An introduction to Leavitt path algebras, with connections to C*-algebras and noncommutative algebraic geometry

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West Coast Operator Algebra Seminar
Denver University November 1, 2014

Overview

General path algebras

K always denotes a field. *Any* field.

Let E be a directed graph. $E = (E^0, E^1, r, s)$

$$\bullet^{s(e)} \xrightarrow{e} \bullet^{r(e)}$$

The *path algebra KE* is the K-algebra with basis $\{p_i\}$ consisting of the directed paths in E. (View vertices as paths of length 0.)

$$p \cdot q = pq \text{ if } r(p) = s(q), \quad 0 \text{ otherwise.}$$

In particular, $s(e) \cdot e = e = e \cdot r(e)$.

Note: E^0 finite $\Leftrightarrow$ KE is unital; then $1_{KE} = \sum_{v \in E^0} v$.

Building Leavitt path algebras

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$$E \quad = \quad \begin{tikzcd} & \bullet^t & \bullet^u \arrow[dl, "h"'] \\ \bullet^v \arrow[ur, "e"] \arrow[r, "f"'] \arrow[r, bend right, "g"'] & \bullet^w \arrow[loop below, "i"] \arrow[r, "j"'] & \bullet^x \end{tikzcd}$$

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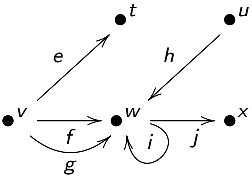

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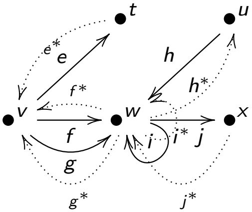

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(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for all $v \in E^0$

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Definition

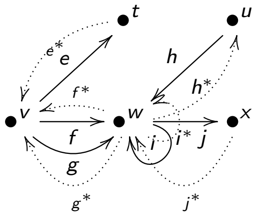
The Leavitt path algebra of E with coefficients in K

$$L_K(E) = K\widehat{E} \ / \ < (CK1), (CK2) >$$

Leavitt path algebras: Examples

Some sample computations in $L_{\mathbb{C}}(E)$ from the Example:

$\widehat{E}\ =$



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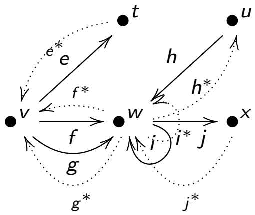

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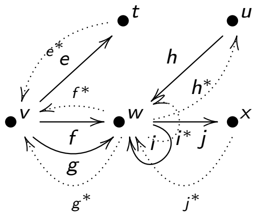

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$$h^*h = w \quad hh^* = u \quad ff^* = \dots \ \text{(no simplification)}$$

$$\text{But } \ (ff^*)^2 = f(f^*f)f^* = f \cdot w \cdot f^* = ff^*.$$

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Then $L_K(E) \cong M_n(K)$.

$$E = \bullet^{v} \circlearrowright x$$

Then $L_K(E) \cong K[x, x^{-1}]$.

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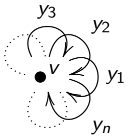
$E = R_n =$ 

Then $L_K(E) \cong L_K(1, n)$, the "Leavitt K-algebra of order n".

(W.G. Leavitt, Transactions. A.M.S. 1962).

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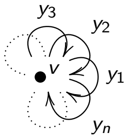
$E = R_n =$ 

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$L_K(1, n)$ is the universal K-algebra R for which ${}_RR \cong {}_RR^n$.

$$L_K(1, n) = \langle x_1, ..., x_n, y_1, ..., y_n \mid x_i y_j = \delta_{i,j} 1_K, \ \sum_{i=1}^{n} y_i x_i = 1_K \rangle$$

Leavitt path algebras

Some general properties of Leavitt path algebras:

1. $L_K(E) = \mathrm{span}_K\{pq^* \mid p, q \text{ paths in } E\}$.
2. $L_K(E) \cong L_K(E)^{op}$.
3. $L_K(E)$ admits a natural $\mathbb{Z}$-grading: $\deg(pq^*) = \ell(p) - \ell(q)$.
4. $J(L_K(E)) = \{0\}$.

Graph C*-algebras

E any directed graph, $\mathcal{H}$ a Hilbert space.

Definition. A **Cuntz-Krieger** E**-family in** $B(\mathcal{H})$ is a collection of mutually orthogonal projections $\{P_v \mid v \in E^0\}$, and partial isometries $\{S_e \mid e \in E^1\}$ with mutually orthogonal ranges, for which:

(CK1) $S_e^* S_e = P_{r(e)}$ for all $e \in E^1$,

(CK2) $\sum_{\{e \mid s(e)=v\}} S_e S_e^* = P_v$ whenever v is a regular vertex, and

(CK3) $S_e S_e^* \leq P_{s(e)}$ for all $e \in E^1$.

The **graph C*-algebra** $C^*(E)$ of E is the universal C*-algebra generated by a Cuntz-Krieger E-family.

Graph C*-algebras

For $\mu = e_1 e_2 \cdots e_n$ a path in E,
let S_μ denote $S_{e_1} S_{e_2} \cdots S_{e_n} \in C^*(E)$.

Proposition: Consider

$$A = \mathrm{span}_{\mathbb{C}}\{P_v, S_\mu S_\nu^* \mid v \in E^0,\ \mu, \nu \text{ paths in } E\} \subseteq C^*(E).$$

Then $L_{\mathbb{C}}(E) \cong A$ as $*$-algebras.

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So it's probably not surprising that there are some close relationships between $L_{\mathbb{C}}(E)$ and $C^*(E)$.

Graph C*-algebras: Examples

Here are the graph C*-algebras which arise from the graphs of the previous examples.

$$E = \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

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Then $C^*(E) \cong M_n(\mathbb{C}) \cong L_{\mathbb{C}}(E)$.

$$E = \bullet^{v} \circlearrowright$$

Then $C^*(E) \cong C(\mathbb{T})$, the continuous functions on the unit circle.

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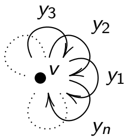
$E = R_n =$ 

Then $C^*(E) \cong \mathcal{O}_n$, the Cuntz algebra of order n.

Brief History

1962: Leavitt defines / investigates $L_K(1, n)$.

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1977: Cuntz defines / investigates $\mathcal{O}_n$.

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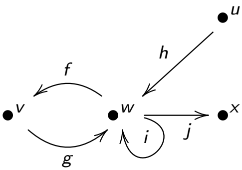
2005: Leavitt path algebras are defined / investigated.

Some graph terminology

Example



1. cycle;

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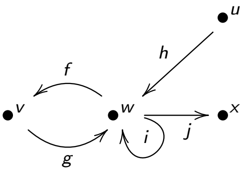

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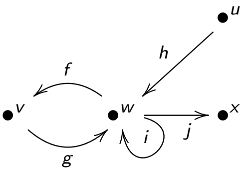

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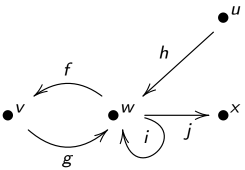

1. cycle; exit for a cycle; Condition (L)
2. downward directed

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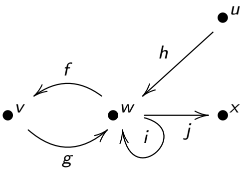

1. cycle; exit for a cycle; Condition (L)
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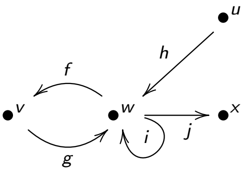

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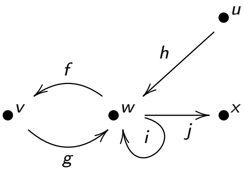

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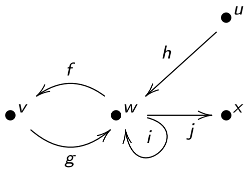

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Standing hypothesis: All graphs are finite (for now) ...

Similarities

We begin by looking at some similarities between the structure of $L_K(E)$ and the structure of $C^*(E)$.

Simplicity

Simplicity:

Algebraic: No nontrivial two-sided ideals.

Analytic: No nontrivial closed two-sided ideals.

Simplicity

Theorem: These are equivalent for any finite graph E:

1. $L_{\mathbb{C}}(E)$ is simple
2. $L_K(E)$ is simple for any field K
3. $C^*(E)$ is (topologically) simple
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Big Question: Can we go 'directly' between

(1) or (2), and (3) or (4) ??

Purely infinite simplicity

Purely infinite simplicity:

Algebraic: R is purely infinite simple in case R is simple and every nonzero right ideal of R contains an infinite idempotent.

Analytic: The simple C*-algebra A is called purely infinite (simple) if for every positive $x \in A$, the subalgebra $\overline{xAx}$ contains an infinite projection.

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(Algebraic) purely infinite simplicity for unital rings is equivalent to: R is not a division ring, and for all nonzero $x \in R$ there exist $\alpha, \beta \in R$ for which $\alpha x \beta = 1$.

(Topological) purely infinite simplicity for unital C*-algebras is equivalent to: $A \neq \mathbb{C}$ and for all nonzero $x \in A$ there exist $\alpha, \beta \in A$ for which $\alpha x \beta = 1$.

Purely infinite simplicity

Theorem: These are equivalent for a finite graph E:

1. $L_{\mathbb{C}}(E)$ is purely infinite simple.
2. $L_K(E)$ is purely infinite simple for any field K.
3. $C^*(E)$ is (topologically) purely infinite simple.
4. $C^*(E)$ is (algebraically) purely infinite simple.
5. E is cofinal, every cycle in E has an exit, and every vertex in E connects to a cycle.

Primitivity

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Algebraic: R is (left) primitive if there exists a simple faithful left R-module.

Analytic: A is (topologically) primitive if there exists a faithful irreducible representation $\pi : A \to B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$.

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Theorem: These are equivalent for a finite graph E:

1. $L_{\mathbb{C}}(E)$ is primitive.
2. $L_K(E)$ is primitive for any field K.
3. $C^*(E)$ is (topologically) primitive.
4. $C^*(E)$ is (algebraically) primitive.
5. E is downward directed and satisfies Conditions (L).

Primitivity

Recently, the primitivity result has been extended to all graphs, both for Leavitt path algebras and graph C*-algebras.

Theorem. (A-, Jason Bell, K.M. Rangaswamy, Trans AMS 2014)
Let E be an arbitrary graph. Then $L_K(E)$ is primitive if and only if

1. E is downward directed, ($\Leftrightarrow L_K(E)$ is prime)
2. E satisfies Condition (L), and
3. there exists a countable set of vertices S in E for which every vertex of E connects to an element of S.
   ("Countable Separation Property")

This result gave a systematic answer to a decades-old question of Kaplansky.

Primitivity

Theorem. (A-, Mark Tomforde, to appear, Münster J. Math) Let E be an arbitrary graph. Then $C^*(E)$ is primitive if and only if the SAME three conditions hold as in the Leavitt path algebra result:

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This result gave a systematic answer to a decades-old question of Dixmier.

Rosetta Stone?

There are many additional examples of this sort of behavior:

For instance:

1. exchange property
2. $\mathcal{V}$-monoid (in particular, $K_0(L_K(E)) \cong K_0(C^*(E))$)
3. possible values of stable rank

But there are no 'direct' proofs for any of them.

Is there some sort of Rosetta Stone ??

Differences

We now look at some differences between the structure of $L_K(E)$ and the structure of $C^*(E)$.

Primeness

Algebraic: R is a prime ring in case $\{0\}$ is a prime ideal of R; that is, in case for any two-sided ideals I, J of R, $I \cdot J = \{0\}$ if and only if $I = \{0\}$ or $J = \{0\}$.

Theorem. K any field, E any graph.
$L_K(E)$ is prime $\Leftrightarrow$ E is downward directed.

Primeness

Analytic: A is a prime C*-algebra in case $\{0\}$ is a prime ideal of A; that is, in case for any closed two-sided ideals I, J of R, $I \cdot J = \{0\}$ if and only if $I = \{0\}$ or $J = \{0\}$.

Theorem: $C^*(E)$ is prime $\Leftrightarrow$ E downward directed **and** satisfies Condition (L).

So for example $L_K(\bullet \circlearrowright)$ is prime, but $C^*(\bullet \circlearrowright)$ is not prime.

Tensor products of graph algebras

Well known: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

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Question: Is the analogous statement true for Leavitt path algebras? i.e., do we have

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Then (early 2011) Answer: No.

Ara & Cortiñas; Dicks; Bell & Bergman

Tensor products of graph algebras

Using Ara / Cortiñas approach, it follows that

$$\otimes^s L_K(1,2) \cong \otimes^t L_K(1,2) \iff s = t.$$

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Using Ara / Cortiñas approach, it follows that

$$\otimes^s L_K(1,2) \cong \otimes^t L_K(1,2) \;\Leftrightarrow\; s = t.$$

Using Dicks' approach, we can show

Proposition. For finite graphs E, F,

$$L_K(E) \otimes L_K(F) \cong L_K(G) \text{ some } G \;\Leftrightarrow\; \text{at least one of } E, F \text{ is acyclic}$$

$L_K(E) \otimes L_K(F) \cong L_K(G) \Leftrightarrow E$ or F acyclic

Sketch of Proof.

1. For any finite E, $L_K(E)$ has $\mathrm{proj.dim.}(L_K(E)) \leq 1$.

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Sketch of Proof.

1. For any finite E, $L_K(E)$ has $\mathrm{proj.dim.}(L_K(E)) \leq 1$.
2. $L_K(E)$ is von Neumann regular $\Leftrightarrow E$ is acyclic.
   (vNr $\Leftrightarrow$ every R-module is flat $\Leftrightarrow \forall a \in R\ \exists x \in R, a = axa$.)

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   (vNr $\Leftrightarrow$ every R-module is flat $\Leftrightarrow \forall a \in R\ \exists x \in R, a = axa$.)
3. So $\mathrm{flatdim.}(L_K(E)) = 1 \Leftrightarrow E$ contains a cycle.

$L_K(E) \otimes L_K(F) \cong L_K(G) \Leftrightarrow E$ or F acyclic

Sketch of Proof.

1. For any finite E, $L_K(E)$ has $\mathrm{proj.dim.}(L_K(E)) \leq 1$.
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4. Old result of Eilenberg et. al.: For K-algebras A, B,
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6. If one of E, F is acyclic (say E), then $L_K(E) \otimes L_K(F)$ is a direct sum of full matrix rings over $L_K(F)$.

Higher K-groups

We mentioned previously that $K_0(L_K(E)) \cong K_0(C^*(E))$. This is true for all E (row-finite).

Notes:

1. $K_0^{\mathrm{top}}(C^*(E)) = K_0^{\mathrm{alg}}(C^*(E))$
2. (for E purely infinite simple) $K_1(C^*(E))$ depends only on A_E, while $K_1(L_K(E))$ depends also on the unit group of K.
3. There is no Bott periodicity for $L_K(E)$.

Similarities

We continue by looking at properties for which

we do not currently know

whether these give similarities or differences between the structure of $L_K(E)$ and the structure of $C^*(E)$.

The isomorphism question

Perhaps the most basic question ...

If $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$, does this imply $C^*(E) \cong C^*(F)$?

And conversely?

(Need to interpret "isomorphism" appropriately.)

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Partial answer: OK in case the graph algebras are simple.
(This uses classification results.)

Answer not known in general.

An algebraic Kirchberg / Phillips Theorem?

Suppose E and F are finite graphs for which $C^*(E)$ and $C^*(F)$ (equivalently, $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$) are simple. Assume that these are also purely infinite.

Note: For E purely infinite simple, $K_0(C^*(E)) \cong K_0(C^*(F))$ implies $K_1(C^*(E)) \cong K_1(C^*(F))$.

A similar result holds for Leavitt path algebras too.

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A similar result holds for Leavitt path algebras too.

A well-known and deep **Theorem**:

$(K_0(C^*(E)), [1_{C^*(E)}]) \cong (K_0(C^*(F)), [1_{C^*(F)}]) \Rightarrow C^*(E) \cong C^*(F)$.

An algebraic Kirchberg / Phillips Theorem?

One approach:

(Step 1) Use results from symbolic dynamics to show that the isomorphism $C^*(E) \cong C^*(F)$ follows in case one also assumes that $\det(I - A_E) = \det(I - A_F)$.

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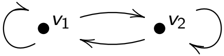
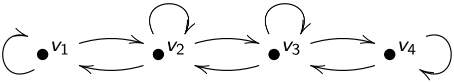
One approach:

(Step 1) Use results from symbolic dynamics to show that the isomorphism $C^*(E) \cong C^*(F)$ follows in case one also assumes that $\det(I - A_E) = \det(I - A_F)$.

(Step 2) Use KK-theory to show that the graph C*-algebras $C^*(E_2)$ and $C^*(E_4)$ are isomorphic:

$E_2 =$ 

and

$E_4 =$ 

(These have identical K-theory, but different determinants.)

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(Step 4) Show such an isomorphism exists.

An algebraic Kirchberg / Phillips Theorem?

A second approach:

Use the Kirchberg / Phillips Theorem.

Remark: The fact that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ is invoked in Phillips' proof ...

An algebraic Kirchberg / Phillips Theorem?

Question: Is there an analogous result for Leavitt path algebras? That is ....

Let K be a field. Suppose E and F are finite graphs for which $L_K(E)$ and $L_K(F)$ are purely infinite simple. Suppose

$$(K_0(L_K(E)), [1_{L_K(E)}]) \ \cong \ (K_0(L_K(F)), [1_{L_K(F)}]).$$

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$$(K_0(L_K(E)), [1_{L_K(E)}]) \ \cong \ (K_0(L_K(F)), [1_{L_K(F)}]).$$

Does this imply that $L_K(E) \cong L_K(F)$?

An algebraic Kirchberg / Phillips Theorem?

For Leavitt path algebras we have:

“Restricted” Algebraic KP Theorem: In this situation, *if we also assume* $\det(I - A_E) = \det(I - A_F)$, then we get $L_K(E) \cong L_K(F)$. (The proof uses the same deep results from symbolic dynamics mentioned above.)

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Is there an explicit isomorphism from $C^*(E_2)$ to $C^*(E_4)$ that we can possibly exploit?

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If it turns out that $L_K(E_2) \cong L_K(E_4)$, it's not clear how one could use this to establish isomorphisms between Leavitt path algebras of different pairs of graphs for which the K-theory matches up but the signs of the determinants do not.

An algebraic Kirchberg / Phillips Theorem?

Algebraic KP Question: Can we drop the determinant hypothesis in the Restricted Algebraic KP Theorem?

Conjecture:

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There are three possibilities: Yes, No, and Sometimes. The answer will be interesting, no matter how things play out.

Connections to noncommutative algebraic geometry

Recently, S. Paul Smith and others have shown that Leavitt path algebras arise naturally in certain algebraic geometry contexts.

Suppose A is a $\mathbb{Z}^+$-graded algebra (i.e., a $\mathbb{Z}$-graded algebra for which $A_n = \{0\}$ for all $n < 0$).

$\mathrm{Gr}(A)$ denotes the category of $\mathbb{Z}$-graded left A-modules (with graded homomorphisms).

$\mathrm{Fdim}(A)$ denotes the full subcategory of $\mathrm{Gr}(A)$ consisting of the graded A-modules which are the sum of their finite dimensional submodules.

Denote by $\mathrm{QGr}(A)$ the quotient category $\mathrm{Gr}(A)/\mathrm{Fdim}(A)$.

Connections to noncommutative algebraic geometry

The category $\mathrm{QGr}(A)$ turns out to be one of the fundamental constructions in noncommutative algebraic geometry.

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Suppose E is a directed graph. Then the path algebra KE is $\mathbb{Z}^+$-graded in the usual way:

$$\deg(v) = 0 \text{ for each vertex } v, \text{and } \deg(e) = 1 \text{ for each edge } e.$$

So we can construct the category $\mathrm{QGr}(KE)$.

Let E^{nss} denote the graph gotten by repeatedly removing all sinks and sources (and their incident edges) from E.

Connections to noncommutative algebraic geometry

Theorem (S.P. Smith, 2012) Let E be a finite graph. Then there is an equivalence of categories

$$\mathrm{QGr}(KE) \sim \mathrm{Gr}(L_K(E^{\mathrm{nss}})).$$

Moreover, since $L_K(E^{\mathrm{nss}})$ is strongly graded, then these categories are also equivalent to $\mathrm{Mod}(L_K(E^{\mathrm{nss}})_0)$, the full category of modules over the zero-component $L_K(E^{\mathrm{nss}})_0$.

So the Leavitt path algebra construction arises naturally in the context of noncommutative algebraic geometry.

Connections to noncommutative algebraic geometry

In general, when the $\mathbb{Z}^+$-graded K-algebra A arises as an appropriate graded deformation of the standard polynomial ring $K[x_0, \ldots, x_n]$, then $\mathrm{QGr}(A)$ shares many similarities with projective n-space $\mathbb{P}^n$; parallels between them have been studied extensively.

However, in general, an algebra of the form KE does not arise in this way; and for these, "it is much harder to see any geometry hiding in $\mathrm{QGr}(KE)$."

In specific situations there are some geometric perspectives available, but the general case is not well understood.

Thank you.

Thanks also to The Simons Foundation.