Quantum Cayley Graphs for Free Groups

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September 8, 2015

Abstract

The spectral theory of self-adjoint operators provides an abstract framework for solving some of the main differential equations of mathematical physics: the heat equation, wave equation, and Schrödinger equation. When the operators are invariant under a group action, a much more detailed analysis is often possible. This work on invariant differential operators on the metric Cayley graphs of free groups throws differential equations, graphs, group actions, functional analysis, algebraic topology, linear algebra, and a pinch of algebraic geometry into the blender. What emerges is a surprisingly satisfying concoction.

Generalities

- This work will look at differential operators on the Cayley graphs of free groups $\mathbb{F}_M$
- Long history of interplay between groups and spectral theory of differential operators.
- Group representations and spectral theory of operators have similar goals: split a 'large' vector space into 'small' invariant pieces, where objects are easier to understand. Use this understanding to solve other problems.
- The interplay of both approaches can be very powerful: (i) representation theory of compact Lie groups, (ii) Fourier series and transforms.

Analysts study differential equations of math physics

- $$\Delta = -\frac{\partial^2}{\partial x^2}, \quad \Delta = -\sum_n \frac{\partial^2}{\partial x_n^2},$$
- Heat/diffusion equation
$$\frac{\partial u}{\partial t} + \Delta u = 0, \quad u(0,x) = f(x).$$
- Wave equation
$$\frac{\partial^2 u}{\partial t^2} + \Delta p = 0, \quad u(0,x) = f(x), \frac{\partial u}{\partial t} = g(x)$$
- Schrodinger equation
$$i\frac{\partial u}{\partial t} + \Delta u = 0, \quad u(0,x) = f(x).$$

Fourier series 1750-1810

- If Δ has a (Hilbert space) basis of eigenvectors $y_n(x)$ with eigenvalues λ_n, the equations will have series solutions. For example the heat equation is solved by

$$f(x) = \sum c_n y_n(x), \quad u(t,x) = \sum c_n e^{-\lambda_n t} y_n(x).$$

- Approach used with Fourier series (Euler-Fourier 1750-1810).

$$y_n(x) = 1, \cos(2\pi nx), \sin(2\pi nx), \quad n = 1, 2, 3, \ldots$$

- The trigonometric functions are linked to the circle group
- There are similar developments using the Fourier transform

$$\hat{f}(\omega) = \int_{\infty}^{\infty} f(x) e^{i\omega x}\, dx.$$

Here the group structure of $\mathbb{R}$ is important.

Further comments

- For heat equation, the low eigenvalues tell you how fast equilibrium is achieved, how big a finite dimensional approximation is needed for desired precision.
- The switch from abstract y_n to concrete trig functions enables other important questions like pointwise convergence, uniform convergence, convergence at jump discontinuities, discrete versions, etc., behavior in more than one dimension, etc.
- More modern continuations include representations of Lie groups and operators invariant under Lie group actions (rotations and ball in $\mathbb{R}^N$)

Sturm-Liouville problems 1836

- Fourier series/transforms for spatially homogeneous problems.
- Spatial inhomogeneities in dim 1 or separable cases lead to eigenvalue equations

$$P(x)y'' + Q(x)y' + R(x)y = \lambda y, \quad + \mathit{bndy\ cond}.$$

- Usually do not have explicit elementary solutions.
- Sturm-Liouville showed eigenvalues and orthogonal eigenfunctions exist, and 'look like' trig case for λ large.
- By a diffeomorphism followed by a conjugation, the equation is reduced to Liouville normal form

$$y'' + q(x)y = \lambda y, \quad + \mathit{bndy\ cond}.$$

Hilbert space and complex variables methods 1910

- Major projects around 1910 highlighted complex variables methods (contour integrals, residues) for eigenfunction expansions.
- Hilbert space l^2 invented
- Birkhoff studied non-self-adjoint Sturm-Liouville problems
- Weyl studied singular self-adjoint SL problems on $L^2[0, \infty)$. Often a continuum of 'eigenvalues', no eigenfunctions in the space; a development more like Fourier transforms needed. Spectral theory strongly linked to measure theory.

Resolvent/Green's function

- The resolvent of operator A is $R_A(\lambda) = (A - \lambda I)^{-1}$.
- For ODEs this amounts to solving

$$-y'' + q(x)y - \lambda y = f$$

for y when f is given. Variation of parameters gives

$$y(x,\lambda) = c_1 y_1 + c_2 y_2 + \int_0^x \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(y_1,y_2)} f(t)\, dt.$$

- Constants c_j are determined by boundary conditions or behavior at endpoints in singular case.

Analytic methods

- The resolvent set is the subset of $\mathbb{C}$ where $R_A(\lambda)$ is bounded and analytic. The spectrum (eigenvalues) is the complement. If A is self-adjoint, the spectrum is part of $\mathbb{R}$.
- We want operator valued functions like $\exp(-t[\Delta + q])$ Riesz-Dunford functional calculus (non-self-adjoint A in Banach spaces 1943): if $f(z)$ is analytic near spec(A) then

$$f(A) = \frac{i}{2\pi} \int_\gamma f(\lambda) R_A(\lambda) \, d\lambda.$$

- For self adjoint Hilbert space operator result is much stronger (Stone 1932). Orthogonal projection onto 'eigenspaces'

$$\frac{1}{2}[P_{[a,b]} + P_{(a,b)}]f = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b [R(\sigma + i\epsilon) - R(\sigma - i\epsilon)]f \, d\sigma.$$

Cayley graphs $\mathbb{F}_M$

- Free group $\mathbb{F}_M$ on M generators
  $\mathbb{F}_M$: equivalence classes of finite length words generated by distinct $s_1, \ldots, s_M$ and inverse symbols $s_1^{-1}, \ldots, s_M^{-1}$. Remove adjacent symbol pairs $s_m s_m^{-1}$ or $s_m^{-1} s_m$ to get equivalent words.
- Cayley graph $\Gamma_{\mathbb{G},\mathbb{S}}$ for the group $\mathbb{G}$ with (minimal) generating set $\mathbb{S}$ is the directed graph with $\mathbb{G}$ elements as vertices and directed edges (v, vs) with $s \in \mathbb{S}$. Forget direction for undirected Γ. $\mathbb{G}$ acts (transitively) by left multiplication on the vertices of Γ; $\mathbb{G}$ also acts on edges, usually not transitive.
- The undirected Cayley graph $\mathcal{T}_M$ of $\mathbb{F}_M$ is a tree whose vertices have degree $2M$.

Quantum Cayley graphs $\mathbb{F}_M$

- Intervals $\{[0, l_e], e \in \mathcal{E}\}$ indexed by the graph edges. Vertices identify interval endpoints for topological graph. Edges (v, vs_m) same length. Extend Euclidean metric from intervals to paths to graph for metric graph.
- $L^2(\Gamma)$ is $\oplus_e L^2[0, l_e]$ with the inner product

$$\langle f, g \rangle = \int_{\mathcal{T}_M} f\overline{g} = \sum_e \int_0^{l_e} f_e(x)\overline{g_e(x)}\, dx.$$

  Self-adjoint $-D^2 + q$ acts component-wise. Continuity plus derivative conditions for domain.
- Functions $q \geq 0$ are even on each edge, same on (v, vs_m).

Problem: analyze $\Delta + q$ on $L^2(\mathcal{T}_M)$

- Like to know (i) spectrum as set, (ii) are there eigenvalues, (iii) can these be determined by 'explicit' formulas or more elementary calculations, (iv) what are consequences for (finite) graphs with tree as universal cover ?
- Previous work - R. Brooks - discrete homogeneous trees - 1991, R. Carlson - quantum homogeneous trees - 1997
- No explicit group theory. Every edge looked the same. Able to use 'Floquet' theory: spectrum is determined by the eigenvalues of a 2×2 analytic matrix function $D(\lambda)$ for $\lambda \in \mathbb{C} \setminus [0, \infty)$ and extension to $[0, \infty)$. Methods/results need extensive rework.

Outline 1

- (i) For $\lambda \in \mathbb{C} \setminus [0, \infty)$, focus on the restricted operator $R_e(\lambda)f$, where $\mathrm{support}(f) \subset e$, with e of type (color) m.
- (ii) By ODEs, we expect

$$R_e(\lambda)f = \int_e R_e(x, t, \lambda) f(t)\, dt,$$

where the kernel has the form

$$R_e(x, t, \lambda) = \begin{matrix} y_-(x, \lambda) y_+(t, \lambda)/W_k(\lambda), & 0 \leq x \leq t \leq l_e, \\ y_-(t, \lambda) y_+(x, \lambda)/W_k(\lambda), & 0 \leq t \leq x \leq l_e. \end{matrix} \tag{0.1}$$

and $y_\pm(x, \lambda)$ extend past e to whole tree.

Outline 2

- (iii) For $\lambda \in \mathbb{C} \setminus [0, \infty)$, functional analysis shows existence of a distinguished one-dimensional space $\mathbb{X}_e$ of functions y_+ on the half-tree $\mathcal{T}_e^+$ (same for $-$).
- (iv) The abelian subgroups $v s_m^k v^{-1}$ act to give a nested family of subtrees $\mathcal{T}_{e(k)}^+$. The induced action on $\mathbb{X}_{e(k)}$ is by complex multiplication by $\mu_m(\lambda)$.

Theorem

Assume $\lambda \in \mathbb{C} \setminus [0, \infty)$, $e = (v, vs_m)$ and $y_+ \in \mathbb{X}_e^+$ with $y_+(v) = 1$. Suppose w is a vertex in $\mathcal{T}_e^+$, and the path from v to w is given by the reduced word $s_m s_{k(1)}^{\pm 1} \dots s_{k(n)}^{\pm 1}$. Then

$$y_+(w) = \mu_m \mu_{k(1)} \dots \mu_{k(n)}. \tag{0.2}$$

By ODE the vertex values of y_+ can be interpolated to the edges.

Outline 3

- Solutions also extend by solving $-y'' + q_m y = \lambda y$, with standard basis

$$C_m(x,\lambda) \sim \cos(\sqrt{\lambda}x), \quad S_m(x,\lambda) \sim \sin(\sqrt{\lambda}x)/\sqrt{\lambda}.$$

Comparison of extensions leads to

Theorem

For $\lambda \in \mathbb{C} \setminus [0,\infty)$ and $m = 1, \dots, M$, the multipliers $\mu_m(\lambda)$ satisfy the system of equations

$$\frac{\mu_m^2(\lambda) - 1}{S_m(l_m,\lambda)\mu_m(\lambda)} - 2\sum_{k=1}^{M} \frac{\mu_k(\lambda) - C_k(l_k,\lambda)}{S_k(l_k,\lambda)} = 0. \tag{0.3}$$

Outline 4

- The equations have solutions $\xi_m(\lambda)$ including $\mu_m(\lambda)$.

Corollary

Suppose $\lambda \in \mathbb{C}$, $S_m(\lambda, l_m) \neq 0$, *and*

$$\xi_m(\lambda)^2 + 1 \neq 0, \quad \sum_{m=1}^{M} \frac{\xi_m^2}{\xi_m^2 + 1} \neq 1/2, \quad m = 1, \ldots, M.$$

Solutions $\xi_m(\lambda)$ *are locally holomorphic* $\mathbb{C}^M$*- valued functions of* λ*.*

Elimination algorithm reduces to $1 - var$ polynomials.

- **Theorem**

  There is a discrete set $Z_0 \subset \mathbb{R}$ *and a positive integer* N *such that for all* $\lambda \in \mathbb{C} \setminus Z_0$ *the equations satisfied by the multipliers* $\mu_m(\lambda)$ *have at most* N *solutions* $\xi_1(\lambda), \ldots, \xi_M(\lambda)$*. For* $\lambda \in \mathbb{C} \setminus Z_0$*, the functions* $\mu_m(\lambda)$ *are solutions of* $1 - var$ *polynomial equations* $p_m(\xi_m) = 0$ *of positive degree, with coefficients entire in* λ*.*

Outline 5

- For $\sigma \in [0, \infty)$ there are limiting values $\mu^{\pm}(\sigma)$.
- Let $Z_m \subset \mathbb{C}$ be the discrete set where the leading coefficient of $p_m(\xi_m)$ vanishes. For $\lambda \in \mathbb{C} \setminus Z_m$ the roots of $p_m(\lambda)$, in particular $\mu_m(\lambda)$, are holomorphic if root is simple. Otherwise the roots extend continuously The limiting values $\mu^{\pm}(\sigma)$ need not agree.

Theorem

Suppose $(\alpha, \beta) \cap Z_0 = \emptyset$. For $m = 1, \ldots, M$ also assume that $(\alpha, \beta) \cap Z_m = \emptyset$ and that $\mu_m^{\pm}(\sigma) \neq \pm 1$ for all $\sigma \in (\alpha, \beta)$. If $[a, b] \subset (\alpha, \beta)$, e is an edge of type m, and $f \in L^2(e)$, then

$$P_{[a,b]} f = \frac{1}{2\pi i} \int_a^b [R_e^+(\sigma) - R_e^-(\sigma)] f \, d\sigma. \tag{0.4}$$

If $\sigma_1 \in (\alpha, \beta)$, then σ_1 is not an eigenvalue of $\Delta + q$.

Outline 6

- The spectrum can be recognized.

Corollary

Assume $\sigma \in [0, \infty) \setminus Z_0$ and for $m = 1, \ldots, M$ suppose $\mu_m^{\pm}(\sigma) \neq \pm 1$. Then σ is in the resolvent set of $\Delta + q$ if and only if $\mu_m^{+}(\sigma)$ is real valued in open neighborhood of σ for $m = 1, \ldots, M$.

Computations 1

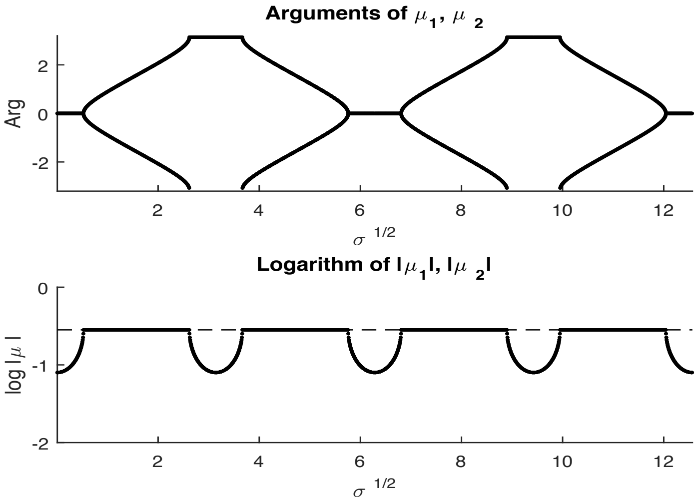

Figure: Case $l_1 = 1$, $l_2 = 1$

Computations 2

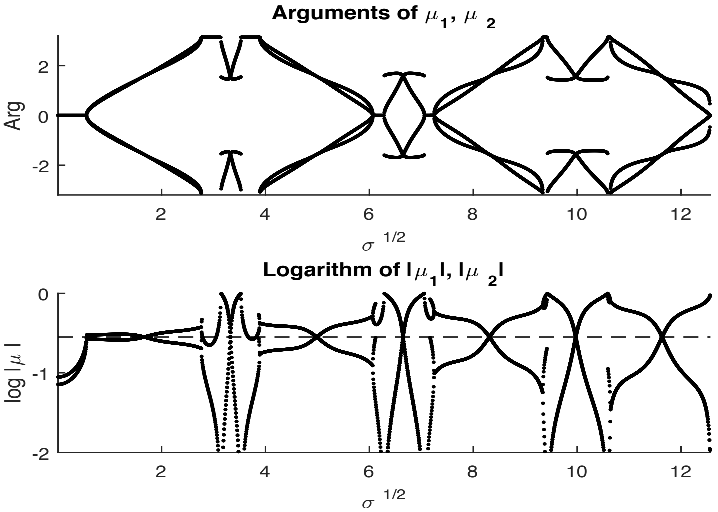

Figure: Case $l_1 = 1$, $l_2 = .89$

Computations 3

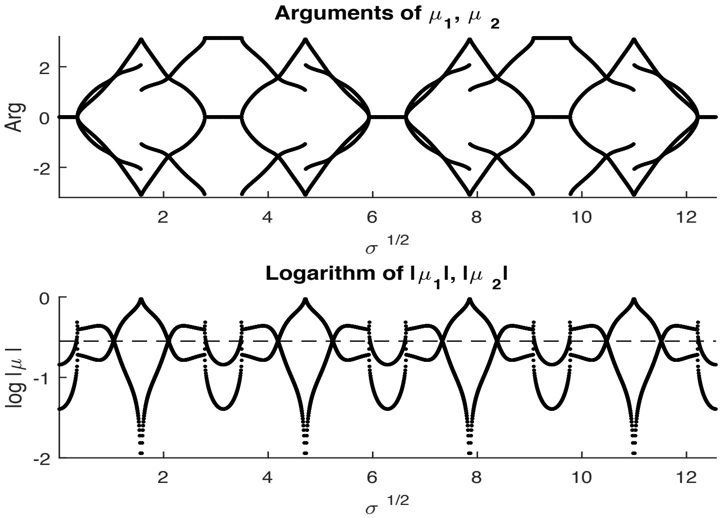

Figure: Case $l_1 = 1$, $l_2 = 2$