Leavitt Path Algebras Having Bases Consisting Solely of Strongly Regular Elements

Daniel Bossaller and Sergio López-Permouth

Ohio University

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Outline

1. Invertible Algebras

2. Locally Invertible Algebras

3. Locally Invertible Leavitt Path Algebras

Background

- A 2013 Paper by López-Permouth and Szabo studied convolutional codes $\mathcal{C}$ over finite fields K which were endowed with the structure of K-algebras. Often generating sets (bases) could be chosen for these codes which consist solely of elements which are of full rank in $\mathcal{C}$ when considered as a vector space.
- This then spawned a dissertation by Moore and two papers by Moore, López-Permouth, Szabo and Pilewski which studied and classified many K-algebras which have this property. Further work by Pilewski studied these algebras in the context of Leavitt path algebras and completely classified all Leavitt path algebras which arise from finite graphs that satisfy this property.

Definitions

Definition

Let A be a ring and K a field such that A is a vector space over K. Then A is an **R-algebra** if for any $k \in K$ and for all $a, b \in A$,

$$k(ab) = (ka)b.$$

Note that this is weaker than the traditional definition of an K-algebra. Usually there is a requirement that the scalars from K should commute with the algebra elements, that is $r(ab) = a(rb)$.

Defintition

An K-algebra A is called **invertible** if there exists a basis $\mathcal{B}$ for A such that $\mathcal{B}$ consists solely of invertible elements. If A is an invertible algebra with basis $\mathcal{B}$, and let $\mathcal{B}^{-1} = \{b^{-1} \mid b \in \mathcal{B}\}$. Then A is **invertible-2 (I2)** if $\mathcal{B}^{-1}$ also forms a basis for A.

Examples

Some examples of invertible and invertible-2 algebras:

- Any field extension E over F is an invertible F-algebra.
- Any factor algebra $K[x]/\left(f(x)\right)$ where $f(0) \neq 0$ is an invertible F-algebra.
- Any group algebra KG is an invertible-2 K-algebra.
- Consequently $K[x, x^{-1}]$ is invertible-2.
- For any $n \geq 2$, any $n \times n$ matrix K-algebra $M_n(K)$ is invertible-2 and for any unital K-algebra A, $M_n(A)$ is an invertible-2 K-algebra.
- Any finite-dimensional K-algebra with $K \neq \mathbb{F}_2$ is invertible.

Why is $\mathbb{F}_2$ a problem? Take $\mathbb{F}_2 \oplus \mathbb{F}_2$, which is an algebra of dimension 2 over $\mathbb{F}_2$, but has only one invertible element. So $\mathbb{F}_2 \oplus \mathbb{F}_2$ cannot be invertible.

As a matter of fact any $\mathbb{F}_2$-algebra A which has a factor isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2$ is not invertible.

Locally Invertible Algebras

Definition

Let A be an R-algebra and e and idempotent. The **corner algebra** generated by e is the set

$$eAe = \{a \in A \mid ea = ae = a\} = \{eae \mid a \in A\}.$$

Let's look again at $A = \mathbb{F}_2 \oplus \mathbb{F}_2$. There exists a basis $\mathcal{B} = \{(1,1),(0,1)\}$ for A. Of course $(1,1)$ is the identity of A so it is invertible, but while $(0,1)$ is not invertible in A, it is invertible in the corner algebra $(0,1)A(0,1)$, so we can say it is "locally invertible" in a sense.

Definition

An K-algebra A is **locally invertible** if there exists a basis $\mathcal{B}$ such that for all $b \in \mathcal{B}$ there exists some idempotent e_b such that b is invertible in the corner algebra e_bAe_b.

Locally Invertible Elements

Let's take a bit to talk about local invertibility from an element-wise point of view.

Definition

An element $a \in A$ is called **locally invertible** if there exists some idempotent e (which we call the **local identity** for a) and element a' (which we call the **local inverse** of a) such that $a, a' \in eAe$ and $aa' = a'a = e$. We will often say that a is locally invertible with respect to e.

This then leads to the question: Can we define a locally invertible-2 property.

Proposed Definition.

An invertible algebra A with locally invertible basis $\mathcal{B}$ is called **locally invertible-2** if $\mathcal{B}' = \{b' \mid b' \text{ is a local inverse of } b \in \mathcal{B}\}$ also forms a basis for A

Locally Invertible-2 Algebras

If local inverses aren't unique, it could be possible that the invertible basis $\mathcal{B}$ has two sets of local inverses, $\mathcal{B}$ and $\mathcal{B}'$, one of which is a basis and of which does not.

Lemma (B. and López-Permouth)

If $a \in A$ is locally invertible, then it has a unique local identity and local inverse.

So the proposed definition is well-defined.

Strongly Regular Elements

It turns out that an element being locally invertible is equivalent to the element being strongly regular.

Definition.

An element $a \in A$ is **(von Neumann) regular** if there exists some $x \in A$ such that $axa = a$. If, furthermore, such an x can be chosen such that $ax = xa$ then a is **strongly regular**. An algebra in which every element is regular is called regular, and an algebra in which every element is strongly regular is called strongly regular.

Lemma (B. and López-Permouth).

An element $a \in A$ is strongly regular if and only if it is locally invertible. Consequently, an algebra is locally invertible if and only if it has a basis consisting of strongly regular elements.

Examples

Some examples of locally invertible algebras:

- Any invertible algebra. Just take $e_b = 1$ for all $b \in \mathcal{B}$.
- Direct sums of locally invertible (locally invertible-2) algebras are locally invertible (locally invertible-2).
- In particular, any free K-algebra is locally invertible-2 since the basis for the K-algebra K is $\{1\}$.

Ex: Upper Triangular Matrices

Example

Let $T_n(\mathbb{F}_2)$ be the algebra of upper triangular matrices with entries taken from $\mathbb{F}_2$, then for $n \geq 2$, $T_n(\mathbb{F}_2)$ is an example of a LI2 algebra which fails to be invertible.

Proof.

For the lack of invertibility consider the ideal J generated by $\{e_{ii} \mid i \geq 3\} \cup \{e_{ij} \mid j > i\}$. Then $T_n(\mathbb{F}_2)/J \simeq \mathbb{F}_2 \oplus \mathbb{F}_2$.
To show local invertibility, one need only produce a basis which is locally invertible. Consider the set $\{e_{ii} \mid 1 \leq i \leq n\}$, as idempotents, each of these elements is locally invertible. Also note that $\sum_{k=1}^{n} e_{kk} = I_n$ the identity matrix for $T_n(\mathbb{F}_2)$. So define a basis

$$\mathcal{B} = \{e_{ii} \mid 1 \leq i \leq n\} \cup \{I_n + e_{ij} \mid j > i\}.$$

□

Exs: Infinite Matrices

Using this notion of local invertibility, we don't need to require that A have a multiplicative identity.Indeed, for an infinite index set,

$$\bigoplus_{i\in I} K$$

is a locally invertible K-algebra which is not unital.
Let $M_\infty(K)$ denote the K-algebra of countably infinite matrices which have finitely many nonzero elements. These are matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

and we will often simply refer to this matrix as A.These matrices are locally invertible-2.

Exs: Infinite Matrices

Example

$M_\infty(K)$ is locally invertible-2 for any field K.

Proof.

Lets construct a locally invertible basis step by step.
Let $\mathcal{B}_1 = (1)$ then define

$$B_2 = \left\{ (1), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

and then $\mathcal{B}_3 = \mathcal{B}_2 \cup P$ where

$$P = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}$$

□

Exs: Infinite Matrices

Proof.

Following this pattern build a basis

$$\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i.$$

By construction this is a linearly independent spanning set, so it is a basis. The LI2 property follows from calculating local inverses. ☐

Analogous to the case of matrix algebras we can show the following:

Theorem.

Let A a unital K-algebra, then $M_\infty(A)$ is a locally invertible-2 K-algebra.

Counterexample: Jacobson-Toeplitz Algebra

A notable counterexample for locally invertible algebras is the so called "Toeplitz-Jacobson" algebra first described by Jacobson in 1950.

Example.

Let $\mathfrak{T} = K\langle x, y \mid xy = 1\rangle$ is an example of a unital algebra which is not locally invertible.

Many of the algebras described above can be realized as the Leavitt path algebra of some graph E.

Leavitt Path Algebras

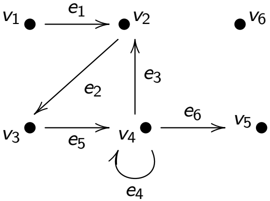
Leavitt Path Algebras

- First investigated independently by Abrams and Aranda Pino in 2005 and by Ara Moreno and Pardo in 2007.
- Generalize the Leavitt algebras, $L(1, n)$, first studied by Leavitt in the 60's during his investigation of the invariant basis number property.
- Purely algebraic analogues of the graph C^*-algebras first investigated by Kumjian, Pask, Raeburn, and Renault in 1998 and 1999.
- These graph C^*-algebras are special cases of the Cuntz-Krieger algebras first studied by Cuntz and Krieger in the 80's.

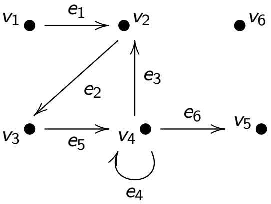
Directed Graphs

A **directed graph** E is a quadruple (E^0, E^1, s, r) where E^0 is the set of vertices of the graph, E^1 is the set of edges, and $r, s : E^1 \to E^0$ are the range and source maps which assign to each edge $e \in E^1$ a source vertex $s(e)$ and range vertex $r(e)$.

Take the following graph E:



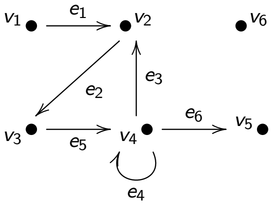
Directed Graphs



Definition.

v_1 is called a **source** because $r^{-1}(v_1) = \emptyset$, and v_5 is called a **sink** since $s^{-1}(v_5) = \emptyset$. v_6 is called an **isolated vertex** since it is both a source and a sink. A vertex v will be called **regular** if $0 < |s^{-1}(v)| < \infty$, and a graph which has only regular vertices will be called **row-finite**. The **out-degree** $\text{Out}(v)$ of a vertex v is $|s^{-1}(v)|$.

Directed Graphs



Definition.

A **path**, α in E is a sequence of edges $\alpha = a_1 a_2 \cdots a_n$ such that $a_i \in E^1$ and $r(a_i) = s(a_{i+1})$. The positive integer n will be called the **length** of the path; the set of all paths in E of any length will be denoted $\mathrm{Path}(E)$. A path α is called a **closed path** if $s(e_1) = r(e_n)$. A closed path such that $r(e_i) = s(e_1)$ only if $i = n$ is called a **closed simple path**. A **cycle** is a closed simple path such that $s(e_i) \neq s(e_j)$ if $i \neq j$. Finally, a cycle of length 1 is called a **loop**. A cycle has an **exit** if there is some vertex v along the cycle such that $\mathrm{Out}(v) > 1$.

Leavitt Path Algebras

Given a directed graph E we adjoin a copy of the edges $(E^1)^*$ such that $s(e^*) = r(e)$ and $r(e^*) = s(e)$ (which we call the set of **ghost edges**) to make the extended graph $\hat{E}$.

Definition.

The **Leavitt path algebra** of the graph E, $L_K(E)$, is formed by taking the free associative algebra $K\langle E^0, E^1, (E^1)^*\rangle$ modulo the ideal generated by the relations

(V) $v \cdot u = \delta_{v,u} v$ where $u, v \in E^0$

$(E1)$ $s(e)e = e = er(e)$ for all $e \in E^1$

$(E2)$ $r(e)e^* = e^* = e^* s(e)$ for all $e \in E^1$

$(CK1)$ $e^* f = \delta_{e,f} r(f)$ for all $e, f \in E^1$

$(CK2)$ $v = \sum_{\{e \in E^1 \mid s(e)=v\}} ee^*$ for all regular vertices v

We will often shorten "Leavitt path algebras" to its acronym LPAs

We have already seen the so-called "primary colors" of Leavitt path algebras in the context of invertible and locally invertible algebras.

Primary Colors of LPAs

Primary Colors

1. Let $R_1 =$ 

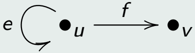
then $L_K(R_1) \simeq K[x, x^{-1}]$, the algebra of Laurent polynomials.

2. Let $R_n =$ 

then $L_K(R_n) \simeq L(1, n)$ the Leavitt algebra first investigated by W.G. Leavitt.

3. Let $A_n = {}_{v_1}\bullet \xrightarrow{e_1} \bullet_{v_2} \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} \bullet_{v_n}$ then $L_K(A_n) \simeq M_n(K)$.

4. Take $A_\infty = {}_{v_1}\bullet \xrightarrow{e_1} \bullet_{v_2} \xrightarrow{e_2} \cdots$ then $L_K(E) \simeq M_\infty(K)$.

5. Take $\mathcal{T} =$ 

then $L_K(\mathcal{T}) \simeq K\langle x, y \mid xy = 1\rangle$.

The Edge-Exclusion Basis

Through a repeated application of the $CK1$ relation one can see that $L_K(E)$ is spanned by the set of monomials of this form:

$$\mathcal{B}' = \{\alpha\beta^* \mid \alpha, \beta \in \mathsf{Path}(E) \text{ and } r(\alpha) = r(\beta)\}.$$

Because we are working over a field K, there is some subset of this set which is a basis. One such subset that we will use is the following:

Definition.

For each regular vertex v in E select some edge f_v such that $s(f_v) = v$. Then let

$$\mathcal{B} = \{\alpha\beta^* \mid r(\alpha) = r(\beta) \text{ and } \alpha\beta^* \neq \alpha' f_v f_v^* \beta'^*\}$$

This set is a basis for $L_K(E)$ and will be called an **edge exclusion basis**.

This basis essentially limits how we can apply the $CK2$ relation when writing elements of the LPA in terms of the basis.

Strongly Regular Leavitt Path Algebras

Lets recall that an element $a \in A$ is called strongly regular if and only if $a = axa$ and $ax = xa$. Another way to formulate this is that there is some element x such that $a^2x = a$ and $xa^2 = a$.

In 2008 Abrams and Rangaswamy proved that a LPA was regular if and only if its underlying graph E was acyclic. This opens a question as to which (if any) Leavitt path algebras are strongly regular.

Lemma (B. and López-Permouth).

A LPA $L_K(E)$ is strongly regular if and only if E consists solely of isolated vertices v_i.

These strongly regular Leavitt path algebras look like

$$L_K(E) \simeq \bigoplus_{i \in I} K$$

Strongly Regular Elements in LPAs

Even though the family of strongly regular LPAs is rather boring. There is some merit to looking at LPAs which have strongly regular bases.
After all, many of the primary colors are invertible or locally invertible. So lets investigate when monomials are strongly regular.

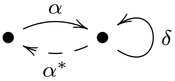
Lemma (B. and López-Permouth).

Let $\lambda\nu^*$ be a nonzero monomial in $L_K(E)$. Then $\lambda\nu^*$ is strongly regular if and only if $(\lambda\nu^*)^2 \neq 0$ and it can be written as

$$\lambda\nu^* = \alpha\alpha^* \text{ or } \lambda\nu^* = \alpha\delta^m\alpha^*$$

were $\alpha \in \mathrm{Path}(E)$, $m \in \mathbb{Z}\backslash\{0\}$, and δ is a cycle without exits.

In other words the strongly regular monomials of $L_K(E)$ look like



Strongly Regular Elements in LPAs

Proof.

If $\lambda\nu^*$ is of the form $\alpha\alpha^*$, it is idempotent thus strongly regular. Now say that $\lambda\nu^*$ is of the form $\alpha\delta^m\alpha^*$ where δ is a cycle without exits. If $m > 0$ then note that $\alpha\delta^{-m}\alpha^*$ is a local inverse for $\alpha\delta^m\alpha^*$.

For the other direction say that $\lambda\nu^*$ is strongly regular. Then there exists some a such that $(\lambda\nu^*)^2 a = \lambda\nu^*$. In particular, $(\lambda\nu^*)^2 \neq 0$ as required. Since $(\lambda\nu^*)(\lambda\nu^*) \neq 0$. The $CK1$ relation states that either $\lambda = \nu\lambda'$ or $\nu = \lambda\nu'$. In the first case, $\lambda\nu^* = \nu\lambda'\nu^*$, and in the second case $\lambda\nu^* = \lambda\nu'^*\lambda^*$.Therefore we can find a path $\alpha \in \text{Path}(E)$ and a closed path γ such that

$$\lambda\nu^* = \alpha\gamma^i\alpha^*$$

for $i \in \{0, \pm 1\}$.

Strongly Regular Elemnts in LPAs

Proof.

If $i = 0$ then $\alpha\gamma^0\alpha^* = \alpha\alpha^*$.
So say that $i = 1$, that is $\lambda\nu^* = \alpha\gamma\alpha^*$ which we will denote by a.Since a is strongly regular, it is locally invertible, so there exists some element $x \in L_K(E)$ such that $ax = xa$ and ax is the left and right identity on a. In other words $a \cdot ax = a$

$$\alpha\gamma^2\alpha^*x = a^2x = a = \alpha\gamma\alpha^*.$$

The middle terms reduce to $\alpha^*x\alpha = \gamma^*$ which implies that

$$\alpha\alpha^*x\alpha\alpha^* = \alpha\gamma^*\alpha^*.$$

Thus the $\alpha\alpha^*L_K(E)\alpha\alpha^*$ coordinate in the Peirce decomposition of x is $\alpha\gamma^*\alpha^*$.

Strongly Regular Elements in LPAs

Proof.

Thus $(\alpha\gamma\alpha^*)(\alpha\gamma^*\alpha^*) = (\alpha\gamma^*\alpha^*)(\alpha\gamma\alpha^*)$; simplifying and canceling on the left and right then gives that $\gamma\gamma^* = s(\gamma)$, and this can only happen if γ is a closed path without exits.

A short argument then gives that $\gamma = \delta^m$ for some $m \in \mathbb{Z}^+$ where δ is a cycle without exits. □

This classification of strongly regular monomials is incredibly useful since the elements of LPAs can all be written as the K-linear span of monomials. The following theorem is a particularly interesting corollary. A graph E satisfies **Condition (NE)** if no cycle in E has an exit.

Theorem (B. and López-Permouth).

Say that E is a graph which satisfies condition (NE), then $L_K(E)$ is locally invertible-2.

LPAs of No Exit Graphs are Locally Invertible

Proof.

Let $\mathcal{B}$ be any edge exclusion basis for $L_K(E)$. That is, for each regular vertex $v \in E^0$, choose an edge $f_v \in E^1$ such that $s(f_v) = v$ and construct

$$\mathcal{B} = \{\alpha\beta^* \mid r(\alpha) = r(\beta) \text{ and } \alpha\beta^* \neq \alpha' f_v f_v^* \beta'^*\}.$$

Note that $E^0 \subseteq \mathcal{B}$.
Then, take $\alpha\beta^* \in \mathcal{B}$ and square it.

- If $(\alpha\beta^*)^2 \neq 0$ then $\alpha\beta^* = \lambda\gamma^n\lambda^*$ where $\lambda \in \text{Path}(E)$, γ a closed path, and $n \in \{-1, 0, 1\}$. Since E is a no-exit graph, this then implies $\alpha\beta^*$ is strongly regular.
- If $\alpha\beta^* = 0$, then construct the element $s(\alpha) + s(\beta) + \alpha\beta^*$. This has a local inverse in $s(\alpha) + s(\beta) - \alpha\beta^*$.

It is a straightforward argument to see that this creates a $LI2$ basis for $L_K(E)$. □

Regular LPAs are Locally invertible

This theorem has two corollaries that show that large families of Leavitt path algebras are locally invertible.

Corollary (B. and López-Permouth).

Every von Neumann regular Leavitt path algebra is locally invertible.

Proof.

If $L_K(E)$ is regular, then by Abrams and Rangaswamy, E is an acyclic graph which trivially satisfies the 'no exit' requirement of the theorem. ☐

A related notion is the notion of a directly finite algebra.

Definition.

A not necessarily unital algebra A is said to be **directly finite** if for every $a, b \in A$ and idempotent e such that $ab = e$ and $ae = ea = a$ and $be = eb = b$, then $ba = e$.

Directly Finite LPAs are locally invertible

In 2015 Vaš gave the following characterization of directly finite LPAS.

Theorem, (Vaš, 2015).

A Leavitt path aglebra is directly finite if and only if every cycle in E has no exits.

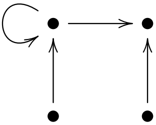
Thus

Corollary (B. and López-Permouth).

Every directly finite Leavitt path aglebra is locally invertible.

Unfortunately, neither regularity or direct finiteness is necessary for local invertibility. Let E be the following graph:

$E =$



, then $L_K(E) \simeq M_2(\mathfrak{T})$.

Further Work

- All regular and directly finite LPAs are locally invertible, is this true in general?
- Provide a necessary and sufficient graph theoretic characterization for local invertibility.
- Recent work by Nagy and Reznikoff showed that normal monomials of the generating set for a graph C^*-algebra, $C^*(E)$, (that is monomials a such that $aa^* = a^*a$) are precisely those which satisfy graph theoretic conditions of the Strongly Regular Monomials lemma. Is there any further connection between the graph C^*-algebras and Leavitt path algebras?

Thanks For Listening

Thank You