A short proof that commutative Artinian rings are Noetherian

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Background

The study of (what is contemporarily known as) chain conditions for an associative ring has its genesis in early work of Emmy Noether and Emil Artin (early 20th century). Rings which satisfy these conditions now bear their names:

Definition

Let R be a commutative ring. Then R is **Noetherian** if there is no infinite, strictly increasing sequence of ideals $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$. R is **Artinian** if there is no infinite, strictly decreasing sequence of ideals $\cdots \subsetneq I_3 \subsetneq I_2 \subsetneq I_1$.

It is not difficult to show that a commutative ring R is Noetherian if and only if every ideal of R is finitely generated. It is also not difficult to find commutative rings with identity which are Noetherian but not Artinian.

Background

Example

The ring $\mathbb{Z}$ of integers is a principal ideal domain, and so every ideal of $\mathbb{Z}$ is finitely generated. However, observe that we have $\cdots \langle 8 \rangle \subsetneq \langle 4 \rangle \subsetneq \langle 2 \rangle$, and so $\mathbb{Z}$ is not Artinian.

If one does not require the presence of a multiplicative identity, then one can find rings which are Artinian but not Noetherian.

Example

Fix a prime number p, and consider the following subgroup of $\mathbb{Q}/\mathbb{Z}$: $C(p^\infty) := \{\mathbb{Z} + \frac{a}{p^n} : a \in \mathbb{Z}, n \in \mathbb{Z}^+\}$. Then $C(p^\infty)$ is an infinite abelian group for which all proper subgroups are finite. One can make $C(p^\infty)$ into a commutative ring R by definition $xy = 0$ for all $x, y \in C(p^\infty)$. Then the subgroups of $C(p^\infty)$ and the ideals of R coincide. In particular, every proper ideal of R is finite, and so R is Artinian. But $C(p^\infty)$ is not finitely generated as an abelian group, and so also is not finitely generated as an ideal (of itself). Thus R is Artinian but not Noetherian.

Background

Curiously, if one requires the presence of a multiplicative identity, then there are no such examples. In other words, if R is a commutative Artinian ring with identity, then R is also Noetherian. This result is commonly known as Akizuki's Theorem. In the noncommutative case, the result still holds (if R is a unital left/right Artinian ring, then R is also left/right Noetherian), and is often referred to as the Hopkins-Levitzki Theorem.
Most published proofs in the literature and textbooks invoke the nilpotency of the Jacobson radical, filtration arguments on finite products of maximal ideals, Nakayama's Lemma, and/or exact sequences. In this talk, we present a short and very elegant proof due to Karamzadeh (1994) using completely elementary methods.

Preliminaries

Throughout, by "ring", we will always mean a commutative ring with identity.

Theorem (First Isomorphism Theorem)

Let R be a ring, and let M and N be R-modules. Suppose that $\varphi\colon M \to N$ is a surjective module homomorphism with kernel $K := \{m \in M\colon \varphi(m) = 0\}$. Then $M/K \cong_R N$.

Next, let M be an R-module. Recall that the **annihilator** of M in R is given by $\mathrm{Ann}_R(M) := \{r \in R\colon rM = \{0\}\}$. One shows easily that $\mathrm{Ann}_R(M)$ is an ideal of R.

Preliminaries

Proposition

Let R be a ring, and let M be an R-module. Further, let $I := \mathrm{Ann}_R(M)$. Then M is naturally an R/I-module via the scalar product $\overline{r} \cdot m := rm$.

Proof.

It suffices to show that the above scalar product is well-defined. Thus, let $r, s \in R$ and $m \in M$ be arbitrary. We must show that if $\overline{r} = \overline{s}$, then $rm = sm$. So assume that $\overline{r} = \overline{s}$. Then $r - s \in \mathrm{Ann}_R(M)$. It follows that $(r - s)m = 0$, i.e., $rm = sm$, as required. □

Preliminaries

Next, we establish two more straightforward results.

Proposition

If D is an Artinian integral domain, then D is a field.

Proof.

Suppose D is an Artinian domain which is not a field, and let $x \in D$ be a nonzero, nonunit. Then one checks that $\cdots \subsetneq \langle x^3 \rangle \subsetneq \langle x^2 \rangle \subseteq \langle x \rangle$, and D is not Artinian, a contradiction. □

Preliminaries

Proposition

Let R be a ring, M be an R-module, and N be an R-submodule of N. If N and M/N are finitely generated, then M is finitely generated.

Proof.

Let $N = \langle n_1, n_2, \ldots, n_k \rangle$ and $M/N = \langle \overline{m}_1, \overline{m}_2, \ldots, \overline{m}_l \rangle$. Now let $m \in M$. Then $\overline{m} = r_1\overline{m}_1 + \cdots + r_l\overline{m}_l$ for some $r_1, \ldots, r_l \in R$. Hence $m - r_1m_1 - \cdots - r_lm_l \in N$. So $m - r_1m_1 - \cdots - r_lm_l = s_1n_1 + \cdots + s_kn_k$ for some $s_1, \ldots, s_k \in R$. But then $m = r_1m_1 + \cdots + r_lm_l + s_1n_1 + \cdots + s_kn_k$, and $m \in \langle m_1, \ldots, m_l, n_1, \ldots, n_k \rangle$. Thus M is finitely generated. □

Finally, we are ready give Karamzadeh's proof of the classical result that every Artinian ring is Noetherian.

The proof

Suppose by way of contradiction that there exists an (commutative, unital) Artinian ring R which is not Noetherian. Among all ideals of R which are not finitely generated, pick a least ideal (relative to set inclusion). Call this ideal I^*. Then observe that I^* satisfies the following property:

I^* is infinitely generated, but all ideals properly contained in I^* are finitely generated.

The proof

Next, we claim that

$$\text{for all } r \in R, \text{ either } rI^* = I^* \text{ or } rI^* = \{0\}. \tag{1}$$

To see this, let $r \in R$ be arbitrary, and consider the map $\varphi \colon I^* \to rI^*$ defined by $\varphi(x) := rx$. Then it is clear that φ is an R-linear surjection from the left R-module I^* to the left R-module rI^*. Let K be the kernel (which is an ideal of R contained in I^*). Then as left R-modules, we have

The proof

$$I^*/K \cong_R rI^*. \tag{2}$$

Now, if $rI^* = I^*$, then (1) is trivially verified. Thus suppose that $rI^* \neq I^*$. Then rI^* is an ideal of R properly contained in I^*, thus is finitely generated by minimality of I^*. Recalling (2) above and the fact that I^* is not finitely generated but rI^* is, we deduce that K is not finitely generated. Now, $K \subseteq I^*$ and I^* is *minimal* with respect to not being finitely generated. We conclude that $K = I^*$, and therefore $rx = 0$ for all $x \in I^*$. We have established (1) (that is, for all $r \in R$, either $rI^* = I^*$ or $rI^* = \{0\}$). Let $\mathrm{Ann}_R(I^*) := \{r \in R \colon rI^* = \{0\}\}$. Observe from (1) that if $r, s \in R \backslash \mathrm{Ann}_R(I^*)$, then $rI^* = I^*$ and $sI^* = I^*$. Therefore, $rsI^* = sI^* = I^*$, so clearly $rs \notin \mathrm{Ann}_R(I^*)$. We have shown that $P := \mathrm{Ann}_R(I^*)$ is a prime ideal of R. Now, $D := R/P$ is an Artinian domain, hence a field. But then $V := I^*$ is naturally a vector space over the field $F := D = R/P$ with scalar product $\bar{r} \cdot x := rx$.

The proof

Because I^* is not finitely generated over R yet every proper ideal of I^* is, it follows that V is not finitely generated over F (that is, V is infinite-dimensional over F), yet every proper F-subspace of V, which is simply an ideal of R properly contained in I^*, is finitely generated over F (that is, every proper F-subspace of V is finite-dimensional over F). However, this is nonsense: since V is infinite-dimensional over F, V has some infinite basis β over F. Choose $\mathbf{v} \in \beta$ arbitrarily and let W be the (proper) subspace of V generated over F by $\beta \backslash \{\mathbf{v}\}$. Then W remains infinite-dimensional over F, a contradiction to the fact that V is minimally infinite-dimensional over F. $\square$

Thanks

Thank you!