

Universal Localization of Piecewise Noetherian Rings

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Acknowledgments

For commutative rings, results on piecewise Noetherian rings are from two joint papers with William D. Weakley, *Piecewise Noetherian Rings*, Comm. Algebra (1984) and *A note on prime ideals which test injectivity*, Comm. Algebra (1987).

In the noncommutative case, most results are from my student Abigail Bailey's dissertation (NIU, 2011) and a joint paper *On noncommutative piecewise Noetherian rings*, Comm. Algebra (2017).

The results on universal localization in this setting are just preliminary results.

The commutative case

Recall: A ring has Noetherian spectrum if it has ACC on prime ideals and each semiprime ideal is a finite intersection of primes.

Definition (1984)

A commutative ring R is called *piecewise Noetherian* if (i) R has Noetherian spectrum and (ii) for each ideal I and each prime ideal P minimal over I , the localized ring R_P/IR_P is Artinian.

Condition (ii) is related to the notion of “ideal length” utilized by Zariski and Samuel, before localization techniques came into use.

Theorem (1984)

The commutative ring R is piecewise Noetherian if and only if it has Noetherian spectrum and ACC on P -primary ideals, for each prime ideal P .

The commutative case

What's missing from the Noetherian case?

Theorem (1984)

The commutative piecewise Noetherian ring R is Noetherian \iff every ideal is a finite intersection of primary ideals.

How is the definition related to other chain conditions?

Theorem (1984)

Let R be a commutative ring.

(a) If R has Krull dimension (as a module), then R is piecewise Noetherian.

(b) If R is piecewise Noetherian, then R has Gabriel dimension.

(c) R is piecewise Noetherian \iff the polynomial ring $R[x]$ is piecewise Noetherian.

More commutative results

Theorem (1984)

If R is piecewise Noetherian, so is R_P , for all prime ideals P .

Theorem (1987)

Over a piecewise Noetherian ring, every module has an associated prime ideal.

Recall that a Prüfer domain is a Dedekind domain if and only if it is Noetherian. Popescu (1984) defined a notion of a *generalized Dedekind domain*.

Theorem (Facchini, 1994)

A Prüfer domain is a generalized Dedekind domain \iff it is piecewise Noetherian.

Background: Goldie's notion of reduced rank

In the noncommutative case, we need the notion of reduced rank.

Goldie (1964): if R is a semiprime Noetherian ring, let $Q_{cl}(R)$ be the semisimple Artinian classical ring of quotients of R .

The *rank* of a finitely generated module ${}_R M$ is the

length of the module $Q_{cl}(R) \otimes_R M$,

denoted by $\text{rank}(M)$. It equals the uniform dimension of $M/\tau(M)$, where $\tau(M)$ is the torsion submodule of M .

More generally, if ${}_R M$ is a finitely generated module over a left Noetherian ring R with prime radical N , whose index of nilpotence is k , then the *reduced rank* of M is defined to be

$$\rho({}_R M) = \sum_{i=1}^k \text{rank}(N^{i-1}M/N^i M).$$

A 1979 paper by Chatters, Goldie, Hajarnavis, and Lenagan publicized the notion by using it to give several new proofs of results in Noetherian rings.

Background: Warfield's generalization

Warfield (1979) observed that in order to define the reduced rank all that is needed is that R/N is a semiprime left Goldie ring and N is nilpotent. The definition of ρ -rank given by McConnell and Robson in **Noncommutative Noetherian Rings** and by Lam in **Lectures on Modules and Rings** is equivalent, but allows computation via other descending chains of submodules.

Warfield proved that the following conditions are equivalent, generalizing Small's theorem:

- (1) the ring R is a left order in a left Artinian ring
- (2) R has finite ρ -rank and satisfies the regularity condition (any element not a zero divisor modulo N is not a zero divisor in R).

Definition (1982)

Let R be a ring with prime radical N , and let γ denote the torsion radical cogenerated by $E(R/N)$. The module ${}_R M$ is said to have finite reduced rank if the module of quotients $Q_\gamma(M)$ has finite length (denoted by $\rho(M)$) in the quotient category $R\text{-Mod}/\gamma$.

Note. If R is a semiprime left Goldie ring, then $\rho(M)$ is the length of $Q_{cl}(R) \otimes_R M$.

Theorem (1982, 2017)

Let N be the prime radical of the ring R . The following conditions are equivalent:

- (1) R has finite reduced rank on the left;*
- (2) the set of left annihilators of subsets of $E(R/N)$ satisfies ACC;*
- (3) R has finitely many minimal prime ideals $\{P_i\}_{i=1}^n$, and for each i the set of left annihilators of $E(R/P_i)$ satisfies DCC.*

Background: torsion-theoretic reduced rank

For an ideal I , let $C(I)$ be the set of elements regular modulo I .

Theorem (2015)

Let N be the prime radical of R . The following are equivalent:

- (1) R has finite reduced rank on the left;*
- (2) R/N is a left Goldie ring, N^k is $C(N)$ -torsion for some integer $k > 0$, and N^i/N^{i+1} has finite reduced rank as an R/N -module, for $i = 1, \dots, k - 1$.*

Note: In this case $\rho({}_R M) = \sum_{i=1}^k \rho(N^{i-1}M/N^iM)$, provided each of the terms is finite. We're almost back to Warfield's definition, except that N need not be nilpotent.

Theorem

The torsion theoretic notion of reduced rank of a ring is Morita invariant.

Theorem (1982)

If R has finite reduced rank on the left, then so does the polynomial ring $R[x]$. Moreover, the reduced rank of $R[x]$ is equal to the reduced rank of R .

I think that the torsion-theoretic notion of reduced rank is worth studying because of the following generalization of the Small-Warfield theorem on orders in Artinian rings.

Theorem (1982)

The following conditions are equivalent for the ring R .

- (1) R is a left order in a left Artinian ring;*
- (2) R has finite reduced rank on the left (in the torsion theoretic sense) and satisfies the regularity condition.*

Main definitions

Definition

The ring R is called *left piecewise Noetherian* if

- (i) for each ideal I of R the factor ring R/I has finite reduced rank on the left (in the torsion theoretic sense) and
- (ii) R has ACC on prime ideals.

We give two definitions, depending on the choice of the definition of reduced rank.

Definition

The ring R is called *strongly piecewise Noetherian on the left* if in addition the prime radical of each factor ring is nilpotent.

Theorem

Every semiprime ideal S of a piecewise Noetherian ring is left Goldie (i.e. R/S is a semiprime left Goldie ring).



Motivation for the terminology

Definition (2017)

Let P be a prime ideal of the ring R . A left ideal $A \subseteq R$ is said to be P -primary if there exists an ideal $I \subseteq A$ such that P is minimal over I and ${}_R R/A$ is $C(P)$ -torsionfree.

Theorem (2017)

The ring R is left piecewise Noetherian if and only if it has Noetherian spectrum and ACC on P -primary left ideals, for each prime ideal P .

Thus the noncommutative definition via reduced rank reduces to the definition given in the commutative case.

Theorem (2017)

The notions of piecewise Noetherian and strongly piecewise Noetherian are Morita invariant.

A familiar example

Example 1. (Small's example is left piecewise Noetherian)

Let $R = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{bmatrix}$. The ring R is well-known to be right but not left Noetherian. (As a left ideal, N is not Noetherian.) Note that ${}_R R$ does not have Krull dimension since the cyclic left R -module $M = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix} / \begin{bmatrix} 0 & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$ does not have finite uniform dimension.

The ring R is strongly piecewise Noetherian on the left: any nonzero ideal I of R must contain N , and R/N is commutative Noetherian; R itself has finite reduced rank on the left because it is a left order in the left Artinian ring $\begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}$.

Extending commutative results

A ring with Krull dimension on the left is strongly left piecewise Noetherian.

I don't know what happens with polynomial rings, or localization.

If the ring is

- (a) strongly left piecewise Noetherian and left fully bounded OR
 - (b) left piecewise Noetherian and every finitely generated module is finitely annihilated, then
- (i) every module has an associated prime ideal;
 - (ii) the Gabriel correspondence between prime ideals and indecomposable injective modules holds;
 - (iii) the ring has Gabriel dimension.

Definition of the universal localization

Let S be a semiprime Goldie ideal, and let $\Gamma(S)$ be the set of all square matrices inverted by the canonical mapping $R \rightarrow R/S \rightarrow Q_{cl}(R/S)$.

Definition (Cohn, 1973, Noetherian case)

The *universal localization* $R_{\Gamma(S)}$ of R at a semiprime Goldie ideal S is the ring universal with respect to inverting all matrices in $\Gamma(S)$.

That is, if $\phi : R \rightarrow T$ inverts all matrices in $\Gamma(S)$, then there exists a unique ring homomorphism ϕ' such that the following diagram commutes.

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & R_{\Gamma(S)} \\ & \searrow \phi & \vdots \phi' \\ & & T \end{array}$$

Note: If S is left localizable, then $R_{\Gamma(S)}$ coincides with the Ore localization R_S defined via elements.

Theorem

Let S be a semiprime Goldie ideal of R .

(a) (Cohn, 1971) The universal localization of R at S exists.

(b) (Cohn, 1971) The canonical mapping $\lambda : R \rightarrow R_{\Gamma(S)}$ is an epimorphism in the category of rings.

(c) (1981) The ring $R_{\Gamma(S)}$ is flat as a right module over R if and only if S is a left localizable ideal.

A characterization of $R_{\Gamma(P)}$

Theorem

Let S be a semiprime Goldie ideal of R .

(a) (Cohn, 1973) $R_{\Gamma(S)}$ modulo its Jacobson radical is naturally isomorphic to $Q_{cl}(R/S)$.

(b) (1981) $R_{\Gamma(S)}$ is universal with respect to the property in (a).

Example 2: (Chain conditions gone awry)

Let $R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$, $P_2 = \begin{bmatrix} 2\mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$, and $P_3 = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 3\mathbb{Z} \end{bmatrix}$.

$R_{\Gamma(P_2 \cap P_3)} / J(R_{\Gamma(P_2 \cap P_3)})$ must be isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, so a

good guess is $\begin{bmatrix} \mathbb{Z}_{(2)} & 0 \\ ? & \mathbb{Z}_{(3)} \end{bmatrix} = \begin{bmatrix} \mathbb{Z}_{(2)} & 0 \\ \mathbb{Q} & \mathbb{Z}_{(3)} \end{bmatrix}$. This turns out to be correct; the ring is *not* Noetherian, but, like Small's example, it is piecewise Noetherian,

Universal localization of piecewise Noetherian rings

Guess: The class of left piecewise Noetherian rings may be the right class in which to study universal localization. At best, the universal localization is “nice” at the top. Perhaps it is like the algebraic closure of a field or the injective envelope of a module: it gives a place to work, but in general asking to compute it is not the right question. One case that *can* be computed is the universal localization at the prime radical.

Theorem (2017)

Let R be left piecewise Noetherian, let N be the prime radical of R , and let $K = \ker(\lambda)$, for $\lambda : R \rightarrow R_{\Gamma(N)}$.

- (a) K is the intersection of all ideals $I \subseteq N$ with $C(N) \subseteq C(I)$.*
- (b) The ring R/K is a left order in a left Artinian ring, and $R_{\Gamma(N)}$ is naturally isomorphic to $Q_{cl}(R/K)$.*

Definition

Let S be a semiprime Goldie ideal of R , with $\lambda : R \rightarrow R_{\Gamma(S)}$. The n^{th} symbolic power of S is $S^{(n)} = \lambda^{-1}(R_{\Gamma(S)}\lambda(S^n)R_{\Gamma(S)})$.

Theorem

If R is left piecewise Noetherian, then the following conditions hold for the symbolic powers of the semiprime ideal S .

- (a) $S^{(n)}$ is the intersection of all ideals I such that $S^n \subseteq I \subseteq S$ and $C(S) \subseteq C(I)$. [Mueller took this as the definition.]*
- (b) $C(S)$ is a left Ore set modulo $S^{(n)}$.*
- (c) $R_{\Gamma(S)}\lambda(S^n)R_{\Gamma(S)} = (J(R_{\Gamma(S)}))^n$, for all $n > 0$.*
- (d) $R/S^{(n)}$ is an order in the left Artinian ring $R_{\Gamma(S)}/(J(R_{\Gamma(S)}))^n$.*

Universal localization and reduced rank

Recall that a commutative ring R is piecewise Noetherian if (i) R has Noetherian spectrum and (ii) for each ideal I and each prime ideal P minimal over I , the localized ring R_P/IR_P is Artinian.

Theorem

If R is left piecewise Noetherian, and P is a prime ideal minimal over the ideal I , then $R_{\Gamma(P)}/I^e R_{\Gamma(P)} \cong (R/I)_{\Gamma(P/I)}$ is a left Artinian ring.

Can we characterize left piecewise Noetherian rings via universal localization? For the prime radical N of R , and a module ${}_R M$, we have the following exact sequences:

$$0 \rightarrow J(R_{\Gamma(N)}) \rightarrow R_{\Gamma(N)} \rightarrow Q_{cl}(R/N) \rightarrow 0 \quad \text{as right } R\text{-modules}$$

$$0 \rightarrow NM \rightarrow M \rightarrow M/NM \rightarrow 0 \quad \text{as left } R\text{-modules}$$

After tensoring

$$\begin{array}{ccccccc} J(R_{\Gamma(N)}) \otimes_R NM & \longrightarrow & R_{\Gamma(N)} \otimes_R NM & \longrightarrow & Q_{cl}(R/N) \otimes_R NM & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ J(R_{\Gamma(N)}) \otimes_R M & \longrightarrow & R_{\Gamma(N)} \otimes_R M & \longrightarrow & Q_{cl}(R/N) \otimes_R M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ J(R_{\Gamma(N)}) \otimes_R M/NM & \longrightarrow & R_{\Gamma(N)} \otimes_R M/NM & \longrightarrow & Q_{cl}(R/N) \otimes_R M/NM & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 & & 0 \end{array}$$

Theorem

Let R be a left piecewise Noetherian ring with prime radical N . If M is a left R -module, then the Goldie rank (or reduced rank) $\rho(M/NM)$ of M/NM is given by the length of the module $(R_{\Gamma(N)} \otimes_R M) / J(R_{\Gamma(N)}) (R_{\Gamma(N)} \otimes_R M)$.

Proof: In the diagram we have the following:

(1) $Q_{cl}(R/N)$ is a right R/N module, so it is annihilated by N , and therefore $Q_{cl}(R/N) \otimes_R NM = 0$.

(2) Since M/NM is a left R/N -module,
 $Q_{cl}(R/N) \otimes_R M/NM = Q_{cl}(R/N) \otimes_{R/N} M/NM$.

(3) The image of the mapping from $J(R_{\Gamma(N)}) \otimes_R M$ into $R_{\Gamma(N)} \otimes_R M$ is $J(R_{\Gamma(N)}) (R_{\Gamma(N)} \otimes_R M)$.

The diagram for the proof of the theorem

$$\begin{array}{ccccccc}
 J(R_{\Gamma(N)}) \otimes_R NM & \longrightarrow & R_{\Gamma(N)} \otimes_R NM & \longrightarrow & 0 & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 J(R_{\Gamma(N)}) \otimes_R M & \longrightarrow & R_{\Gamma(N)} \otimes_R M & \longrightarrow & Q_{cl}(R/N) \otimes_R M & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \cong & & \downarrow \\
 J(R_{\Gamma(N)}) \otimes_R M/NM & \longrightarrow & R_{\Gamma(N)} \otimes_R M/NM & \longrightarrow & Q_{cl}(R/N) \otimes_{R/N} M/NM & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0 & & 0
 \end{array}$$

Conclusion:

$$(R_{\Gamma(N)} \otimes_R M) / J(R_{\Gamma(N)}) (R_{\Gamma(N)} \otimes_R M) \cong Q_{cl}(R/N) \otimes_{R/N} M/NM$$

Cautionary examples

Example 3. (Symbolic powers may be of no help)

$$\text{Let } R = \begin{bmatrix} \mathbb{Z}/2\mathbb{Z} & 0 & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z} & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 \end{bmatrix}.$$

Then $N = \ker(\lambda)$, $R_{\Gamma(N)} = Q_{cl}(R/N)$, and so $N^{(2)} = N^{(3)} = N$.

This is a standard example of a ring that is not an order in an Artinian ring. Using $R_{\Gamma(N)}$ to compute $\rho(R)$ is possible but messy.

Example 4. ($R_{\Gamma(P)}$ is Artinian but P is not minimal)

$$\text{Let } R = \begin{bmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix} \text{ and } P = \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}. \text{ Then } R \text{ is Noetherian}$$

and $P^2 = P$, so $P = \ker(\lambda)$ and $R_{\Gamma(P)} = R/P \cong \mathbb{Z}/2\mathbb{Z}$. Thus $R_{\Gamma(P)}$ is an Artinian ring, but P is not a minimal prime ideal, since R is a prime ring. This differs from the commutative case.

Theorem (Forster-Swan, first part)

Let R be commutative Noetherian and ${}_R M$ be finitely generated. The minimal number of generators of M is $\leq \max_{P \text{ prime}} \{ \text{the minimal number of generators of } M_P + \text{the Krull dimension of } R/P \}$.

The minimal number of generators of M_P can be calculated as the dimension of the vector space $M_P/J(R_P)M_P$ over $R_P/J(R_P)$.

The noncommutative case (roughly):

- (1) Stafford replaced the ideal theoretic Krull dimension of R/P with the module theoretic Krull dimension of R/P .
- (2) He replaced $M_P/J(R_P)M_P$ by $Q_{cl}(R/P) \otimes_R M/PM$.
- (3) He then divided the dimension of $Q_{cl}(R/P) \otimes_R M/PM$ by the uniform dimension of $Q_{cl}(R/P)$, since $Q_{cl}(R/P)$ is simple Artinian rather than a division ring, and used the greatest integer function.

Some questions and future directions

Comment: The universal localization $R_{\Gamma(P)}$ provides the language to avoid Stafford's work-around in (2).

Question: Is the universal localization of a left piecewise Noetherian ring again left piecewise Noetherian?

Question: Are there conditions under which the kernel of $R \rightarrow R_{\Gamma(P)}$ is the intersection of the symbolic powers of P , as in the commutative case? Fully bounded Noetherian?

Future directions:

Enveloping algebras? McConnell [1968] obtained some results on Goldie's localization for certain enveloping algebras.

Group rings? Some topologists have been interested in universal localization at the augmentation ideal.

Thank you!

Some information about the kernel

Theorem (1976)

If ${}_R K \subseteq S$ is finitely generated, with $SK = K$, then $K \subseteq \ker(\lambda)$.

Proof.

Let $K = \sum_{i=1}^n Rx_i$, for $x_1, \dots, x_n \in R$. Since $K = SK$, we have $K = \sum_{i=1}^n Sx_i$. For $\mathbf{x} = (x_1, \dots, x_n)$ we have $\mathbf{x}^t = A\mathbf{x}^t$, where the $n \times n$ matrix A has entries in S . Thus $(I_n - A)\mathbf{x}^t = \mathbf{0}^t$. But $I_n - A$ is invertible modulo S , so it certainly belongs to $\Gamma(S)$. Therefore the entries of \mathbf{x} must belong to $\ker(\lambda)$, and so $K \subseteq \ker(\lambda)$. \square

Corollary

$R_{\Gamma(P)}$ can be determined for a prime ideal P of an hereditary Noetherian prime ring, since in HNP rings each prime ideal is either localizable or idempotent.