Constructing classes of prime, non-primitive, von Neumann regular algebras

Gene Abrams
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Assume $R$ at least has enough idempotents: $RR = \bigoplus_{i \in I} Re_i$.

If $R$ is not unital, denote by $R_1$ the (standard) unital $K$-algebra for which $\dim_K (R_1 / R) = 1$. (Dorroh extension)
Prime rings

Definition: $R$ is prime if the product of any two nonzero two-sided ideals of $R$ is nonzero.
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Note: Definition of primeness makes sense for nonunital rings.

Lemma: \( R \) nonunital. Then \( R \) prime \( \iff \) \( R_1 \) prime.
Primitive rings

Definition: $R$ is left primitive if $R$ admits a faithful simple (i.e. irreducible) left $R$-module; i.e. if there exists $R M$ simple for which $\text{Ann}_R(M) = \{0\}$.

Examples: any simple $K$-algebra.
Note: a set of enough idempotents can be used to build irreducibles.

NON-Examples: many, e.g. $K[x, x^{-1}]$.

Note: Definition of primitivity makes sense for non-unital rings.
If $R$ is prime, then $R$ is primitive $\iff R_1$ is primitive.

Well-known (and easy) **Proposition**: Every primitive ring is prime.
Leavitt path algebras

Let $E = (E^0, E^1, r, s)$ be any directed graph, and $K$ any field.

$$s(e) \xrightarrow{e} r(e)$$

Construct the *double graph* (or *extended graph*) $\hat{E}$, and then the path algebra $K\hat{E}$. 
Leavitt path algebras

Let $E = (E^0, E^1, r, s)$ be any directed graph, and $K$ any field.

![Diagram](bullet_s(e) \xrightarrow{e} \bullet r(e))

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Leavitt path algebras

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\[ \bullet s(e) \xrightarrow{e} \bullet r(e) \]

Construct the double graph (or extended graph) $\hat{E}$, and then the path algebra $K\hat{E}$. Impose these relations in $K\hat{E}$:

(CK1) \hspace{1cm} e^* e = r(e); \hspace{1cm} f^* e = 0 \text{ for } f \neq e \text{ in } E^1; \text{ and }

(CK2) \hspace{1cm} v = \sum_{\{e \in E^1 \mid s(e) = v\}} ee^* \text{ for all } v \in E^0

(just at those vertices $v$ which are regular: $0 < |s^{-1}(v)| < \infty$)
Leavitt path algebras

Let $E = (E^0, E^1, r, s)$ be any directed graph, and $K$ any field.

\[
\bullet s(e) \xrightarrow{e} \bullet r(e)
\]

Construct the double graph (or extended graph) $\tilde{E}$, and then the path algebra $K\tilde{E}$. Impose these relations in $K\tilde{E}$:

(CK1) $e^*e = r(e)$; $f^*e = 0$ for $f \neq e$ in $E^1$; and

(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for all $v \in E^0$

(just at those vertices $v$ which are regular: $0 < |s^{-1}(v)| < \infty$)

Then the Leavitt path algebra of $E$ with coefficients in $K$ is:

\[
L_K(E) = K\tilde{E} / < (CK1), (CK2)>
\]
Leavitt path algebras: Examples

Example 1.

\[ E = \bullet v_1 \xrightarrow{e_1} \bullet v_2 \xrightarrow{e_2} \bullet v_3 \xrightarrow{\ldots} \bullet v_{n-1} \xrightarrow{e_{n-1}} \bullet v_n \]

Then \( L_K(E) \cong M_n(K) \).

Example 2.

\[ E = \bullet v_1 \xrightarrow{e_1} \bullet v_2 \xrightarrow{e_2} \bullet v_3 \xrightarrow{\ldots} \]

Then \( L_K(E) \cong F_{M\mathbb{N}}(K) \).

Example 3.

\[ E = \bullet v_1 \xrightarrow{(\mathbb{N})} \bullet v_2 \]

Then \( L_K(E) \cong F_{M\mathbb{N}}(K)_1 \).
Example 4.

\[ E = R_1 \overset{v}{\rightarrow} x \]

Then \( L_K(E) \cong K[x, x^{-1}] \).

Example 5.

\[ E = R_n \overset{v}{\rightarrow} y_1 \rightarrow y_2 \rightarrow \cdots \rightarrow y_n \]

Then \( L_K(E) \cong L_K(1, n) \), the Leavitt algebra of type \((1, n)\).
1. \( L_K(E) \) has enough idempotents. \( L_K(E) \) is unital if and only if \( E^0 \) is finite; in this case \( 1_{L_K(E)} = \sum_{v \in E^0} v \).
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2. Every element of $L_K(E)$ can be expressed as $\sum_{i=1}^n k_i \alpha_i \beta_i^*$ where $k_i \in K$ and $\alpha_i, \beta_i$ are paths for which $r(\alpha_i) = r(\beta_i)$. (This is not generally a basis.)
Leavitt path algebras: basic properties

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3. An exit $e$ for a cycle $c = e_1 e_2 \cdots e_n$ based at $v$ is an edge for which $s(e) = s(e_i)$ for some $1 \leq i \leq n$, but $e \neq e_i$.

   If every cycle in $E$ has an exit (Condition (L)), then every nonzero ideal of $L_K(E)$ contains a vertex.

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Leavitt path algebras: basic properties

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   If every cycle in $E$ has an exit (Condition (L)), then every nonzero ideal of $L_K(E)$ contains a vertex.

4. If $c$ is a cycle based at $v$ for which $c$ has no exit, then $vL_K(E)v \cong K[x, x^{-1}]$. 
Prime Leavitt path algebras

Notation: $u \geq v$ means either $u = v$ or there exists a path $p$ in $E$ for which $s(p) = u$, $r(p) = v$. $u$ connects to $v$.

Lemma. If $I$ is a two-sided ideal of $L_{K}(E)$, and $u \in E_{0}$ has $u \in I$, and $u \geq v$, then $v \in I$. 
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Prime Leavitt path algebras

**Theorem.** (Aranda Pino, Pardo, Siles Molina 2009) $E$ arbitrary. Then

$L_K(E)$ is prime $\iff$ for each pair $v, w \in E^0$ there exists $u \in E^0$ with $v \geq u$ and $w \geq u$. 

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**Downward Directed or Condition (MT3)**
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*Downward Directed or Condition (MT3)*

**Idea of Proof.**

$(\Rightarrow) \langle v \rangle \langle w \rangle \neq \{0\}$.

$(\Leftarrow)$ Use previous Lemma.
The Countable Separation Property

**Definition.** Let $E$ be any directed graph. $E$ has the *Countable Separation Property* (CSP) if there exists a countable set of vertices $S$ in $E$ for which every vertex of $E$ connects to an element of $S$.

$E$ has the *Countable Separation Property with respect to $S$.*
The Countable Separation Property

So trivially, if $E^0$ is countable, then $E$ has CSP.
The Countable Separation Property

So trivially, if $E^0$ is countable, then $E$ has CSP.

**Example:** $X$ uncountable, $S$ the set of finite subsets of $X$. Define the graph $E_X$:

1. vertices indexed by $S$, and
2. edges induced by proper subset relationship.

Then $E_X$ does not have CSP.
Primitive Leavitt path algebras

Note: Since $L_K(E) \cong L_K(E)^{op}$, left primitivity and right primitivity coincide for Leavitt path algebras.
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**Theorem.** (A-, Bell, Rangaswamy, to appear, *Trans. A.M.S.*)

$L_K(E)$ is primitive $\iff$

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**Theorem.** (A-, Bell, Rangaswamy, to appear, *Trans. A.M.S.*)

$L_K(E)$ is primitive $\iff$

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Primitive Leavitt path algebras

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**Theorem.** (A-, Bell, Rangaswamy, to appear, *Trans. A.M.S.*)

$L_K(E)$ is primitive $\iff$

1. $L_K(E)$ is prime (i.e., $E$ is downward directed),
2. every cycle in $E$ has an exit (i.e., $E$ has Condition (L)),
3. $E$ has the Countable Separation Property.
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$L_K(E)$ primitive $\iff$ $E$ has (MT3), (L), and CSP

Strategy of Proof:

1. (Easy) A unital ring $R$ is left primitive if and only if there is a left ideal $N \neq R$ of $R$ such that for every nonzero two-sided ideal $I$ of $R$, $N + I = R$. 

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2. Embed a prime $L_K(E)$ in a unital algebra $L_K(E)_1$ in the usual way; primitivity is preserved.
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L_K(E) primitive ⇔ E has (MT3), (L), and CSP

Strategy of Proof:

1. (Easy) A unital ring R is left primitive if and only if there is a left ideal N ≠ R of R such that for every nonzero two-sided ideal I of R, N + I = R.

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3. Show that CSP allows us to build a left ideal in L_K(E)_1 with the desired properties.

4. Then show that the lack of the CSP implies that no such left ideal can exist in L_K(E)_1.
$L_K(E)$ primitive $\iff$ $E$ has (MT3), (L), and CSP

($\iff$). Suppose $E$ downward directed, $E$ has Condition (L), and $E$ has CSP.

Suffices to establish primitivity of $L_K(E)_1$. Let $T$ denote a set of vertices w/resp. to which $E$ has CSP.

$T$ is countable: label the elements $T = \{v_1, v_2, \ldots\}$.

Using downward directedness of $E$, inductively define a sequence $\lambda_1, \lambda_2, \ldots$ of paths in $E$ for which, for each $i \in \mathbb{N}$,

1. $\lambda_i$ is an initial subpath of $\lambda_j$ whenever $i \leq j$, and
2. $v_i \geq r(\lambda_i)$. 

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Note: Each $\lambda_i\lambda_i^*$ is idempotent in $L_K(E)_1$.

And since $\lambda_i$ is an initial subpath of $\lambda_t$ for all $i \leq t$, we get that

$$(1 - \lambda_i\lambda_i^*)(1 - \lambda_t\lambda_t^*) = 1 - \lambda_i\lambda_i^* \quad \text{for } i \leq t.$$
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\]

Define \( N = \sum_{i=1}^{\infty} L_K(E)_1(1 - \lambda_i \lambda_i^*) \).
Easily \( N \neq L_K(E)_1 \).
Claim: Every nonzero two-sided ideal $I$ of $L_K(E)_1$ contains some $\lambda_n\lambda^*_n$. 

Idea: $E$ is downward directed, so $L_K(E)$, and therefore $L_K(E)_1$, is prime. Since $L_K(E)$ embeds in $L_K(E)_1$ as a two-sided ideal, we get $I \cap L_K(E)$ is a nonzero two-sided ideal of $L_K(E)_1$. So Condition (L) gives that $I$ contains some vertex $w$. Then $w \geq v_n$ for some $n$ by CSP. But $v_n \geq r(\lambda_n)$ by construction, so $w \geq r(\lambda_n)$. So $w \in I$ gives $r(\lambda_n) \in I$, so $\lambda_n\lambda^*_n = \lambda_n \cdot r(\lambda_n) \cdot \lambda^*_n \in I$. 

Now we're done. Show $N + I = L_K(E)_1$ for every nonzero two-sided ideal $I$ of $L_K(E)_1$. But $1 - \lambda_n\lambda^*_n \in N$ (all $n \in N$) and $\lambda_n\lambda^*_n \in I$ (some $n \in N$) gives $1 \in N + I$. □
Claim: Every nonzero two-sided ideal $I$ of $L_K(E)_1$ contains some $\lambda_n\lambda^*_n$.

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$L_K(E)$ primitive $\iff E$ has (MT3), (L), and CSP

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$$\lambda_n \lambda_n^* = \lambda_n \cdot r(\lambda_n) \cdot \lambda_n^* \in I.$$
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Then $w \geq v_n$ for some $n$ by CSP. But $v_n \geq r(\lambda_n)$ by construction, so $w \geq r(\lambda_n)$. So $w \in I$ gives $r(\lambda_n) \in I$, so

$$\lambda_n\lambda^*_n = \lambda_n \cdot r(\lambda_n) \cdot \lambda^*_n \in I.$$ 

Now we’re done. Show $N + I = L_K(E)_1$ for every nonzero two-sided ideal $I$ of $L_K(E)_1$. But $1 - \lambda_n\lambda^*_n \in N$ (all $n \in \mathbb{N}$) and $\lambda_n\lambda^*_n \in I$ (some $n \in \mathbb{N}$) gives $1 \in N + I$.  

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$L_K(E)$ primitive $\Rightarrow E$ has (MT3), (L), and CSP

For the converse:

1) If $E$ is not downward directed then $\text{L}_K(E)$ is not prime, so that $\text{L}_K(E)$ is not primitive.

2) General ring theory result: If $R$ is primitive and $0 \neq f = f^2 \in R$ then $fRf$ is primitive. So if $E$ fails to have Condition (L), then $E$ contains a cycle $c$ (based at $v$) without exit, so that $vL_K(E)v \sim K[x, x-1]$, which is not primitive, and thus $L_K(E)$ is not primitive.
\( L_K(E) \) primitive \( \implies \) \( E \) has (MT3), (L), and CSP

For the converse:

1) If \( E \) is not downward directed then \( L_K(E) \) not prime, so that \( L_K(E) \) not primitive.
$L_K(E)$ primitive $\Rightarrow$ $E$ has (MT3), (L), and CSP

For the converse:

1) If $E$ is not downward directed then $L_K(E)$ not prime, so that $L_K(E)$ not primitive.

2) General ring theory result:

If $R$ is primitive and $0 \neq f = f^2 \in R$ then $fRf$ is primitive.

So if $E$ fails to have Condition (L), then $E$ contains a cycle $c$ (based at $v$) without exit, so that $vL_K(E)v \cong K[x, x^{-1}]$, which is not primitive, and thus $L_K(E)$ is not primitive.
$L_K(E)$ primitive $\Rightarrow$ $E$ has (MT3), (L), and CSP

3) (The hard part.) Show if $E$ does not have CSP then $L_K(E)$ is not primitive.
$L_K(E)$ primitive $\Rightarrow$ $E$ has (MT3), (L), and CSP

3) (The hard part.) Show if $E$ does not have CSP then $L_K(E)$ is not primitive.

Use this easy Lemma:
Let $N$ be a left ideal of a unital ring $A$. If there exist $x, y \in A$ with

$$1 + x \in N, \quad 1 + y \in N, \quad \text{and} \quad xy = 0,$$

then $N = A$. 
\( L_K(E) \) primitive \( \Rightarrow \) \( E \) has (MT3), (L), and CSP

We show that if \( E \) does not have CSP, then there does NOT exist a left ideal \( N \neq L_K(E)_1 \) for which \( N + I = L_K(E)_1 \) for all two-sided ideals \( I \) of \( L_K(E)_1 \).

To do this: assume \( N \) is such an ideal, show \( N = L_K(E)_1 \).
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We show that if \( E \) does not have CSP, then there does NOT exist a left ideal \( N \neq L_K(E)_1 \) for which \( N + I = L_K(E)_1 \) for all two-sided ideals \( I \) of \( L_K(E)_1 \).

To do this: assume \( N \) is such an ideal, show \( N = L_K(E)_1 \).

Strategy: If \( N \) has this property, then for each \( v \in E^0 \) we have \( N + \langle v \rangle = L_K(E)_1 \).

This gives a set \( \{ x_v \mid v \in E^0 \} \subseteq L_K(E)_1 \) for which \( x_v \in \langle v \rangle \), and \( 1 + x_v \in N \) for all \( v \in E^0 \).
$L_K(E)$ primitive $\Rightarrow$ $E$ has (MT3), (L), and CSP

Now show that the lack of CSP in $E^0$ forces the existence of a pair of vertices $v, w$ for which $x_v \cdot x_w = 0$. (This is the technical part.)

Then use the Lemma.
$L_K(E)$ primitive $\Rightarrow$ $E$ has (MT3), (L), and CSP

Key pieces of the technical part:

1. Every element $\ell$ of $L_K(E)$ can be written as $\sum_{i=1}^{n} k_{i} \alpha_{i} \beta_{i}^{*}$ for some $n = n(\ell)$, and paths $\alpha_{i}, \beta_{i}$. In particular, we can cover all elements of $L_K(E)$ by specifying $n$ and lengths of paths. This is a countable covering of $L_K(E)$. (Not a partition.)
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2. Collect up the $x_v$ according to this covering. Since $E$ does not have CSP, then some specific subset in the cover does not have CSP.
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2. Collect up the $x_v$ according to this covering. Since $E$ does not have CSP, then some specific subset in the cover does not have CSP.

3. Show that, in this specific subset $Z$, there exists $v \in Z$ for which the set

$$\{ w \in Z \mid x_v x_w = 0 \}$$

does not have CSP. In particular, this set is nonempty. Pick such $v$ and $w$. Then we are done by the Lemma. \qed
von Neumann regular rings

Definition: $R$ is von Neumann regular (or just regular) in case

$$\forall a \in R \ \exists \ x \in R \text{ with } a = axa.$$
von Neumann regular rings

Definition: $R$ is *von Neumann regular* (or just *regular*) in case

$$\forall a \in R \ \exists \ x \in R \ \text{with} \ a = axa.$$ 

$R$ is not required to be unital.

$R$ is regular $\iff R_1$ is regular.
Kaplansky’s Question

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Is every regular prime algebra primitive?
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Is every regular prime algebra primitive?

Answered in the negative (Domanov, 1977), a group-algebra example. (Clever, but very ad hoc.)
Kaplansky’s Question

**Theorem.** (A-, K.M. Rangaswamy 2010)

\[ L_K(E) \text{ is von Neumann regular } \iff E \text{ is acyclic.} \]
Kaplansky’s Question

**Theorem.** (A-, K.M. Rangaswamy 2010)

$L_K(E)$ is von Neumann regular $\iff$ $E$ is acyclic.

**Idea of Proof:**

$(\Leftarrow)$ If $E$ contains a cycle $c$ based at $v$, can show that $a = v + c$ has no regular inverse.

$(\Rightarrow)$ Show that if $E$ is acyclic then every element of $L_K(E)$ can be trapped in a subring of $L_K(E)$ which is isomorphic to a finite direct sum of finite matrix rings.
Application to Kaplansky’s question

It’s not hard to find acyclic graphs $E$ for which $L_K(E)$ is prime but for which C.S.P. fails.

**Example** (described previously): $X$ uncountable, $S$ the set of finite subsets of $X$. Define the graph $E_X$:
- vertices indexed by $S$, and
- edges induced by proper subset relationship.

Then for the graph $E_X$,

1. $L_K(E_X)$ is regular ($E$ is acyclic)
2. $L_K(E_X)$ is prime ($E$ is downward directed)
3. $L_K(E_X)$ is not primitive ($E$ does not have CSP).
By using uncountable sets of different cardinalities, we get an infinite class of algebras which answer Kaplansky’s question in the negative.

**Theorem:** For any field $K$, there exists an infinite class of $K$-algebras (of the form $L_K(E_X)$) which are von Neumann regular and prime, but not primitive.
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**Theorem:** For any field $K$, there exists an infinite class of $K$-algebras (of the form $L_K(E_X)$) which are von Neumann regular and prime, but not primitive.

**Remark:** These examples are also *Cohn path algebras.*
Application to Kaplansky’s question

For these graphs $E$, embedding $L_K(E)$ in $L_K(E)_1$ in the usual way gives unital, prime, non-primitive, von Neumann regular algebras. So we get

**Theorem:** For any field $K$, there exists an infinite class of unital $K$-algebras (of the form $L_K(E_X)_1$) which are prime, non-primitive, and von Neumann regular.
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Remark: The algebras $L_K(E_X)_1$ are never Leavitt path algebras.
Application to Kaplansky’s question

Note: There are additional classes of graphs $E$ which are
- acyclic,
- and downward directed,
- but don’t have C.S.P.

From these we get additional examples of (both unital and nonunital) von Neumann regular, prime, non-primitive algebras.
An intriguing connection:

Let $E$ be any graph. Then $C^*(E)$ is primitive $\iff$

1. $E$ is downward directed,
2. $E$ satisfies Condition (L), and
3. $E$ satisfies the Countable Separation Property.

$\iff L_K(E)$ is primitive for every field $K$.

This theorem yields an infinite class of examples of prime, nonprimitive $C^*$-algebras.

Proofs of the sufficiency direction for $L_C(E)$ and $C^*(E)$ results are dramatically different.
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Gene Abrams

Constructing classes of prime, non-primitive, von Neumann regular algebras
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Questions?