Mad Veterinarian Puzzles:

Grids, Groups, Graphs; Gerbils, Groundhogs, Giraffes?

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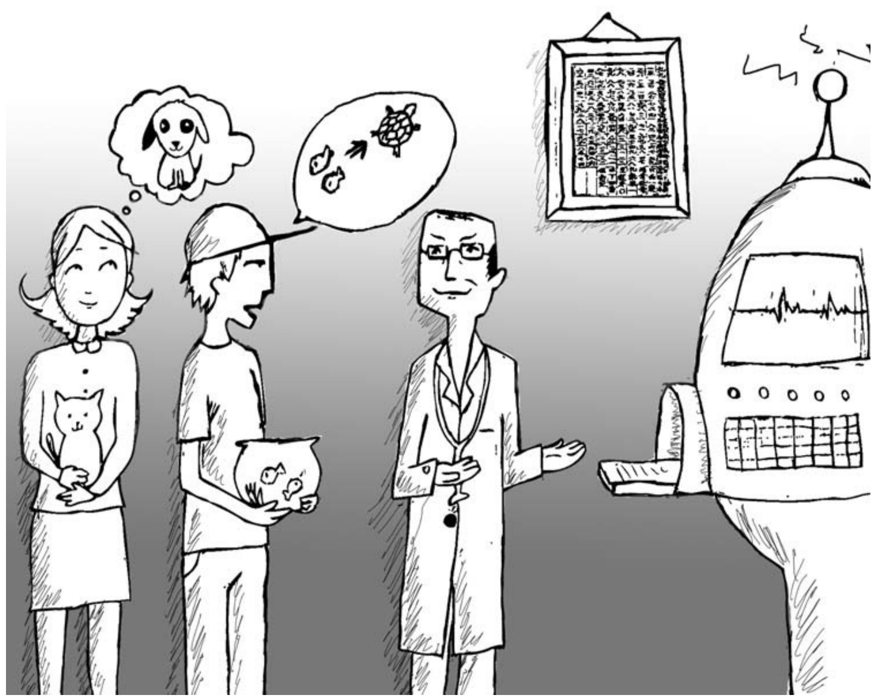
There is a collection of problems that have come to be known as "Mad Veterinarian Puzzles", for reasons which will soon become obvious.

As a warm-up, we will look at some of the "usual" questions that are asked in this context Then we will ask some new types of questions. The goal is to see just how much structure is contained in these ideas, and really how beautiful these ideas can be!

If we have some time at the end, we will mention how these puzzles arise in some current mathematical research areas.

This collection of activities will be posted on the website www.uccs.edu/gabrams . Feel free to modify / use it with your own students!



(graphic courtesy of the Mathematical Association of America.)

We'll start by filling in some grids.

Grid 1

max	1	2	3	4
1				
2				
3				
4				

Grid 2

min	1	2	3	4
1				
2				
3				
4				

Grid 3

$+_{mod4}$	1	2	3	4
1				
2				
3				
4				

Grid 4

$\times_{mod4}$	1	2	3	4
1				
2				
3				
4				

Grid 5

$\times$ (each slot)	(1,1)	(-1,1)	(1,-1)	(-1,-1)
(1,1)				
(-1,1)				
(1,-1)				
(-1,-1)				

Grid 6

$+_{mod2}$ (each slot)	(2,2)	(1,2)	(2,1)	(1,1)
(2,2)				
(1,2)				
(2,1)				
(1,1)				

Three of these grids share a property that the other three do not. Can you find such a property?

Here are a few more grids to try, in case you happen to be familiar with the germane notation

Grid 7 (Here i = $\sqrt{-1}$ in **C**omplex numbers)

x	1	i	-1	-i
1				
i				
-1				
-i				

Grid 8

$\cup$	$\varnothing$	{a}	{b}	{a,b}
$\varnothing$				
{a}				
{b}				
{a,b}				

Grid 9

matrix mult.	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$
$\begin{pmatrix}1&0\\0&1\end{pmatrix}$				
$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$				
$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$				
$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$				

Grid 10

matrix mult.	$\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
$\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$				
$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}$				
$\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$				
$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$				

Grid 11

$\times_{\mathbf{mod8}}$	1	3	5	7
1				
3				
5				
7				

Grid 12

$x * y = y$	a	b	c	d
a				
b				
c				
d				

Puzzle 1

Here's our first example of a Mad Veterinarian Puzzle. (The wording is similar in flavor to the wording of the original and most famous Mad Vet Puzzle, which you can find at http://www.bumblebeagle.org/madvet/index.html .)

A Mad Veterinarian has created three animal transmogrifying machines.

Place a cat in the input bin of the first machine, press the button, and .... whirrr .... bing! Open the output bins to find that the cat has transformed into a dog! The second machine can convert a dog into one cat, one dog, and one mouse. The third machine can convert a mouse into a cat and a dog. Each machine can also operate in reverse (e.g., if you've got one cat, one dog, and one mouse, you can convert them into a dog. (So, in shortened notation: C --- > D D --- > C D M M --- > C D)

Question 1.0 Suppose this Mad Vet starts with three cats. See if you can find a sequence of machines that will turn her three cats into six cats (and no other animals).

Question 1.1 Suppose this Mad Vet starts with one cat. See if you can find a sequence of machines that will turn her one cat into four cats (and no other animals).

Question 1.2 Now that you know that there is a way to turn one cat into four cats, what other numbers of (only) cats can you (easily) conclude (without specifying the machines being used) into which she can turn her one cat?

Question 1.3 Suppose now that the Mad Vet starts with 20 cats (and no other animals). What's the smallest number of (only) cats she can produce from this collection of 20 cats?

Question 1.4 Repeat the previous question, but this time start with 25 cats. Repeat it again, but now start with 30 cats.

Question 1.5 Now suppose the Mad Vet has N cats in her clinic. Give a rule / formula / algorithm / procedure which determines the smallest number of (only) cats into which she can turn these N cats.

(Side question: How do you KNOW that your answers to the previous three questions are really the smallest?)

Question 1.6 It's easy to see that if the Mad Vet has one dog, then she can turn it into one cat (just use Machine 1 in reverse). So as a result, any collection that consists only of dogs and cats can be turned into a collection consisting only of cats. If the Mad Vet starts with 8 cats and 23 dogs, what's the smallest number of (only) cats into which she can turn this collection?

Question 1.7 Find a sequence of machines that will turn one mouse into a collection of (only) cats. What is the smallest number of cats into which this mouse can be transmogrified?

Question 1.8 Suppose the Mad Vet starts with a collection of 7 cats, 5 dogs, and 10 mice. Find the smallest number of (only) cats into which she can turn this collection.

Question 1.9 Suppose the Mad Vet starts with a collection of c cats, d dogs, and m mice. Describe a procedure for finding the smallest number of (only) cats into which she can turn this collection.

Puzzle 1, in shortened notation: **C --- > D** **D --- > C D M** **M --- > C D**

Classes of animals

We say that two collections of animals are in the same "class" if you can use the Mad Vet machines to transform one collection to the other. (Because the machines are allowed to run in reverse, if you can transform one collection to another, then you can transform the second collection back to the first.)

For example: With the machines of Puzzle 1, the collection consisting of {2Cats, 1Dog} is in the same class as the collection consisting of {4Cats, 1Mouse}, because

{2Cats, 1Dog} --- (Machine 2) ---- > {3Cats, 1Dog, 1Mouse} ---- (Machine 1 reverse) --- > {4Cats, 1Mouse}

We sometimes use square brackets to denote "classes". So, for instance, [2Cats, 1Dog] means all of the collections of animals which are in the same class as the collection {2Cats, 1Dog} . We just showed that

[2Cats, 1Dog] = [3Cats, 1Dog, 1Mouse] = [4Cats, 1Mouse].

Question 1.10 Justify why [2Cats, 1Dog] = [3Cats, 2Dogs, 2Mice]

Question 1.11 Find a number N for which [2Cats, 1Dog] = [NCats]

Question 1.12 Find a DIFFERENT number M for which [2Cats, 1Dog] = [MCats]

MAJOR Question 1.13 How many different classes of animals are there in this Mad Vet's clinic? Specifically:
Give a list of classes having the property that EVERY possible collection of animals can be transformed to exactly one class on your list. Justify why your list is a complete list of classes, and try to explain why there are no repeated classes on your list. Call this list the *List of All Classes*.
(Here's a Major Start to the Answer to Major Question 1.13: [1Cat] is a class. So is [2Cats].)

Question 1.14 To which class on your list from Major Question 1.13 is the class [7Cats, 5Dogs, 10Mice] equal ?

Puzzle 1, in shortened notation: C --- > D D --- > C D M M --- > C D

Combining Classes

There is an ‘arithmetic’ of classes. Here’s one way to think about it. Suppose the Mad Vet knows the List of All Classes in her clinic (in other words, suppose the Mad Vet knows the answer to Major Question 1.13).

Now suppose the Mad Vet starts with some collection of animals. For the Mad Vet’s birthday, a friend brings her a present: some more animals! Now the Mad Vet has a new collection of animals. But which class in the List of All Classes is the new collection equal to?

For example: Suppose the Mad Vet has two cats. The Mad Vet’s friend brings the Mad Vet two more cats as a present. So now the Mad Vet has {4Cats} in her clinic. To which class on the List of All Classes is [4Cats] equal ?? You showed previously (way back in the very first Question!) that [1Cat] = [4Cats].

So: [2Cats] + [2Cats] = [4Cats] = [1Cat]

We now make a grid of ALL possible combinations of classes. We list all three of the classes as the heading of each row, and each column. One entry in the grid has already been filled in, it’s the information we just verified, namely, that [2Cats] + [2Cats] = [1Cat].

Question 1.15 Make a grid of ALL possible combinations of classes. Specifically, determine what the sum of each column entry with each row entry is, and record your answer in that row-column intersection.

+	[1Cat]	[2Cats]	[3Cats]
[1Cat]			
[2Cats]		[1 Cat]	
[3Cats]			

Puzzle 2

A second Mad Veterinarian has these three animal transmogrifying machines.

Machine #1 turns one cat into one dog and one mouse

Machine #2 turns one dog into one cat and one mouse

Machine #3 turns one mouse into one cat and one dog

(So, in shortened notation: C --- > D M D --- > C M M --- > C D)

Question 2.1 How many different classes of animals are there in Puzzle 2?

Specifically: Give a list of classes having the property that EVERY possible collection of animals can be transformed to exactly one class on your list. Justify why your list is a complete list of classes. (Try to explain why there are no repeated classes on your list.)

Question 2.2 Make a grid of ALL possible combinations of classes. Specifically, list out all the different classes along the top row, and again along the left hand column. Then determine what the sum of each column entry with each row entry is, and record your answer in that row-column intersection. What can you say about each of the rows and columns of the lower-right portion of the table?

+	

Puzzle 3

A third Mad Veterinarian has these two animal transmogrifying machines.

Machine #1 turns one cat into two cats

Machine #2 turns one dog into two dogs

(So, in shortened notation: C --- > 2C D --- > 2D)

Question 3.1 How many different classes of animals are there in Puzzle 3?

Specifically: Give a list of classes having the property that EVERY possible collection of animals can be transformed to exactly one class on your list. Justify why your list is a complete list of classes. (Try to explain why there are no repeated classes on your list.)

Question 3.2 Make a table of ALL possible combinations of classes. Specifically, list out all the different classes along the top row, and again along the left hand column. Then determine what the sum of each column entry with each row entry is, and record your answer in that row-column intersection. What can you say about each of the rows and columns of the lower-right portion of the table?

+	

Puzzle 4

A fourth Mad Veterinarian has these three animal transmogrifying machines.

Machine #1 turns one cat into one cat, one dog and one mouse
Machine #2 turns one dog into one cat and one mouse
Machine #3 turns one mouse into one cat and one dog

(So, in shortened notation: C --- > C D M   D --- > C M   M --- > C D)

Question 4.1 How many different classes of animals are there in Puzzle 4?
Specifically: Give a list of classes having the property that EVERY possible collection of animals can be transformed to exactly one class on your list. Justify why your list is a complete list of classes. (Try to explain why there are no repeated classes on your list.)

Question 4.2 Make a table of ALL possible combinations of classes. Specifically, list out all the different classes along the top row, and again along the left hand column. Then determine what the sum of each column entry with each row entry is, and record your answer in that row-column intersection. What can you say about each of the rows and columns of the lower-right portion of the table?

+	

Mad Veterinarian Puzzles: Food for Thought

Food For Thought 1) There is a technique to actually PROVE that a List of All Classes has no repeated classes on it. We can use what's called an *invariant*. For instance, in Puzzle 1: If we assign dollar values to animals in each species by valuing Cats at $1 apiece, Dogs at $1 apiece, and Mice at $2 apiece, then we can show that the three classes [C], [2C], and [3C] are different. With what we saw in Puzzle 1, this gives a valid proof that there are exactly three classes of animals corresponding to these three machines.

The specific invariant that you'll use to prove that there are no repeats on the List of All Classes for other Puzzles will probably be different than the one we used for Puzzle 1. See if you can find invariants for the other three puzzles.

For more info about invariants in Mad Vet puzzles, see e.g. Joshua Zucker's activity from December 1, 2010 at
http://www.marinmathcircle.org/archives.html

Food For Thought 2) Grids in which every member of the List of All Classes appears exactly once in each row and each column of the lower right square are very special in mathematics: those grids correspond to what are called *groups*. Intuitively (but not technically correctly, at least not yet), a group is a system in which you can always get from any thing in the system to any other thing in the system by combining the first thing with something in the system. For instance, the set of *all* whole numbers (positive, negative, and 0), where combining operation is addition, has this property; on the other hand, the set of *positive* whole numbers, where the combining operation is addition, does not. (Can you see why?)

Food For Thought 3) There are many additional questions which arise in the context of Mad Veterinarian Puzzles. For instance, which Mad Vet Puzzles yield groups in the grid for the List of All Classes? And, in such situations, is there some way to 'easily' determine *which* group it is?

Abelian groups

(Remark: For those of you who have seen the idea of a group before … the approach we will take here might be somewhat different than the one you've already seen, but both approaches lead us to the same place.)

If S is a set with an associative, commutative binary operation + then (S,+) is an *abelian group* in case this property is satisfied:

For every x,y in S there exists z in S for which x + z = y.

In words: in case this property is satisfied:
For any choice of elements x and y in S we can 'get from x to y' by adding some element of S to x.

As a grid: in case this property is satisfied:
Each element of S appears in each row (and in each column) of the grid for S

Examples of abelian groups:

1) **Z** (the set of all integers) with usual addition operation + .
2) Let n be a positive integer. Let $\mathbf{Z_n}$ denote the set {1,2,…,n}, and let $+_n$ be the operation "addition mod n".
3) Let m and n be positive integers. Let $\mathbf{Z_m} \times \mathbf{Z_n}$ denote the set of pairs (a,b) where a is in {1,2,…,m}, and b is in {1,2,…,n}. Define an operation + on these pairs "coordinatewise" (so use $+_m$ for the left hand numbers, and use $+_n$ for the right hand numbers).
4) A generalization of the previous example: There's no need to just use two groups when you form the Cartesian product (i.e., the 'pairs') as we did above, you can use any number of groups you want to form the product, and you will still get an abelian group.

Examples of associative, commutative binary operations on set which are NOT abelian groups:

1) **N** (the set of all positive integers) with usual operation + .
2) $\mathbf{Z}^+$ (the set of all non-negative integers) with usual operation +.
3) **Z** (the set of all integers) with usual operation x .
4) The set {1,2,…,n} with operation "multiplication mod n".
5) The set {1,2,…,n} with operation "max", or operation "min".

Comments:

1) $\mathbf{Z_m} \times \mathbf{Z_n}$ is the 'same' as $\mathbf{Z_{mn}}$ **ONLY IN CASE g.c.d.(m,n)=1.** So for example the abelian group $\mathbf{Z_2} \times \mathbf{Z_3}$ is the 'same' as the abelian group $\mathbf{Z_6}$ **.** (Essentially, this means that the grips corresponding to the two groups, if relabeled appropriately, would be identical.) On the other hand, if m and n share at least one common factor (>1), then $\mathbf{Z_m} \times \mathbf{Z_n}$ is NOT the 'same' as $\mathbf{Z_{mn}}$. So for example the abelian group $\mathbf{Z_2} \times \mathbf{Z_2}$ is NOT the 'same' as the abelian group $\mathbf{Z_4}$**.**
2) The symbol $\mathbf{Z_1}$ denotes the group {0} having just one element. On the other hand, the symbol $\mathbf{Z_0}$ denotes the group **Z** (the group of all integers with operation +).

Possible activity: On a separate sheet of paper, see if you and your team can write down the group grids for these three abelian groups:

(1) $\mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_2}$ (2) $\mathbf{Z_2} \times \mathbf{Z_4}$ (3) $\mathbf{Z_8}$

Directed graphs

If someone uses the word *graph* in a mathematical context, the first thing that might pop into your head is a picture of a function in the xy-plane. But there is another mathematical meaning for the word *graph*.

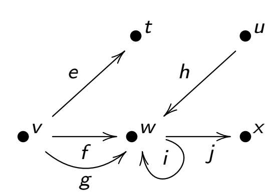
Somewhat informally (but hopefully rather clearly), a *graph* is a collection of nodes (the "vertices" of the graph) and connections (the "edges" of the graph). This mathematical concept comes up a lot in the real world, for instance, to model computer networks or communication systems. This idea also comes up quite often in various branches of mathematics; indeed, *graph theory* is a robust area of mathematics research in its own right.

Here's a note to those of you who might be familiar with Euler's "Bridges of Konigsberg" problem: graphs are used to solve this! (We apply "Euler's Theorem for Graphs" to this scenario.) Indeed, Euler is considered to be the original developer of graph theory.

A somewhat more specialized type of graph is a *directed graph*. A directed graph also has vertices and edges, but each of the edges is given an orientation. (That is, each edge is assumed to go "from" one vertex "to" another.)

In the directed graphs we will consider, it is totally acceptable to have an edge that goes from a vertex to itself. (We call such an edge a *loop*.) It's also completely fine to have more than one edge that goes from one vertex to another. (Some people call these *parallel edges*.) Here's an example of a directed graph E.

$E =$



Activity. For any Mad Vet Puzzle, there is a natural way to associate with it a directed graph. The directed graph encodes information about how the Mad Vet machines work in that puzzle. (The graph only describes the 'forward' direction of the machines.)

On a separate sheet of paper, see if you and your team can come up with a way to draw the directed graph associated with each of the four Mad Vet Puzzles we've considered so far. We'll call such a graph the *Mad Vet Graph* of the Puzzle.

The Smith normal form of a matrix

Here are six operations that you will be allowed to do on a matrix.

1) Switch any two rows with each other.
2) Switch any two columns with each other.
3) Multiply all the entries in any row by -1.
4) Multiply all the entries in any column by -1.
5) Add any integer-multiple of one row to another row.
6) Add any integer-multiple of one column to another column.

Now let B be any $n \times n$ (square) matrix whose entries are whole numbers. Using some sequence of the allowable operations, you can always change B into a matrix S where:

1) All of the entries that are not on the main diagonal (upper left to lower right) of S are 0.
2) Let s_1, s_2, … , s_n denote the entries on the main diagonal of S. If any of these entries equal 0, then we put those at the end of the list. For each of the consecutive nonzero entries, that entry is a divisor of the next entry on the list.

The resulting matrix S having these two properties is called the **Smith normal form** of the matrix B. It turns out that every whole number matrix B has a unique Smith normal form. For example: If B is the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

then the Smith normal form of B turns out to be the matrix S:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Possible Activity: See if you and your team can get from the indicated matrix B to the matrix S using the allowable operations. (You'll definitely need some extra paper to do that …)

Two Big Punchlines:
The Mad Vet Group Theorem,

and a nice way to answer the question

What group is it?

Here's the key result that allows us to easily determine which Mad Vet Puzzles yield (abelian) groups.

"Mad Vet Group" Theorem: Draw the Mad Vet Graph of the Mad Vet Puzzle. Call this graph E. Then the Mad Vet grid gives an (abelian) group precisely when E has all three of these properties:

1) E contains at least one cycle,
2) There is a directed path from every vertex of E to every cycle of E, and
3) Every cycle of E has a "fork". (In other words: whenever you are travelling along a cycle in the graph, at some vertex along the way you have at least one "choice" of which edge to take.) (The technical term for this is: an *exit* for the cycle.)

[Remarks: (1) The Mad Vet Group Theorem plays a role in the context of Mad Vet Puzzles identical to the role played by Euler's Theorem in the context of the Bridges of Konigsberg question. (2) The proof of the Mad Vet Group Theorem is somewhat long, but only undergraduate level mathematics ideas are used in it.]

"What group is it?" Theorem: Suppose that you are in a situation where the Mad Vet Puzzle gives a group. (You would apply the Mad Vet Group Theorem to determine whether or not you are actually in such a situation.) In such a situation, here's how to figure out what group it is.

Step 1) Form the adjacency matrix A of the Mad Vet Graph.
Step 2) Form the matrix $B = I - A$. (Here I means the identity matrix of the appropriate size.)
Step 3) Compute the Smith normal form S of the matrix B.
Step 4) Write the diagonal entries of S as $s_1, s_2, \ldots, s_n$.

Then the Mad Vet Group is the group

$$\mathbf{Z}_{s1} \times \mathbf{Z}_{s2} \times \ldots \times \mathbf{Z}_{sn}.$$

Possible Activity: Now reconsider each of the four Mad Vet Puzzles, and see if you and your team can use the Mad Vet Group Theorem to determine (or re-determine) which Puzzles give groups. For those puzzles that do give groups, use the *What group is it?* Theorem to determine exactly which group it is.

Mad Veterinarian Puzzles: More Food for Thought

Food For Thought 4) If you have a specific abelian group in mind, can you build a Mad Vet Puzzle whose grid *is* that group?

Food For Thought 5) Is it possible to have a Mad Vet Puzzle for which the List of All Classes is infinite? Moreover, can such a Puzzle give a group?

Food For Thought 6) So who cares about these Mad Vet Puzzles anyway?? It turns out that there is a very strong connection between these puzzles and some current lines of mathematics research, called *Leavitt path algebras* and *graph C*-algebras*. [Note: This connection was realized during a June 2008 Workshop on Math Teachers' Circles, sponsored by and held at the American Institute of Mathematics in Palo Alto. Thanks A.I.M.!]

Food For Thought 7) More info about these ideas can be found in the article *The graph menagerie: Abstract algebra and the Mad Veterinarian* (by Gene Abrams and Jessica Sklar), Mathematics Magazine **83**(3), 2010, 168-179. (A link to it can be found at www.uccs.edu/gabrams .)

Maybe YOU can come up with some additional Food For Thought questions?