Infinite-Dimensional Triangularization

Zak Mesyan

University of Colorado
Colorado Springs

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Triangular Matrices

An upper-triangular $n \times n$ matrix is of the following form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix},$$

where the a_{ij} are coefficients in a field k.

- The eigenvalues, determinant, and trace of an upper-triangular (or lower-triangular) matrix are very easy to compute.
- Taking products and powers of upper-triangular (or lower-triangular) matrices is substantially easier than those of arbitrary matrices.
- Proofs of many linear algebra results rely on putting matrices in some kind of upper-triangular (or lower-triangular) form (e.g., Jordan canonical form) to simplify computations.

Classical Triangularization Theorem

Let k be a field, V a finite-dimensional k-vector space, and T a linear transformation of V. Then the following are equivalent.

(1) T has an upper-triangular representation as a matrix with respect to some basis for V.

(1′) T has a lower-triangular representation as a matrix with respect to some basis for V.

(2) There is a polynomial $p(x) \in k[x] \setminus k$ that factors into linear terms in $k[x]$, such that $p(T) = 0$.

(3) There exists a well-ordered set of T-invariant subspaces of V, which is maximal as a well-ordered set of subspaces of V.

(3′) There exists a totally ordered set of T-invariant subspaces of V, which is maximal as a totally ordered set of subspaces of V.

(4) T has a representation as a matrix in Jordan canonical form with respect to some basis for V.

Definitions and Notation

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- Let $(\mathbb{B}, \leq)$ be a partially ordered basis for V. Then $T \in \mathrm{End}_k(V)$ is *triangular with respect to* $(\mathbb{B}, \leq)$ if $T(v) \in \langle \{u \in \mathbb{B} \mid u \leq v\} \rangle$ for all $v \in \mathbb{B}$, and T is *strictly triangular with respect to* $(\mathbb{B}, \leq)$ if $T(v) \in \langle \{u \in \mathbb{B} \mid u < v\} \rangle$ for all $v \in \mathbb{B}$.

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- If $T \in \mathrm{End}_k(V)$ is triangular, respectively strictly triangular, with respect to some *well-ordered* basis for V, then we say that T is *triangularizable*, respectively *strictly triangularizable*.

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- If $T \in \mathrm{End}_k(V)$ is triangular, respectively strictly triangular, with respect to some *well-ordered* basis for V, then we say that T is *triangularizable*, respectively *strictly triangularizable*.
- If $\dim_k(V) = n < \aleph_0$, then $T \in \mathrm{End}_k(V) = \mathbb{M}_n(k)$ being "triangularizable" is equivalent to there being an invertible matrix $S \in \mathbb{M}_n(k)$ such that STS^{-1} is upper-triangular.

Infinite-Dimensional Triangularization Theorem

The following are equivalent for any $T \in \mathrm{End}_k(V)$.

(1) T is triangularizable.

(2) For every finite-dimensional subspace W of V there is a polynomial $p(x) \in k[x] \setminus k$ that factors into linear terms in $k[x]$, such that $p(T)$ annihilates W.

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(4) $V = \bigoplus_{a \in k} \bigcup_{i=1}^{\infty} \ker((T - a \cdot 1)^i)$.

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(5) There is a partially ordered basis $(\mathbb{B}, \preceq)$ for V such that T is triangular with respect to $(\mathbb{B}, \preceq)$ and $\{u \in \mathbb{B} \mid u \preceq v\}$ is finite for all $v \in \mathbb{B}$.

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If k is algebraically closed, then these are also equivalent to the following.

(6) Every finite-dimensional subspace of V is contained in a finite-dimensional T-invariant subspace of V.

(7) V is locally artinian, viewed as a $k[x]$-module, where x acts on V as T. (I.e., every finitely-generated submodule of V is artinian.)

Proof

(1) $\Rightarrow$ (6) Let $W \subseteq V$ be a finite-dimensional subspace, and let $U_1 \subseteq \mathbb{B}$ be finite such that $W \subseteq \langle U_1 \rangle$. For each $i > 1$ ($i \in \mathbb{Z}^+$) define recursively

$$U_i = \{v \in \mathbb{B} \mid \pi_v T(U_{i-1}) \neq 0\} \cup U_{i-1},$$

where $\pi_v \in \mathrm{End}_k(V)$ is the projection onto $\langle v \rangle$ with kernel $\langle \mathbb{B} \setminus \{v\} \rangle$. Since U_1 is finite, so is every U_i, by induction. Also, $T(U_i) \subseteq \langle U_{i+1} \rangle$ for all $i \in \mathbb{Z}^+$.

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Suppose that the chain $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ does not stabilize. Then for each $i \in \mathbb{Z}^+$ let $v_i \in \mathbb{B}$ be the maximal element, with respect to $\leq$, such that $v_i \in U_i \setminus U_{i-1}$ (where $U_0 = \emptyset$). This is well-defined since each $U_i \setminus U_{i-1}$ is finite but nonempty. It follows from T being triangular with respect to $\mathbb{B}$ that $v_1 > v_2 > v_3 > \cdots$ is a strictly descending chain, contradicting $\mathbb{B}$ being well-ordered.

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Hence $U_n = U_{n+1}$ for some $n \in \mathbb{Z}^+$, and therefore $T(U_n) \subseteq \langle U_{n+1} \rangle = \langle U_n \rangle$. Thus $T(\langle U_n \rangle) \subseteq \langle U_n \rangle$, where $W \subseteq \langle U_n \rangle$ and $\langle U_n \rangle$ is finite-dimensional.

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(3) There exists a well-ordered set of T-invariant subspaces of V, which is maximal as a well-ordered set of subspaces of V.

(4) $V = \bigoplus_{a \in k} \bigcup_{i=1}^{\infty} \ker((T - a \cdot 1)^i)$.

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If k is algebraically closed, then these are also equivalent to the following.

(6) Every finite-dimensional subspace of V is contained in a finite-dimensional T-invariant subspace of V.

(7) V is locally artinian, viewed as a $k[x]$-module, where x acts on V as T. (I.e., every finitely-generated submodule of V is artinian.)

Proof

(6) $\Rightarrow$ (2) Suppose that k is algebraically closed, and let W be a finite-dimensional subspace of V. Then, by (6), there is a finite-dimensional T-invariant subspace W' of V containing W. Viewing the restriction of T to W' as a (finite) matrix, there is a polynomial $p(x) \in k[x] \setminus k$ such that $p(T)$ annihilates W' (by the Cayley-Hamilton theorem), and hence also W. Since k is algebraically closed, $p(x)$ factors into linear terms in $k[x]$, proving (2).

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(7) $\Rightarrow$ (6) Let W be a finite-dimensional subspace of V. Then, by (7), the $k[x]$-submodule $M = k[x]W$ of V is artinian. Since M is a T-invariant subspace of V, it suffices to show that M is finite-dimensional. But since $k[x]$ is a principal ideal domain and M is a finitely-generated $k[x]$-module,

$$M \cong k[x]^r \oplus k[x]/\langle f_1(x)\rangle \oplus \cdots \oplus k[x]/\langle f_n(x)\rangle,$$

where $r \in \mathbb{N}$, $f_1(x), \ldots, f_n(x) \in k[x] \setminus \{0\}$, and $\langle f_i(x)\rangle$ is the ideal of $k[x]$ generated by $f_i(x)$. Since M is artinian, $r = 0$, and hence M is a finite-dimensional k-vector space.

Inverses

Suppose that $T \in \mathrm{End}_k(V)$ triangular with respect to a well-ordered basis $(\mathbb{B}, \leq)$ for V. Also for each $v \in \mathbb{B}$ let $\pi_v \in \mathrm{End}_k(V)$ be the projection onto $\langle v \rangle$ with kernel $\langle \mathbb{B} \setminus \{v\} \rangle$. Then the following are equivalent.

(1) T is invertible.

(2) The restriction of T to any finite-dimensional T-invariant subspace of V is invertible.

(3) T is injective.

(4) $T(\langle \{u \in \mathbb{B} \mid u \leq v\} \rangle) = \langle \{u \in \mathbb{B} \mid u \leq v\} \rangle$ for all $v \in \mathbb{B}$.

(5) $\pi_v T \pi_v \neq 0$ for all $v \in \mathbb{B}$.

Moreover, if T is invertible, then its inverse is triangular with respect to $(\mathbb{B}, \leq)$.

Strictly Triangularizable Transformations

The following are equivalent for any $T \in \mathrm{End}_k(V)$.

(1) T is strictly triangularizable.

(2) $V = \bigcup_{i=1}^{\infty} \ker(T^i)$.

(3) T is triangularizable, and if $(\mathbb{B}, \leq)$ is a well-ordered basis for V with respect to which T is triangular, then T is strictly triangular with respect to $(\mathbb{B}, \leq)$.

(4) T is triangularizable, and $\ker(T - a \cdot 1) \neq 0$ if and only if $a = 0$, for all $a \in k$.

Simultaneous Triangularization

Theorem (Frobenius, 1878)

If k is algebraically closed, and $T_1, T_2 \in \mathbb{M}_n(k)$ commute with each other, then there is an invertible matrix $S \in \mathbb{M}_n(k)$ such that ST_1S^{-1} and ST_2S^{-1} are both upper-triangular.

Since $\mathbb{M}_n(k)$ is finite-dimensional, it follows that any commuting set of matrices in $\mathbb{M}_n(k)$ is simultaneously triangularizable.

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Theorem

Let $X \subseteq \mathrm{End}_k(V)$ a finite commutative collection of transformations. If each element of X is triangularizable, then there exists a well-ordered basis for V with respect to which every element of X is triangular.

Function Topology

Definition

A basis of open sets for the *function topology* on $\mathrm{End}_k(V)$ is given by the sets

$$\{T \in \mathrm{End}_k(V) \mid T(x_1) = y_1, \ldots, T(x_n) = y_n\},$$

with $x_1, \ldots, x_n, y_1, \ldots, y_n \in V$.

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Facts

(1) $R = \mathrm{End}_k(V)$ is a topological ring with respect to the function topology. That is, $\cdot : R \times R \to R$, $+ : R \times R \to R$, and $- : R \to R$ are continuous.

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(2) The function topology on $\mathrm{End}_k(V)$ is Hausdorff and complete.

(3) If V is finite-dimensional, then the function topology on $\mathrm{End}_k(V) = \mathbb{M}_n(k)$ is the discrete topology.

The Countable Case

Suppose that V is countably infinite-dimensional, with basis $\{v_i \mid i \in \mathbb{Z}^+\}$.

- A typical basic open set in $\mathrm{End}_k(V)$ consists column-finite matrices of the form
$$\begin{pmatrix} A & * \\ 0 & * \end{pmatrix},$$
where $A \in \mathbb{M}_{nm}(k)$ is a fixed $n \times m$ matrix.

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where $A \in \mathbb{M}_{nm}(k)$ is a fixed $n \times m$ matrix.
- The function topology on $\mathrm{End}_k(V)$ is induced by the following metric d. Given $T, S \in \mathrm{End}_k(V)$, let
$$d(T,S) = \begin{cases} 0 & \text{if } T = S \\ 2^{-(i+1)} & \text{if } T \neq S \end{cases},$$
where $i \in \mathbb{N}$ is the least number such that $T(v_i) \neq S(v_i)$.

Schur's Theorem

Theorem (Schur, early 1900s)

For every $T \in \mathbb{M}_n(\mathbb{C})$ there exists a unitary $U \in \mathbb{M}_n(\mathbb{C})$ (i.e., $U^* = U^{-1}$) such that UTU^* is upper-triangular.

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Theorem

Define $\mathbb{T} \subseteq \mathrm{End}_k(V)$ to be the subset of all triangularizable transformations, and let $\overline{\mathbb{T}} \subseteq \mathrm{End}_k(V)$ be the closure of $\mathbb{T}$ in the function topology.

Then for all $T \in \mathrm{End}_k(V)$, we have $T \in \overline{\mathbb{T}}$ if and only if the restriction of T to any finite-dimensional T-invariant subspace of V is triangularizable.

In particular, if k is algebraically closed, then $\overline{\mathbb{T}} = \mathrm{End}_k(V)$.

Topologically Nilpotent Transformations

Theorem (Levitzki, 1931)

Every nilpotent multiplicative subsemigroup of $\mathbb{M}_n(k)$ is triangularizable.

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Definition

$X \subseteq \mathrm{End}_k(V)$ is *topologically nilpotent* if $(T_i \cdots T_2 T_1)_{i=1}^{\infty}$ converges to 0 in the function topology on $\mathrm{End}_k(V)$, given any infinite list $T_1, T_2, T_3, \ldots \in X$. Equivalently, for all $T_1, T_2, T_3, \ldots \in X$ and every finite-dimensional subspace W of V, there exists $n \in \mathbb{Z}^+$ such that $T_n \cdots T_2 T_1(W) = 0$.

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Theorem

Let $X \subseteq \mathrm{End}_k(V)$. Then X is (simultaneously) strictly triangularizable if and only if X is topologically nilpotent.

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Theorem

Let $X \subseteq \mathrm{End}_k(V)$. Then X is (simultaneously) strictly triangularizable if and only if X is topologically nilpotent.

Corollary

If $X \subseteq \mathrm{End}_k(V)$ is topologically nilpotent, then so is the nonunital k-subalgebra of $\mathrm{End}_k(V)$ generated by X.

Triangularizable Algebras

Theorem (McCoy, 1936)

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Theorem

The following are equivalent for any k-subalgebra R of $\mathrm{End}_k(V)$.

(1) R is triangularizable.

(2) R is contained in a k-subalgebra A of $\mathrm{End}_k(V)$ such that $A/\mathrm{rad}(A) \cong k^{\Omega}$ as topological k-algebras for some set Ω, and $\mathrm{rad}(A)$ is topologically nilpotent.

Moreover, if (2) holds and R is closed, then $\mathrm{rad}(R) = R \cap \mathrm{rad}(A)$.

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The following are equivalent for any k-subalgebra R of $\mathrm{End}_k(V)$.

(1) R is triangularizable.

(2) There exist a k-subalgebra $R \subseteq A \subseteq \mathrm{End}_k(V)$ and a set Ω such that $A/\mathrm{rad}(A) \cong k^{\Omega}$ as topological k-algebras, and $\mathrm{rad}(A)$ is topologically nilpotent.

Corollary

The following are equivalent for any k-subalgebra R of $\mathbb{M}_n(k)$.

(1) R is triangularizable.

(2) $R/\mathrm{rad}(R) \cong k^m$ as k-algebras, for some $m \in \mathbb{Z}^+$.

If k is algebraically closed, then these are also equivalent to the following.

(3) $R/\mathrm{rad}(R)$ is commutative.

Thank you!