Constructing classes of prime, non-primitive, von Neumann regular algebras

Gene Abrams

UCCSSignature.png

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If R is not unital, denote by R_1 the (standard) unital K-algebra for which $\dim_K(R_1/R) = 1$. (Dorroh extension)

Prime rings

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Note: Definition of primeness makes sense for nonunital rings.

Lemma: R nonunital. Then R prime $\Leftrightarrow$ R_1 prime.

Primitive rings

Definition: R is *left primitive* if R admits a faithful simple (= irreducible) left R-module;

i.e. if there exists ${}_RM$ simple for which $\mathrm{Ann}_R(M) = \{0\}$.

Examples: any simple K-algebra.

Note: a set of enough idempotents can be used to build irreducibles.

NON-Examples: many, e.g. $K[x, x^{-1}]$.

Note: Definition of primitivity makes sense for non-unital rings.

If R is prime, then R is primitive $\Leftrightarrow$ R_1 is primitive.

Well-known (and easy) **Proposition**: Every primitive ring is prime.

Leavitt path algebras

Let $E = (E^0, E^1, r, s)$ be any directed graph, and K any field.

$$\bullet^{s(e)} \xrightarrow{\ e\ } \bullet^{r(e)}$$

Construct the *double graph* (or *extended graph*) $\widehat{E}$, and then the path algebra $K\widehat{E}$.

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(CK1) $e^*e = r(e)$; $f^*e = 0$ for $f \neq e$ in E^1; and

(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for all $v \in E^0$

(just at those vertices v which are *regular*: $0 < |s^{-1}(v)| < \infty$)

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Then the *Leavitt path algebra of E with coefficients in K* is:

$$L_K(E) = K\widehat{E} \ / \ < (CK1), (CK2) >$$

Leavitt path algebras: Examples

Example 1.

$$E = \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

Then $L_K(E) \cong \mathrm{M}_n(K)$.

Example 2.

$$E = \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \longrightarrow \cdots$$

Then $L_K(E) \cong \mathrm{FM}_{\mathbb{N}}(K)$.

Example 3.

$$E = \bullet^{v_1} \xrightarrow{(\mathbb{N})} \bullet^{v_2}$$

Then $L_K(E) \cong \mathrm{FM}_{\mathbb{N}}(K)_1$.

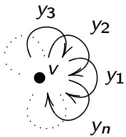
Leavitt path algebras: Examples

Example 4.

$$E = R_1 =$$ 

Then $L_K(E) \cong K[x, x^{-1}]$.

Example 5.

$$E = R_n =$$ 

Then $L_K(E) \cong L_K(1, n)$, the *Leavitt algebra of type* $(1, n)$.

Leavitt path algebras: basic properties

1. $L_K(E)$ has enough idempotents. $L_K(E)$ is unital if and only if E^0 is finite; in this case $1_{L_K(E)} = \sum_{v \in E^0} v$.

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2. Every element of $L_K(E)$ can be expressed as $\sum_{i=1}^{n} k_i \alpha_i \beta_i^*$ where $k_i \in K$ and α_i, β_i are paths for which $r(\alpha_i) = r(\beta_i)$. (This is not generally a basis.)

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3. An *exit* e for a cycle $c = e_1 e_2 \cdots e_n$ based at v is an edge for which $s(e) = s(e_i)$ for some $1 \leq i \leq n$, but $e \neq e_i$.

If every cycle in E has an exit (*Condition (L)*), then every nonzero ideal of $L_K(E)$ contains a vertex.

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If every cycle in E has an exit (*Condition (L)*), then every nonzero ideal of $L_K(E)$ contains a vertex.

4. If c is a cycle based at v for which c has no exit, then $vL_K(E)v \cong K[x, x^{-1}]$.

Prime Leavitt path algebras

Notation: $u \geq v$ means either $u = v$ or there exists a path p in E for which $s(p) = u, r(p) = v$. *u connects to v.*

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Lemma. If I is a two-sided ideal of $L_K(E)$, and $u \in E^0$ has $u \in I$, and $u \geq v$, then $v \in I$.

Theorem. (Aranda Pino, Pardo, Siles Molina 2009) E arbitrary. Then

$$L_K(E) \text{ is prime} \quad \Longleftrightarrow$$

for each pair $v, w \in E^0$ there exists $u \in E^0$ with $v \geq u$ and $w \geq u$.

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Downward Directed or *Condition (MT3)*

Idea of Proof.

$(\Rightarrow)$ $\langle v \rangle \langle w \rangle \neq \{0\}$.

$(\Leftarrow)$ Use previous Lemma.

Definition. Let E be any directed graph. E has the *Countable Separation Property* (CSP) if there exists a countable set of vertices S in E for which every vertex of E connects to an element of S.

E has the *Countable Separation Property with respect to* S.

The Countable Separation Property

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1. vertices indexed by S, and
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Then E_X does not have CSP.

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3. E has the Countable Separation Property.

$L_K(E)$ primitive $\Leftrightarrow$ E has (MT3), (L), and CSP

Strategy of Proof:

1. (Easy) A unital ring R is left primitive if and only if there is a left ideal $N \neq R$ of R such that for every nonzero two-sided ideal I of R, $N + I = R$.

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3. Show that CSP allows us to build a left ideal in $L_K(E)_1$ with the desired properties.

4. Then show that the lack of the CSP implies that no such left ideal can exist in $L_K(E)_1$.

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($\Leftarrow$). Suppose E downward directed, E has Condition (L), and E has CSP.

Suffices to establish primitivity of $L_K(E)_1$. Let T denote a set of vertices w/resp. to which E has CSP.

T is countable: label the elements $T = \{v_1, v_2, ...\}$.

Using downward directedness of E, inductively define a sequence $\lambda_1, \lambda_2, ...$ of paths in E for which, for each $i \in \mathbb{N}$,

1. λ_i is an initial subpath of λ_j whenever $i \leq j$, and
2. $v_i \geq r(\lambda_i)$.

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Note: Each $\lambda_i\lambda_i^*$ is idempotent in $L_K(E)_1$.

And since λ_i is an initial subpath of λ_t for all $i \leq t$, we get that

$$(1-\lambda_i\lambda_i^*)(1-\lambda_t\lambda_t^*) = 1-\lambda_i\lambda_i^* \quad \text{for } i \leq t.$$

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Define $N = \sum_{i=1}^{\infty} L_K(E)_1(1 - \lambda_i\lambda_i^*)$.

Easily $N \neq L_K(E)_1$.

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Claim: Every nonzero two-sided ideal I of $L_K(E)_1$ contains some $\lambda_n \lambda_n^*$.

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Idea: E is downward directed, so $L_K(E)$, and therefore $L_K(E)_1$, is prime. Since $L_K(E)$ embeds in $L_K(E)_1$ as a two-sided ideal, we get $I \cap L_K(E)$ is a nonzero two-sided ideal of $L_K(E)$. So Condition (L) gives that I contains some vertex w.

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Then $w \geq v_n$ for some n by CSP. But $v_n \geq r(\lambda_n)$ by construction, so $w \geq r(\lambda_n)$. So $w \in I$ gives $r(\lambda_n) \in I$, so

$$\lambda_n\lambda_n^* = \lambda_n \cdot r(\lambda_n) \cdot \lambda_n^* \in I.$$

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$$\lambda_n\lambda_n^* = \lambda_n \cdot r(\lambda_n) \cdot \lambda_n^* \in I.$$

Now we're done. Show $N + I = L_K(E)_1$ for every nonzero two-sided ideal I of $L_K(E)_1$. But $1 - \lambda_n\lambda_n^* \in N$ (all $n \in \mathbb{N}$) and $\lambda_n\lambda_n^* \in I$ (some $n \in \mathbb{N}$) gives $1 \in N + I$.

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

For the converse:

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

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For the converse:

1) If E is not downward directed then $L_K(E)$ not prime, so that $L_K(E)$ not primitive.

2) General ring theory result:

If R is primitive and $0 \neq f = f^2 \in R$ then fRf is primitive.

So if E fails to have Condition (L), then E contains a cycle c (based at v) without exit, so that $vL_K(E)v \cong K[x, x^{-1}]$, which is not primitive, and thus $L_K(E)$ is not primitive.

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3) (The hard part.) Show if E does not have CSP then $L_K(E)$ is not primitive.

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Use this easy Lemma:

Let N be a left ideal of a unital ring A. If there exist $x, y \in A$ with

$$1 + x \in N, \ 1 + y \in N, \ \text{and } xy = 0,$$

then $N = A$.

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

We show that if E does not have CSP, then there does NOT exist a left ideal $N \neq L_K(E)_1$ for which $N + I = L_K(E)_1$ for all two-sided ideals I of $L_K(E)_1$.

To do this: assume N is such an ideal, show $N = L_K(E)_1$.

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To do this: assume N is such an ideal, show $N = L_K(E)_1$.

Strategy: If N has this property, then for each $v \in E^0$ we have $N + \langle v \rangle = L_K(E)_1$.

This gives a set $\{x_v \mid v \in E^0\} \subseteq L_K(E)_1$ for which $x_v \in \langle v \rangle$, and $1 + x_v \in N$ for all $v \in E^0$.

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

Now show that the lack of CSP in E^0 forces the existence of a pair of vertices v, w for which $x_v \cdot x_w = 0$. (This is the technical part.)

Then use the Lemma.

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

Key pieces of the technical part:

1. Every element ℓ of $L_K(E)$ can be written as $\sum_{i=1}^{n} k_i \alpha_i \beta_i^*$ for some $n = n(\ell)$, and paths α_i, β_i. In particular, we can cover all elements of $L_K(E)$ by specifying n and lengths of paths. This is a countable covering of $L_K(E)$. (Not a partition.)

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2. Collect up the x_v according to this covering. Since E does not have CSP, then some specific subset in the cover does not have CSP.

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2. Collect up the x_v according to this covering. Since E does not have CSP, then some specific subset in the cover does not have CSP.
3. Show that, in this specific subset Z, there exists $v \in Z$ for which the set
$$\{w \in Z \mid x_v x_w = 0\}$$
does not have CSP. In particular, this set is nonempty. Pick such v and w. Then we are done by the Lemma. □

von Neumann regular rings

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$$\forall a \in R \ \exists \ \ x \in R \ \text{ with } a = axa.$$

R is not required to be unital.

R is regular $\Leftrightarrow$ R_1 is regular.

Kaplansky's Question

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I. Kaplansky, *Algebraic and analytic aspects of operator algebras*, AMS, 1970.

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Is every regular prime algebra primitive?

Answered in the negative (Domanov, 1977), a group-algebra example. (Clever, but very ad hoc.)

Kaplansky's Question

Theorem. (A-, K.M. Rangaswamy 2010)

$$L_K(E) \text{ is von Neumann regular} \Leftrightarrow E \text{ is acyclic.}$$

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Idea of Proof: ($\Leftarrow$) If E contains a cycle c based at v, can show that $a = v + c$ has no regular inverse.

($\Rightarrow$) Show that if E is acyclic then every element of $L_K(E)$ can be trapped in a subring of $L_K(E)$ which is isomorphic to a finite direct sum of finite matrix rings.

Application to Kaplansky's question

It's not hard to find acyclic graphs E for which $L_K(E)$ is prime but for which C.S.P. fails.

Example (described previously): X uncountable, S the set of finite subsets of X. Define the graph E_X:

- vertices indexed by S, and

- edges induced by proper subset relationship.

Then for the graph E_X,

1. $L_K(E_X)$ is regular (E is acyclic)
2. $L_K(E_X)$ is prime (E is downward directed)
3. $L_K(E_X)$ is not primitive (E does not have CSP).

Application to Kaplansky's question

By using uncountable sets of different cardinalities, we get an infinite class of algebras which answer Kaplansky's question in the negative.

Theorem: For any field K, there exists an infinite class of K-algebras (of the form $L_K(E_X)$) which are von Neumann regular and prime, but not primitive.

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Remark: These examples are also *Cohn path algebras*.

Application to Kaplansky's question

For these graphs E, embedding $L_K(E)$ in $L_K(E)_1$ in the usual way gives unital, prime, non-primitive, von Neumann regular algebras. So we get

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Theorem: For any field K, there exists an infinite class of unital K-algebras (of the form $L_K(E_X)_1$) which are prime, non-primitive, and von Neumann regular.

Remark: The algebras $L_K(E_X)_1$ are never Leavitt path algebras.

Application to Kaplansky's question

Note: There are additional classes of graphs E which are

- acyclic,

- and downward directed,

- but don't have C.S.P.

From these we get additional examples of (both unital and nonunital) von Neumann regular, prime, non-primitive algebras.

Primitive graph C*-algebras

An intriguing connection:

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Theorem. (A-, Mark Tomforde, submitted)

Let E be any graph. Then $C^*(E)$ is primitive $\iff$

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$\iff L_K(E)$ is primitive for every field K.

Primitive graph C*-algebras

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Theorem. (A-, Mark Tomforde, submitted)

Let E be any graph. Then $C^*(E)$ is primitive $\Longleftrightarrow$

1. E is downward directed,
2. E satisfies Condition (L), and
3. E satisfies the Countable Separation Property.

$\Longleftrightarrow$ $L_K(E)$ is primitive for every field K.

This theorem yields an infinite class of examples of prime, nonprimitive C*-algebras.

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Proofs of the sufficiency direction for $L_{\mathbb{C}}(E)$ and $C^*(E)$ results are dramatically different.

Questions?