Rings between $\mathbb{Z}$ and $\mathbb{Q}$

Luke Harmon

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Hooks to interest the listener

Subgroups of $\mathbb{Q}$.
Some of examples of additive subgroups of $\mathbb{Q}$ are: $\mathbb{Z}$, $\left\{ \frac{m}{3^n} \middle| m, n \in \mathbb{Z} \right\}$, and $2\mathbb{Z}$.

It turns out that there are $card(\mathbb{R}) = 2^{\aleph_0}$ additive subgroups of $\mathbb{Q}$. These subgroups were classified by Baer in 1937.

Interesting Note: The additive subgroups of $\mathbb{Q} \times \mathbb{Q}$ remain unclassified.

Our goal is to classify all of the subrings of $\mathbb{Q}$ that contain $\mathbb{Z}$ as a subring.

Preliminaries

R will always denote a commutative, unital ring.

Definitions.

A proper ideal P of R is *prime* if $ab \in P \implies a \in P$ or $b \in P$.

Example 1. The principal ideal (3) is a prime ideal of $\mathbb{Z}$.

A subset S of a ring R is *multiplicative* provided:
(i) $0_R \notin S$, (ii) $1_R \in S$, (iii) $x, y \in S \implies xy \in S$.

Example 2. $S = \{2^n | n \geq 0\}$ is a multiplicative subset of $\mathbb{Z}$.

A multiplicative set S is *saturated* if $xy \in S \implies x, y \in S$.

The *saturated closure* $\widehat{S}$ of a multiplicative set S is the set of all $r \in R$ for which there exists $t \in R$ such that $rt \in S$. Intuitively, $\widehat{S}$ consists of all "divisors" of elements of S.

Example 3. The saturated closure (in $\mathbb{Z}$) of $S = \{2^n | n \geq 0\}$ is $\widehat{S} = \{\pm 2^n | n \geq 0\}$, since each divisor of 2^n is of the form $\pm 2^k$ for $0 \leq k \leq n$.

Preliminaries

Lemma. For any multiplicative set $S \subseteq R$, the saturated closure $\widehat{S}$ is a saturated set.

Proof. $1_R^2 = 1_R \in S$, so $1_R \in \widehat{S}$. If $0_R \in \widehat{S}$, then $t0_R = 0_R \in S$ for some $t \in R$, a contradiction. So $0_R \notin \widehat{S}$. If $a, b \in \widehat{S}$, then there exist $t, t' \in R$ such that $at, bt' \in S$. But S is multiplicative and R is commutative, so $(at)(bt') = (ab)(tt') \in S$. By definition, $ab \in \widehat{S}$. So $\widehat{S}$ is multiplicative. Finally, $ab \in \widehat{S}$ means there exists $t \in R$ such that $(ab)t = a(bt) = b(at) \in S$. Hence $a, b \in \widehat{S}$, which shows that $\widehat{S}$ is saturated. Note that $S \subseteq \widehat{S}$, since R is unital.

Proposition 1. If $S \subseteq R$ is multiplicative and I is an ideal of R with $I \cap S = \varnothing$, then I is contained in a prime ideal P with $P \cap S = \varnothing$.

Sketch of proof. Use Zorn's Lemma to prove that the collection $\mathfrak{I} := \{J | J \text{ an ideal of } R \text{ that contains } I, \text{ and is disjoint from } S\}$, partially ordered by $\subseteq$, has a maximal element M. Then show that M is prime.

Example 4. Let $R = \mathbb{Z}$ and $S = \{2^n | n \geq 0\}$. The ideal $I = (6)$ is disjoint from S and $I \subseteq (3)$, which is prime.

Proposition 2. If S is saturated, then S is the complement of a union of prime ideals.

Proof. Suppose S is a saturated subset of R. By definition of multiplicative, $0_R \notin S$, so $S^c \neq \varnothing$. Choose $x \in S^c$ and consider the principal ideal (x). Claim that $(x) \cap S = \varnothing$. Otherwise, $rx \in S$ for some $r \in R$. Since S is saturated, $x \in S$, a contradiction. So $(x) \cap S = \varnothing$. By *Proposition 1*, (x) is contained in a prime ideal $P_x \subseteq S^c$. Invoke the Axiom of Choice to pick a prime ideal P_x, which contains x, for each $x \in S^c$. So $S^c = \bigcup_{x \in S^c} P_x$. Thus $S = (\bigcup_{x \in S^c} P_x)^c$.

The Classification

Recall that our goal is to classify all rings R for which $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$.

Example 5. The set $R = \left\{ \frac{m}{2^n} \middle| m \in \mathbb{Z}, n \geq 0 \right\}$ is a ring under the usual addition and multiplication of fractions that contains $\mathbb{Z}$ as a proper subring and is itself a proper subring of $\mathbb{Q}$.

Definition. Let D be an integral domain. The *field of fractions* of D is $Frac(D) = \left\{ \frac{r}{s} \middle| r, s \in D, s \neq 0_D \right\}$, where $\frac{r}{s} = \frac{r'}{s'} \iff rs' - r's = 0$, with operations $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$ and $\frac{r}{s} + \frac{r'}{s'} = \frac{rs'+r's}{ss'}$.

Definition. Let S be a multiplicative subset of an integral domain D. The *ring of fractions* of D with respect to S is the subring of $Frac(D)$ given by $D_S = \left\{ \frac{r}{s} \middle| r \in D, s \in S \right\}$

Notice that the ring R in *Example 5* is the ring of fractions $\mathbb{Z}_S$, where $S = \{2^n | n \geq 0\}$.

The Classification

Proposition 3. Every ring R that is a subring of $\mathbb{Q}$ and contains $\mathbb{Z}$ as a subring is of the form $\mathbb{Z}_S$ for some multiplicative set $S \subseteq \mathbb{Z}$.

Sketch of proof. Define $S := \left\{ q | \frac{p}{q} \in R,\ gcd(p,q) = 1 \right\}$. Choose $\frac{p}{q}, \frac{p'}{q'} \in R$ with $gcd(p,q) = gcd(p',q') = 1$. By Bézout's identity, there are $\alpha, \beta \in \mathbb{Z}$ such that $\alpha p + \beta q = 1$. Dividing both sides by q yields $\frac{1}{q} = \alpha \left(\frac{p}{q} \right) + \beta \in R$. Similarly, $\frac{1}{q'} \in R$. Hence $\frac{1}{qq'} \in R$, and $qq' \in S$. Observe that $0 \notin S$ and $1 \in S$, so S is multiplicative. We claim that $R = \mathbb{Z}_S$. $R \subseteq \mathbb{Z}_S$, since each element of R can be written as $\frac{p}{q}$ with $gcd(p,q) = 1$. For the opposite containment, pick $\frac{a}{b} \in \mathbb{Z}_S$. Another argument using Bézout's identity shows that $\frac{1}{b} \in R$. Whence $a \left(\frac{1}{b} \right) = \frac{a}{b} \in R$. So our claim is true.

The Classification

Proposition 4. If $S \subseteq \mathbb{Z}$ is multiplicative, then $\mathbb{Z}_S = \mathbb{Z}_{\widehat{S}}$.

Sketch of proof. Choose $x \in \mathbb{Z}_{\widehat{S}}$, so $x = \frac{r}{s}$ for some $r \in \mathbb{Z}$ and $s \in \widehat{S}$. Then there exists $s' \in \mathbb{Z}$ such that $ss' \in S$. Since $0 \notin S$, $s' \neq 0$, and so $\frac{r}{s} = \frac{rs'}{ss'} \in \mathbb{Z}_S$. Hence $\mathbb{Z}_{\widehat{S}} \subseteq \mathbb{Z}_S$. The opposite containment is an immediate consequence of $\widehat{S}$ containing S.

The Classification

We are now ready to classify all of the rings between $\mathbb{Z}$ and $\mathbb{Q}$.

Theorem. Every subring of $\mathbb{Q}$ that contains $\mathbb{Z}$ as a subring is of the form $\mathbb{Z}_S$ for some saturated set $S \subseteq \mathbb{Z}$.

Proof. Immediate from *Proposition 3* and *Proposition 4*.

Consider $\mathbb{Z}_S$, where $S \subseteq \mathbb{Z}$ is saturated. By *Proposition 2* (saturated sets are the complements of unions of prime ideals) and the fact that all of the nonzero prime ideals of $\mathbb{Z}$ are the principal ideals generated by prime numbers, $S = \left(\bigcup_{p\in\mathfrak{p}}(p)\right)^c$, where $\mathfrak{p}$ is a set of primes. It follows that each element of S is not a multiple of any element of $\mathfrak{p}$. Whence $p \nmid s$ for each $p \in \mathfrak{p}$ and $s \in S$.
In particular, each ring between $\mathbb{Z}$ and $\mathbb{Q}$ is a set of fractions whose denominators are not divisible by the elements of some set of prime numbers.

The Classification

As with the additive subgroups of $\mathbb{Q}$, there are $2^{\aleph_0}$ rings between $\mathbb{Z}$ and $\mathbb{Q}$.

References

T. W. Hungerford. Algebra. Graduate Texts in Mathematics, **142-147**. Springer-Verlag, New York, 1974.

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Questions?