Leavitt path algebras of Cayley graphs arising from cyclic groups

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The monoid $\mathcal{V}(R)$

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Using operation $\oplus$, $\mathcal{V}(R)$ is a conical monoid, with 'distinguished' element $[R]$.

Examples:

1) $R = K$, a field. Then $\mathcal{V}(R) = \mathbb{Z}^+$. Note $[R] \mapsto 1$.

2) $R = \mathrm{M}_2(K)$. Then $\mathcal{V}(R) = \mathbb{Z}^+$. Note $[R] \mapsto 2$.

The monoid $\mathcal{V}(R)$

3) $R = L_K(1, n)$, the Leavitt algebra of order n.

R is generated by $x_1, ..., x_n, y_1, ..., y_n$, with relations

$$y_i x_j = \delta_{i,j} 1_R \quad \text{and} \quad \sum_{i=1}^{n} x_i y_i = 1_R.$$

R has $R \cong R^n$ as left R-modules. In this case

$$\mathcal{V}(R) = \{0, x, 2x, \ldots, (n-1)x\},$$

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Notes:

(1) For any R, $K_0(R)$ is the universal group of $\mathcal{V}(R)$.

(2) If $R \cong R'$ then there is an isomorphism of monoids $\varphi : \mathcal{V}(R) \to \mathcal{V}(R')$ for which $\varphi([R]) = [R']$.

Bergman's Theorem

Theorem

(George Bergman, Trans. A.M.S. 1975) Let K be a field. Let S be a finitely generated conical monoid S with a distinguished element I, and choose a set of relations $\mathcal{R}$ for S. Then there exists a K-algebra $B = B(\mathcal{R})$ for which $\mathcal{V}(B) \cong S$, and for which, under this isomorphism, $[B] \mapsto I$.

The construction is explicit, uses amalgamated products.

Bergman included the algebras $L_K(1, n)$ as examples of these universal algebras. $L_K(1, n)$ is the algebra B corresponding to the monoid with generator x and relation $x = nx$

The monoid M_E

Let E be a directed graph. $E = (E^0, E^1, r, s)$ (Today: E finite)

$$\bullet^{s(e)} \xrightarrow{\ e\ } \bullet^{r(e)}$$

Construct the abelian monoid M_E:

$$\text{generators} \quad \{a_v \mid v \in E^0\}$$

$$\text{relations} \quad a_v = \sum_{r(e)=w} a_w \qquad \text{(for } v \text{ not a sink)}$$

In M_E, define $x = \sum_{v \in E^0} a_v$. Easily, x is distinguished.

In M_E, denote the zero element by z.

The monoid M_E

M_E: $\{a_v \mid v \in E^0\}$; $a_v = \sum_{r(e)=w} a_w$; $x = \sum_{v \in E^0} a_v$.

1) Example: $E = \bullet$ Then $M_E = \mathbb{Z}^+$, and $x = 1$.

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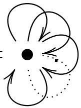
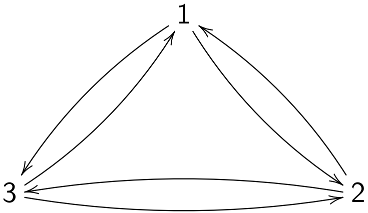
3) Example: $E = R_n =$  $(n \geq 2)$

Then $M_E = \{z, a, 2a, ..., (n-1)a\}$, with $na = a$.

Note: $M_E \setminus \{z\} = \mathbb{Z}_{n-1}$.

4) Example The graph $E = C_3^{-1}$



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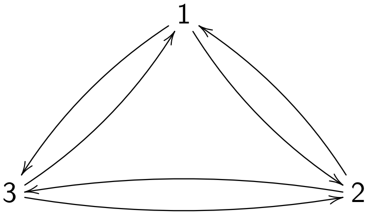

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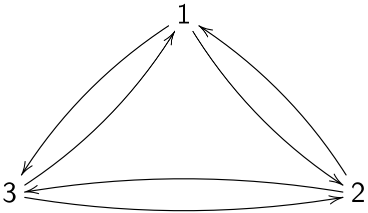

Not hard to show: $M_E = \{z,\ a_1,\ a_2,\ a_3,\ a_1 + a_2 + a_3\}$

Note: $M_E \setminus \{z\} \ \cong\ \mathbb{Z}_2 \times \mathbb{Z}_2$.

Here $x = a_1 + a_2 + a_3 \mapsto (0,0) \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

The Leavitt path algebra of a graph

Let E be a finite graph, and K any field.

We *define* $L_K(E)$, the *Leavitt path algebra of* E *with coefficients in* K, as the universal K-algebra arising from Bergman's theorem, corresponding to the monoid M_E (using the above generators and relations). In particular,

$$\mathcal{V}(L_K(E)) \cong M_E.$$

Under this isomorphism, $[L_K(E)] \mapsto \sum_{v \in E^0} a_v$.

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(Note: This is historically not how things began ...)

The Leavitt path algebra of a graph

Example: $L_K(\bullet) = K$.

Example: $L_K(\bullet \to \bullet) = \mathrm{M}_2(K)$.

Example: $L_K(R_n) = L_K(1, n)$ for $n \geq 2$.

The Leavitt path algebra of a graph

Example: For each $n \in \mathbb{N}$ let C_n denote the "directed cycle" graph with n vertices.

Then it's easy to show that $M_{C_n} = \mathbb{Z}^+$, and $x = n$.

The corresponding Leavitt path algebra is $\mathrm{M}_n(K[x, x^{-1}])$.

purely infinite simple rings

Definition: An idempotent $e \in R$ is *infinite* in case $Re \cong Rf \oplus Rg$ where f, g are idempotents for which $Re \cong Rf$, and $Rg \neq \{0\}$.

Example: $1 \in R = L_K(1, n) = L_K(R_n)$ is infinite, as $R1 = R \cong R^n = R1 \oplus Rg$ for an appropriate idempotent g.

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Proposition: (Pardo, posted online 2011) If $R = L_K(E)$, then R is purely infinite simple if and only if $\mathcal{V}(L_K(E)) \setminus \{[0]\}$ is a group.

purely infinite simple Leavitt path algebras

Theorem: (A-, Aranda Pino, 2006): $L_K(E)$ is purely infinite simple if and only if E has:

1. every vertex in E connects to every cycle in E,
2. every cycle in E has an *exit*, and
3. E contains at least one cycle.

So $L_K(E)$ is purely infinite simple for $E = R_n$ ($n \geq 2$).
Also $L_K(E)$ is purely infinite simple for $E = C_3^{-1}$.

Note $L_K(E)$ is not purely infinite simple for $E = \bullet$, or for $E = \bullet \rightarrow \bullet$, or for any of the C_n graphs.

purely infinite simple Leavitt path algebras

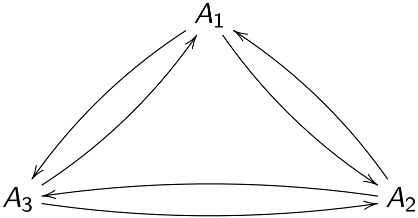
When $L_K(E)$ is purely infinite simple, the K_0 groups are easily described in terms of the adjacency matrix A_E of E. Let $n = |E^0|$. View $I_n - A_E^t$ as a linear transformation $\mathbb{Z}^n \to \mathbb{Z}^n$. Then

$$K_0(L_K(E)) \cong \mathrm{Coker}(I_n - A_E^t).$$

Moreover, $\mathrm{Coker}(I_n - A_E^t)$ can be computed by finding the Smith normal form of $I_n - A_E^t$.

purely infinite simple Leavitt path algebras

$E = C_3^{-1}$



$$I_3 - A_E^t = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \text{ whose Smith normal form is: } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Conclude that $K_0(L_K(E)) \cong \mathrm{Coker}(I_3 - A_E^t) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

And under this isomorphism, $[L_K(E)] \mapsto (0, 0)$.

The Restricted Algebraic KP Theorem

Using some very powerful and deep results from symbolic dynamics, we can show

Theorem: (A- / Louly / Pardo / Smith 2011): Suppose $L_K(E)$ and $L_K(F)$ are purely infinite simple. If

$$K_0(L_K(E)) \cong K_0(L_K(F))$$

via an isomorphism φ for which $\varphi([L_K(E)]) = [L_K(F)]$,

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via an isomorphism φ for which $\varphi([L_K(E)]) = [L_K(F)]$,

$$\text{and} \quad \text{sign}(\det(I - A_E^t)) = \text{sign}(\det(I - A_F^t)),$$

then $L_K(E) \cong L_K(F)$.

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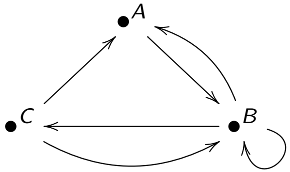
$E =$ 

1. $K_0(L_K(E)) \cong \mathbb{Z}_3$
2. under this isomorphism, $[L_K(E)] \mapsto 1$
3. $\det(I - A_E^t) = -3 < 0$.

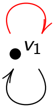
But $L_K(R_4)$ has this same data. So $L_K(E) \cong L_K(R_4) = L_K(1,4)$.

Using the Restricted Algebraic KP Theorem

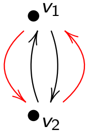
Now apply the Goal to an infinite class of graphs.

The graphs C_n^{-1}:

$C_1^{-1} =$



$C_2^{-1} =$



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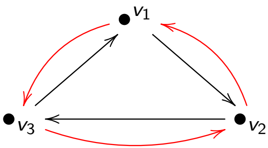

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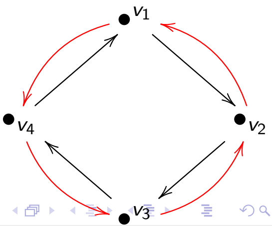

Using The Restricted Algebraic KP Theorem

Let E_n denote C_n^{-1}, with vertices labeled $1, 2, ..., n$.

Note that E_n satisfies the conditions of the Purely Infinite Simple Theorem, so that $M_{E_n} \setminus \{z\}$ is a group (necessarily $K_0(L_K(E_n))$).

In $M_{E_n} \setminus \{z\}$ we have, for each $1 \leq i \leq n$,

$$a_{i+1} = a_i + a_{i+2}$$

(interpret subscripts mod n). So in particular

$$a_{i+1} = a_i + (a_{i+1} + a_{i+3}).$$

So (using that $M_{E_n} \setminus \{z\}$ is a group) we get $0 = a_i + a_{i+3}$, i.e., that

$$a_i = -a_{i+3}$$

in $M_{E_n} \setminus \{z\}$. This gives

$$a_i = a_{i+6}$$

Using The Restricted Algebraic KP Theorem

Using this idea, one can show

Proposition: If $m \equiv n \bmod 6$, then $M_{E_n} \setminus \{z\} \cong M_{E_m} \setminus \{z\}$.

Rephrased: If $m \equiv n \bmod 6$, then $K_0(L_K(E_n)) \cong K_0(L_K(E_m))$.

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Here are those K_0 groups:

$n \bmod 6$	1	2	3	4	5	6
$K_0(L_K(E_n)) \cong$	$\{0\}$	$\mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_3$	$\{0\}$	$\mathbb{Z} \times \mathbb{Z}$

Using The Restricted Algebraic KP Theorem

Since $a_i = a_{i-1} + a_{i+1}$ for all $1 \leq i \leq n$ in $M_{E_n} \setminus \{z\}$, we get that

$$x = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (a_{i-1} + a_{i+1}) = \sum_{i=1}^{n} a_{i-1} + \sum_{i=1}^{n} a_{i+1} = x + x,$$

so that $x = 0$ in the group $M_{E_n} \setminus \{z\}$.

Using The Restricted Algebraic KP Theorem

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Using a formula for the determinant of a circulant matrix (involving roots of unity in $\mathbb{C}$), one can show that

$$\det(I_n - A^t_{E_n}) = \prod_{j=0}^{n-1}(1 - 2\cos\frac{2\pi}{n}j).$$

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Elementary computations then give: $\det(I_n - A_{E_n}^t) \leq 0$ for all n.

Using The Restricted Algebraic KP Theorem

We now have all the ingredients in place to achieve our main result.

Theorem: (A-, Schoonmaker; to appear) Up to isomorphism the collection of Leavitt path algebras $\{L_K(C_n^{-1}) \mid n \in \mathbb{N}\}$ is completely described by the following four pairwise non-isomorphic classes of K-algebras.

1. $L_K(C_n^{-1}) \cong L_K(C_m)$ in case $m \equiv 1$ or $5 \mod 6$ and $n \equiv 1$ or $5 \mod 6$.

   In this case, these algebras are isomorphic to $L_K(1,2)$.

2. $L_K(C_n^{-1}) \cong L_K(C_m)$ in case $m \equiv 2$ or $4 \mod 6$ and $n \equiv 2$ or $4 \mod 6$.

   In this case, these algebras are isomorphic to $\mathrm{M}_3(L_K(1,4))$.

3. $L_K(C_n^{-1}) \cong L_K(C_m)$ in case $m, n \equiv 3 \mod 6$.
4. $L_K(C_n^{-1}) \cong L_K(C_m)$ in case $m, n \equiv 6 \mod 6$.

What's next?

For each $n \in \mathbb{N}$, let C_n be the "basic cycle graph" with n vertices.

For $0 \leq i \leq n-1$, let C_n^j be the graph gotten by taking C_n and adding, at each vertex v_i, an edge from v_i to v_{i+j}.

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Interestingly, we have not seen any sort of cyclic pattern in the K_0 groups of $L_K(C_n^{n-2})$.

Can we drop the determinant hypothesis?

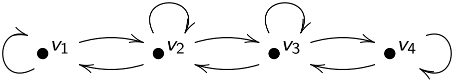
Algebraic KP Question:

Can we drop the hypothesis
on the sign of the determinants
in the Restricted Algebraic KP Theorem?

Can we drop the determinant hypothesis?

Here's the "smallest" example of a situation of interest. Consider the Leavitt path algebras $L(R_2)$ and $L(E_4)$, where

$R_2 =$ 

and $E_4 =$ 

It is not hard to establish that

$$(K_0(L(R_2)), [1_{L(R_2)}]) = (\{0\}, 0) = (K_0(L(E_4)), [1_{L(E_4)}]);$$

$$\det(I - A^t_{R_2}) = -1; \quad \text{and} \quad \det(I - A^t_{E_4}) = 1.$$

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Here's the "smallest" example of a situation of interest. Consider the Leavitt path algebras $L(R_2)$ and $L(E_4)$, where

$$R_2 = \begin{tikzcd} \bullet^{v} \arrow[loop above] \arrow[loop right] \end{tikzcd} \quad \text{and} \quad E_4 = \begin{tikzcd} \bullet^{v_1} \arrow[loop left] \arrow[r, bend left] & \bullet^{v_2} \arrow[loop above] \arrow[l, bend left] \arrow[r, bend left] & \bullet^{v_3} \arrow[loop above] \arrow[l, bend left] \arrow[r, bend left] & \bullet^{v_4} \arrow[loop right] \arrow[l, bend left] \end{tikzcd}$$

It is not hard to establish that

$$(K_0(L(R_2)), [1_{L(R_2)}]) = (\{0\}, 0) = (K_0(L(E_4)), [1_{L(E_4)}]);$$

$$\det(I - A_{R_2}^t) = -1; \quad \text{and} \quad \det(I - A_{E_4}^t) = 1.$$

Question: Is $L_K(R_2) \cong L_K(E_4)$?

Questions?

Thanks to the Simons Foundation