each of the coefficients $\delta_{j_2,\cdots,j_n}$ is assumed to be a totally antisymmetric function of its indices). A similar set of conditions obviously holds for the boundary data $\tilde{\delta}_{j_{n+1},\ldots,N}$. Consequently, the number of scattering coefficients does not grow factorially with the number N of components, in contrast to what (3.16b) and (3.17b) might seem to suggest.

The components v_j and w_j of the decomposition (3.19) are not uniquely defined. To fix the decomposition, one could impose $\mathbf{J}$-orthogonality conditions on the factors. That is, one could require $v_j^\dagger \mathbf{J}\, v_n = 0$ and $w_j^\dagger \mathbf{J}\, w_n = 0$ for all $j \neq n$, excluding $j = 1$ and $n = N+1$, or vice-versa. Then, by choosing $v_1 = f_1 = \mu_1$ and $w_{N+1} = g_{N+1} = \tilde{\mu}_{N+1}$ we would have, for all $n = 2, \ldots, N$,

$$f_{n-1} \wedge [(\partial_x - ik\mathbf{J} - \mathbf{Q} + i\lambda_n)\, v_n] = 0\,, \tag{3.21a}$$
$$[(\partial_x - ik\mathbf{J} - \mathbf{Q} + i\lambda_n)\, w_n] \wedge g_{n+1} = 0\,. \tag{3.21b}$$

The factors v_n and w_n so defined would therefore be "weak" eigenfunctions, in the sense that they do not satisfy the differential equations (3.6), but rather equations (3.21), where the differential operator is wedged by f_{n-1} and g_{n+1}, respectively. The columns of a fundamental solution, however, must satisfy the differential equations in the usual sense. To obtain strong solutions of the scattering problem from the fundamental tensor family, we therefore rely instead on the following:

Lemma 3.3 *Given a tensor family $\{f_n, g_n\}_{n=1,\ldots,N+1}$ and for all $k \in \hat{\mathbb{C}} \setminus \Sigma$ such that $f_{n-1} \wedge g_n \neq 0$, there exist two unique sets of meromorphic functions $\{m_n(x,k)\}_{n=1,\ldots,N+1}$ and $\{\tilde{m}_n(x,k)\}_{n=1,\ldots,N+1}$ such that, for all $n = 1, \ldots, N+1$,*

$$f_n = f_{n-1} \wedge m_n\,, \qquad m_n \wedge g_n = 0\,, \tag{3.22a}$$
$$g_{n-1} = \tilde{m}_{n-1} \wedge g_n\,, \qquad f_n \wedge \tilde{m}_n = 0\,. \tag{3.22b}$$

Note that, according to (3.12), the condition $f_{n-1} \wedge g_n \neq 0$ in Lemma 3.3 holds for generic $k \in \hat{\mathbb{C}} \setminus \Sigma$. The location of the exceptional points is discussed in detail in Theorem 3.4 below.

We note that the (sectional) analyticity of all members of the fundamental tensor family is, by itself, insufficient to guarantee that the factors from which they are composed are themselves analytic. Nonanalytic terms in the factors can be "killed" by the wedge product. Nonetheless, we show immediately below that the vector valued functions defined by (3.22) contain only analytic terms, and are, at the same time, solutions of the differential equations (3.6).

Theorem 3.4 *(Analytic eigenfunctions. I) For all $n = 1, \ldots, N+1$, the functions $m_n(x,k)$ and $\tilde{m}_n(x,k)$ are uniquely defined by Lemma 3.3 for all $k \in \hat{\mathbb{C}} \setminus \Sigma$ such that $\Delta_{n-1}(k) \neq 0$ and $\Delta_n(k) \neq 0$, respectively, and they satisfy the differential equations*

$$[\partial_x - \mathbf{L}_- + i\lambda_n \mathbf{I} - (\mathbf{Q} - \mathbf{Q}_-)]\; m_n = 0\,, \tag{3.23a}$$
$$[\partial_x - \mathbf{L}_+ + i\lambda_n \mathbf{I} - (\mathbf{Q} - \mathbf{Q}_+)]\; \tilde{m}_n = 0\,, \tag{3.23b}$$

together with the weak boundary conditions

$$\lim_{x\to-\infty} e_1^- \wedge \cdots \wedge e_{n-1}^- \wedge m_n = e_1^- \wedge \cdots \wedge e_n^-\,, \tag{3.24a}$$
$$\lim_{x\to+\infty} \tilde{m}_n \wedge e_{n+1}^+ \wedge \cdots \wedge e_{N+1}^+ = e_n^+ \wedge \cdots \wedge e_{N+1}^+\,, \tag{3.24b}$$

and with the following asymptotic behavior at the opposite infinity:

$$\lim_{x\to\infty} m_n \wedge e_{n+1}^+ \wedge \cdots \wedge e_{N+1}^+ = \frac{\gamma_n}{\gamma_{n-1}}\, \frac{\Delta_n}{\Delta_{n-1}}\, e_n^+ \wedge \cdots \wedge e_{N+1}^+\,. \tag{3.25a}$$

$$\lim_{x\to-\infty} e_1^- \wedge \cdots \wedge e_{n-1}^- \wedge \tilde{m}_n = \frac{\gamma_{n-1}}{\gamma_n}\,\frac{\Delta_{n-1}}{\Delta_n} e_1^- \wedge \cdots \wedge e_n^- \,. \tag{3.25b}$$

Away from the zeros of $\Delta_{n-1}(k)$ and $\Delta_n(k)$, respectively, the functions $m_n(x,k)$ and $\tilde{m}_n(x,k)$ depend smoothly on x and are analytic functions of k.

In particular, (3.23a) and (3.23b) in Theorem 3.4 imply that, for all k off the cut for which they are defined, the vector functions $m_n(x,k)$ and $\tilde{m}_n(x,k)$ are analytic eigenfunctions of the modified scattering problem (3.6). We next consider their limiting values on the cut.

Theorem 3.5 *(Analytic eigenfunctions. II) For all $n = 1,\dots,N+1$, the eigenfunctions $m_n(x,k)$ and $\tilde{m}_n(x,k)$ admit admit smooth extension on Σ, which we denote respectively by $m_n^\pm(x,k)$ and by $\tilde{m}_n^\pm(x,k)$ where the plus/minus sign denotes whether the limit is taken from the upper or the lower sheet. For $k \in \Sigma$, the functions $m_n^\pm(x,k)$ and $\tilde{m}_n^\pm(x,k)$ are also both solution of the differential equation* (3.23a)*, with weak boundary conditions:*

$$\lim_{x\to-\infty} \left(e_1^-\right)^\pm \wedge \cdots \wedge \left(e_{n-1}^-\right) \wedge m_n^\pm = (e_1^-)^\pm \wedge \cdots \wedge (e_n^-)^\pm \,, \tag{3.26a}$$

$$\lim_{x\to+\infty} \tilde{m}_n^\pm \wedge \left(e_{n+1}^+\right)^\pm \wedge \cdots \wedge \left(e_{N+1}^+\right)^\pm = \left(e_n^+\right)^\pm \wedge \cdots \wedge \left(e_{N+1}^+\right)^\pm \,, \tag{3.26b}$$

and with the asymptotic behavior:

$$\lim_{x\to\infty} m_n^\pm \wedge \left(e_{n+1}^+\right)^\pm \wedge \cdots \wedge \left(e_{N+1}^+\right)^\pm = \frac{\gamma_n^\pm}{\gamma_{n-1}^\pm}\,\frac{\Delta_n^\pm}{\Delta_{n-1}^\pm} \left(e_n^+\right)^\pm \wedge \cdots \wedge \left(e_{N+1}^+\right)^\pm \,, \tag{3.27a}$$

$$\lim_{x\to-\infty} \left(e_1^-\right)^\pm \wedge \cdots \wedge \left(e_{n-1}^-\right)^\pm \wedge \tilde{m}_n^\pm = \frac{\gamma_{n-1}^\pm}{\gamma_n^\pm}\,\frac{\Delta_{n-1}^\pm}{\Delta_n^\pm} \left(e_1^-\right)^\pm \wedge \cdots \wedge \left(e_n^-\right)^\pm \,, \tag{3.27b}$$

and (as above) the superscripts $\pm$ denote the limiting values of the corresponding functions on the cut.

The results of Theorem 3.4 can be strengthened off the cut, as far as the boundary conditions are concerned. In fact, taking into account the asymptotic behavior of the fundamental tensors obtained in Theorem 3.3, the following holds:

Corollary 3.1 *For $k \notin \Sigma$, the functions $m_j(x,k)$ and $\tilde{m}_j(x,k)$ are solutions of the differential equations* (3.6) *with the following boundary conditions:*

$$\lim_{x\to-\infty} m_1 = e_1^- \,, \qquad \lim_{x\to\infty} m_1 = \eta_{11}\Delta_1 e_1^+ \,, \tag{3.28a}$$

$$\lim_{x\to-\infty} m_{N+1} = e_{N+1}^- \,, \qquad \lim_{x\to\infty} m_{N+1} = \frac{e^{i\Delta\theta}}{\eta_{11}\Delta_N} e_{N+1}^+ \,, \tag{3.28b}$$

$$\lim_{x\to\infty} \tilde{m}_{N+1} = e_{N+1}^+ \,, \qquad \lim_{x\to-\infty} \tilde{m}_{N+1} = e^{-i\Delta\theta}\eta_{11}\Delta_N e_{N+1}^- \,, \tag{3.28c}$$

$$\lim_{x\to\infty} \tilde{m}_1 = e_1^+ \,, \qquad \lim_{x\to-\infty} \tilde{m}_1 = \frac{1}{\eta_{11}\Delta_1} e_1^- \,, \tag{3.28d}$$

and, for all $n = 2,\dots,N$,

$$\lim_{x\to-\infty} m_n = \sum_{j=2}^{n} \alpha_{nj}\, e_j^- \,, \qquad \lim_{x\to+\infty} m_n = \sum_{j=n}^{N} \beta_{nj}\, e_j^+ \,, \tag{3.29a}$$

$$\lim_{x\to+\infty}\tilde{m}_n = \sum_{j=n}^{N}\tilde{\beta}_{nj}\,e_j^+\,,\qquad \lim_{x\to-\infty}\tilde{m}_n = \sum_{j=2}^{n}\tilde{\alpha}_{nj}\,e_j^-\,,\tag{3.29b}$$

where, for all $n = 2,\ldots,N$ and all $j = 2,\ldots,n$,

$$\alpha_{nj} = \frac{\tilde{\delta}_{j,n+1,\ldots,N}}{\tilde{\delta}_{n,\ldots,N}}\,,\qquad \tilde{\alpha}_{nj} = \frac{\tilde{\delta}_{j,n+1,\ldots,N}}{\tilde{\delta}_{n+1,\ldots,N}}\,,\tag{3.30a}$$

whereas, for all $n = 2,\ldots,N$ and all $j = n,\ldots,N$,

$$\beta_{nj} = \frac{\delta_{2,\ldots,n-1,j}}{\delta_{2,\ldots,n-1}}\,,\qquad \tilde{\beta}_{nj} = \frac{\delta_{2,\ldots,n-1,j}}{\delta_{2,\ldots,n}}\,.\tag{3.30b}$$

From Theorem 3.3 it then follows that, for all $n = 2,\ldots,N$ and all $j = 2,\ldots,n$, the coefficients α_{nj} and $\tilde{\alpha}_{nj}$ are meromorphic functions of k, and, according to (3.18), their poles are located respectively at the zeros of Δ_{n-1} and at those of Δ_n, independently of j. Similarly, for all $n = 2,\ldots,N$ and $j = n,\ldots,N$, the coefficients β_{nj} and $\tilde{\beta}_{nj}$ are meromorphic functions of k, and their poles are located at the zeros of Δ_{n-1} and at those of Δ_n, respectively, independently of j. Note also that (3.18) implies, for all $n = 1,\ldots,N+1$,

$$\alpha_{nn} = \tilde{\beta}_{nn} = 1\,,\qquad \tilde{\alpha}_{nn} = \tilde{\delta}_{n,\cdots,N}/\tilde{\delta}_{n+1,\cdots,N} = \Delta_{n-1}/\Delta_n\,,\qquad \beta_{nn} = \delta_{2,\cdots,n}/\delta_{2,\cdots,n-1} = \Delta_n/\Delta_{n-1}\,.$$

The following corollary summarizes the results on the analyticity of the eigenfunctions m_n and $\tilde{m}_n$, and, together with the previous one, paves the way for the reconstruction of the fundamental matrices $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$.

Corollary 3.2 *The $(N{+}1)\times(N{+}1)$ matrices*

$$\mathbf{m}(x,k) = (m_1,\ldots,m_{N+1})\,,\qquad \tilde{\mathbf{m}}(x,k) = (\tilde{m}_1,\ldots,\tilde{m}_{N+1})\,,\tag{3.31}$$

are defined $\forall k\in\hat{\mathbb{C}}\setminus(\Sigma\cup Z)$, where Z is the discrete set

$$Z = \bigcup_{n=1}^{N} Z_n\,,\qquad Z_n = \{k\in\hat{\mathbb{C}}\setminus\Sigma : \Delta_n(k) = 0\}\,.\tag{3.32}$$

Moreover, $\forall k\in\hat{\mathbb{C}}\setminus(\Sigma\cup Z)$ it is

$$\mathbf{m}(x,k) = \tilde{\mathbf{m}}(x,k)\mathbf{d}(k)\,,\tag{3.33}$$

where

$$\mathbf{d}(k) = \operatorname{diag}\left(\frac{\gamma_1}{\gamma_0}\frac{\Delta_1}{\Delta_0},\frac{\gamma_2}{\gamma_1}\frac{\Delta_2}{\Delta_1}\ldots,\frac{\gamma_N}{\gamma_{N-1}}\frac{\Delta_N}{\Delta_{N-1}},\frac{\gamma_{N+1}}{\gamma_N}\frac{\Delta_{N+1}}{\Delta_N}\right).\tag{3.34}$$

For all $x\in\mathbb{R}$, $\mathbf{m}(x,k)$ and $\tilde{\mathbf{m}}(x,k)$ are meromorphic on $\hat{\mathbb{C}}\setminus(\Sigma\cup Z)$, with a pole at each point of Z. Finally, $\mathbf{m}(x,k)$ and $\tilde{\mathbf{m}}(x,k)$ extend smoothly to Σ from either sheet. We denote these limits as $\mathbf{m}^\pm(x,k)$ and $\tilde{\mathbf{m}}^\pm(x,k)$, respectively.

Note that (3.33) amounts to saying that the columns of the two matrices $\mathbf{m}$ and $\tilde{\mathbf{m}}$ differ only up to a normalization, and therefore either one of the two matrices will be enough by itself to formulate the inverse problem.

Finally, the above results allow us to reconstruct the fundamental matrices $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$.

Corollary 3.3 *For generic $k \in \hat{\mathbb{C}}/\Sigma$, the fundamental matrix $\boldsymbol{\mu}(x,k) = (\mu_1, \dots, \mu_{N+1})$ defined in* (3.4) *is given by:*

$$\mu_1 \equiv m_1 \,, \qquad \mu_{N+1} \equiv m_{N+1} \tag{3.35a}$$

[as follows from (3.28a) *and* (3.28b)*] and the remaining fundamental eigenfunctions μ_n for $n = 2, \cdots, N$ are obtained recursively as follows:*

$$\mu_2 := m_2 \,, \qquad \mu_n := m_n - \sum_{j=2}^{n-1} \alpha_{n\,j} \mu_j \,. \tag{3.35b}$$

A similar argument applies to the fundamental matrix $\tilde{\boldsymbol{\mu}}$. Eqs. (3.28d) *and* (3.28c) *show that*

$$\tilde{\mu}_1 \equiv \tilde{m}_1 \,, \qquad \tilde{\mu}_{N+1} \equiv \tilde{m}_{N+1} \,. \tag{3.36a}$$

The fundamental eigenfunctions $\tilde{\mu}_n$ for $n = 2, \cdots, N$ are obtained recursively as follows:

$$\tilde{\mu}_N := \tilde{m}_N \,, \qquad \tilde{\mu}_n := \tilde{m}_n - \sum_{j=n+1}^{N} \tilde{\beta}_{n\,j} \tilde{\mu}_j \,. \tag{3.36b}$$

From the definitions (3.35b), (3.36b) and the first of (3.22), one can easily show by induction that the fundamental eigenfunctions $\{\mu_n\}_{n=1}^{N+1}$, $\{\tilde{\mu}_n\}_{n=1}^{N+1}$ also satisfy

$$f_1 = \mu_1 \,, \qquad f_j = f_{n-1} \wedge \mu_n \,, \qquad n = 2, \dots, N+1 \,, \tag{3.37a}$$

$$g_{N+1} = \tilde{\mu}_{N+1} \,, \qquad g_{n-1} = \tilde{\mu}_{n-1} \wedge g_n \,, \qquad n = 2, \dots, N+1 \,. \tag{3.37b}$$

In general, the matrices $\mathbf{m}$ and $\boldsymbol{\mu}$ (and correspondingly, $\tilde{\mathbf{m}}$ and $\tilde{\boldsymbol{\mu}}$) do not coincide, but the above relations establish the one-to-one correspondence between the two complete sets of eigenfunctions. $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$ are fundamental matrices in that off Σ they have the simple asymptotic behavior as either $x \to -\infty$ or as $x \to \infty$ prescribed in (3.4) and (3.5). However, their analyticity properties in k (specifically, the location of the poles of each column vector) are in general more involved than those of $\mathbf{m}$ and $\tilde{\mathbf{m}}$, as follows from (3.35b) and (3.36b). Conversely, $\mathbf{m}$ and $\tilde{\mathbf{m}}$ have a more complicated asymptotic behavior for large $|x|$, but their analyticity properties are simpler (cf. Section 3.4 regarding the location of the poles). In the formulation of the inverse problem it is more convenient to deal with $\mathbf{m}$ and $\tilde{\mathbf{m}}$, while determining the asymptotic behavior of the eigenfunctions with respect to the scattering parameter via a WKB expansion is more easily achieved for the fundamental matrices $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$ (see Section 3.5). For this reason, in the following, we will keep both sets of eigenfunctions, use either one or the other depending on the calculation to be performed, and invoke relations (3.35b) and (3.36b) to go from one set to the other. The construction obviously simplifies significantly when $\mathbf{m} \equiv \boldsymbol{\mu}$ (and, correspondingly, $\tilde{\mathbf{m}} \equiv \tilde{\boldsymbol{\mu}}$), which is equivalent to the conditions $\tilde{\delta}_{j,n+1,\dots,N} \equiv 0$ for $n = 3, \dots, N$ and $j = 2, \dots, n-1$, and $\delta_{2,\dots,n-1,j} \equiv 0$ for $n = 3, \dots, N$ and $j = n+1, \dots, N$.

3.4 Characterization of the scattering data

We take the scattering data to be the data which describe the poles and the jump relations across Σ of the piecewise meromorphic matrix $\mathbf{m}$. One can restrict the singularities of $\mathbf{m}$ by restricting the zeros of the functions $\{\Delta_n\}_{n=1}^{N}$.

Definition 3.3 *The scattering operator* $\mathbf{L}$ *will be called generic if, for each component $\mathbb{C}_{\mathrm{I}}$ and $\mathbb{C}_{\mathrm{II}}$ of $\hat{\mathbb{C}}$, the functions $\Delta_1, \dots, \Delta_N$: (i) have no common zeros and no multiple zeros; (ii) and they have no zeros on Σ.*

Note that since they are sectionally analytic, with $\Delta_n \to 1$ as $|k| \to \infty$ [cf. (3.55)], each Δ_n has only finitely many zeros.

The following theorem relates the residues of the meromorphic eigenfunction m_n ($n = 2, \ldots, N+1$) at any of its poles to the values of the eigenfunction m_{n-1} at the same point:

Theorem 3.6 *(Residues) Suppose $\Delta_{n-1}(k)$ has a simple zero at $k_o \in \hat{\mathbb{C}}/\Sigma$, and suppose $\Delta_n(k_o)\Delta_{n-2}(k_o) \neq 0$. Then $m_n(x,k)$ has a simple pole at $k = k_o$ with residue which satisfies*

$$\mathrm{Res}_{k=k_o} m_n(x,k) = c_o\, e^{i(\lambda_{n-1}(k_o)-\lambda_n(k_o))x} m_{n-1}(x,k_o) \tag{3.38}$$

for some constant $c_0 \neq 0$.

The behavior of the fundamental matrix at each of such poles is then characterized as follows:

Theorem 3.7 *(Discrete spectrum) For each $k_o \in Z_{n-1}$, there exists a unique constant $c_o \in \hat{\mathbb{C}} \setminus 0$ such that*

$$\mathbf{m}(x,k) \left(\mathbf{I} - \frac{c_o}{k-k_o}\, e^{i\mathbf{\Lambda}(k_o)x} \mathbf{D}_{n-1} e^{-i\mathbf{\Lambda}(k_o)x} \right) \tag{3.39}$$

is regular at $k = k_o$, where $\mathbf{D}_{n-1} = (\delta_{n-1,n})$ and $\delta_{n,n'}$ is the Kronecker delta.

It is then clear that the elements of the set Z in (3.32) play the role of the discrete eigenvalues, and that $c_o \mathbf{D}_{n-1}$ is the (matrix) norming constant associated with the discrete eigenvalue $k_o \in Z_{n-1}$.

Finally, we derive the jump condition across Σ of the piecewise meromorphic matrix $\mathbf{m}$, which will be used in Section 4 to formulate the inverse problem for the fundamental eigenfunctions.

Theorem 3.8 *(Scattering matrix) For all $k \in \Sigma$, there exists a unique matrix $\mathbf{S}(k)$ such that*

$$\mathbf{m}^+(x,k)\, \boldsymbol{\pi} = \mathbf{m}^-(x,k)\, e^{i\mathbf{\Lambda}^-(k)x} \mathbf{S}(k)\, e^{-i\mathbf{\Lambda}^-(k)x}, \tag{3.40}$$

where $\mathbf{\Lambda}^-(k) = \lim \mathbf{\Lambda}(k)$ as $k \to \Sigma$ from $\mathbb{C}_{\mathrm{II}}$, and $\boldsymbol{\pi}$ was defined in (2.18).

Note that from the definition of the matrix $\boldsymbol{\pi}$ it follows that $\mathbf{m}^+ \boldsymbol{\pi}\, e^{ix\mathbf{\Lambda}^-} = \mathbf{m}^+ e^{ix\mathbf{\Lambda}^+} \boldsymbol{\pi}$, and therefore we can write the jump condition defining the scattering data as

$$\boldsymbol{\varphi}^+(x,k) = \boldsymbol{\varphi}^-(x,k) \hat{\mathbf{S}}(k) \tag{3.41}$$

with $\hat{\mathbf{S}}(k) = \mathbf{S}(k)\boldsymbol{\pi}^{-1} \equiv \mathbf{S}(k)\boldsymbol{\pi}$, $\boldsymbol{\varphi}(x,k) = \mathbf{m}(x,k)\, e^{i\mathbf{\Lambda}(k)x}$ and the subscript $\pm$ denotes the limit for $k \in \Sigma$ from the upper/lower sheet of the Riemann surface. In particular, for $k \in \Sigma$ one has $\boldsymbol{\varphi}^{\pm}(x,k) = \mathbf{m}^{\pm}(x,k)\, e^{i\mathbf{\Lambda}^{\pm}(k)x}$. Eq. (3.41) yields the matrix elements of $\hat{\mathbf{S}}(k)$ as

$$\hat{S}_{ij}(k) = \frac{\mathrm{Wr}\left[\varphi_1^-, \ldots, \varphi_{j-1}^-, \varphi_i^+, \varphi_{j+1}^-, \cdots \varphi_{N+1}^-\right]}{\mathrm{Wr}\left[\varphi_1^-, \ldots, \varphi_{N+1}^-\right]}.$$

[Here again the superscript denotes the limiting value from above or below Σ of the corresponding vectors.] In what follows, we express the elements of the jump matrix $\mathbf{S}$ (or, equivalently, $\hat{\mathbf{S}}$) in terms of the boundary data derived in Sections 3.2 and 3.3.

As we pointed out in Section 3.3, for $k \in \Sigma$ the asymptotic behavior of the eigenfunctions acquires contributions from the various subdominant terms, according to the weak asymptotic conditions (3.26a) and (3.27a). Specifically, one has

$$\mathbf{m}^{\pm}(x,k) \sim \mathbf{E}_{-}^{\pm}\, e^{i\boldsymbol{\Lambda}^{\pm}x} \left(\boldsymbol{\alpha}^{\pm}\right)^{T} e^{-i\boldsymbol{\Lambda}^{\pm}x} \qquad \text{as } x \to -\infty\,, \quad k \in \Sigma\,, \tag{3.42a}$$

$$\mathbf{m}^{\pm}(x,k) \sim \mathbf{E}_{+}^{\pm}\, e^{i\boldsymbol{\Lambda}^{\pm}x} \left(\boldsymbol{\beta}^{\pm}\right)^{T} e^{-i\boldsymbol{\Lambda}^{\pm}x} \qquad \text{as } x \to +\infty\,, \quad k \in \Sigma\,, \tag{3.42b}$$

where the superscripts $^{+}$ and $^{-}$ denote respectively the limits as $k \to \Sigma$ from $\mathbb{C}_{\mathrm{I}}$ and $\mathbb{C}_{\mathrm{II}}$, and where $\boldsymbol{\alpha}^{\pm}(k)$, $\boldsymbol{\beta}^{\pm}(k)$ are $(N+1) \times (N+1)$ square matrices defined on Σ, with

$$\alpha_{nn}^{\pm} = 1\,, \qquad \beta_{nn}^{\pm} = \frac{\gamma_n^{\pm}}{\gamma_{n-1}^{\pm}} \frac{\Delta_n^{\pm}}{\Delta_{n-1}^{\pm}}\,, \qquad n = 1, \ldots, N+1\,, \tag{3.43}$$

and $\alpha_{nj}^{\pm}$ (resp. $\beta_{nj}^{\pm}$) for $n = 2, \ldots, N$ and $j = 2, \ldots, n$ (resp. $j = n, \ldots, N$) are the limiting values on Σ of the meromorphic functions α_{nj} (resp. β_{nj}) in (3.29a). Equations (3.26a) and (3.27a) imply that $\boldsymbol{\alpha}^{\pm}(k)$ is lower triangular, while $\boldsymbol{\beta}^{\pm}(k)$ is upper triangular. The above relations (3.43) then yield

$$\det \boldsymbol{\alpha}^{\pm}(k) = 1\,, \qquad \det \boldsymbol{\beta}^{\pm}(k) = \prod_{j=1}^{N+1} \frac{\gamma_j^{\pm}}{\gamma_{j-1}^{\pm}} \frac{\Delta_j^{\pm}}{\Delta_{j-1}^{\pm}} \equiv e^{i\Delta\theta}\,. \tag{3.44}$$

Taking into account (3.42), the limiting values as $|x| \to \infty$ of (3.40) give the following relationships between boundary (or scattering) data [expressing the behavior of the eigenfunctions as $x \to \pm\infty$ on Σ, cf. (3.42)] and spectral data [providing the jump of the sectionally meromorphic fundamental matrix across Σ, as given in (3.40)]:

$$\mathbf{S} = \boldsymbol{\pi}\boldsymbol{\beta}^{+}\boldsymbol{\pi} \left(\boldsymbol{\beta}^{-}\right)^{-1} = \boldsymbol{\pi}\boldsymbol{\alpha}^{+}\boldsymbol{\pi} \left(\boldsymbol{\alpha}^{-}\right)^{-1}\,, \tag{3.45}$$

where we have used $\left(\mathbf{E}_{\pm}^{-}\right)^{-1} \mathbf{E}_{\pm}^{+} = \boldsymbol{\pi} \left(\mathbf{E}_{\pm}^{+}\right)^{-1} \mathbf{E}_{\pm}^{+} \equiv \boldsymbol{\pi}$, as well as $e^{-i\boldsymbol{\Lambda}^{-}x}\boldsymbol{\pi}e^{i\boldsymbol{\Lambda}^{+}x} \equiv \boldsymbol{\pi}$.

The asymptotic behavior at large x of the matrices $\mathbf{m}^{\pm}(x,k)$ given in (3.42), suggests the introduction of eigenfunctions with fixed boundary conditions on Σ as, say, $x \to -\infty$, by means of the following definition:

$$\mathbf{m}^{\pm}(x,k) = \mathbf{M}^{\pm}(x,k) \left(\boldsymbol{\alpha}^{\pm}(k)\right)^{T}\,, \qquad k \in \Sigma\,. \tag{3.46}$$

The matrices $\mathbf{M}^{\pm}(x,k)$, which are (up to normalizations, see Section 6) the analogues of the ones introduced in [16] for the 2-component case, in general *do not* admit analytic extension off Σ. Nonetheless, on Σ the eigenfunctions $\boldsymbol{\phi}^{\pm}(x,k) = \mathbf{M}^{\pm}(x,k)e^{i\boldsymbol{\Lambda}^{\pm}(k)x}$ are such that

$$\boldsymbol{\phi}^{\pm}(x,k) \sim \mathbf{E}_{-}^{\pm}(k)e^{i\boldsymbol{\Lambda}^{\pm}(k)x} \qquad \text{as } x \to -\infty\,, \tag{3.47a}$$

$$\boldsymbol{\phi}^{\pm}(x,k) \sim \mathbf{E}_{+}^{\pm}(k)e^{i\boldsymbol{\Lambda}^{\pm}(k)x}\mathbf{A}^{\pm}(k) \qquad \text{as } x \to +\infty\,, \tag{3.47b}$$

with

$$\mathbf{A}^{\pm}(k) = \left(\boldsymbol{\beta}^{\pm}(k)\right)^{T} \left(\boldsymbol{\alpha}^{\pm}(k)^{T}\right)^{-1}\,. \tag{3.48}$$

The matrices $\mathbf{A}^{\pm}$ are therefore the analogues of the traditional scattering matrices (up to the switching of the first and last eigenfunctions when crossing the cut).

3.5 Asymptotics of eigenfunctions and scattering data as $z \to \infty$ and $z \to 0$

Here we summarize the results on the asymptotic behavior of the fundamental eigenfunctions with respect to the uniformization variable z introduced in Section 2.3. As clarified in Section 3.4, in order to formulate the inverse problem we only need one of the two fundamental matrices, say $\boldsymbol{\mu}(x,z)$. Therefore, we will derive the asymptotic behavior of the fundamental eigenfunctions $\mu_n(x,z)$, $n = 1,\dots,N+1$, both for $z \to \infty$ and for $z \to 0$. Due to the ordering of the eigenvalues, for the eigenfunctions $\mu_1(x,z)$ and $\mu_{N+1}(x,z)$ it will be necessary to specify in which half-plane ($\operatorname{Im} z > 0$ or $\operatorname{Im} z < 0$) the asymptotic expansion is being considered. It is worth pointing out there is no conceptual distinction between the points $z = 0$ and $z = \infty$ in the z-plane, as they are both images of $k \to \infty$ on either sheet of the Riemann surface, and one can change one into the other by simply defining the uniformization variable as $z = k - \lambda$ instead of $z = k + \lambda$. The details of the calculations, performed using suitable WKB expansions, are given in Appendix B. The results are the following:

$$\mu_1(x,z) = \begin{pmatrix} z + O(1) \\ i\mathbf{r}(x) + O(1/z) \end{pmatrix} \qquad z \to \infty, \quad \operatorname{Im} z > 0, \tag{3.49a}$$

$$\mu_1(x,z) = \begin{pmatrix} \mathbf{q}^T(x)\mathbf{r}_-/z + O(1/z^2) \\ i\mathbf{r}_- + O(1/z) \end{pmatrix} \qquad z \to \infty, \quad \operatorname{Im} z < 0, \tag{3.49b}$$

$$\mu_1(x,z) = \begin{pmatrix} z\,\mathbf{q}^T(x)\mathbf{r}_-/q_0^2 + O(z^2) \\ i\mathbf{r}_- + O(z) \end{pmatrix} \qquad z \to 0, \quad \operatorname{Im} z > 0, \tag{3.50a}$$

$$\mu_1(x,z) = \begin{pmatrix} q_0^2/z + O(1) \\ i\mathbf{r}(x) + O(z) \end{pmatrix} \qquad z \to 0, \quad \operatorname{Im} z < 0. \tag{3.50b}$$

For $n = 2,\dots,N$ the behavior of the eigenfunctions is the same in both half-planes, and given by:

$$\mu_n(x,z) = \begin{pmatrix} \mathbf{q}^T(x)\mathbf{r}_{0,n-1}^{\perp}/z + O(1/z^2) \\ i\mathbf{r}_{0,n-1}^{\perp} + O(1/z) \end{pmatrix} \qquad z \to \infty, \tag{3.51}$$

$$\mu_n(x,z) = \begin{pmatrix} z\,\mathbf{q}^T(x)\mathbf{r}_{0,n-1}^{\perp}/q_0^2 + O(z^2) \\ i\mathbf{r}_{0,n-1}^{\perp} + O(z) \end{pmatrix} \qquad z \to 0, \tag{3.52}$$

where $\mathbf{r}_{0,1}^{\perp},\dots,\mathbf{r}_{0,N-1}^{\perp}$ denote the columns of the matrix $\mathbf{R}_0^{\perp}$ defined by (2.12). Finally,

$$\mu_{N+1}(x,z) = \begin{pmatrix} \mathbf{q}^T(x)\mathbf{r}_-/z + O(1/z^2) \\ i\mathbf{r}_- + O(1/z) \end{pmatrix} \qquad z \to \infty, \quad \operatorname{Im} z > 0, \tag{3.53a}$$

$$\mu_{N+1}(x,z) = \begin{pmatrix} z + O(1) \\ i\mathbf{r}(x) + O(1/z) \end{pmatrix} \qquad z \to \infty, \quad \operatorname{Im} z < 0, \tag{3.53b}$$

$$\mu_{N+1}(x,z) = \begin{pmatrix} q_0^2/z + O(1) \\ i\mathbf{r}(x) + O(z) \end{pmatrix} \qquad z \to 0, \quad \operatorname{Im} z > 0, \tag{3.54a}$$

$$\mu_{N+1}(x,z) = \begin{pmatrix} z\,\mathbf{q}^T(x)\mathbf{r}_-/q_0^2 + O(z^2) \\ i\mathbf{r}_- + O(z) \end{pmatrix} \qquad z \to 0, \quad \operatorname{Im} z < 0. \tag{3.54b}$$

Taking into account the constraint (2.5) on the boundary values of the potentials, (2.22) and the first of (3.37a), and assuming that the limits as $x \to \infty$ and as $z \to \infty$ [or $z \to 0$] can be interchanged, Eqs. (3.16a) and (3.49) [or (3.50)] yield:

$$\delta_{\emptyset}(z) \sim \begin{cases} 1 & z \to \infty, \quad \operatorname{Im} z > 0, \\ e^{i\Delta\theta} & z \to \infty, \quad \operatorname{Im} z < 0, \end{cases}$$

while

$$\delta_{\emptyset}(z) \sim \begin{cases} e^{i\Delta\theta} & z \to 0\,, \quad \operatorname{Im} z > 0\,, \\ 1 & z \to 0\,, \quad \operatorname{Im} z < 0\,. \end{cases}$$

From the second of Eqs. (3.18) for $n = 1$, and the limiting values of η_{11} in (2.23), we then obtain $\Delta_1(z) \to 1$ both as $z \to \infty$ and as $z \to 0$. We can then use (3.37a), (3.16b) and the above-mentioned asymptotic behavior to show by induction that, provided the limits as $x \to \infty$ and as $z \to \infty$ [resp. $z \to 0$] can be interchanged, for all $n = 2, \ldots, N$:

$$\delta_{2,\ldots,n}(z) \sim \begin{cases} 1 & z \to \infty\,, \quad \operatorname{Im} z > 0\,, \\ e^{i\Delta\theta} & z \to \infty\,, \quad \operatorname{Im} z < 0\,, \end{cases}$$

and

$$\delta_{2,\ldots,n}(z) \sim \begin{cases} e^{i\Delta\theta} & z \to 0\,, \quad \operatorname{Im} z > 0\,, \\ 1 & z \to 0\,, \quad \operatorname{Im} z < 0\,, \end{cases}$$

while

$$\delta_{j_2,\ldots,j_n}(z) \to 0 \quad \text{as } z \to \infty \text{ and as } z \to 0\,,$$

for any $\{2 \le j_2 < j_3 < \cdots < j_n \le N\} \ne \{2, \ldots, n\}$. The dual result for the boundary data $\tilde{\delta}_{j_n,\ldots,j_N}$, i.e.

$$\tilde{\delta}_{j_n,\ldots,j_N}(z) \to 0 \quad \text{as } z \to \infty \text{ and as } z \to 0\,,$$

for any $\{2 \le j_n < \cdots < j_N \le N\} \ne \{n, \ldots, N\}$ is proved analogously. As a consequence, (3.18) imply that for all $n = 1, \ldots, N$ it is:

$$\Delta_n(z) \to 1 \qquad \text{as } z \to \infty \text{ and as } z \to 0\,. \tag{3.55}$$

Altogether, the above asymptotic expansions and the definitions (3.30a) show that for all $n = 3, \ldots, N$ and all $j = 2, \ldots, n-1$

$$\alpha_{n\,j}(z) \to 0 \qquad \text{as } z \to \infty \text{ and as } z \to 0\,. \tag{3.56}$$

Therefore, according to (3.35b), the matrix $\mathbf{m}(x,z)$ has the same asymptotic behavior in z as $\boldsymbol{\mu}(x,z)$, i.e., as $z \to \infty$, with $\operatorname{Im} z > 0$:

$$\mathbf{m}(x,z) \sim \begin{pmatrix} z & \mathbf{q}^T(x)\mathbf{r}_{0,1}^{\perp}/z & \ldots & \mathbf{q}^T(x)\mathbf{r}_{0,N-1}^{\perp}/z & \mathbf{q}^T(x)\mathbf{r}_-/z \\ i\mathbf{r}(x) & i\mathbf{r}_{0,1}^{\perp} & \ldots & i\mathbf{r}_{0,N-1}^{\perp} & i\mathbf{r}_- \end{pmatrix}. \tag{3.57a}$$

Similarly, as $z \to 0$ with $\operatorname{Im} z > 0$:

$$\mathbf{m}(x,z) \sim \begin{pmatrix} z\,\mathbf{q}^T(x)\mathbf{r}_-/q_0^2 & z\,\mathbf{q}^T(x)\mathbf{r}_{0,1}^{\perp}/q_0^2 & \ldots & z\,\mathbf{q}^T(x)\mathbf{r}_{0,N-1}^{\perp}/q_0^2 & q_0^2/z \\ i\mathbf{r}_- & i\mathbf{r}_{0,1}^{\perp} & \ldots & i\mathbf{r}_{0,N-1}^{\perp} & i\mathbf{r}(x) \end{pmatrix}. \tag{3.57b}$$

As usual, the first and last columns of (3.57) are interchanged when either $z \to \infty$ or $z \to 0$ with $\operatorname{Im} z < 0$.

3.6 Symmetries

As in the scalar and 2-component case, the scattering problem admits two symmetries, which relate the value of the eigenfunctions on different sheets of the Riemann surface. As usual, these symmetries translate into compatibility conditions (constraints) on the scattering data, and play a fundamental role in the solution of the inverse problem.

First symmetry: upper/lower-half plane. Consider the transformation $(k,\lambda) \to (k^*,\lambda^*)$, i.e., $z \to z^*$: When the potential satisfies the symmetry condition $\mathbf{r} = \mathbf{q}^*$, i.e., one has $\mathbf{Q}^\dagger = \mathbf{Q}$, and (2.14) implies

$$\partial_x \left[\boldsymbol{\varphi}^\dagger(x,z^*)\mathbf{J}\,\boldsymbol{\varphi}(x,z)\right] = 0\,.$$

Evaluating the asymptotic values of the bilinear form $\boldsymbol{\varphi}^\dagger(x,z^*)\mathbf{J}\,\boldsymbol{\varphi}(x,z)$ as $x \to -\infty$ and as $x \to \infty$ then yields, for $z \in \mathbb{R}$:

$$\begin{aligned}\left[\boldsymbol{\alpha}^\mp(z)\right]^* e^{-i\boldsymbol{\Lambda}^\mp(z)x}\left[\mathbf{E}_-^\mp(z)\right]^\dagger \mathbf{J}\mathbf{E}_-^\pm(z)e^{i\boldsymbol{\Lambda}^\pm(z)x}\left[\boldsymbol{\alpha}^\pm(z)\right]^T \\ = \left[\boldsymbol{\beta}^\mp(z)\right]^* e^{-i\boldsymbol{\Lambda}^\mp(z)x}\left[\mathbf{E}_+^\mp(z)\right]^\dagger \mathbf{J}\mathbf{E}_+^\pm(z)e^{i\boldsymbol{\Lambda}^\pm(z)x}\left[\boldsymbol{\beta}^\pm(z)\right]^T\,. \end{aligned} \tag{3.58}$$

Note that

$$\left[\mathbf{E}_-^\mp(z)\right]^\dagger \mathbf{J}\mathbf{E}_-^\pm(z) = \begin{pmatrix} 0 & \mathbf{0}_{1\times(N-1)} & q_0^2 - q_0^4/z^2 \\ \mathbf{0}_{(N-1)\times 1} & q_0^2\mathbf{I}_{N-1} & \mathbf{0}_{(N-1)\times 1} \\ q_0^2 - z^2 & \mathbf{0}_{1\times(N-1)} & 0 \end{pmatrix} \tag{3.59}$$

and

$$e^{-i\boldsymbol{\Lambda}^\mp(z)x}\left[\mathbf{E}_-^\mp(z)\right]^\dagger \mathbf{J}\mathbf{E}_-^\pm(z)\,e^{i\boldsymbol{\Lambda}^\pm(z)x} = q_0^2\,e^{-i\Lambda^\mp(z)x}\,\boldsymbol{\Gamma}(z)\,e^{i\boldsymbol{\Lambda}^\pm(z)x} = q_0^2\boldsymbol{\Gamma}(z)\,, \tag{3.60}$$

where

$$\boldsymbol{\Gamma}(z) = \boldsymbol{\pi}\,\mathrm{diag}\left(1 - z^2/q_0^2, 0, \ldots, 0, 1 - q_0^2/z^2\right) = \begin{pmatrix} 0 & \mathbf{0}_{1\times(N-1)} & 1 - q_0^2/z^2 \\ \mathbf{0}_{(N-1)\times 1} & \mathbf{I}_{N-1} & \mathbf{0}_{(N-1)\times 1} \\ 1 - z^2/q_0^2 & \mathbf{0}_{1\times(N-1)} & 0 \end{pmatrix}\,. \tag{3.61}$$

It then follows that, $\forall z \in \mathbb{R}$,

$$\left[\boldsymbol{\alpha}^\mp(z)\right]^* \boldsymbol{\Gamma}(z)\left[\boldsymbol{\alpha}^\pm(z)\right]^T = \left[\boldsymbol{\beta}^\mp(z)\right]^* \boldsymbol{\Gamma}(z)\left[\boldsymbol{\beta}^\pm(z)\right]^T\,. \tag{3.62}$$

This symmetry is the generalization to arbitrary N of the one that was obtained in [16] using the "adjoint" problem. For all analytic scattering coefficients, the above symmmetry is also extended off the cut in the usual way.

Second symmetry: inside/outside the circle. The scattering problem also admits another symmetry that relates values of eigenfunctions and scattering coefficients at points (k,λ) and $(k,-\lambda)$ on the two sheets of $\hat{\mathbb{C}}$ or across the cut. In terms of the uniform variable z, this symmetry corresponds to $z \to q_0^2/z$, which couples points inside and outside the circle C_0, centered at the origin and of radius q_0. Indeed, the scattering problem is manifestly invariant with respect to the exchange $(k,\lambda) \to (k,-\lambda)$. By looking at the boundary conditions off the real axis we thus have immediately

$$\boldsymbol{\varphi}(x,z) = \boldsymbol{\varphi}(x,q_0^2/z)\boldsymbol{\pi}. \tag{3.63}$$

Then, when $z \in \mathbb{R}$, the comparison of the asymptotic values as $x \to -\infty$ yields

$$\boldsymbol{\alpha}^\pm(z) = \boldsymbol{\pi}\,\boldsymbol{\alpha}^\mp(q_0^2/z), \qquad \boldsymbol{\beta}^\pm(z) = \boldsymbol{\pi}\,\boldsymbol{\beta}^\mp(q_0^2/z). \tag{3.64}$$

Discrete eigenvalues. The combination of the two symmetries implies that discrete eigenvalues appear in symmetric quartets:

$$\{z_j, z_j^*, q_0^2/z_j, q_0^2/z_j^*\}, \qquad j = 1,\ldots,J.$$

In particular, in the scalar case, discrete eigenvalues can only exist on the circle C_0.

4 Inverse problem

The starting point for solving the inverse problem is the jump condition (3.41), which we can write in terms of the uniformization variable as:

$$\boldsymbol{\varphi}^+(x,z) = \boldsymbol{\varphi}^-(x,z)\,\hat{\mathbf{S}}(z) \qquad \forall z \in \mathbb{R}\,, \tag{4.1}$$

where $\boldsymbol{\varphi}(x,z) = \mathbf{m}(x,z)e^{i\boldsymbol{\Lambda}(z)x}$ and superscripts $\pm$ denote limits from the upper/lower half plane of z. In the following, we will assume, in agreement with the genericity hypothesis in Definition 3, that each function $\Delta_j(z)$, for $j = 1,\dots,N$, has simple zeros at points $\{z_{j,i}\}_{i=1}^{\bar{n}_j}$ in the upper half-plane and $\{\bar{z}_{j,i}\}_{i=1}^{\bar{n}_j}$ in the lower half-plane. As a consequence, from Theorems 4 and 6, and from the asymptotic behavior as $z \to 0,\infty$ in (3.57), we have the following: (*i*) $\varphi_1(x,z)e^{-i\lambda_1(z)x}$ is a sectionally analytic function for all $z \in \mathbb{C}$, with a jump across the real z-axis and a (simple) pole at $z = \infty$; (*ii*) $\varphi_j(x,z)e^{-ik(z)x}$ for all $j = 2,\dots,N$ are sectionally meromorphic functions of z, with (simple) poles at the zeros of $\Delta_{j-1}(z)$; (*iii*) $\varphi_{N+1}(x,z)e^{-i\lambda_{N+1}(z)x}$ is a sectionally meromorphic function of z, with (simple) poles at the zeros of $\Delta_N(z)$ and a (simple) pole at $z = 0$.

Eq. (4.1) then defines a matrix Riemann-Hilbert problem (RHP) with poles. In order to suitably normalize the problem, we rewrite the jump condition for each vector eigenfunction by subtracting out the asymptotic behavior of the functions in the right-hand side as $z \to \infty$ as well as the residue at $z = 0$ in the upper half-plane:

$$\frac{\varphi_1^+}{z} - \begin{pmatrix}1\\ \mathbf{0}\end{pmatrix} e^{-i\lambda x} - \frac{1}{z}\begin{pmatrix}0\\ i\mathbf{r}_-\end{pmatrix} e^{-i\lambda x} = -\begin{pmatrix}1\\ \mathbf{0}\end{pmatrix} e^{-i\lambda x} - \frac{1}{z}\begin{pmatrix}0\\ i\mathbf{r}_-\end{pmatrix} e^{-i\lambda x} + \frac{\varphi_{N+1}^-}{z} + \sum_{i=1}^{N+1} \varphi_i^- \frac{V_{i,1}}{z} \tag{4.2a}$$

$$\varphi_j^+ - \begin{pmatrix}0\\ i\mathbf{r}_{0,j-1}^\perp\end{pmatrix} e^{ikx} = -\begin{pmatrix}0\\ i\mathbf{r}_{0,j-1}^\perp\end{pmatrix} e^{ikx} + \varphi_j^- + \sum_{i=1}^{N+1} \varphi_i^- V_{i,j} \qquad j = 2,\dots,N \tag{4.2b}$$

$$\varphi_{N+1}^+ - \begin{pmatrix}1\\ i\mathbf{r}_-\end{pmatrix} e^{i\lambda x} - \frac{1}{z}\begin{pmatrix}q_0^2\\ \mathbf{0}\end{pmatrix} e^{i\lambda x} = -\begin{pmatrix}1\\ i\mathbf{r}_-\end{pmatrix} e^{i\lambda x} - \frac{1}{z}\begin{pmatrix}q_0^2\\ \mathbf{0}\end{pmatrix} e^{i\lambda x} + \varphi_1^- + \sum_{i=1}^{N+1} \varphi_i^- V_{i,N+1} \tag{4.2c}$$

where $\mathbf{V}(z) = \big(V_{i,j}(z)\big) = \hat{\mathbf{S}}(z) - \boldsymbol{\pi}$, and $\boldsymbol{\pi}$ is the permutation matrix defined in (2.18).

We then introduce the Cauchy projectors:

$$P^\pm[f](z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - (z \pm i0)} d\zeta\,, \tag{4.3}$$

which are well defined for any function f that is integrable on the real line, and they are such that $P^\pm[f^\pm](z) = \pm f(z)$, $P^\mp[f^\pm](z) = 0$ for any function $f^\pm(z)$ that is analytic for $\pm\,\mathrm{Im}\,z \geq 0$, and decaying as $z \to \infty$ (e.g., see [1]). We then consider, for instance, $\mathrm{Im}\,z > 0$ and apply the projector P^+ to both sides of the above equations. Taking into account the analyticity of the eigenfunctions and equation (3.38), expressing the residue of a given eigenfunction φ_j at each pole in terms of values of the previous eigenfunction φ_{j-1}, we obtain for $\mathrm{Im}\,z > 0$:

$$\varphi_1(x,z) = \begin{pmatrix}z\\ i\mathbf{r}_-\end{pmatrix} e^{-i\lambda(z)x} + \sum_{i=1}^{\bar{n}_N} \bar{c}_{N,i}\, e^{-i(\lambda(\bar{z}_{N,i})+k(\bar{z}_{N,i}))x} \frac{z\varphi_N(x,\bar{z}_{N,i})}{z-\bar{z}_{N,i}} + \frac{1}{2\pi i}\sum_{i=1}^{N+1}\int \frac{z\,\varphi_i^-(x,\zeta)V_{i\,1}(\zeta)}{\zeta(\zeta-z)}d\zeta \tag{4.4a}$$

$$\varphi_2(x,z) = \begin{pmatrix}0\\ i\mathbf{r}_{0,1}^\perp\end{pmatrix} e^{ik(z)x} + \sum_{i=1}^{n_1} c_{1,i}\, e^{i(k(\bar{z}_{1,i})+\lambda(\bar{z}_{1,i}))x} \frac{\varphi_1(x,z_{1,i})}{z-z_{1,i}} + \sum_{i=1}^{\bar{n}_1} \bar{c}_{1,i}\, e^{i(k(\bar{z}_{1,i})-\lambda(\bar{z}_{1,i}))x} \frac{\varphi_1(x,\bar{z}_{1,i})}{z-\bar{z}_{1,i}}$$

$$+\frac{1}{2\pi i}\sum_{i=1}^{N+1}\int\frac{\varphi_i^-(x,\zeta)V_{i\,2}(\zeta)}{\zeta-z}d\zeta\,, \tag{4.4b}$$

$$\varphi_j(x,z)=\begin{pmatrix}0\\ i\mathbf{r}_{0,j-1}^{\perp}\end{pmatrix}e^{ik(z)x}+\sum_{i=1}^{n_{j-1}}c_{j-1,i}\,\frac{\varphi_{j-1}(x,z_{j-1,i})}{z-z_{j-1,i}}+\sum_{i=1}^{\bar{n}_{j-1}}\bar{c}_{j-1,i}\,\frac{\varphi_{j-1}(x,\bar{z}_{j-1,i})}{z-\bar{z}_{j-1,i}}$$
$$+\frac{1}{2\pi i}\sum_{i=1}^{N+1}\int\frac{\varphi_i^-(x,\zeta)V_{i\,j}(\zeta)}{\zeta-z}d\zeta\qquad j=3,\ldots,N\,, \tag{4.4c}$$

$$\varphi_{N+1}(x,z)=\begin{pmatrix}q_0^2/z\\ i\mathbf{r}_-\end{pmatrix}e^{i\lambda(z)x}+\sum_{i=1}^{n_N}c_{N,i}\,e^{i(\lambda(z_{N,i})-k(z_{N,i}))x}\frac{\varphi_N(x,z_{N,i})}{z-z_{N,i}}+\frac{1}{2\pi i}\sum_{i=1}^{N+1}\int\frac{\varphi_i^-(x,\zeta)V_{i\,N+1}(\zeta)}{\zeta-z}d\zeta\,, \tag{4.4d}$$

where $c_{i,j}$ (resp. $\bar{c}_{i,j}$) $i=1,\ldots,N$, $j=1,\ldots n_i$, is the *norming constants* associated to the discrete eigenvalue $z_{i,j}$ (resp. $\bar{z}_{i,j}$) in the upper (resp. lower) half-plane, as defined in (3.38).

Similarly, when $\operatorname{Im} z<0$ we can apply a P^- projector to both sides of the jump equations and obtain:

$$\varphi_{N+1}(x,z)=\begin{pmatrix}z\\ i\mathbf{r}_-\end{pmatrix}e^{-i\lambda(z)x}+\sum_{i=1}^{\bar{n}_N}\bar{c}_{N,i}\,e^{-i(\lambda(\bar{z}_{N,i})+k(\bar{z}_{N,i}))x}\frac{z\varphi_N(x,z_{N,i})}{z-\bar{z}_{N,i}}+\frac{1}{2\pi i}\sum_{i=1}^{N+1}\int\frac{z\,\varphi_i^-(x,\zeta)V_{i\,1}(\zeta)}{\zeta(\zeta-z)}d\zeta\,, \tag{4.5a}$$

$$\varphi_2(x,z)=\begin{pmatrix}0\\ i\mathbf{r}_{0,1}^{\perp}\end{pmatrix}e^{ik(z)x}+\sum_{i=1}^{n_1}c_{1,i}\,e^{i(k(\bar{z}_{1,i})+\lambda(\bar{z}_{1,i}))x}\frac{\varphi_1(x,z_{1,i})}{z-z_{1,i}}+\sum_{i=1}^{\bar{n}_1}\bar{c}_{1,i}\,e^{i(k(\bar{z}_{1,i})-\lambda(\bar{z}_{1,i}))x}\frac{\varphi_1(x,\bar{z}_{1,i})}{z-\bar{z}_{1,i}}$$
$$+\frac{1}{2\pi i}\sum_{i=1}^{N+1}\int\frac{\varphi_i^-(x,\zeta)V_{i\,2}(\zeta)}{\zeta-z}d\zeta\,, \tag{4.5b}$$

$$\varphi_j(x,z)=\begin{pmatrix}0\\ i\mathbf{r}_{0,j-1}^{\perp}\end{pmatrix}e^{ik(z)x}+\sum_{i=1}^{n_{j-1}}c_{j-1,i}\frac{\varphi_{j-1}(x,z_{j-1,i})}{z-z_{j-1,i}}+\sum_{i=1}^{\bar{n}_{j-1}}\bar{c}_{j-1,i}\,\frac{\varphi_{j-1}(x,\bar{z}_{j-1,i})}{z-\bar{z}_{j-1,i}}$$
$$+\frac{1}{2\pi i}\sum_{i=1}^{N+1}\int\frac{\varphi_i^-(x,\zeta)V_{i\,j}(\zeta)}{\zeta-z}d\zeta\qquad j=3,\ldots,N\,, \tag{4.5c}$$

$$\varphi_1(x,z)=\begin{pmatrix}q_0^2/z\\ i\mathbf{r}_-\end{pmatrix}e^{i\lambda(z)x}+\sum_{i=1}^{n_N}c_{N,i}\,e^{i(\lambda(z_{N,i})-k(z_{N,i}))x}\frac{\varphi_N(x,z_{N,i})}{z-z_{N,i}}+\frac{1}{2\pi i}\sum_{i=1}^{N+1}\int\frac{\varphi_i^-(x,\zeta)V_{i\,N+1}(\zeta)}{\zeta-z}d\zeta\,. \tag{4.5d}$$

The linear-algebraic system can be closed by evaluating the above equations at the various discrete eigenvalues that appear in the rhs. The potential is then reconstructed, for instance, by the large-z expansion of $\varphi_1(x,z)e^{-i\lambda(z)x}$ in the upper half-plane of z, which corresponds to the first column in (3.57a), whose last N-components allow one to recover $\mathbf{r}(x)$.

5 Time evolution

To deal more effectively with the NZBCs, it is convenient to define a rotated field as $\mathbf{q}'(x,t)=\mathbf{q}(x,t)\,e^{-2iq_0^2t}$. It is then easy to see that the asymptotic values of the potential $\mathbf{q}'_\pm=\lim_{x\to\pm\infty}\mathbf{q}'$ are now time-independent, and that $\mathbf{q}'$ solves the modified defocusing VNLS equation

$$i\mathbf{q}'_t=\mathbf{q}'_{xx}+2(q_0^2-\|\mathbf{q}'\|^2)\mathbf{q}'\,. \tag{5.1}$$

Equation (5.1) is the compatibility condition of the modified Lax pair

$$v_x=\mathbf{L}'v\,,\qquad v_t=\mathbf{T}'v\,, \tag{5.2a}$$

where $\mathbf{L}'$ has the same expression as $\mathbf{L}$ in (2.2a) except that $\mathbf{Q}$ is replaced by $\mathbf{Q}'$, and where

$$\mathbf{T}'(x,t,k) = i(q_0^2 - 2k^2)\mathbf{J} - i\mathbf{J}\mathbf{Q}'^2 - 2k\mathbf{Q}' - i\mathbf{J}\mathbf{Q}'_x\,. \tag{5.3}$$

Since the scattering problem is the same as in Sections 3 and 4, the formalism developed the direct and inverse problem remains valid. Nonetheless, the change allows one to obtain the time dependence of the eigenfunctions very easily, as we show next. For simplicity we will drop the primes in the rest of this section.

Since $\mathbf{Q} \to \mathbf{Q}_\pm$ as $x \to \pm\infty$, we expect that, asymptotically, the time dependence of the scattering eigenfunctions will be given by

$$\mathbf{T}'_\pm(k) = \lim_{x\to\pm\infty} \mathbf{T}'(x,t,k) = -2k(ik\mathbf{J} + \mathbf{Q})\,.$$

Since $\mathbf{T}'_\pm = -2k\mathbf{L}_\pm$, however, the eigenvector matrices $\mathbf{E}_\pm(k)$ of $\mathbf{L}_\pm$ [cf. (2.13)], are also the eigenvector matrices of the asymptotic time evolution operator $\mathbf{T}'_\pm$. We can therefore take into account the time evolution of the Jost solutions Φ and $\tilde{\Phi}$ defined in (3.1) and (3.2) by simply replacing the boundary conditions (3.1b) and (3.2b) respectively with

$$\lim_{x\to-\infty} \Phi(x,t,k)e^{-i\mathbf{\Lambda}(k)(x-2kt)} = \mathbf{E}_-(k)\,, \tag{5.4a}$$

$$\lim_{x\to+\infty} \tilde{\Phi}(x,t,k)\, e^{-i\mathbf{\Lambda}(k)(x-2kt)} = \mathbf{E}_+(k)\,. \tag{5.4b}$$

In other words, with the above definitions $\Phi(x,t,k)$ and $\tilde{\Phi}(x,t,k)$ are *simultaneous solutions of both parts of the Lax pair*. If one changes the definition of the fundamental matrix solutions correspondingly, replacing (3.3) with

$$\Phi(x,t,k) = \boldsymbol{\mu}(x,t,k)\, e^{i\mathbf{\Lambda}(k)(x-2kt)}, \tag{5.5a}$$

$$\tilde{\Phi}(x,t,k) = \tilde{\boldsymbol{\mu}}(x,t,k)\, e^{i\mathbf{\Lambda}(k)(x-2kt)}\,, \tag{5.5b}$$

it should then be clear that all the results in Section 3 will carry through when $t \neq 0$ with only trivial changes. In particular the scattering relation (3.41) [which expresses the proportionality relation between two fundamental solutions of the scattering problem] remains valid, as long as the definition of the sectionally meromorphic eigenfunctions is changed, as appropriate, to

$$\boldsymbol{\varphi}(x,t,k) = \mathbf{m}(x,t,k)\, e^{i\mathbf{\Lambda}(k)(x-2kt)}\,.$$

Correspondingly, (3.40) becomes simply

$$\mathbf{m}^+(x,t,k)\,\boldsymbol{\pi} = \mathbf{m}^-(x,t,k)\, e^{i\mathbf{\Lambda}^-(k)(x-2kt)}\mathbf{S}(k)\, e^{-i\mathbf{\Lambda}^-(k)(x-2kt)}\,. \tag{5.6}$$

It is then immediate to see that, with these definitions, all scattering coefficients contained in $\mathbf{S}(k)$ are independent of time. And, as a result, so are the discrete eigenvalues and the norming constants.

Similar changes allow one to carry over the time dependence to the inverse problem. In particular, it is straightforward to see that all the equations in Section 4 remain valid for $t \neq 0$ as long as x is replaced by $x - 2kt$ wherever it appears explicitly.

6 Comparison with the "adjoint problem" formulation of the IST for the 2-component NLS

In [16] scattering eigenfunctions were introduced for $k \in \Sigma$, defined by the following boundary conditions:

$$\phi_1(x,k) \sim w_1^-(k)\, e^{-i\lambda x}, \qquad \phi_2(x,k) \sim w_2^-(k)\, e^{ikx}, \qquad \phi_3(x,k) \sim w_3^-(k)\, e^{i\lambda x}, \qquad x \to -\infty \tag{6.1a}$$

$$\psi_1(x,k) \sim w_1^+(k)\, e^{-i\lambda x}, \qquad \psi_2(x,k) \sim w_2^+(k)\, e^{ikx}, \qquad \psi_3(x,k) \sim w_3^+(k)\, e^{i\lambda x}, \qquad x \to +\infty \quad (6.1b)$$

with

$$w_1^\pm(k) = \begin{pmatrix} \lambda + k \\ i\mathbf{r}_\pm \end{pmatrix}, \qquad w_2^\pm(k) = \begin{pmatrix} 0 \\ -i\mathbf{r}_\pm^\perp \end{pmatrix}, \qquad w_3^\pm(k) = \begin{pmatrix} \lambda - k \\ -i\mathbf{r}_\pm \end{pmatrix}, \tag{6.2}$$

where $\mathbf{r}_\pm = \big(r_\pm^{(1)}, r_\pm^{(1)}\big)^T$ and $\mathbf{q}_\pm = \big(q_\pm^{(1)}, q_\pm^{(2)}\big)^T$ are 2-component vectors, and $\mathbf{r}_\pm^\perp = \big(q_\pm^{(2)}, -q_\pm^{(1)}\big)^T$.

The solutions with fixed (with respect to x) boundary conditions were denoted by:

$$M_1(x,k) = e^{i\lambda x}\phi_1(x,k)\,, \qquad M_2(x,k) = e^{-ikx}\phi_2(x,k), \qquad M_3(x,k) = e^{-i\lambda x}\phi_3(x,k)\,, \tag{6.3a}$$

$$N_1(x,k) = e^{i\lambda x}\psi_1(x,k)\,, \qquad N_2(x,k) = e^{-ikx}\psi_2(x,k)\,, \qquad N_3(x,k) = e^{-i\lambda x}\psi_3(x,k)\,, \tag{6.3b}$$

and it was shown that $M_1(x,\cdot)$, $N_3(x,\cdot)$ can be analytically continued on the upper sheet of the Riemann surface, $M_3(x,\cdot)$, $N_1(x,\cdot)$ on the lower, while $M_2(x,\cdot)$, $N_2(x,\cdot)$ in general do not admit analytic continuation for k off Σ. The eigenfunctions $\mathbf{M} = (M_1, M_2, M_3)$ do not precisely coincide with the ones introduced in (3.46), because of slightly different normalization choices, but they are morally the same objects, in the sense that they are scattering eigenfunctions, defined for $k \in \Sigma$ by fixing their behavior as $x \to -\infty$.

On the other hand, recall that, when $N = 2$, the construction in Section 3 provides the 3×3 fundamental matrices $\boldsymbol{\mu}(x,k)$ and $\tilde{\boldsymbol{\mu}}(x,k)$ for all $k \notin \Sigma$ and with boundary conditions as $x \to -\infty$ and as $x \to +\infty$, respectively. In particular, μ_1 [the first column of $\boldsymbol{\mu}$] and $\tilde{\mu}_3$ [the last column of $\tilde{\boldsymbol{\mu}}$] are analytic everywhere on $\hat{\mathbb{C}} \setminus \Sigma$, with a discontinuity across Σ. The remaining columns are, in general, sectionally meromorphic functions of k, also with a discontinuity across Σ. Nonetheless, when $N = 2$ there is no additional term in any of equations (3.35b) and therefore (3.36b), and as a result $\boldsymbol{\mu}(x,k) = \mathbf{m}(x,k)$. The comparison of the boundary conditions (6.1–6.3) and (3.4–3.5) then gives

$$\mu_1(x,k) = \begin{cases} M_1(x,k) & k \in \mathbb{C}_1 \setminus \Sigma, \\ -M_3(x,k) & k \in \mathbb{C}_2 \setminus \Sigma, \end{cases} \qquad \tilde{\mu}_3(x,k) = \begin{cases} -N_3(x,k) & k \in \mathbb{C}_1 \setminus \Sigma, \\ N_1(x,k) & k \in \mathbb{C}_2 \setminus \Sigma, \end{cases} \tag{6.4}$$

which means that for the limiting values on Σ from either sheet the following relations hold: for all $k \in \Sigma$,

$$\mu_1^+(x,k) = M_1(x,k)\,, \quad \mu_1^-(x,k) = -M_3(x,k)\,, \quad \tilde{\mu}_3^+(x,k) = -N_3(x,k)\,, \quad \tilde{\mu}_3^-(x,k) = N_1(x,k)\,. \tag{6.5}$$

Moreover, for $k \in \Sigma$, the asymptotic behavior of $\tilde{\mu}_3$ as $x \to +\infty$ coincides that of the following eigenfunctions:

$$\tilde{\mu}_3(x,k) = \begin{cases} -N_3(x,k) \sim \begin{pmatrix} k - \lambda \\ i\mathbf{r}_+ \end{pmatrix} & \text{as } x \to +\infty\,, \qquad k \in \mathbb{C}_\mathrm{I}\,, \\ N_1(x,k) \sim \begin{pmatrix} \lambda + k \\ i\mathbf{r}_+ \end{pmatrix} & \text{as } x \to +\infty\,, \qquad k \in \mathbb{C}_\mathrm{II}\,, \end{cases}$$

while from (3.28c) it follows

$$\tilde{\mu}_3(x,k) = \begin{cases} -N_3(x,k) \sim e^{-i\Delta\theta}\eta_{11}\Delta_2(k) \begin{pmatrix} k - \lambda \\ i\mathbf{r}_- \end{pmatrix} & \text{as } x \to -\infty\,, \qquad k \in \mathbb{C}_\mathrm{I} \setminus \Sigma\,, \\ N_1(x,k) \sim e^{-i\Delta\theta}\eta_{11}\Delta_2(k) \begin{pmatrix} \lambda + k \\ i\mathbf{r}_- \end{pmatrix} & \text{as } x \to -\infty\,, \qquad k \in \mathbb{C}_\mathrm{II} \setminus \Sigma\,. \end{cases}$$

Note that the last equations acquires subdominant terms when $k \in \Sigma$. Explicitly, according to (3.33) and (3.42), one has

$$\frac{e^{i\Delta\theta}}{\eta_{11}^\pm \Delta_2^\pm(k)}\tilde{\mu}_3^\pm(x,k) \sim \alpha_{31}^\pm \begin{pmatrix} k \pm \lambda \\ i\mathbf{r}_- \end{pmatrix} e^{\mp 2i\lambda x} + \alpha_{32}^\pm \begin{pmatrix} 0 \\ i\mathbf{r}_0^\perp \end{pmatrix} e^{-i(\pm\lambda - k)x} + \begin{pmatrix} k \mp \lambda \\ i\mathbf{r}_- \end{pmatrix} \quad \text{as } x \to -\infty\,. \tag{6.6}$$

Similarly, the behavior of μ_1 is fixed for $x \to -\infty$ (also on Σ); specifically,

$$\mu_1(x,k) = \begin{cases} M_1(x,k) \sim \begin{pmatrix} \lambda + k \\ i\mathbf{r}_- \end{pmatrix} & \text{as } x \to -\infty\,, \quad k \in \mathbb{C}_{\mathrm{I}}\,, \\ -M_3(x,k) \sim \begin{pmatrix} k - \lambda \\ i\mathbf{r}_- \end{pmatrix} & \text{as } x \to -\infty\,, \quad k \in \mathbb{C}_{\mathrm{II}}\,, \end{cases} \tag{6.7}$$

while from (3.28a) it follows

$$\mu_1(x,k) = \begin{cases} M_1(x,k) \sim \eta_{11}\Delta_1(k) \begin{pmatrix} k + \lambda \\ i\mathbf{r}_+ \end{pmatrix} & \text{as } x \to +\infty\,, \quad k \in \mathbb{C}_{\mathrm{I}} \setminus \Sigma\,, \\ -M_3(x,k) \sim \eta_{11}\Delta_1(k) \begin{pmatrix} k - \lambda \\ i\mathbf{r}_+ \end{pmatrix} & \text{as } x \to +\infty\,, \quad k \in \mathbb{C}_{\mathrm{II}} \setminus \Sigma\,, \end{cases} \tag{6.8}$$

and again the limiting value of these last two relations on Σ will contain subdominant terms. Explicitly, according to (3.42), one has

$$\mu_1^{\pm}(x,k) \sim \beta_{11}^{\pm}(k) \begin{pmatrix} k \pm \lambda \\ i\mathbf{r}_+ \end{pmatrix} + \beta_{12}^{\pm}(k) \begin{pmatrix} 0 \\ i\mathbf{r}_0^{\perp} \end{pmatrix} e^{i(\pm\lambda + k)x} + \beta_{13}^{\pm}(k) \begin{pmatrix} k \mp \lambda \\ i\mathbf{r}_+ \end{pmatrix} e^{\pm 2i\lambda x} \quad \text{as } x \to +\infty\,. \tag{6.9}$$

Regarding the other columns, from (3.29a) we have

$$\mu_2(x,k) \sim \begin{cases} \begin{pmatrix} 0 \\ i\mathbf{r}_0^{\perp} \end{pmatrix} & \text{as } x \to -\infty\,, \quad k \notin \Sigma\,, \\ \dfrac{\Delta_2(k)}{\Delta_1(k)} \begin{pmatrix} 0 \\ i\mathbf{r}_0^{\perp} \end{pmatrix} & \text{as } x \to +\infty\,, \quad k \notin \Sigma\,, \end{cases} \tag{6.10}$$

while

$$\mu_2^{\pm}(x,k) \sim \beta_{22}^{\pm}(k) \begin{pmatrix} 0 \\ i\mathbf{r}_0^{\perp} \end{pmatrix} + \beta_{23}^{\pm}(k) \begin{pmatrix} k \mp \lambda \\ i\mathbf{r}_+ \end{pmatrix} e^{-i(\pm\lambda - k)x} \quad \text{as } x \to +\infty\,, \quad k \in \Sigma\,, \tag{6.11}$$

and

$$\mu_2^{\pm}(x,k) \sim \begin{pmatrix} 0 \\ i\mathbf{r}_0^{\perp} \end{pmatrix} + \alpha_{21}^{\pm} \begin{pmatrix} k \mp \lambda \\ i\mathbf{r}_- \end{pmatrix} e^{i-(\pm\lambda + k)x} \quad \text{as } x \to -\infty\,, \quad k \in \Sigma\,. \tag{6.12}$$

For the remaining columns of $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$, i.e., μ_3 and $\tilde{\mu}_1$, $\tilde{\mu}_2$, we can use (3.33) to obtain

$$\mu_3(x,k) = e^{i\Delta\theta} \frac{1}{\eta_{11}\Delta_2(k)} \tilde{\mu}_3(x,k)\,, \tag{6.13a}$$

$$\tilde{\mu}_1(x,k) = \frac{1}{\eta_{11}\Delta_1(k)} \mu_1(x,k)\,, \qquad \tilde{\mu}_2(x,k) = \frac{\Delta_1(k)}{\Delta_2(k)} \mu_2(x,k)\,, \tag{6.13b}$$

valid for all k for which $\Delta_1(k)\Delta_2(k) \neq 0$.

On the other hand, recall that additional analytic eigenfunctions $\chi(x,k)$ and $\bar{\chi}(x,k)$ were obtained in [16] via wedge products of analytic adjoint eigenfunctions. These eigenfunctions, analytical, respectively, in the upper and lower sheet of the Riemann surface, satisfy on Σ the following relations:

$$\begin{aligned} \chi(x,k)e^{-ikx} &= 2\lambda b_{33}(k) N_2(x,k) - 2\lambda b_{23}(k) e^{i(\lambda - k)x} N_3(x,k) \\ &= 2\lambda a_{11}(k) M_2(x,k) - 2\lambda a_{21}(k) e^{-i(\lambda + k)x} M_1(x,k) \\ \bar{\chi}(x,k)e^{-ikx} &= 2\lambda b_{21}(k) e^{-i(\lambda + k)x} N_1(x,k) - 2\lambda b_{11}(k) N_2(x,k) \end{aligned} \tag{6.14a}$$

$$= 2\lambda a_{23}(k) e^{i(\lambda-k)x} M_3(x,k) - 2\lambda a_{33}(k) M_2(x,k) \tag{6.14b}$$

where $\mathbf{A}(k) = (a_{ij}(k))$ is a matrix of scattering data such that

$$\boldsymbol{\phi}(x,k) = \boldsymbol{\psi}(x,k)\mathbf{A}^T(k)\,, \tag{6.15}$$

and $\mathbf{B}(k) = (b_{ij}(k)) \equiv \mathbf{A}^{-1}(k)$. The coefficients $a_{11}(k)$ and $b_{33}(k)$ [resp. $a_{33}(k)$ and $b_{11}(k)$] were shown to be analytic on the upper [resp. lower] sheet of the Riemann surface.

Defining $\mathbf{r}_{\pm}^{\perp} = \mathbf{r}_0^{\perp} e^{i\theta_{\pm}^{\perp}}$ to account for the phase difference in the normalizations (2.17) and (6.2), and comparing the asymptotic behavior as $x \to -\infty$ of $\mu_1^{\pm}$ in (6.9) and of M_1 and M_3 as given by (6.1), (6.3) and (6.15), according to (6.5) we obtain

$$a_{11}(k) = \beta_{11}^{+}(k) \equiv \eta_{11}^{+}\Delta_1^{+}(k)\,, \qquad a_{12}(k) = -\beta_{12}^{+}(k)e^{-i\theta_{-}^{\perp}}\,, \qquad a_{13}(k) = -\beta_{13}^{+}(k)\,, \tag{6.16}$$

as well as

$$a_{33}(k) = \beta_{11}^{-}(k) \equiv \eta_{11}^{-}\Delta_1^{-}(k)\,, \qquad a_{32}(k) = -\beta_{12}^{-}(k)e^{-i\theta_{-}^{\perp}}\,, \qquad a_{31}(k) = -\beta_{13}^{-}(k)\,, \tag{6.17}$$

showing that indeed the zeros of $\Delta_1(k)$ in each sheet play the same roles of the zeros of $a_{11}(k)$ on $\mathbb{C}_1$ and of $a_{33}(k)$ on $\mathbb{C}_2$ (and are therefore discrete eigenvalues, in the sense of [16]). On the other hand, comparing the asymptotic behavior of $\mu_3^{\pm}$ and N_1, N_3 as $x \to -\infty$ yields, according to (6.6), (6.1), (6.3) and the inverse of (6.15),

$$b_{33}(k) = e^{-i\Delta\theta}\eta_{11}^{+}\Delta_2^{+}(k)\,, \qquad b_{11}(k) = e^{-i\Delta\theta}\eta_{11}^{-}\Delta_2^{-}(k)\,, \tag{6.18}$$

which then proves that the zeros of $\Delta_1(k)$ and $\Delta_2(k)$ are paired, according to the symmetry relations derived in [16]. Note that, in terms of the uniformization variable z, the genericity assumption in Definition 3 corresponds to the requirement that for each quartet of discrete eigenvalues inside and outside the unit circle $\{z_n, z_n^*, q_0^2/z_n, q_0^2/z_n^*\}$, each of them is a simple zero of one (and only one) of the functions $a_{11}(z)$ (UHP of z), $a_{33}(z)$ (LHP of z), $b_{33}(z)$ (UHP of z) and $b_{11}(z)$ (LHP of z).

Finally, comparing the asymptotic behavior of $\mu_2^{\pm}$ as $x \to -\infty$ in (6.12) and the relations (6.14), we obtain

$$\mu_2(x,k) = \begin{cases} -\dfrac{\chi(x,k)e^{-ikx}}{2\lambda a_{11}(k)} e^{-i\theta_{-}^{\perp}}\,, & k \in \mathbb{C}_{\mathrm{I}} \\ \dfrac{\bar{\chi}(x,k)e^{-ikx}}{2\lambda a_{33}(k)} e^{-i\theta_{-}^{\perp}}\,, & k \in \mathbb{C}_{\mathrm{II}} \end{cases}\,,$$

which shows the correspondence between the analytic eigenfunctions constructed via the adjoint problem, and the meromorphic eigenfunction provided by the construction via tensors described in this article.

7 Concluding remarks

The general methodology developed and presented in this paper works regardless of the number of components. Even in the 2-component case, however, the present approach allows one to establish more rigorously various results that were only conjectured in [16], or were not adequately addressed there. Among them are the functional class of potentials for which the scattering eigenfunctions are well-defined, and the analyticity of the scattering data.

On the theoretical side, a few issues remain that still need to be clarified. For example, the behavior of the eigenfunctions and scattering coefficients at the branch points must still be rigorously established, as well as the limiting behavior of the eigenfunctions at the opposite space limit as resulting from the Volterra integral equations. On the other hand, on a more practical side the results of this work open up a number of interesting problems:

(i) Detailed analysis of the 3-component case, which is the simplest case that was previously unsolved.
(ii) Derivation of explicit solutions and study of the resulting soliton interactions.
(iii) In particular, an interesting question is whether solutions exist that exhibit a non-trivial polarization shift upon interaction, like in the focusing case [2].
(iv) Study of the long-time asymptotics of the solutions using the non-linear steepest descent method [6, 7].

All of these issues are left for future work.

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Appendix A: Proofs

Proof of Lemma 3.1: Suppose that $\boldsymbol{\mu}(x,k)$ and $\boldsymbol{\mu}'(x,k)$ are two solutions of (3.4). Since $\det\boldsymbol{\mu}(x,k)$ is a non-zero constant, the matrix $\boldsymbol{\mu}(x,k)$ is invertible for all x. A simple computation then shows that

$$\frac{\partial}{\partial x}(\boldsymbol{\mu}^{-1}\boldsymbol{\mu}') = \left[i\boldsymbol{\Lambda},\boldsymbol{\mu}^{-1}\boldsymbol{\mu}'\right].$$

The solution of the above matrix differential equation is readily obtained as

$$\boldsymbol{\mu}^{-1}(x,k)\boldsymbol{\mu}'(x,k) = e^{i\boldsymbol{\Lambda}x}\mathbf{A}\,e^{-i\boldsymbol{\Lambda}x}\,, \tag{A.1}$$

where $\mathbf{A} = \boldsymbol{\mu}^{-1}(0,k)\boldsymbol{\mu}'(0,k)$. One could also solve (A.1) for $\mathbf{A}$ in terms of $\boldsymbol{\mu}^{-1}\boldsymbol{\mu}'$. It is not possible to directly evaluate the resulting expression in the limit $x\to-\infty$, since some of the entries of $e^{\pm i\boldsymbol{\Lambda}x}$ diverge in that limit. On the other hand, since $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ are bounded for all x, so are $\boldsymbol{\mu}^{-1}$ and the product $\boldsymbol{\mu}^{-1}\boldsymbol{\mu}'$. Hence any terms in the right-hand side of (A.1) that diverge either as $x\to-\infty$ or as $x\to\infty$ must have a zero coefficient. It is easy to see that this implies that $\mathbf{A}$ must be block-diagonal with the same block structure as $\boldsymbol{\Lambda}$. Then, taking the limit as $x\to-\infty$ and using the boundary conditions (3.4b) yields $\mathbf{A}=\mathbf{I}$, and therefore $\boldsymbol{\mu}\equiv\boldsymbol{\mu}'$. The proof of the uniqueness of the solution of (3.5) is obtained following similar arguments. ■

Proof of Theorem 3.1: It is straightforward to see that the columns μ_1 and $\tilde{\mu}_{N+1}$ of the fundamental matrices $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$ can be written as solutions of the following Volterra integral equations:

$$\mu_1(x,k) = e_1^- + \int_{-\infty}^{x} e^{(x-y)(\mathbf{L}_- - i\lambda_1\mathbf{I})}\left[\mathbf{Q}(y)-\mathbf{Q}_-\right]\mu_1(y,k)dy\,,$$

$$\tilde{\mu}_{N+1}(x,k) = e_{N+1}^+ - \int_{x}^{\infty} e^{(x-y)(\mathbf{L}_+ - i\lambda_{N+1}\mathbf{I})}\left[\mathbf{Q}(y)-\mathbf{Q}_+\right]\tilde{\mu}_{N+1}(y,k)dy\,.$$

Standard Neumann series arguments show that, due to the ordering of the eigenvalues, the above integral equations have a unique solution, and such solution is an analytic function of k, if the potentials $\mathbf{q}-\mathbf{q}_-$ and $\mathbf{q}-\mathbf{q}_+$ are respectively in the functional classes $L^1(-\infty,c)$ and $L^1(c,\infty)$ for all $c\in\mathbb{R}$ [cf. (2.1)]. This estabilishes the analyticity of $f_1=\mu_1$ and $g_{N+1}=\tilde{\mu}_{N+1}$ for all $k\in\hat{\mathbb{C}}\setminus\Sigma$, with well-defined limits to Σ from either sheet, including the branch points $\pm q_0$. We next show that, for all $n=2,\dots,N+1$, the forms f_n and g_n are also solutions of Volterra integral equations that are well-defined $\forall k\in\hat{\mathbb{C}}\setminus\Sigma$ with well-defined limits to Σ.

The operators appearing in the extended differential equations (3.10), namely,

$$\mathbf{A}_n(k)=\mathbf{L}_-^{(n)}-i\,(\lambda_1+\cdots+\lambda_n)\,\mathbf{I}\,,\qquad \mathbf{B}_n(k)=\mathbf{L}_+^{(N-n+2)}-i\,(\lambda_n+\cdots+\lambda_{N+1})\,\mathbf{I}\,,$$

can be diagonalized as follows:

$$\mathbf{A}_n(k)=\mathbf{E}_-\tilde{\mathbf{A}}_n\,(\mathbf{E}_-)^{-1}\,,\qquad \mathbf{B}_n(k)=\mathbf{E}_+\tilde{\mathbf{B}}_n\,(\mathbf{E}_+)^{-1}\,,\tag{A.2}$$

where the matrix multiplication is performed according to (3.8), and $\tilde{\mathbf{A}}_n$ and $\tilde{\mathbf{B}}_n$ are the normal operators

$$\tilde{\mathbf{A}}_n=\mathbf{\Lambda}^{(n)}-i\,(\lambda_1+\cdots+\lambda_n)\,\mathbf{I}\,,\qquad \tilde{\mathbf{B}}_n=\mathbf{\Lambda}^{(N-n+2)}-i\,(\lambda_n+\cdots+\lambda_{N+1})\,\mathbf{I}\,.$$

The relations (A.2) follow from the definition of the extensions $\mathbf{L}_\pm^{(n)}$ and $\mathbf{\Lambda}^{(n)}$ and from (2.13). Moreover, it is easy to check that the standard basis tensors

$$\left\{e_{j_1}\wedge\cdots\wedge e_{j_n}:1\le j_1<j_2<\cdots<j_n\le N+1\right\}$$

for $\bigwedge^n(\mathbb{C}^{N+1})$ [where as before $e_1,\dots,e_{N+1}$ are the vectors of the canonical basis of $\mathbb{C}^{N+1}$] are eigenvectors of $\tilde{\mathbf{A}}_n$ and $\tilde{\mathbf{B}}_n$, and that the spectrum of $\tilde{\mathbf{A}}_n$ and $\tilde{\mathbf{B}}_n$ is given by, respectively

$$\begin{aligned}\operatorname{spec}(\tilde{\mathbf{A}}_n)&=\left\{\lambda_{j_1}+\cdots+\lambda_{j_n}-\lambda_1-\cdots-\lambda_n:j_1<j_2<\cdots<j_n\right\}\,,\\ \operatorname{spec}(\tilde{\mathbf{B}}_n)&=\left\{\lambda_{j_n}+\cdots+\lambda_{j_{N+1}}-\lambda_n-\cdots-\lambda_{N+1}:j_n<j_{n+1}<\cdots<j_{N+1}\right\}\,.\end{aligned}$$

Due to the ordering of the eigenvalues, the real part of the spectrum of $\tilde{\mathbf{A}}_n$ is therefore always non-positive, whereas the real part of the spectrum of $\tilde{\mathbf{B}}_n$ is always nonnegative.

To take advantage of the above diagonalization, it is convenient to introduce the following transformation of the fundamental tensor families:

$$f_n^\#=(\mathbf{E}_-)^{-1}f_n\,,\qquad g_n^\#=(\mathbf{E}_+)^{-1}g_n\,.$$

It should be clear that f_n and g_n are solutions of the differential problem (3.10) if and only if $f_n^\#$ and $g_n^\#$ are solutions of

$$\partial_x f_n^\#=\tilde{\mathbf{A}}_n f_n^\#+\mathbf{Q}_n^{\#,-}f_n^\#\,,\qquad \lim_{x\to-\infty}f_n^\#=e_1\wedge\cdots\wedge e_n\,,\tag{A.3a}$$

and

$$\partial_x g_n^\#=\tilde{\mathbf{B}}_n g_n^\#+\mathbf{Q}_{N-n+2}^{\#,+}g_n^\#\,,\qquad \lim_{x\to\infty}g_n^\#=e_n\wedge\cdots\wedge e_{N+1}\,,\tag{A.3b}$$

where

$$\mathbf{Q}_n^{\#,\pm}=(\mathbf{E}_\pm)^{-1}\big[\mathbf{Q}^{(n)}-\mathbf{Q}_\pm^{(n)}\big]\mathbf{E}_\pm\,,\tag{A.4}$$

and again the matrix multiplication is performed according to (3.8). Because of its k-dependence, the term $\mathbf{Q}_n^{\#,\pm}$ becomes an energy-dependent potential.

The $2(N+1)$ problems (A.3) above can all be analyzed via a *single abstract model* [namely, (A.6) below] for a normal operator in a finite-dimensional Hermitian vector space V with norm $\|\cdot\|$. More precisely, let A be a normal operator on such a vector space, which plays the role of either $\tilde{\mathbf{A}}_n$ or $\tilde{\mathbf{B}}_n$, and let $q(x,z)$ be a linear operator on V of the form

$$q(x,z) = \frac{z}{z^2 - q_0^2} \sum_{j=-2n}^{2n} z^j q_j(x)\,, \tag{A.5}$$

which plays the role of the energy-dependent potential (A.4). Here and below, z is the uniformization variable introduced in Section 2.3. The explicit z-dependent factor in front of the summation in (A.5) reflects the presence of a factor $1/\lambda$ in $\mathbf{E}_\pm^{-1}$ coming from the determinant of $\mathbf{E}_\pm$, while the summation reflects the fact that all remaining terms in $\mathbf{E}_\pm$ and its inverse have degree no larger than 1 and no less than -1 in z [cf. (2.22)]. Moreover, the fact that $\mathbf{q} - \mathbf{q}_- \in L^1(-\infty, c)$ and $\mathbf{q} - \mathbf{q}_+ \in L^1(c, \infty)$ implies similar properties for the $q_j(x)$'s. Then, for a fixed $z \in \mathbb{C} \setminus \Sigma$ and a fixed $u_0 \in \ker A$, consider the "model problem"

$$\partial_x u = A(z)u + q(x,z)u\,, \qquad \lim_{x\to-\infty} u(x) = u_0\,, \tag{A.6}$$

The problem for g_n can be brought to this form by simply changing x to $-x$. Correspondingly, u plays the role of either $f_n^\#$ or $g_n^\#$. If u satisfies the differential equation in (A.6), for any real s and x it is also a solution of the linear Volterra integral equation

$$u(x) = e^{(x-s)A(z)}u(s) + \int_s^x e^{(x-y)A(z)} q(y,z)u(y)\,dy.$$

The result of the theorem then follows from the fact that $A(z)$ is a normal operator with non positive real part, and therefore $e^{A(z)t}$ has norm less than or equal to 1 for all $t \geq 0$. We may thus take the limit of the above integral equation as $s \to -\infty$ and apply the boundary conditions in (A.6) to obtain

$$u(x) = u_0 + \int_{-\infty}^x e^{(x-y)A(z)} q(y,z)u(y)\,dy. \tag{A.7}$$

Conversely, any solution of the Volterra integral equation (A.7), which is bounded as $x \to -\infty$, solves (A.6). Applying the usual Picard iteration procedure then proves the existence, uniqueness and analyticity in z of the solution. ■

Proof of Theorem 3.2: Consider the maximal-rank tensors $h_n = f_n \wedge g_{n+1}$, for all $n = 1, \dots, N$. Each of these tensors satisfies a differential equation of the form

$$\partial_x h_n = \left[\mathbf{L}_\pm^{(N+1)} - ik(N-1)\mathbf{I} + \left(\mathbf{Q}^{(N+1)} - \mathbf{Q}_\pm^{(N+1)}\right)\right] h_n\,.$$

For any linear operator A acting on $\mathbb{C}^{N+1}$, the extension $A^{(N+1)}$ is equivalent to scalar multiplication by the trace of A. Because the trace of the operator on the right-hand side of the above equation is zero, we have that each of the h_n is a function of k only. Moreover, in $\mathbb{C}^{N+1}$, any $(N+1)$-form can be written as $C\, e_1 \wedge \cdots \wedge e_{N+1}$ for some scalar C. Therefore, (3.12) defines uniquely a function $\Delta_n(k)$ wherever the function $\gamma_n(k)$ is non-zero.

Analogously, (3.14) hold because both f_{N+1} and g_1 are maximal rank tensors, which, with similar arguments as above, can be shown to be independent of x. Therefore, their value must coincide with their asymptotic limit as either $x \to -\infty$ or $x \to \infty$.

The specific value of $\gamma_n(k)$ follows from the fact that, thanks to our choice of normalization for $\mathbf{R}_0^{\perp}$, the $N \times N$ matrix $(\mathbf{R}_0^{\perp}, \mathbf{r}_{\pm})$ has mutually orthogonal columns, each with norm q_0. Finally, the $\Delta_n(k)$ defined by (3.12) are analytic on $\hat{\mathbb{C}} \setminus \Sigma$ because the f_n's and g_n's are analytic there, and for the same reason they admit smooth extensions to $\Sigma \setminus \{\pm q_0\}$ from each sheet. ■

Proof of Theorem 3.3: As a solution of (3.10a), the tensor f_n is in the kernel of the operator $\partial_x - \mathbf{L}^{(n)} + i(\lambda_1 + \cdots + \lambda_n)\mathbf{I}$. But since $\mathbf{Q} \to \mathbf{Q}_+$ as $x \to \infty$, in this limit f_n is asymptotically in the kernel of

$$\partial_x - \mathbf{L}_+^{(n)} + i(\lambda_1 + \cdots + \lambda_n)\mathbf{I}\,.$$

Since this kernel is spanned by the collection of all tensors of the form $e_1^+ \wedge e_{j_2}^+ \wedge \cdots \wedge e_{j_n}^+$ for $2 \le j_2 < \cdots < j_n \le N$, equations (3.16) then follow.

By similar arguments we have that, as $x \to -\infty$, the tensor g_n must asymptotically be in the kernel of the operator

$$\partial_x - \mathbf{L}_-^{(N-n+2)} + i(\lambda_n + \cdots + \lambda_{N+1})\mathbf{I}\,,$$

which is spanned by the collection of all tensors of the form $e_{j_n}^- \wedge \cdots \wedge e_{j_N}^- \wedge e_{N+1}^-$, for $2 \le j_n < \cdots < j_N \le N$. Equations (3.17) then follow.

To prove (3.18), note that (3.12) and (3.16) imply $h_n = \Delta_n\, e_1^- \wedge \cdots \wedge e_n^- \wedge e_{n+1}^+ \wedge \cdots \wedge e_{N+1}^+$ for all $n = 1, \ldots, N$. Since h_n is independent of x, however, it equals its limits as $x \to \pm\infty$. That is,

$$\begin{aligned}\lim_{x\to-\infty} h_n = e_1^- \wedge \cdots \wedge e_n^- \wedge \left[\sum_{2\le j_{n+1}<j_{n+2}<\cdots<j_N\le N} \tilde{\delta}_{j_{n+1},\ldots,j_N}\, e_{j_{n+1}}^- \wedge \cdots \wedge e_{j_N}^- \wedge e_{N+1}^-\right] \\ = \lim_{x\to\infty} h_n = \left[\sum_{2\le j_2<j_3<\cdots<j_n\le N} \delta_{j_2,\ldots,j_n}\, e_1^+ \wedge e_{j_2}^+ \wedge \cdots \wedge e_{j_n}^+\right] \wedge e_{n+1}^+ \wedge \cdots \wedge e_{N+1}^+\,.\end{aligned}$$

All terms but one in the sums above have at least one repeated vector. Hence, taking into account the decompositions (2.19), we conclude that relations (3.18) follow. ■

Proof of Lemma 3.2: Recall first that if $u_1, \ldots, u_n$ are vectors in $\mathbb{C}^{N+1}$ such that $u_1 \wedge \cdots \wedge u_n \neq 0$ and $g \in \Lambda^{n+1}\left(\mathbb{C}^{N+1}\right)$, then, the equation

$$u_1 \wedge \cdots \wedge u_n \wedge v = g$$

has a solution v if and only if $u_j \wedge g = 0$ for all $j = 1, \ldots, n$. Using the above result, we next prove by induction that there exist unique smooth functions, $v_1, \ldots, v_{N+1} : \mathbb{R} \to \mathbb{C}^{N+1}$ such that, for all $n = 1, \ldots, N$,

$$v_1 \wedge \cdots \wedge v_n = f_n\,, \tag{A.8a}$$

$$v_n \wedge f_j = 0\,, \qquad \forall j = n, \ldots, N+1\,, \tag{A.8b}$$

$$f_{n-1} \wedge [(\partial_x - ik\mathbf{J} - \mathbf{Q} + i\lambda_n)\, v_n] = 0\,. \tag{A.8c}$$

The induction is anchored with the choice $v_1 = f_1$, which implies that (A.8a) and (A.8c) are satisfied trivially. We therefore need to show that (A.8b) holds for $n = 1$ and $j = 1, \ldots, N+1$. To this end, note that, for any $j = 1, \ldots, N+1$, the product $v_1 \wedge f_j$ is the solution of the following homogeneous differential equation:

$$\left[\partial_x - \mathbf{L}_-^{(j+1)} + i\lambda_1 + i\left(\lambda_1 + \cdots + \lambda_j\right) - (\mathbf{Q}-\mathbf{Q}_-)^{(j+1)}\right](v_1 \wedge f_j)$$
$$= \{[\partial_x - \mathbf{L}_- + i\lambda_1 - (\mathbf{Q}-\mathbf{Q}_-)]\, v_1\} \wedge f_j + v_1 \wedge \left[\partial_x - \mathbf{L}_-^{(j)} + i\left(\lambda_1 + \cdots + \lambda_j\right) - (\mathbf{Q}-\mathbf{Q}_-)^{(j)}\right] f_j = 0\,,$$

because $v_1 = f_1$ and f_j both satisfy (3.10a). Moreover, $v_1 \wedge f_j$ satisfies the zero boundary condition:

$$\lim_{x\to-\infty} v_1(x,k) \wedge f_j(x,k) = e_1^- \wedge e_1^- \wedge e_2^- \wedge \cdots \wedge e_j^- = 0\,.$$

Then the problem has the unique solution $v_1 \wedge f_j \equiv 0$. The induction is thus anchored. Suppose now that $v_1, \ldots, v_{n-1}$ have been determined according to (A.8). Note that $f_{n-1} = v_1 \wedge \cdots \wedge v_{n-1}$ is generically nonzero, since it solves a first order differential equation with nonzero boundary condition. The equation $f_{n-1} \wedge v = f_n$ then has a solution v_n provided $v_s \wedge f_n = 0$ for all $s = 1, \ldots, n-1$. But this condition is part of the induction assumption, so there exists a solution v_n that satisfies (A.8a). We thus need to show that such v_n also satisfies (A.8b) and (A.8c). First, note that

$$f_{n-1} \wedge [\partial_x - \mathbf{L}_- + i\lambda_n - (\mathbf{Q}-\mathbf{Q}_-)]\, v_n = \left[\partial_x - \mathbf{L}_-^{(n)} + i(\lambda_1 + \cdots + \lambda_n) - (\mathbf{Q}-\mathbf{Q}_-)^{(n)}\right] f_n$$
$$- \left\{\left[\partial_x - \mathbf{L}_-^{(n-1)} + i(\lambda_1 + \cdots + \lambda_{n-1}) - (\mathbf{Q}-\mathbf{Q}_-)^{(n-1)}\right] f_{n-1}\right\} \wedge v_n = 0\,,$$

where we used that $f_n = f_{n-1} \wedge v_n$ and equations (3.10a). This proves (A.8c). Moreover, as a consequence we have, for all $x \in \mathbb{R}$,

$$[\partial_x - \mathbf{L}_- + i\lambda_n - (\mathbf{Q}-\mathbf{Q}_-)]\, v_n \in \operatorname{span}\{v_1, \ldots, v_{n-1}\}\,. \tag{A.9}$$

Finally, for all $j = n, \ldots, N+1$ then we have

$$\left[\partial_x - \mathbf{L}_-^{(j+1)} + i(\lambda_n + \lambda_1 + \cdots + \lambda_l) - (\mathbf{Q}-\mathbf{Q}_-)^{(j+1)}\right](v_n \wedge f_j)$$
$$= \{[\partial_x - \mathbf{L}_- + i\lambda_n - (\mathbf{Q}-\mathbf{Q}_-)]\, v_n\} \wedge f_j + v_n \wedge \left[\partial_x - \mathbf{L}_-^{(j)} + i(\lambda_1 + \cdots + \lambda_j) - (\mathbf{Q}-\mathbf{Q}_-)^{(j)}\right] f_j\,. \tag{A.10a}$$

The first term in the right-hand side of (A.10), however, vanishes because of (A.9) and the induction assumption (A.8b). And the second term vanishes because the f_j's satisfy (3.10a). Moreover,

$$\lim_{x\to-\infty} v_n(x,k) \wedge f_j(x,k) = \lim_{x\to-\infty} v_n \wedge e_1^- \wedge \cdots \wedge e_j^- = \lim_{x\to-\infty} v_n \wedge f_n \wedge e_{n+1}^- \wedge \cdots \wedge e_j^-$$
$$= \lim_{x\to-\infty} v_n \wedge v_1 \wedge \cdots \wedge v_n \wedge e_{n+1}^- \wedge \cdots \wedge e_j^- = 0\,, \tag{A.10b}$$

where we used (A.8a) as well as the boundary conditions (A.8a). The unique solution of the homogeneous differential equation plus boundary conditions (A.10) is therefore $v_n \wedge f_j = 0$, which proves (A.8b) for $j \geq n$ and thus completes the induction. The proof of existence for the w_n's can be carried out analogously. ■

Proof of Lemma 3.3: For a fixed $n = 1, \ldots, N+1$ we know that f_{n-1} and g_n are point-wise decomposable. We can write these decompositions as

$$f_{n-1} = u_1 \wedge \cdots \wedge u_{n-1}, \qquad g_n = u_n \wedge \cdots \wedge u_{N+1}.$$

For instance, we can take $u_j = v_j$ for $j = 1, \ldots, n-1$ and $u_j = w_j$ for $j = n, \ldots, N+1$, where v_n and w_n are the vectors in Lemma 3.2. Then, if $f_{n-1} \wedge g_n \neq 0$, the $N+1$ vectors $\{u_n\}_{n=1,\ldots,N+1}$ are linearly independent and thus form a basis of $\mathbb{C}^{N+1}$. We can therefore express v_n as a linear combination of them:

$$v_n = \sum_{j=1}^{N+1} c_j u_j \,,$$

for some unique choice of coefficients $c_1, \ldots, c_{N+1}$. Since $f_n = f_{n-1} \wedge v_n$, imposing the first condition in (3.22a) [namely $f_n = f_{n-1} \wedge m_n$] gives

$$u_1 \wedge \cdots \wedge u_{n-1} \wedge \left(\sum_{j=n}^{N+1} c_j u_j \right) = u_1 \wedge \ldots u_{n-1} \wedge m_n \,.$$

In turn, this implies

$$m_n - \sum_{j=n}^{N+1} c_j u_j \in \operatorname{span}\{u_1, \ldots, u_{n-1}\} \,.$$

So we can express m_n as

$$m_n = \sum_{j=n}^{N+1} c_j u_j + \sum_{j=1}^{n-1} b_j u_j \,,$$

with coefficients $b_1, \ldots, b_{n-1}$ to be determined. But imposing now the second of (3.22a) [namely the condition $m_n \wedge g_n = 0$] implies $b_1 = \cdots = b_{n-1} = 0$ (due to the decomposition of g_n). Therefore we have determined the unique solution of (3.22a).

To show that such a m_n is a meromorphic function of k, we use the canonical basis and the standard inner product on $\bigwedge \left(\mathbb{C}^{N+1}\right)$, and express (3.22a) as

$$\begin{gathered} \left\langle (f_n - f_{n-1} \wedge m_n), e_{i_1} \wedge \cdots \wedge e_{i_n} \right\rangle = 0 \,, \\ \left\langle m_n \wedge g_n, e_{i_1} \wedge \cdots \wedge e_{i_{N-n+3}} \right\rangle = 0 \,. \end{gathered}$$

These conditions provide an over-determined linear system of equations for the coefficients of m_n with respect to the standard basis of $\mathbb{C}^{N+1}$. Since it has a unique solution, some subset of $N+1$ equations among these has nonzero determinant. Solving this subsystem gives the coefficients of m_j locally as rational functions of those of f_{n-1}, f_n, and g_n, which, according to Theorem 3.2, are analytic functions of k on $\hat{\mathbb{C}}/\Sigma$.

The dual results for the $\tilde{m}_n$'s are proved in a similar way. ■

Proof of Theorem 3.4: For $n = 1$, equations (3.23a), (3.24a) and (3.25a) follow from the fact that $m_1 = f_1$. So consider a fixed $n = 2, \ldots, N+1$. According to Lemma 3.3, there exists a unique $m_n(x,k)$ such that (3.22a) holds. Since such an m_n depends smoothly on f_{n-1} and g_n, it is a smooth bounded function of x. Together with (3.10a), equations (3.22a) give [using similar methods to those in Lemma 3.2]

$$\begin{gathered} f_{n-1} \wedge \left[\partial_x - i\mathbf{L}_- + i\lambda_n - (\mathbf{Q} - \mathbf{Q}_-)\right] m_n = 0 \,, \\ g_n \wedge \left[\partial_x - i\mathbf{L}_- + i\lambda_n - (\mathbf{Q} - \mathbf{Q}_-)\right] m_n = 0 \,. \end{gathered}$$

But wherever $f_{n-1} \wedge g_n \neq 0$, these equations imply (3.23a) [because the term to the right of both wedge signs must be in the span of two linearly independent set of vectors, and therefore is identically zero]. Moreover, the limit as $x \to -\infty$ of $f_{n-1} \wedge m_n = f_n$ gives (3.24a) for $n = 2, \ldots, N+1$.

To prove (3.25a), we first show below that the second of (3.22a) implies, for $n = 2, \ldots, N$,

$$m_n \wedge g_{n+1} = c_n g_n \,, \tag{A.11}$$

where $c_n(x,k)$ is a scalar function. In fact, the decomposition $g_n = w_n \wedge \cdots \wedge w_{N+1}$ [from Lemma 3.2] and the condition $m_n \wedge g_n = 0$ imply $m_n = \sum_{j=n}^{N+1} c_j w_j$, and therefore

$$m_n \wedge g_{n+1} = \left(\sum_{j=n}^{N+1} c_j w_j \right) \wedge w_{n+1} \wedge \cdots \wedge w_{N+1} \,,$$

from which (A.11) follows trivially. Thus, for all $n = 2, \ldots, N$,

$$c_n f_{n-1} \wedge g_n = f_{n-1} \wedge m_n \wedge g_{n+1} = f_n \wedge g_{n+1} \,, \tag{A.12}$$

and since both $f_{n-1} \wedge g_n$ and $f_n \wedge g_{n+1}$ are independent of x, we conclude that c_n must also be independent of x. (But in general it depends on k, obviously.) If we now consider the limit of (A.12) as $x \to -\infty$, taking into account (3.12) we obtain $c_n \Delta_{n-1} e_1^- \wedge \cdots \wedge e_{n-1}^- \wedge e_n^+ \wedge \ldots e_{N+1}^+ = \Delta_n e_1^- \wedge \cdots \wedge e_n^- \wedge e_{n+1}^+ \wedge \ldots e_{N+1}^+$, i.e., for all $n = 2, \ldots, N$,

$$c_n \Delta_{n-1} \gamma_{n-1} = \Delta_n \gamma_n \,. \tag{A.13}$$

From (A.11) and (A.13) we therefore conclude

$$m_n \wedge g_{n+1} = \frac{\gamma_n}{\gamma_{n-1}} \frac{\Delta_n}{\Delta_{n-1}} g_n \qquad n = 2, \ldots, N \,. \tag{A.14}$$

[Note that $\Delta_{n-1} \neq 0$ because of (3.12) and $f_{n-1} \wedge g_n \neq 0$.] The limit of the above equation as $x \to +\infty$ yields (3.25a) for $n = 2, \ldots, N$.

To establish (3.25a) for $n = N+1$ [for which (A.11) obviously does not apply], we first observe that, putting together the results of Lemmas 3.2 and 3.3, for the first and the last of the m_n's and $\tilde{m}_s$'s one has the following. Since $\tilde{m}_1 \wedge f_1 = 0$ and $m_1 \equiv f_1$, it follows that $\tilde{m}_1 \wedge m_1 = 0$, i.e.,

$$m_1 = d_1 \tilde{m}_1 \,, \tag{A.15a}$$

where d_1 a scalar function depending, in principle, on both x and k. Note however that m_1 and $\tilde{m}_1$ satisfy the same differential equation [this uses (3.23b) in Theorem 3.4, which is proved exactly as above], implying that $d_1(k)$ is independent of x. Similarly, $m_{N+1} \wedge g_{N+1} = 0$ and $\tilde{m}_{N+1} = g_{N+1}$ imply $\tilde{m}_{N+1} \wedge m_{N+1} = 0$, i.e.,

$$m_{N+1} = d_{N+1} \tilde{m}_{N+1} \tag{A.15b}$$

with d_{N+1} again a scalar function of k only (for the same reasons as above). Using (A.15b), as well as (3.17a) and (3.18), we then obtain

$$\lim_{x \to +\infty} m_{N+1} = d_{N+1}(k) e_{N+1}^+ \,, \qquad \lim_{x \to -\infty} m_{N+1} = d_{N+1}(k) e^{-i\Delta\theta} \eta_{11} \Delta_N \, e_{N+1}^- \,.$$

On the other hand, since (3.24a) is also valid for $n = N+1$, it is $e^{-i\Delta\theta} \eta_{11} d_{N+1} \Delta_N = 1$. Then, recalling $\Delta_{N+1} = 1$, we have $d_{N+1} = e^{i\Delta\theta} / [\eta_{11} \Delta_N] = [\gamma_{N+1} \Delta_{N+1}] / [\gamma_N \Delta_N]$, which completes the proof of (3.25a).

The analyticity of m_n wherever $f_{n-1} \wedge g_n \neq 0$ was already established in Lemma 3.3. Finally, (3.12) implies immediately that the only points $k \notin \Sigma$ such that $f_{n-1}(x,k) \wedge g_n(x,k) = 0$ but $\Delta_{n-1}(k) \neq 0$ are those for which $\gamma_1(k) = \cdots = \gamma_N(k) = 0$. As discussed earlier, these are points $k = q_0 \cos(\Delta\theta/2)$ on each sheet, where $\eta_{11}(k) = 0$. It is relatively easy to see, however, that at these two points one can simply define m_n by analytic continuation.

Similar arguments allow one to prove (3.23b) and the corresponding boundary conditions and asymptotic behavior for the $\tilde{m}_n$'s and to establish their analyticity properties. ■

The proof of Theorem 3.5 follows similar methods as that of Theorem 3.4 above.

Proof of Corollary 3.1: The results follow by taking the limits as $x \to \pm\infty$ of (3.22a), using the boundary conditions established in Theorem 3.3 and solving the resulting over-determined linear system. [The solvability conditions of the system correspond to Plücker relations such as (3.20).] ■

Proof of Corollary 3.2: The analyticity properties of the matrices $\mathbf{m}$ and $\tilde{\mathbf{m}}$ are an immediate consequence of their construction, together with the earlier results about the fundamental tensors f_j, g_j. The relation (3.33), defining the transition matrix $\mathbf{d}$, follows from the fact that for $k \in \hat{\mathbb{C}}/\Sigma$, and for each $j = 1, \ldots, N+1$, m_j and $\tilde{m}_j$ satisfy the same differential equation, and their boundary values are proportional to each other (cf. corollary 3.1). ■

Proof of Corollary 3.3: The result is a straightforward consequence of Corollaries 3.1 and **??**, and of the definitions (3.4) and (3.5). ■

Proof of Theorem 3.6: We start by proving that, under the given hypotheses, there exists a complex constant $b \neq 0$ such that

$$\tilde{m}_j(x, k_o) = b\, e^{i(\lambda_{j-1}(k_o) - \lambda_j(k_o))x} m_{j-1}(x, k_o)\,. \tag{A.16}$$

[It is worth noting that if $k_o \notin \Sigma$, (A.16) implies that m_{j-1} and $\tilde{m}_j$ decay exponentially as $x \to +\infty$ and as $x \to -\infty$, respectively.] In what follows, all functions of k will be meant as evaluated at $k = k_o$, but we will omit the dependence on k_o for brevity. Let $\{v_j, w_j\}$ be as in Lemma 3.2. We know (cf. proof of Lemma 3.3) that

$$m_{j-1} \in \operatorname{span}\{w_{j-1}, w_j, \ldots, w_{N+1}\}\ ,$$

and

$$0 \equiv f_{j-1} \wedge g_j = f_{j-2} \wedge m_{j-1} \wedge g_j = c_{j-1} f_{j-2} \wedge g_{j-1}\,,$$

where $c_{j-1} \equiv c_{j-1}(k_o)$ is a scalar function. [The above form is 0 because the first term is proportional to $\Delta_{j-1}(k_o)$, and we are assuming k_o is a zero of Δ_{j-1}; also, the last identity follows from (A.11)]. Since $f_{j-2} \wedge g_{j-1} \neq 0$ (we are assuming $\Delta_{j-2}(k_o) \neq 0$), then this means that necessarily $c_{j-1}(k_o) = 0$. On the other hand, from $m_{j-1} \wedge g_j = c_{j-1} g_{j-1}$ it follows that $m_{j-1} - c_{j-1} w_{j-1} \in \operatorname{span}\{w_j, \ldots, w_{N+1}\}$ and therefore at a point where $c_{j-1}(k)$ vanishes one has

$$m_{j-1} \in \operatorname{span}\{w_j, \ldots, w_{N+1}\}\ .$$

Moreover, $\operatorname{span}\{w_j, w_{j+1}, \ldots\} = \operatorname{span}\{\tilde{m}_j, w_{j+1}, \ldots\}$, because $\tilde{m}_j \wedge g_{j+1} = g_j$ and since $g_j = w_j \wedge \cdots \wedge w_{N+1}$ and $g_{j+1} = w_{j+1} \wedge \cdots \wedge w_{N+1}$, this implies that $\tilde{m}_j \in \operatorname{span}\{w_j, \ldots, w_{N+1}\}$]. Therefore we finally conclude that

$$m_{j-1} \in \operatorname{span}\{\tilde{m}_j, w_{j+1}, \ldots\}\ .$$

In a similar way one can show that

$$\tilde{m}_j \in \operatorname{span}\{v_1, \ldots, v_{j-2}, m_{j-1}\}\ .$$

It then follows that there are functions $b(x)$ and $b_1(x)$ such that

$$\tilde{m}_j - b\, m_{j-1} \in \operatorname{span}\{v_1, \ldots, v_{j-2}\}\ , \qquad \tilde{m}_j - b_1 m_{j-1} \in \operatorname{span}\{w_{j+1}, \ldots, w_{N+1}\}\,,$$

i.e.,

$$\tilde{m}_j - b\, m_{j-1} = \sum_{\ell=1}^{j-2} d_\ell v_\ell, \qquad \tilde{m}_j - b_1 m_{j-1} = \sum_{\ell=j+1}^{N+1} \tilde{d}_\ell w_\ell. \tag{A.17}$$

Taking the wedge product of both these vectors with $f_{j-2} \wedge g_{j+1}$, we obtain

$$\left(\tilde{m}_j - b\, m_{j-1}\right) \wedge f_{j-2} \wedge g_{j+1} = \left(\sum_{\ell=1}^{j-2} d_\ell v_\ell\right) \wedge f_{j-2} \wedge g_{j+1} = \left(\sum_{\ell=1}^{j-2} d_\ell v_\ell\right) \wedge v_1 \wedge \dots v_{j-2} \wedge g_{j+1} = 0$$

$$\left(\tilde{m}_j - b_1 m_{j-1}\right) \wedge f_{j-2} \wedge g_{j+1} = \left(\sum_{\ell=j+1}^{N+1} \tilde{d}_\ell w_\ell\right) \wedge f_{j-2} \wedge g_{j+1} = \left(\sum_{\ell=j+1}^{N+1} \tilde{d}_\ell w_\ell\right) \wedge f_{j-2} \wedge w_{j+1} \wedge \cdots \wedge w_{N+1} = 0$$

so that $\tilde{m}_j \wedge f_{j-2} \wedge g_{j+1} = b\, m_{j-1} \wedge f_{j-2} \wedge g_{j+1}$ and $\tilde{m}_j \wedge f_{j-2} \wedge g_{j+1} = b_1 m_{j-1} \wedge f_{j-2} \wedge g_{j+1}$ and therefore $b\, m_{j-1} \wedge f_{j-2} \wedge g_{j+1} = b_1 m_{j-1} \wedge f_{j-2} \wedge g_{j+1}$, i.e.,

$$b\, f_{j-1} \wedge g_{j+1} = b_1 f_{j-1} \wedge g_{j+1}$$

Now, $f_{j-1} \wedge v_j \wedge g_{j+1} = f_j \wedge g_{j+1} \neq 0$, so $f_{j-1} \wedge g_{j+1} \neq 0$ and $b_1 = b$. Together with (A.17), this implies that $\tilde{m}_j - b\, m_{j-1} \in \operatorname{span}\left\{v_1, \dots, v_{j-2}\right\}$ and also $\tilde{m}_j - b\, m_{j-1} \in \operatorname{span}\left\{w_{j+1}, \dots, w_{N+1}\right\}$ and since these two sets are linearly independent, it follow that

$$\tilde{m}_j(x,k) = b(x) m_{j-1}(x,k)\,.$$

The differential equations (3.23a) and (3.23b) satisfied by m_{j-1} and $\tilde{m}_j$ imply that $b(x)$ has form given in (A.16). Eq. (3.38) then follows from (A.16), Theorems 3.3 and 3.4 and Corollaries 3.1 and 3.2. ■

Proof of Theorem 3.7: By the genericity assumption, Δ_{j-1} is the only one of the Δ's which vanishes at k_o, and it has a simple zero. The results of the above sections show that the columns m_ℓ are holomorphic near k_o for $\ell \neq j$, while m_j has a simple pole. Theorem 3.6 shows that (3.39) is regular at k_o. Suppose $\mathbf{C}_o$ is another matrix such that $\mathbf{m}\left(0,k\right)\left[\mathbf{I} - \left(k-k_o\right)^{-1}\mathbf{C}_o\right]$ is regular at k_o (since it is true for all x, we can take $x=0$ without loss of generality). Now $\mathbf{I} - \left(k-k_o\right)^{-1} c_o \mathbf{D}_{n-1}$ has determinant equal to 1, therefore

$$\begin{aligned}&\left[\mathbf{m}(0,k)\left(\mathbf{I} - \left(k-k_o\right)^{-1} k_o \mathbf{D}_{n-1}\right)\right]^{-1}\left[\mathbf{m}\left(0,k\right)\left(\mathbf{I} - \left(k-k_o\right)^{-1}\mathbf{C}_o\right)\right]\\ &= \left(\mathbf{I} + \left(k-k_o\right)^{-1} c_o \mathbf{D}_{n-1}\right)\left(\mathbf{I} - \left(k-k_o\right)^{-1}\mathbf{C}_o\right) = \mathbf{I} + \left(k-k_o\right)^{-1}\left(c_o\mathbf{D}_{n-1} - \mathbf{C}_o\right) - \left(k-k_o\right)^{-2} c_o \mathbf{D}_{n-1}\mathbf{C}_o\end{aligned}$$

is regular at $k = k_o$. This implies $c_o \mathbf{D}_{n-1} = \mathbf{C}_o$ (which provides the uniqueness result we wanted to prove), together with the constraint $c_o \mathbf{D}_{n-1}\mathbf{C}_o = 0$, which is satisfied by $\mathbf{C}_o$ of the form specified above. ■

Proof of Theorem 3.8: The argument is the same as for the basic uniqueness result. Indeed, the choice of $\boldsymbol{\pi}$ gives

$$\boldsymbol{\Lambda}^- \boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{\Lambda}^+\,, \qquad \boldsymbol{\pi}\boldsymbol{\Lambda}^- = \boldsymbol{\Lambda}^+\boldsymbol{\pi}\,. \tag{A.18}$$

Therefore, both $\mathbf{m}^+\boldsymbol{\pi}$ and $\mathbf{m}^-$ satisfy the differential equation

$$\partial_x \mathbf{v} = ik\mathbf{J}\,\mathbf{v} - i\mathbf{v}\,\boldsymbol{\Lambda}^- - \mathbf{Q}\,\mathbf{v}\,,$$

so that

$$\partial_x\left[\left(\mathbf{m}^-\right)^{-1}\mathbf{m}^+\boldsymbol{\pi}\right] = i\left[\boldsymbol{\Lambda}^-\,,\,\left(\mathbf{m}^-\right)^{-1}\mathbf{m}^+\boldsymbol{\pi}\right]\,,$$

which implies $\left(\mathbf{m}^-\right)^{-1}\mathbf{m}^+\boldsymbol{\pi}$ has the form $e^{i\boldsymbol{\Lambda}^-(k)x}\mathbf{S}(k)e^{-i\boldsymbol{\Lambda}^-(k)x}$. ■

Appendix B: WKB expansion

Let us consider the scattering problem (2.1a) for the fundamental eigenfunction μ_1. In terms of the uniform variable z introduced in Section 2.3, the system of differential equations becomes

$$\partial_x \mu_1^{(1)}(x,z) = -i\frac{q_0^2}{z}\mu_1^{(1)}(x,z) + \sum_{j=1}^{N} q^{(j)}(x)\mu_1^{(j+1)}(x,z) \qquad \operatorname{Im} z > 0\,, \tag{B.19a}$$

$$\partial_x \mu_1^{(j)}(x,z) = iz\,\mu_1^{(j)}(x,z) + r^{(j-1)}(x)\mu_1^{(1)}(x,z), \qquad \operatorname{Im} z > 0\,, \quad j = 2,\ldots,N+1, \tag{B.19b}$$

where superscript $^{(j)}$, as usual, denotes the j-th component of the eigenfunction and $q^{(j)}, r^{(j)}$ the j-th components of the N-component vectors $\mathbf{q}$ and $\mathbf{r}$.

Let us start with the following ansatz for the expansion of $\mu_1(x,z)$ as $z \to \infty$, $\operatorname{Im} z > 0$:

$$\mu_1^{(1)}(x,z) = z\,\mu_{1,\infty}^{(1),-1}(x) + \mu_{1,\infty}^{(1),0}(x) + \frac{\mu_{1,\infty}^{(1),1}(x)}{z}\ldots \qquad z \to \infty\,, \quad \operatorname{Im} z > 0\,,$$

$$\mu_1^{(j)}(x,z) = \mu_{1,\infty}^{(j),0}(x) + \frac{\mu_{1,\infty}^{(j),1}(x)}{z} + \ldots \qquad z \to \infty\,, \quad \operatorname{Im} z > 0\,.$$

Substituting into (B.19a) and matching the terms with the same order in z^{-n}, we obtain

$$\partial_x \mu_{1,\infty}^{(1),-1}(x) = 0 \qquad \Rightarrow \qquad \mu_{1,\infty}^{(1),-1}(x) = \text{const} \equiv 1 \tag{B.20}$$

(the value of the constant is fixed by knowledge of the asymptotic behavior as $x \to -\infty$) and

$$\partial_x \mu_{1,\infty}^{(1),0}(x) = -iq_0^2 + \sum_{j=1}^{N} q^{(j)}(x)\mu_{1,\infty}^{(j),0}(x).$$

Substituting the expansions into (B.19b) yields

$$\mu_{1,\infty}^{(j+1),0}(x) = ir^{(j)}(x), \qquad j = 1,\ldots,N \tag{B.21}$$

and

$$\partial_x \mu_{1,\infty}^{(j+1),0}(x) = i\mu_{1,\infty}^{(j+1),1}(x) + r^{(j)}(x)\mu_{1,\infty}^{(1),0}(x) \qquad j = 1,\ldots,N$$

so that

$$\partial_x \mu_{1,\infty}^{(1),0}(x) = -iq_0^2 + i\sum_{j=1}^{N} q^{(j)}(x) r^{(j)}(x)\,,$$

i.e.

$$\mu_{1,\infty}^{(1),0}(x) = i\int_{-\infty}^{x} \left[\|\mathbf{q}(x')\|^2 - q_0^2\right] dx'\,,$$

where again we fixed the boundary so that the constant of integration is zero. Then

$$\mu_{1,\infty}^{(j+1),1}(x) = \partial_x r^{(j)}(x) - r^{(j)}(x)\int_{-\infty}^{x} \left[\|\mathbf{q}(x')\|^2 - q_0^2\right] dx' \qquad j = 1,\ldots,N.$$

Proceeding iteratively one can, in principle, determine all the coefficients of the asymptotic expansion.

In the lower-half z-plane, the differential equations satisfied by $\mu_1(x,z)$ are

$$\partial_x \mu_1^{(1)}(x,z) = -iz\,\mu_1^{(1)}(x,z) + \sum_{j=1}^{N} q^{(j)}(x)\mu_1^{(j+1)}(x,z) \qquad \operatorname{Im} z < 0\,, \tag{B.22a}$$

$$\partial_x \mu_1^{(j)}(x,z) = i\frac{q_0^2}{z}\mu_1^{(j)}(x,z) + r^{(j-1)}(x)\mu_1^{(1)}(x,z), \qquad \operatorname{Im} z < 0\,, \quad j = 2,\dots,N+1. \tag{B.22b}$$

The ansatz for the expansion as $z \to \infty$, $\operatorname{Im} z < 0$ will be:

$$\mu_1^{(1)}(x,z) = \frac{\mu_{1,\infty}^{(1),1}(x)}{z} + \dots \qquad z \to \infty, \quad \operatorname{Im} z < 0$$
$$\mu_1^{(j)}(x,z) = \mu_{1,\infty}^{(j),0}(x) + \frac{\mu_{1,\infty}^{(j),1}(x)}{z} + \dots \qquad z \to \infty, \quad \operatorname{Im} z < 0\,.$$

Substituting into (B.22) and matching the terms with the same order in z^{-n} we get for the leading order coefficients:

$$\mu_{1,\infty}^{(j),0}(x) = ir_-^{(j-1)} \qquad j = 2,\dots,N+1$$
$$\mu_{1,\infty}^{(1),1}(x) = \sum_{l=1}^{N} q^{(l)}(x)r_-^{(l)} \equiv \mathbf{q}^T(x)\mathbf{r}_-\,.$$

In order to properly formulate the inverse problem, we need to compute the behavior of the eigenfunctions as $z \to 0$. In this case, we use the ansatz

$$\mu_1^{(1)}(x,z) = z\,\mu_{1,0}^{(1),1}(x) + z^2\mu_{1,0}^{(1),2}(x) + \dots \qquad z \to 0, \quad \operatorname{Im} z > 0$$
$$\mu_1^{(j)}(x,z) = \mu_{1,0}^{(j),0}(x) + z\,\mu_{1,0}^{(j),1}(x) + \dots \qquad z \to 0, \quad \operatorname{Im} z > 0\,.$$

Substituting into (B.19) yields for the leading order coefficients of the expansion:

$$\partial_x \mu_{1,0}^{(j+1),0}(x) = 0 \qquad \Rightarrow \qquad \mu_{1,0}^{(j+1),0}(x) = \text{const} \equiv ir_-^{(j)} \qquad j = 1,\dots,N \tag{B.23}$$

[where, again, the value of the constant is fixed by knowledge of the behavior as $x \to -\infty$] and

$$\mu_{1,0}^{(1),1}(x) = \frac{\sum_{j=1}^{N} q^{(j)}(x)r_-^{(j)}}{q_0^2} \equiv \frac{\mathbf{q}^T(x)\mathbf{r}_-}{q_0^2}\,. \tag{B.24}$$

In the lower half z-plane, the ansatz for the expansion of $\mu_1(x,z)$ about $z = 0$ will be

$$\mu_1^{(1)}(x,z) = \frac{1}{z}\mu_{1,0}^{(1),-1}(x) + \mu_{1,0}^{(1),0}(x) + z\,\mu_{1,0}^{(1),1}(x) + \dots \qquad z \to 0, \quad \operatorname{Im} z < 0$$
$$\mu_1^{(j)}(x,z) = \mu_{1,0}^{(j),0}(x) + z\,\mu_{1,0}^{(j),1}(x) + \dots \qquad z \to 0, \quad \operatorname{Im} z < 0 \qquad j = 2,\dots,N+1$$

and replacing into the differential equations (B.22) and matching the corresponding powers of z yields

$$\mu_{1,0}^{(1),-1}(x) = q_0^2\,, \qquad \mu_{1,0}^{(j),0}(x) = ir^{(j-1)}(x) \qquad j = 2,\dots,N+1. \tag{B.25}$$

The eigenfunction $\mu_{N+1}(x,z)$ satisfies the following system of ODEs:

$$\partial_x \mu_{N+1}^{(1)} = -iz\,\mu_{N+1}^{(1)} + \sum_{j=1}^{N} q^{(j)}\mu_{N+1}^{(j+1)} \qquad \operatorname{Im} z > 0 \tag{B.26a}$$

$$\partial_x \mu_{N+1}^{(j+1)} = i\frac{q_0^2}{z}\mu_{N+1}^{(j)} + r^{(j)}(x)\mu_{N+1}^{(1)}, \qquad \operatorname{Im} z > 0\,, \qquad j = 1, \ldots, N, \tag{B.26b}$$

and

$$\partial_x \mu_{N+1}^{(1)} = -i\frac{q_0^2}{z}\mu_{N+1}^{(1)} + \sum_{j=1}^{N} q^{(j)}\mu_{N+1}^{(j+1)} \qquad \operatorname{Im} z < 0 \tag{B.27a}$$

$$\partial_x \mu_{N+1}^{(j+1)} = iz\,\mu_{N+1}^{(j)} + r^{(j)}(x)\mu_{N+1}^{(1)}, \qquad \operatorname{Im} z < 0\,, \qquad j = 1, \ldots, N. \tag{B.27b}$$

We then make the following ansatz for the behavior of $\mu_{N+1}(x,z)$ as $z \to 0$ with $\operatorname{Im} z > 0$

$$\mu_{N+1}^{(1)}(x,z) = \frac{\mu_{N+1,0}^{(1),-1}(x)}{z} + \mu_{N+1,0}^{(1),0}(x) + z\,\mu_{N+1,0}^{(1),1}(x) + \ldots \qquad z \to 0, \quad \operatorname{Im} z > 0 \tag{B.28a}$$

$$\mu_{N+1}^{(j+1)}(x,z) = \mu_{N+1,0}^{(j+1),0}(x) + z\,\mu_{N+1,0}^{(j+1),1}(x) + \ldots \qquad z \to 0, \quad \operatorname{Im} z > 0 \qquad j = 1, \ldots, N. \tag{B.28b}$$

Substituting these expansions into (B.26) and matching the terms of the corresponding powers of z, we get from the first few orders

$$\partial_x \mu_{N+1,0}^{(1),-1}(x) = 0 \qquad \Rightarrow \qquad \mu_{N+1,0}^{(1),-1}(x) = \text{const} \tag{B.29}$$

$$\partial_x \mu_{N+1,0}^{(1),0}(x) = -i\mu_{N+1,0}^{(1),-1}(x) + \sum_{j=1}^{N} q^{(j)}(x)\mu_{N+1,0}^{(j+1),0}(x) \qquad j = 1, \ldots, N, \tag{B.30}$$

and

$$\mu_{N+1,0}^{(j+1),0}(x) = \frac{i\mu_{N+1,0}^{(1),-1}}{q_0^2} r^{(j)}(x)$$

$$\partial_x \mu_{N+1,0}^{(j+1),0}(x) = iq_0^2\mu_{N+1,0}^{(j+1),1}(x) + r^{(j)}(x)\mu_{N+1,0}^{(1),0}, \qquad j = 1, \ldots, N.$$

We fix the constant value of $\mu_{N+1,0}^{(1),-1}(x)$ to be

$$\mu_{N+1,0}^{(1),-1}(x) = q_0^2 \tag{B.31}$$

so that the leading order term of the expansion coincides with the constant boundary as $x \to -\infty$, and then it follows that the previous system of equations can be explicitly solved giving

$$\mu_{N+1,0}^{(j+1),0}(x) = ir^{(j)}(x), \qquad j = 1, \ldots, N. \tag{B.32}$$

The ansatz for the behavior of μ_{N+1} as $z \to 0$ with $\operatorname{Im} z < 0$ will be:

$$\mu_{N+1}^{(1)}(x,z) = z\,\mu_{N+1,0}^{(1),1}(x) + \ldots \qquad z \to 0, \quad \operatorname{Im} z < 0 \tag{B.33a}$$

$$\mu_{N+1}^{(j+1)}(x,z) = \mu_{N+1,0}^{(j+1),0}(x) + z\,\mu_{N+1,0}^{(j+1),1}(x) + \ldots \qquad z \to 0, \quad \operatorname{Im} z < 0\,, \qquad j = 1, \ldots, N. \tag{B.33b}$$

Substituting these expansions into (B.27) and matching the terms of the corresponding powers of z, we get

$$\partial_x \mu_{N+1,0}^{(j),0} = 0\,, \qquad \mu_{N+1,0}^{(1),1}(x) = -\frac{i}{q_0^2}\sum_{j=1}^{N} q^{(j)}(x)\mu_{N+1,0}^{(j+1),0}(x)\,.$$

Taking into account the asymptotic behavior of μ_{N+1} as $x \to -\infty$, the first condition yields

$$\mu_{N+1,0}^{(j+1),0}(x) = ir_-^{(j)} \qquad j = 1, \dots, N \tag{B.34a}$$

and the second one gives

$$\mu_{N+1,0}^{(1),1}(x) = \frac{1}{q_0^2} \sum_{j=1}^{N} q^{(j)}(x) r_-^{(j)} \equiv \mathbf{q}^T(x)\mathbf{r}_-/q_0^2 . \tag{B.34b}$$

Similarly, if we consider the expansion about $z = \infty$, $\operatorname{Im} z > 0$, with the ansatz

$$\mu_{N+1}^{(1)}(x,z) = \frac{\mu_{N+1,\infty}^{(1),1}(x)}{z} + \frac{\mu_{N+1,\infty}^{(1),2}(x)}{z^2} \dots \qquad z \to \infty, \quad \operatorname{Im} z > 0 \tag{B.35a}$$

$$\mu_{N+1}^{(j+1)}(x,z) = \mu_{N+1,\infty}^{(j+1),0}(x) + \frac{m_{N+1,\infty}^{(j+1),1}(x)}{z} + \dots \qquad z \to \infty, \quad \operatorname{Im} z > 0, \qquad j = 1, \dots, N. \tag{B.35b}$$

Substituting into (B.26) and matching the coefficients of the powers of z we obtain

$$\partial_x \mu_{N+1,\infty}^{(j+1),0}(x) = 0 \qquad \Rightarrow \qquad \mu_{N+1,\infty}^{(j+1),0}(x) = \text{const} \equiv ir_-^{(j)} \tag{B.36}$$

(the value of the constant once again is fixed by the behavior as $x \to -\infty$) and

$$\mu_{N+1,\infty}^{(1),1}(x) = \sum_{j=1}^{N} r_-^{(j)} q^{(j)}(x) \equiv \mathbf{q}^T(x)\mathbf{r}_- . \tag{B.37}$$

The ansatz for the behavior of μ_{N+1} as $z \to \infty$ on $\operatorname{Im} z < 0$ is:

$$\mu_{N+1}^{(1)}(x,z) = z\, \mu_{N+1,\infty}^{(1),-1}(x) + \mu_{N+1,\infty}^{(1),0}(x) + \dots \qquad z \to \infty, \quad \operatorname{Im} z < 0 \tag{B.38a}$$

$$\mu_{N+1}^{(j+1)}(x,z) = \mu_{N+1,\infty}^{(j+1),0}(x) + \frac{1}{z}\mu_{N+1,\infty}^{(j+1),1}(x) + \dots \qquad z \to \infty, \quad \operatorname{Im} z < 0 \qquad j = 1, \dots, N. \tag{B.38b}$$

Replacing into the differential equations (B.27) yields

$$\partial_x \mu_{N+1,\infty}^{(1),-1}(x) = 0 \qquad \Rightarrow \qquad \mu_{N+1,\infty}^{(1),-1}(x) = \text{const} \equiv 1 \tag{B.39a}$$

(the value of the constant being fixed by comparison with the asymptotic behavior as $x \to -\infty$) and

$$\mu_{N+1,\infty}^{(j),0}(x) = ir^{(j-1)}(x), \qquad j = 2, \dots, N. \tag{B.39b}$$

For the eigenfunctions $\mu_\ell(x,k)$ with $\ell = 2, \dots, N$, the differential equations are the same on both half-planes:

$$\partial_x \mu_\ell^{(1)}(x,k) = -i(z + q_0^2/z)\mu_\ell^{(1)}(x,k) + \sum_{j=1}^{N} q^{(j)}(x)\mu_\ell^{(j+1)}(x,k)$$

$$\partial_x \mu_\ell^{(j+1)}(x,k) = r^{(j)}(x)\mu_\ell^{(1)}(x,k), \qquad j = 1, \dots, N$$

We make the ansatz

$$\mu_\ell^{(1)}(x,z) = \mu_{\ell,\infty}^{(1),0}(x) + \frac{\mu_{\ell,\infty}^{(1),1}(x)}{z} + \frac{\mu_{\ell,\infty}^{(2),1}(x)}{z^2} \dots \qquad z \to \infty$$

$$\mu_\ell^{(j+1)}(x,z) = \mu_{\ell,\infty}^{(j+1),0}(x) + \frac{\mu_{\ell,\infty}^{(j+1),1}(x)}{z} + \dots \qquad z \to \infty, \qquad j = 1, \dots, N.$$

Substituting into the system of ODE's and matching we get

$$\mu_{\ell,\infty}^{(1),0}(x) = 0 \tag{B.40}$$

in accordance with the large x behavior, and

$$i\mu_{\ell,\infty}^{(1),1}(x) = \sum_{j=1}^{N} q^{(j)}(x)\mu_{\ell,\infty}^{(j+1),0}(x) \tag{B.41}$$

$$\partial_x \mu_{\ell,\infty}^{(1),1}(x) = \sum_{j=1}^{N} q^{(j)}(x)\mu_{\ell,\infty}^{(j+1),1}(x) - i\mu_{\ell,\infty}^{(1),2}(x) \tag{B.42}$$

as well as

$$\mu_{\ell,\infty}^{(j+1),0}(x) = \text{const} \equiv i r_{0,\ell}^{\perp,(j)} \tag{B.43}$$

[as usual, we used the large x behavior to fix the values of the constants.] Then we substitute (B.43) into (B.41) to obtain

$$\mu_{\ell,\infty}^{(1),1}(x) = \sum_{j=1}^{N} q^{(j)}(x) r_{0,\ell}^{\perp,(j)} \equiv \mathbf{q}^T(x)\mathbf{r}_{0,\ell}^{\perp}\,. \tag{B.44}$$

The ansatz for the behavior at $z = 0$ (in both half-planes) will be

$$\begin{aligned} \mu_\ell^{(1)}(x,z) &= \mu_{\ell,0}^{(1),0}(x) + z\,\mu_{\ell,0}^{(1),1}(x) + z^2\mu_{\ell,0}^{(1),2}(x) + \dots \qquad z \to 0 \\ \mu_\ell^{(j+1)}(x,z) &= \mu_{\ell,0}^{(j+1),0}(x) + z\,\mu_{\ell,0}^{(j+1),1}(x) + \dots \qquad z \to 0, \qquad j = 1,\dots,N,\ \ell = 2,\dots,N \end{aligned}$$

and the substitution into the system of ODE's yields

$$\mu_{\ell,0}^{(1),0}(x) = 0 \tag{B.45}$$

as expected, and

$$i q_0^2 \mu_{\ell,0}^{(1),1}(x) = \sum_{j=1}^{N} q^{(j)}(x)\mu_{\ell,0}^{(j+1),0}(x)\,. \tag{B.46}$$

From the equations for the other components one finally obtains

$$\mu_{\ell,0}^{(j+1),0}(x) = \text{const} \equiv i r_{0,\ell}^{\perp,(j)} \tag{B.47}$$

and

$$\partial_x \mu_{\ell,0}^{(j+1),1}(x) = r^{(j)}(x)\mu_{\ell,0}^{(1),1}(x). \tag{B.48}$$

Taking into account (B.47), (B.46) finally gives

$$\mu_{\ell,0}^{(1),1}(x) = \sum_{j=1}^{N} q^{(j)}(x) r_{0,\ell}^{\perp,(j)}/q_0^2 \equiv \mathbf{q}^T(x)\mathbf{r}_{0,\ell}^{\perp}\,. \tag{B.49}$$

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