Chen simple modules and Prüfer modules over Leavitt path algebras

Gene Abrams

(joint work with F. Mantese and A. Tonolo)

Vietnam Institute for Advanced Study in Mathematics
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Overview

- Brief review of Leavitt path algebras
- Chen simple modules
- $\text{Ext}^1_{L_K(E)}(S, T)$ for various simple $L_K(E)$-modules $S, T$
- Prüfer modules
- Injective modules over $L_K(E)$
The algebra $L_K(E)$

Throughout, $K$ is a field.
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Let $E = (E^0, E^1, s, r)$ be a directed graph.

The extended graph of $E$ is the graph $\hat{E} = (E^0, E^1 \cup (E^1)^*, s', r')$, with $(E^1)^* = \{e^* | e \in E^1\}$, $r' | E^1 = r$, $s' | E^1 = s$, $r'(e^*) = s(e)$, $s'(e^*) = r(e)$.

The Leavitt path algebra $L_K(E)$ of $E$ over $K$ is the $K$-path algebra $K\hat{E}$ modulo the relations:

$$e^* e' = \delta_{e e'},$$

$v = \sum_{\{e \in E^1 | s(e) = v\}} e^*$ (for any $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$).

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- $e^* e' = \delta_{e,e'} r(e)$ for any $e, e' \in E^1$
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Chen simple modules and Prüfer modules over Leavitt path algebras
A path $\sigma = e_1 e_2 \cdots e_n$ in $E$ is closed if $r(e_n) = s(e_1)$. 

A closed path $\sigma$ is basic if $\sigma \neq \beta^m$ for any closed path $\beta$ and integer $m \geq 2$. 

If $\alpha \in \text{Path}(E)$, the element $\alpha \in L_K(E)$ is called a real path. 

If $\beta = e_1 e_2 \cdots e_n \in \text{Path}(E)$, the element $\beta^* = e_n^* \cdots e_2^* e_1^* \in L_K(E)$ is called a ghost path. 

Let $M$ be a left $L_K(E)$-module and $m \in M$. Denote by $\hat{\rho}_m: L_K(E) \rightarrow M, r \mapsto rm$. 

For a vertex $v \in E_0$, denote by $\rho_m: L_K(E) \rightarrow M, x \mapsto xm$. 

Note: Every $x \in L_K(E)$ can be written as $x = \sum_{i=1}^{n} k_i \alpha_i^* \beta_i$, where $0 \neq k_i \in K$ and $\alpha_i, \beta_i \in \text{Path}(E)$. 

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Notation

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Infinite paths

An infinite path in $E$ is a sequence $p = e_1 e_2 e_3 \cdots$, where $e_i \in E^1$ for all $i \in \mathbb{N}$, and for which $s(e_{i+1}) = r(e_i)$ for all $i \in \mathbb{N}$. 

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- Let $c$ be a closed path in $E$. Denote $ccc \cdots$ by $c^\infty$. 

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- If $p = e_1 e_2 e_3 \cdots \in E^\infty$ and $n \in \mathbb{N}$, denote by $\tau_{>n}(p)$ the infinite path $e_{n+1} e_{n+2} \cdots$. 
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- $p \in E^\infty$ is rational if $p \sim c^\infty$ for some closed path $c$. $p \in E^\infty$ is irrational if it is not rational.
Example

Let $R_2$ denote the graph

\[ e \xrightarrow{\bullet} v \xrightarrow{f} \bullet \]

Any path of the form $ef^i$ for $i \in \mathbb{Z}^+$ is a basic closed path in $\text{Path}(R_2)$.
Example

Let $R_2$ denote the graph

$$
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
e
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
f
\end{array}
\end{array}
\end{array}
$$

Any path of the form $ef^i$ for $i \in \mathbb{Z}^+$ is a basic closed path in $\text{Path}(R_2)$.

For any $i \in \mathbb{Z}^+$, $c_i = (ef^i)_{\infty}$ is a rational infinite path. Note that $c_i \sim c_j$ if and only if $i = j$. 

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\[ \begin{array}{c}
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  \bullet \\
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\end{array} \]

- Any path of the form $ef^i$ for $i \in \mathbb{Z}^+$ is a basic closed path in $\text{Path}(R_2)$.
- For any $i \in \mathbb{Z}^+$, $c_i = (ef^i)^\infty$ is a rational infinite path. Note that $c_i \sim c_j$ if and only if $i = j$.
- $q = efefefeffeffe \cdots$ is an irrational infinite path in $R_2^\infty$. 
Let $p \in E^\infty$. Let $V_p$ denote the $K$-vector space with basis the distinct elements of $E^\infty$ which are tail-equivalent to $p$. 

The $K$-linear extension of this action endows $V_p$ with the structure of a left $L_K(E)$-module.
Chen simple modules

Let \( p \in E^\infty \). Let \( V[p] \) denote the \( K \)-vector space with basis the distinct elements of \( E^\infty \) which are tail-equivalent to \( p \). For any \( v \in E^0, e \in E^1 \), and \( q = f_1f_2f_3\cdots \) with \( q \sim p \), define

\[
\begin{align*}
  v \cdot q & = \begin{cases} q & \text{if } v = s(f_1) \\ 0 & \text{otherwise} \end{cases} \\
  e \cdot q & = \begin{cases} eq & \text{if } r(e) = s(f_1) \\ 0 & \text{otherwise,} \end{cases} \\
  e^* \cdot q & = \begin{cases} \tau_{1}(q) & \text{if } e = f_1 \\ 0 & \text{otherwise} \end{cases}
\end{align*}
\]

The \( K \)-linear extension of this action endows \( V[p] \) with the structure of a left \( L_K(E) \)-module.
Theorem: Let $p \in E^\infty$. Then the left $L_K(E)$-module $V[p]$ is simple. If $p, q \in E^\infty$, then $V[p] \cong V[q]$ as left $L_K(E)$-modules if and only if $p \sim q$, if and only if $V[p] = V[q]$.

Idea: A linear combination of distinct paths tail equivalent to $p$ can be reduced to a single nonzero term by appropriate multiplication. Then any path tail equivalent to $p$ can be generated from this single term via the module action.


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Note: Let $w \in E^0$ be a sink. We consider $w = w^\infty$ as an element in $E^\infty$. The Chen simple module $V_{[w^\infty]}$ coincides with the ideal $L_K(E)w$. 
Example

Consider the graph $R_2$

\[ V[e^\infty], V[f^\infty], V[ef^i\infty] \text{ for any } i \in \mathbb{Z}^+ \text{ are Chen simple modules generated by a rational infinite path.} \]
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\[
\begin{array}{c}
\circ \quad \circ \quad \circ \\
\downarrow \quad \downarrow \quad \downarrow \\
\circ \quad \circ \quad \circ \\
\end{array}
\quad v
\]

- $V_{[e^\infty]}$, $V_{[f^\infty]}$, $V_{[ef^i\infty]}$ for any $i \in \mathbb{Z}^+$ are Chen simple modules generated by a rational infinite path.
- For $q = efefffeffffe \cdots$, $V_{[q]}$ is a Chen simple module generated by an irrational infinite path.
Reminder: For a left $R$-module $M$, a projective resolution of $M$ is an exact sequence

$$\cdots P_n \to P_{n-1} \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where each $P_i$ is a projective left $R$-module.
Projective resolutions of Chen simple modules

**Aim:** To construct a projective resolution of any Chen simple module $V_{[p]}$. We have three cases:

1. $V_{[w]} \sim LK(E_w)$ where $w$ is a sink,
2. $V_{[c]}$ where $c$ is a basic closed path;
3. $V_{[q]}$ where $q$ is an irrational infinite path.

Remark: Type (1) is trivial, since $w$ is an idempotent and so the left ideal $LK(E_w)$ is a projective left $LK(E)$-module. Type (3) is interesting, but we won't need it in the rest of the lecture, so discussion omitted.
Aim: To construct a projective resolution of any Chen simple module \( V_p \). We have three cases:

1. \( V_{[w, \infty]} \cong L_K(E)w \) where \( w \) is a sink,

Remark: Type (1) is trivial, since \( w \) is an idempotent and so the left ideal \( L_K(E)w \) is a projective left \( L_K(E) \)-module. Type (3) is interesting, but we won't need it in the rest of the lecture, so discussion omitted.
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Projective resolutions of Chen simple modules

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Type (2)

Theorem: Let $c$ be a basic closed path in $E$, with $v = s(c)$.

A projective resolution of $V[c^\infty]$ is given by

$$0 \longrightarrow L_K(E)v \overset{\rho_{c-v}}{\longrightarrow} L_K(E)v \overset{\rho_{c^\infty}}{\longrightarrow} V[c^\infty] \longrightarrow 0$$
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2. If $E$ is a finite graph, an alternate projective resolution of $V_{[c^\infty]}$ is given by

$$0 \longrightarrow L_K(E) \overset{\hat{\rho}_{c^{-1}}}{\longrightarrow} L_K(E) \overset{\hat{\rho}_{c^\infty}}{\longrightarrow} V_{[c^\infty]} \longrightarrow 0$$
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**Theorem:** Let $c$ be a basic closed path in $E$, with $v = s(c)$.

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In particular, the Chen simple module $V_{[c^\infty]}$ is finitely presented.
Example

Consider the Toeplitz graph

\[
\begin{array}{c}
e \circlearrowleft \bullet \\
v \rightarrow f \\
w
\end{array}
\]

and the Chen simple module \( V_{[e^\infty]} \). Then

\[
0 \longrightarrow L_K(E)v \overset{\rho_{e^v}}{\longrightarrow} L_K(E)v \overset{\rho_{e^\infty}}{\longrightarrow} V_{[e^\infty]} \longrightarrow 0
\]

\[
0 \longrightarrow L_K(E) \overset{\hat{\rho}_{e^{-1}}}{\longrightarrow} L_K(E) \overset{\hat{\rho}_{e^\infty}}{\longrightarrow} V_{[e^\infty]} \longrightarrow 0
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are projective resolutions of the finitely presented module \( V_{[e^\infty]} \).
Proof

Main points of the proof:

- Since \((c - v)c^\infty = c^\infty - c^\infty\), we get \(L_K(E)(c - v) \subseteq \text{Ker}(\rho c^\infty)\).
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- Since \((c - \nu)c^\infty = c^\infty - c^\infty\), we get \(L_K(E)(c - \nu) \subseteq \text{Ker}(\rho_{c^\infty})\).
- The inclusion \(\text{Ker}(\rho_{c^\infty}) \subseteq L_K(E)(c - \nu)\) follows analyzing the shape of the standard form monomials in \(\text{Ker}(\rho_{c^\infty})\).
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- Since \((c - v)c^\infty = c^\infty - c^\infty\), we get \(L_K(E)(c - v) \subseteq \text{Ker}(\rho_{c^\infty})\).
- The inclusion \(\text{Ker}(\rho_{c^\infty}) \subseteq L_K(E)(c - v)\) follows analyzing the shape of the standard form monomials in \(\text{Ker}(\rho_{c^\infty})\).
- By a degree argument, we get \(r(c - v) = 0\) if and only if \(r = 0\). So the map \(\rho_{c-v} : L_K(E)v \to L_K(E)v\) is a monomorphism of left \(L_K(E)\)-modules.

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Aim: To describe $\text{Ext}^1_{L_K(E)}(S, T)$, where $S$ and $T$ are Chen simple modules over $L_K(E)$, for $E$ finite.
The Ext groups

Aim: To describe $\text{Ext}^1_{L_K(E)}(S, T)$, where $S$ and $T$ are Chen simple modules over $L_K(E)$, for $E$ finite.

Remarks:

- the abelian group $\text{Ext}^1_{L_K(E)}(S, T)$ has a natural structure of $K$-vector space
The $\text{Ext}$ groups

**Aim:** To describe $\text{Ext}^1_{L_K(E)}(S, T)$, where $S$ and $T$ are Chen simple modules over $L_K(E)$, for $E$ finite.

**Remarks:**

- the abelian group $\text{Ext}^1_{L_K(E)}(S, T)$ has a natural structure of $K$-vector space
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**The $\text{Ext}$ groups**

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**Remarks:**
- The abelian group $\text{Ext}^1_{L_K(E)}(S, T)$ has a natural structure of $K$-vector space.
- $\text{Ext}^1_{L_K(E)}(S, T) \neq 0$ if and only if there exists a non-splitting short exact sequence $0 \rightarrow T \rightarrow N \rightarrow S \rightarrow 0$.
- If $w$ is a sink, then $\text{Ext}^1_{L_K(E)}(V[w], M) = 0$ for any $M$. 

Gene Abrams (joint work with F. Mantese and A. Tonolo)

Chen simple modules and Prüfer modules over Leavitt path algebras
When $S$ is of type (2)

Let $T$ be a Chen simple module. Let $U(T) := \{ v \in E^0 \mid vT \neq \{0\} \}$. 

Theorem: (A-, Mantese, Tonolo, 2015) Let $E$ be a finite graph. Let $d$ be a basic closed path in $E$ and let $T$ be a Chen simple module. Then the following are equivalent:

1. $\text{Ext}^1_L K(E)(V[d]_{\infty}, T) \neq 0$.
2. $s(d) \in U(T)$.

Corollary: Let $E$ be a finite graph. Let $d$ be a basic closed path. Then $\text{Ext}^1_L K(E)(V[d]_{\infty}, V[d]_{\infty}) \neq 0$. In particular, $V[d]_{\infty}$ is neither projective, nor injective.
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Consider the graph $R_2$:

![Graph](image)

Let $q \in R_2^{\infty}$ and let $T = V[q]$. Let $d$ be a basic closed path in $R_2$. Since $v = s(d) \in U(T) = \{v\}$, the previous theorem applies and hence $\text{Ext}^1_{L_K(R_2)}(V[d^{\infty}], T) \neq 0$. 
Proof: main points

Let $E$ be a finite graph. Let $d$ be a basic closed path in $E$ and let $T$ be a Chen simple module. Consider the projective resolution

\[ 0 \longrightarrow L_K(E) \xrightarrow{\hat{\rho}_{d-1}} L_K(E) \xrightarrow{\hat{\rho}_d} V_{[d\infty]} \longrightarrow 0 \] and the resulting standard long exact sequence

\[ \text{Hom}_{L_K(E)}(V_{[d\infty]}, T) \xrightarrow{\hat{\rho}_d\ast} \text{Hom}_{L_K(E)}(L_K(E), T) \xrightarrow{\hat{\rho}(d-1)\ast} \text{Hom}_{L_K(E)}(L_K(E), T) \]
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Let $E$ be a finite graph. Let $d$ be a basic closed path in $E$ and let $T$ be a Chen simple module. Consider the projective resolution

$$0 \longrightarrow L_K(E) \xrightarrow{\hat{\rho}_{d-1}} L_K(E) \xrightarrow{\hat{\rho}_{d}} V[d\infty] \longrightarrow 0$$

and the resulting standard long exact sequence

$$\pi \longrightarrow \text{Ext}^1_{L_K(E)}(V[d\infty], T) \xrightarrow{\hat{\rho}_{d\infty}^*} \text{Hom}_{L_K(E)}(L_K(E), T) \xrightarrow{\hat{\rho}(d-1)^*} \text{Hom}_{L_K(E)}(L_K(E), T) \longrightarrow \cdots$$

So for $t \in T$, $\pi(\hat{\rho}_{d-1}^* t) = 0 \iff \hat{\rho}_{d-1}^* f = \hat{\rho}_t$ for some $f = \hat{\rho}_X \in \text{Hom}_{L_K(E)}(L_K(E), T) \iff$ the equation $(d-1)X = t$ has a solution in $T$. 

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Chen simple modules and Prüfer modules over Leavitt path algebras
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and the resulting standard long exact sequence

$$\pi \rightarrow \text{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T) \xrightarrow{\pi} \text{Ext}^1_{L_K(E)}(L_K(E), T) \xrightarrow{\pi} \text{Ext}^1_{L_K(E)}(L_K(E), T) (=0) \rightarrow \cdots$$

So for $t \in T$,

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$$0 \to L_K(E) \xrightarrow{\hat{\rho}_{d^{-1}}} L_K(E) \xrightarrow{\hat{\rho}_d} V_{[d, \infty]} \to 0$$

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$$\begin{align*}
\pi & \quad \text{Ext}^1_{L_K(E)}(V_{[d, \infty]}, T) \\
\hat{\rho}_{d^{-1}} & \quad \text{Hom}_{L_K(E)}(L_K(E), T) \\
\hat{\rho}_d & \quad \text{Hom}_{L_K(E)}(L_K(E), T)
\end{align*}$$

So for $t \in T$,

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Let $E$ be a finite graph. Let $d$ be a basic closed path in $E$ and let $T$ be a Chen simple module. Consider the projective resolution

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$$\hom_{L_K(E)}(V_{[d]}, T) \xrightarrow{\hat{\rho}_{d-1}*} \hom_{L_K(E)}(L_K(E), T) \xrightarrow{\hat{\rho}_(d-1)*} \hom_{L_K(E)}(L_K(E), T)$$

$$\pi \to \ext_{L_K(E)}^1(V_{[d]}, T) \to \ext_{L_K(E)}^1(L_K(E), T) \to 0 \to \cdots$$

So for $t \in T$,

$$\pi(\hat{\rho}_t) = 0 \iff$$

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Proof: main points

So we get:

**Proposition:** \( \text{Ext}^1_{L_K(E)}(V_{d^\infty}, T) = 0 \) if and only if 
\((d - 1)X = t\) has a solution in \(T\) for every \(t \in T\).
Proof: main points

So we get:

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But then it’s not hard to show:

**Lemma:**

1) Let \(T = V_{[q]}\), with \(V_{[q]} \neq V_{[d^{\infty}]}\). Suppose \(s(d) \in U(T)\). Let \(t \in T\) be “not divisible” by \(d\). Then the equation \((d - 1)X = t\) has no solution in \(T\)
Proof: main points

So we get:

Proposition: $\text{Ext}_{L_K(E)}^1(V_{[d^\infty]}, T) = 0$ if and only if $(d - 1)X = t$ has a solution in $T$ for every $t \in T$.

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Prüfer modules

In particular, we have

**Corollary:** For $d$ a cycle in $E$, the left $L_K(E)$-module $V_{[d^\infty]}$ is (simple and) not injective.

**Question:** What is the injective hull of $V_{[d^\infty]}$?

Recall: $\hat{\rho}_{d-1}: L_K(E) \to L_K(E)$ is a monomorphism. (In other words, $d-1$ is not a right zero-divisor in $L_K(E)$.) Moreover, $V_{[d^\infty]} \cong L_K(E)/L_K(E)(d-1)$. 

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\[
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\]
Prüfer modules

We look at the standard Prüfer abelian groups for guidance.

\[ p \] denotes a prime in \( \mathbb{Z} \).

\[ \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}/p^2\mathbb{Z} \hookrightarrow \mathbb{Z}/p^3\mathbb{Z} \hookrightarrow \cdots \]

The embedding is \[ a + p^i\mathbb{Z} \mapsto pa + p^{i+1}\mathbb{Z} \]

The Prüfer \( p \)-group is

\[ \mathbb{Z}(p^\infty) = \bigcup_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z} \]

Another point of view: \[ \mathbb{Z}(p^\infty) = \left\{ \frac{a}{p^i} \mid i \in \mathbb{N} \right\} \], with addition mod \( \mathbb{Z} \).
Prüfer modules

Well-known properties of $\mathbb{Z}(p^\infty)$:

1) $\mathbb{Z}(p^\infty)$ is divisible as a $\mathbb{Z}$-module: for every $z \in \mathbb{Z}$ and $t \in \mathbb{Z}(p^\infty)$ the equation $zX = t$ has a solution in $\mathbb{Z}(p^\infty)$. In particular, $\mathbb{Z}(p^\infty)$ is injective as a $\mathbb{Z}$-module.

2) The only proper subgroups of $\mathbb{Z}(p^\infty)$ are the $\mathbb{Z}/p^i\mathbb{Z}$ ($i \in \mathbb{N}$). In particular, $\mathbb{Z}(p^\infty)$ has d.c.c., but not a.c.c., on submodules.

3) Each of the quotients $\mathbb{Z}/p^i+1\mathbb{Z}/\mathbb{Z}/p^i\mathbb{Z}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

4) $\mathbb{Z}(p^\infty)/\mathbb{Z}/p^i\mathbb{Z} \cong \mathbb{Z}(p^\infty)$ for all $i \in \mathbb{N}$.

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Prüfer modules

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Prüfer modules

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Well-known properties of $\mathbb{Z}(p^\infty)$:

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Prüfer modules

6) \( \text{End}_\mathbb{Z}(\mathbb{Z}(p^\infty)) \) is the ring of \( p \)-adic integers; think of this as "formal power series in \( p \)", with coefficients in \( \{0, 1, \ldots, p - 1\} \).
6) $\text{End}_\mathbb{Z}(\mathbb{Z}(p^\infty))$ is the ring of $p$-adic integers; think of this as “formal power series in $p$”, with coefficients in $\{0, 1, \ldots, p - 1\}$.

OR, think of it as an inverse limit of the rings / maps

$$\cdots \to \mathbb{Z}/p^3\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p^1\mathbb{Z}.$$
Prüfer modules

We can do this in general.

**Proposition**: Suppose $a \in R$ has these two properties:

(1) $R/Ra$ is a simple left $R$-module, and

(2) for every $i \in \mathbb{N}$, the equation $aX = 1 + Ra^i$ has no solution in $R/Ra^i$.

Then the direct limit $U_{R,a}$ of the sequence

$$R/Ra \hookrightarrow R/Ra^2 \hookrightarrow R/Ra^3 \hookrightarrow \cdots$$

has structural properties analogous to those for $\mathbb{Z}(p^\infty)$ given above.
Prüfer modules

Now we apply these ideas to the specific case where

\[ R = L_K(E), \ a = c - 1 \]

where \( c \) is a cycle in the finite graph \( E \).

\[ L_K(E)/L_K(E)(c-1) \hookrightarrow L_K(E)/L_K(E)(c-1)^2 \hookrightarrow L_K(E)/L_K(E)(c-1)^3 \hookrightarrow \cdots \]

Denote the direct limit of this sequence by \( U_{E,c-1} \).
Prüfer modules

We already have property (1):

\[ L_K(E)/L_K(E)(c - 1) \] is a simple left \( L_K(E) \)-module, because it is isomorphic to \( V_{[c^\infty]} \).

For property (2):

**Proposition**: For any basic closed path \( c \) in \( E \), the equation

\[ (c - 1)X = 1 + L_K(E)(c - 1)^n \]

has NO solution in \( L_K(E)/L_K(E)(c - 1)^n \).
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For property (2):

**Proposition**: For any basic closed path $c$ in $E$, the equation

$$(c - 1)X = 1 + L_K(E)(c - 1)^n$$

has NO solution in $L_K(E)/L_K(E)(c - 1)^n$.

**Idea of proof**: Establish a “Division Algorithm by $c - 1$” inside $L_K(E)$. (Messy, but relatively straightforward.)
Proposition: Let $E$ be a finite graph, let $c$ be a basic closed path in $E$ based at $v$, and let $U_{E,c^{-1}}$ be the Prüfer module associated to $c$. Suppose that there exists a cycle $d \neq c$ which connects to $v$. Then $U_{E,c^{-1}}$ is not injective.
Prüfer modules

**Proof**: By work on $\text{Ext}^1$ groups described previously (using the hypothesis that $d$ connects to $v$),

$$\text{Ext}^1(V_{[d]}^{\infty}, V_{[c]}^{\infty}) \neq 0.$$

Let $\alpha_1$ denote $1 + L_K(E)(c - 1)$. We get

$$0 \to V_{[c]}^{\infty} \cong L_K(E) \alpha_1 \to U_{E,c-1} \to U_{E,c-1}/L_K(E) \alpha_1 \cong U_{E,c-1} \to 0$$

But $\text{Hom}(V_{[d]}^{\infty}, U_{E,c-1}) = 0$, because the only simple submodule of $U_{E,c-1}$ is isomorphic to $V_{[c]}^{\infty} \not\cong V_{[d]}^{\infty}$. 
Prüfer modules

This gives the resulting long exact sequence

\[
\begin{align*}
\text{Hom}_{L_K(E)}(V_{d\infty}, V_{c\infty}) & \longrightarrow \text{Hom}_{L_K(E)}(V_{d\infty}, U_{E,c-1}) & \longrightarrow \text{Hom}_{L_K(E)}(V_{d\infty}, U_{E,c-1}) &= 0
\end{align*}
\]
This gives the resulting long exact sequence

\[
\begin{align*}
\text{Hom}_{L_K(E)}(V_{[d \infty]}, V_{[c \infty]}) & \to \text{Hom}_{L_K(E)}(V_{[d \infty]}, U_{E, c-1}) \to \text{Hom}_{L_K(E)}(V_{[d \infty]}, U_{E, c-1}) \quad (=0) \\
\pi \to \text{Ext}^1_{L_K(E)}(V_{[d \infty]}, V_{[c \infty]}) \quad (\neq 0) & \to \text{Ext}^1_{L_K(E)}(V_{[d \infty]}, U_{E, c-1}) \to \text{Ext}^1(V_{[d \infty]}, U_{E, c-1})
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\text{Hom}_{L_K(E)}(V_{d\infty}, V_{c\infty}) & \longrightarrow \text{Hom}_{L_K(E)}(V_{d\infty}, U_{E, c-1}) \longrightarrow \text{Hom}_{L_K(E)}(V_{d\infty}, U_{E, c-1}) (=0) \\
\pi & \longrightarrow \text{Ext}_1^{L_K(E)}(V_{d\infty}, V_{c\infty}) (\neq 0) \longrightarrow \text{Ext}_1^{L_K(E)}(V_{d\infty}, U_{E, c-1}) \longrightarrow \text{Ext}_1(V_{d\infty}, U_{E, c-1})
\end{align*}
\]

Consequently, \(\text{Ext}_1^{L_K(E)}(V_{d\infty}, U_{E, c-1}) \neq 0\), so that \(U_{E, c-1}\) is not injective.
On the other hand what happens when there is NO cycle \( d \) which connects to \( c \)?

Call such a cycle \( c \) \textit{maximal}. 

**Example:** The Toeplitz graph 

\[
\begin{array}{cccc}
\ast & \ast & \ast \\
\uparrow & \uparrow & \Rightarrow \\
\ast & \ast & \ast \\
\end{array}
\]

(The Leavitt path algebra \( L_K(T) \) is isomorphic to the Jacobson algebra \( K\langle X, Y | XY = 1 \rangle \).)
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Example: The Toeplitz graph

$$T = c \circlearrowright \bullet \longrightarrow \bullet$$

(The Leavitt path algebra $L_K(T)$ is isomorphic to the Jacobson algebra $K\langle X, Y | XY = 1 \rangle$.)
**Main Theorem:** Let $E$ be a finite graph and let $c$ be a basic closed path in $E$. Let $U_{E,c-1}$ be the Prüfer module associated to $c$. Then $U_{E,c-1}$ is injective if and only if $c$ is a maximal cycle.

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Chen simple modules and Prüfer modules over Leavitt path algebras
Main Theorem: Let $E$ be a finite graph and let $c$ be a basic closed path in $E$. Let $U_{E,c^{-1}}$ be the Prüfer module associated to $c$. Then $U_{E,c^{-1}}$ is injective if and only if $c$ is a maximal cycle.

Moreover, in case $U_{E,c^{-1}}$ is injective, then:

(1) $U_{E,c^{-1}}$ is the injective envelope of the Chen simple module $V_{[c^\infty]}$, and

(2) $\text{End}_{L_K(E)}(U_{E,c^{-1}})$ is isomorphic to the ring $K[[x]]$ of formal power series in $x$. 

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Prüfer modules

One direction?  Done above.

Other direction?
Prüfer modules

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Other direction?

Two steps: Reduce to the case when $c$ is a source loop. Then prove the result in this case.
Prüfer modules

**Proposition:**

1) Source elimination is a Morita equivalence, and preserves Prüfer modules.

2) Reduction of a source cycle to a source loop is a Morita equivalence, and preserves Prüfer modules.

**Proof:** Omitted. Not too difficult.
Prüfer modules

We analyze specific elements.

**Proposition:** Let \( c \) be a source loop. Let \( j \in \text{Ann}_{\mathbb{K}}(E) \)(\( U_E, c^{-1} \)). Then there exists \( n \in \mathbb{N} \) such that \( c^n j = 0 \).

**Proof:** It is not hard to show that any nonzero \( j \in \text{Ann}_{\mathbb{K}}(E) \)(\( U_E, c^{-1} \)) is a \( \mathbb{K} \)-linear combination of elements of the form

\[
\alpha \beta^* w \gamma \delta^* \neq 0,
\]

where \( w \neq s(c) \). Now consider cases.

1) If \( \alpha \beta^* w = w \) then \( c^* \alpha \beta^* w \gamma \delta^* = c^* w \gamma \delta^* = 0 \).

2) If \( \alpha \beta^* w = \beta^* w \neq w \) then \( s(\beta^*) = r(\beta) \neq s(c) \), otherwise \( \beta \) would be a path which starts in \( w \) and ends at \( s(c) \), contrary to \( c \) being a source loop. Then \( c^* \alpha \beta^* w \gamma \delta^* = c^* \beta^* w \gamma \delta^* = 0 \).
3) In all the other cases $\alpha = c^t \eta_1 \cdots \eta_s$ with $c \neq \eta_1 \in E^1$, $t \geq 0$ and $s \geq 1$. Then

$$(c^{t+1})^* \alpha \beta^* w \gamma \delta^* = (c^{t+1})^* c^t \eta_1 \cdots \eta_s \beta^* w \gamma \delta^* = c^* \eta_1 \cdots \eta_s \beta^* w \gamma \delta^* = 0.$$ 

Since $j$ is a finite sum of terms of the form $\alpha \beta^* w \gamma \delta^*$, the result follows.
Prüfer modules

**Proposition:** For any \( \ell \in L_K(E) \setminus \text{Ann}_{L_K(E)}(U_{E,c-1}) \) and for any \( u \in U_{E,c-1} \), there exists \( X \in U_{E,c-1} \) such that \( \ell X = u \). That is, \( u \) is divisible by any element in \( L_K(E) \setminus \text{Ann}_{L_K(E)}(U_{E,c-1}) \).
Prüfer modules

**Proposition:** For any \( \ell \in L_K(E) \setminus \text{Ann}_{L_K(E)}(U_{E,c-1}) \) and for any \( u \in U_{E,c-1} \), there exists \( X \in U_{E,c-1} \) such that \( \ell X = u \). That is, \( u \) is divisible by any element in \( L_K(E) \setminus \text{Ann}_{L_K(E)}(U_{E,c-1}) \).

**Idea of Proof:** It can be shown that

\[
\text{Ann}_{L_K(E)}(U_{E,c-1}) = \bigcap_{n \geq 1} L_K(E)(c - 1)^n = \langle E^0 \setminus s(c) \rangle.
\]

Then using the “Division Algorithm” for \( c - 1 \) (and some computation) yields the result.
**Corollary**: If $0 \neq u \in U_{E, c-1}$ then $(c^*)^m u \neq 0$ for all $m \in \mathbb{N}$. 
Prüfer modules

Corollary: If $0 \neq u \in U_{E,c-1}$ then $(c^*)^m u \neq 0$ for all $m \in \mathbb{N}$.

Proof: Since $c \notin L_K(E)(c-1) \supseteq \text{Ann}_{L_K(E)}(U_{E,c-1})$, by previous Proposition there exists $0 \neq x \in U_{E,c-1}$ with

$$cx = u.$$  

We may assume that $s(c)x = x$. Then

$$0 \neq x = s(c)x = c^* cx = c^* u.$$  

Repeating the same argument for $0 \neq c^* u \in U_{E,c-1}$, we get $(c^*)^2 u \neq 0$. Now continue.
Prüfer modules

**Key Proposition:** Let $c$ be a source loop in $E$. Let $I_f$ be a finitely generated left ideal of $L_K(E)$, and let $\varphi : I_f \to U_{E,c^{-1}}$ be a $L_K(E)$-homomorphism. Then there exists $\psi : L_K(E) \to U_{E,c^{-1}}$ such that $\psi|_{I_f} = \varphi$. Consequently,

$$\text{Ext}^1(L_K(E)/I_f, U_{E,c^{-1}}) = 0.$$
Key Proposition: Let \( c \) be a source loop in \( E \). Let \( I_f \) be a finitely generated left ideal of \( L_K(E) \), and let \( \varphi : I_f \to U_{E,c-1} \) be a \( L_K(E) \)-homomorphism. Then there exists \( \psi : L_K(E) \to U_{E,c-1} \) such that \( \psi|_{I_f} = \varphi \). Consequently,

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\text{Ext}^1(L_K(E)/I_f, U_{E,c-1}) = 0.
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Proof: By the result presented in this morning’s lecture, we know that \( L_K(E) \) is a Bézout ring, i.e., that every finitely generated left ideal of \( L_K(E) \) is principal.
Prüfer modules

**Key Proposition**: Let \( c \) be a source loop in \( E \). Let \( I_f \) be a finitely generated left ideal of \( L_K(E) \), and let \( \varphi : I_f \to U_{E,c-1} \) be a \( L_K(E) \)-homomorphism. Then there exists \( \psi : L_K(E) \to U_{E,c-1} \) such that \( \psi|_{I_f} = \varphi \). Consequently,

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**Proof**: By the result presented in this morning’s lecture, we know that \( L_K(E) \) is a Bézout ring, i.e., that every finitely generated left ideal of \( L_K(E) \) is principal.

So \( I_f = L_K(E)\ell \) for some \( \ell \in I_f \).
Assume on one hand that $\ell \in \text{Ann}_{L_K(E)}(U_{E,c-1})$, and hence $I_f \leq \text{Ann}_{L_K(E)}(U_{E,c-1})$.

But we know these two things:

1) Any element of $\text{Ann}_{L_K(E)}(U_{E,c-1})$ is annihilated by some $c^*N$,

and

2) $c^*n u \neq 0$ for all $0 \neq u \in U_{E,c-1}$ and $n \in \mathbb{N}$. 

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But then for $\varphi \in \text{Hom}_{L_K(E)}(I_f, U_E, c-1)$ we see that $\varphi(\ell) = 0$. Here’s why:

Otherwise, if $\varphi(\ell) \neq 0$, then $(c^*)^n\varphi(\ell) \neq 0$ for all $n$; but $\ell \in \text{Ann}_{L_K(E)}(U_E, c-1)$ gives $(c^*)^N\ell = 0$ for some $N$, so that $0 = \varphi((c^*)^N\ell) = (c^*)^N \varphi(\ell)$, a contradiction.

And $\varphi(\ell) = 0$ gives $\varphi = 0$, because $I_f$ is generated by $\ell$. Thus in this case we must have $\text{Hom}_{L_K(E)}(I_f, U_E, c-1) = 0$, and the conclusion follows trivially.
Assume on the other hand that $\ell \notin \text{Ann}_{L_K(E)}(U_{E,c-1})$. But then there exists $x \in U_{E,c-1}$ for which $\ell x = \varphi(\ell)$.

Let $\psi : L_K(E) \to U_{E,c-1}$ be the map $\rho_x$. Then, for each $i = r\ell \in I_f$, we have

$$
\psi(i) = \psi(r\ell) = r\ell \psi(1) = r\ell x = \varphi(\ell) = \varphi(r\ell) = \varphi(i),
$$

and so $\varphi$ extends in this case as well.
Prüfer modules

**Proposition:** Let $E$ be a finite graph, and $c$ a source loop in $E$. Then the endomorphism ring of the left $L_K(E)$-module $U_{E,c^{-1}}$ is isomorphic to the ring of formal power series $K[[x]]$.

Proof omitted, but it’s not too hard.
Prüfer modules

We need one more tool.

We know the entire lattice of proper submodules of $U_{E,c-1}$ as a left $L_K(E)$-module, it consists precisely of the $L_K(E)/L_K(E)(c-1)^i$.

Proposition: Each $L_K(E)/L_K(E)(c-1)^i$ is a right $S$-submodule of $U_{E,c-1}$, and these are ALL the right $S$-submodules of $U_{E,c-1}$.

In particular, $(U_{E,c-1})_S$ is artinian.

Proof: Not hard.
Prüfer modules

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We know the entire lattice of proper submodules of $U_{E,c-1}$ as a left $L_K(E)$-module, it consists precisely of the $L_K(E)/L_K(E)(c-1)^i$.

But $U_{E,c-1}$ is a right module over its endomorphism ring $S$, which is isomorphic to $K[[x]]$.

**Proposition**: Each $L_K(E)/L_K(E)(c-1)^i$ is a right $S$-submodule of $U_{E,c-1}$, and these are ALL the right $S$-submodules of $U_{E,c-1}$. In particular, $(U_{E,c-1})_S$ is artinian.

**Proof**: Not hard.
Here’s why we care about the right $S$-structure of $U_{E,c-1}$:

This property implies that the functor $\text{Ext}^1(-, U_{E,c-1})$ sends direct limits to inverse limits.

(More details: If a module is linearly compact over its endomorphism ring, then it is algebraically compact and hence pure-injective. But for a pure-injective left $R$-module $M$, the functor $\text{Ext}^1(-, M)$ sends direct limits to inverse limits.)
Finally, we get the result.

**Theorem:** Let $E$ be a finite graph with source loop $c$. Then the Prüfer module $U_{E,c-1}$ is injective. Indeed, $U_{E,c-1}$ is the injective envelope of $V_{[c\infty]}$. 
Prüfer modules  (Key Prop.)  $\text{Ext}^1(L_K(E)/I_f, U_{E,c-1}) = 0$.

**Proof:** In order to check the injectivity of $U_{E,c-1}$, we apply Baer’s Lemma; that is, we need only check that $U_{E,c-1}$ is injective relative to any short exact sequence of the form

$$0 \to I \to L_K(E) \to L_K(E)/I \to 0.$$

This is equivalent to showing that $\text{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1}) = 0$ for any left ideal $I$ of $L_K(E)$. 

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This is equivalent to showing that $\text{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1}) = 0$ for any left ideal $I$ of $L_K(E)$.

Write $I = \varinjlim I_\lambda$, where the $I_\lambda$ are the finitely generated submodules of $I$. It is standard that

$$L_K(E)/I = \varinjlim L_K(E)/I_\lambda.$$
Prüfer modules (Key Prop.) \( \text{Ext}^1(L_K(E)/I_f, U_{E,c-1}) = 0. \)

So now applying the functor \( \text{Ext}^1_{L_K(E)}(\_, U_{E,c-1}) \), we get:

\[
\text{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1})
\]
Prüfer modules  

(Key Prop.)  \[ \operatorname{Ext}^1(L_K(E)/I_f, U_{E,c-1}) = 0. \]

So now applying the functor \( \operatorname{Ext}^1_{L_K(E)}(-, U_{E,c-1}) \), we get:

\[
\operatorname{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1}) = \operatorname{Ext}^1_{L_K(E)}(\lim_{\lambda} L_K(E)/I_\lambda, U_{E,c-1}) = 0.
\]

(by Key Proposition)

Since \( L_K(E) \alpha_1 \) is an essential submodule of \( U_{E,c-1} \), the last statement follows.
Prüfer modules  

(Key Prop.) \( \text{Ext}^1(L_K(E)/I_f, U_{E,c-1}) = 0. \)

So now applying the functor \( \text{Ext}^1_{L_K(E)}(-, U_{E,c-1}) \), we get:

\[
\text{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1}) \\
= \text{Ext}^1_{L_K(E)}(\lim_{\lambda} L_K(E)/I_\lambda, U_{E,c-1}) \\
= \lim_{\lambda} \text{Ext}^1(L_K(E)/I_\lambda, U_{E,c-1}) \quad \text{(by Proposition above)}
\]
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So now applying the functor \( \Ext^1_{L_K(E)}(-, U_{E,c-1}) \), we get:

\[
\Ext^1_{L_K(E)}(L_K(E)/I, U_{E,c-1}) = \Ext^1_{L_K(E)}(\lim_{\lambda \to} L_K(E)/I_{\lambda}, U_{E,c-1}) \\
= \lim_{\leftarrow} \Ext^1(L_K(E)/I_{\lambda}, U_{E,c-1}) \quad \text{(by Proposition above)} \\
= \lim_{\leftarrow} 0 = 0. \quad \text{(by Key Proposition)}
\]
Prüfer modules  (Key Prop.) \( \text{Ext}^1(L_K(E)/I_f, U_{E,c-1}) = 0. \)

So now applying the functor \( \text{Ext}^1_{L_K(E)}(-, U_{E,c-1}) \), we get:

\[
\text{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1})
\]
\[
= \text{Ext}^1_{L_K(E)}(\lim_{\to} L_K(E)/I_\lambda, U_{E,c-1})
\]
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= \lim_{\leftarrow} \text{Ext}^1(L_K(E)/I_\lambda, U_{E,c-1}) \quad \text{(by Proposition above)}
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\[
= \lim_{\leftarrow} 0 = 0. \quad \text{(by Key Proposition)}
\]

Since \( L_K(E)\alpha_1 \) is an essential submodule of \( U_{E,c-1} \), the last statement follows.