The graph menagerie: Abstract algebra and The Mad Veterinarian

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Bob's Mad Vet Puzzle Page

http://www.bumblebeagle.org/madvet/index.html

Welcome to Bob's Mad Veterinarian Puzzle Page

In September of 1998, after fiddling with this puzzle format for about a decade, I posted the first Mad Veterinarian puzzle to the rec.puzzles newsgroup:

Bob's Mad Vet Puzzle Page

A mad veterinarian has created three animal transmogrifying machines.
Place a cat in the input bin of the first machine, press the button, and *whirr… bing!* Open the output bins to find two dogs and five mice.
The second machine can convert a dog into three cats and three mice, and the third machine can convert a mouse into a cat and a dog. Each machine can also operate in reverse (e.g. if you've got two dogs and five mice, you can convert them into a cat).

You have one cat.

1. Can you convert it into seven mice?
2. Can you convert it into a pack of dogs, with no mice or cats left over?

Bob's Mad Vet Puzzle Page

Puzzle solvers discovered that it was impossible to convert a single cat into seven mice, nor to a lonesome pack of dogs.

However, they posed and answered followup questions, such as *how many mice can be created from a single cat?* and *what's the smallest number of cats that can be turned into just dogs?*

Bob's Mad Vet Puzzle Page

Below, I've set up several puzzles of this type, and a java applet that lets you solve them. Each applet deals with one set of machines and poses several conversions for you to try to solve.

How To Solve Mad Veterinarian Puzzles

Easy Three Animal Labratory Mar/17/2003

Original Three Animal Labratory Mar/17/2003

Hard Four Animal Labratory Mar/17/2003

Harder Four Animal Labratory Apr/1/2003

Schoolhouse Jelly Beans Apr/2/2003

Mad Vet puzzles and ...

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Leavitt path algebras !!

Mad Vet scenarios

A *Mad Vet scenario* posits a Mad Veterinarian in possession of a finite number of transmogrifying machines, where

1. Each machine transmogrifies a single animal of a given species into a finite nonempty collection of animals from any number of species;
2. Each machine can also operate in reverse; and
3. There is one machine corresponding to each species in the menagerie.

Mat Vet Scenario #1

Scenario #1. Suppose a Mad Veterinarian has three machines with the following properties.

Machine 1 turns one ant into one beaver;
Machine 2 turns one beaver into one ant, one beaver and one cougar;
Machine 3 turns one cougar into one ant and one beaver.

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Let's do some transmogrification !!

Mad Vet graphs

Given any Mad Vet scenario, its corresponding *Mad Vet graph* is the directed graph with

$$V = \{A_1, A_2, \ldots, A_n\},$$

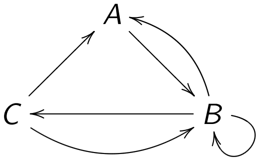
and having, for each A_i, A_j in V, exactly

$$d_{i,j} \text{ edges with initial vertex } A_i \text{ and terminal vertex } A_j,$$

where the machine corresponding to species A_i produces $d_{i,j}$ animals of species A_j.

Mad Vet graphs

Example. Mad Vet scenario #1 has the following Mad Vet graph.



Recall:

Machine 1: Ant $\rightarrow$ Beaver
Machine 2: Beaver $\rightarrow$ Ant, Beaver, and Cougar
Machine 3: Cougar $\rightarrow$ Ant, Beaver

Mad Vet equivalence

Key idea: Let's say there are n different species. Let

$$\mathbb{Z}^+ \text{ denote } \{0, 1, 2, \ldots\}.$$

A *menagerie* is an element of the set

$$S = (\mathbb{Z}^+)^n \setminus \{(0, 0, \ldots, 0)\}.$$

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There is a natural correspondence between menageries and nonempty collections of animals from species $A_1, A_2, \ldots, A_n$.

For instance, in Scenario $\#1$ a collection of two beavers and five cougars would correspond to $(0, 2, 5)$ in S.

Mad Vet equivalence

There is a naturally arising relation $\sim$ on S:

Given $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in S, we write

$$a \sim b$$

if there is a sequence of Mad Vet machines that will transmogrify the collection of animals associated with menagerie a into the collection of animals associated with menagerie b.

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Using the three properties of a Mad Vet scenario, it is straightforward to show that $\sim$ is an equivalence relation on S.

Mad Vet equivalence

We focus on the set

$$W = \{[a] \, : \, a \in S\}$$

of equivalence classes of S under $\sim$.

Example. Suppose that our Mad Vet of Scenario #1 starts with the menagerie $(1,0,0)$.

(Recall: Machine 1: A $\rightarrow$ B Machine 2: B $\rightarrow$ A, B, C Machine 3: C $\rightarrow$ A,B)

Then, for example,

$$(1,0,0) \sim (0,1,0) \sim (1,1,1) \sim (2,2,0) \sim (4,0,0).$$

Rewritten,

$$[(1,0,0)] = [(0,1,0)] = [(1,1,1)] = [(2,2,0)] = [(4,0,0)] \text{ in } W.$$

Mad Vet equivalence

(Recall: Machine 1: A $\longrightarrow$ B Machine 2: B $\longrightarrow$ A, B, C Machine 3: C $\longrightarrow$ A,B)

Claim. W is the 3-element set

$$\{[(1,0,0)],[(2,0,0)],[(3,0,0)]\}.$$

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Claim. W is the 3-element set

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Reason. It's not hard to see that any (a,b,c) is equivalent to one of the menageries $(1,0,0)$, $(2,0,0)$, or $(3,0,0)$.

Mad Vet equivalence

(Recall: Machine 1: A $\longrightarrow$ B Machine 2: B $\longrightarrow$ A, B, C Machine 3: C $\longrightarrow$ A,B)

Why are these classes not equal to each other? Given a menagerie $m = (a, b, c)$, define the sum

$$s_m = a + b + 2c.$$

(Intuitively: s_m is the dollar value of menagerie m, where an Ant costs \$1, a Beaver \$1, and a Couger \$2.)

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Then Machines 1 and 3 leave s_m the same, while Machine 2 increases s_m by 3 (and running Machine 2 in reverse decreases s_m by 3). So any application of any machine to any menagerie leaves the total value of the menagerie *invariant* mod 3. So the three classes are distinct.

Semigroups, monoids, and groups

Reminder / review of notation.

1. *semigroup*: associative operation.

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3. *group*: monoid, for which each element has an inverse.
   e.g. $\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, ...\}$ under addition.

Mad Vet semigroups

Start with a Mad Vet scenario. Define addition on W (the set of equivalence classes of menageries) by setting

$$[x] + [y] = [x + y].$$

Interpret as "unions" of menageries.

This operation is well defined.

"Mad Vet semigroup."

Mad Vet semigroups

(Recall: Machine 1: A $\longrightarrow$ B Machine 2: B $\longrightarrow$ A, B, C Machine 3: C $\longrightarrow$ A,B)

Example.

$$W = \{[(1,0,0)], [(2,0,0)], [(3,0,0)]\}.$$

We get, for instance,

$$[(1,0,0)] + [(1,0,0)] = [(1+1,0,0)] = [(2,0,0)],$$

as we'd expect.

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$$[(1,0,0)] + [(3,0,0)] = [(4,0,0)] = [(1,0,0)].$$

So $[(3,0,0)]$ behaves like an identity element with respect to the element $[(1,0,0)]$ in W.

Mad Vet semigroups

Similarly

$$[(2,0,0)]+[(3,0,0)] = [(2,0,0)], \text{ and } [(3,0,0)]+[(3,0,0)] = [(3,0,0)].$$

Mad Vet semigroups

Similarly

$$[(2,0,0)]+[(3,0,0)] = [(2,0,0)], \text{ and } [(3,0,0)]+[(3,0,0)] = [(3,0,0)].$$

So for this Mad Vet scenario the Mad Vet semigroup W is a monoid with identity $[(3,0,0)]$.

Mad Vet semigroups

Actually, since

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in W, every element in W has an inverse.

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in W, every element in W has an inverse.

So W is in fact a group, necessarily $\mathbb{Z}_3$.

Mad Vet semigroups

Scenario #2. Suppose the same Mad Vet has replaced two of her machines with new machines.

Machine 1 still turns one ant into one beaver;
Machine 2 now turns one beaver into one ant and one cougar;
Machine 3 now turns one cougar into two cougars.

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$$W = \{[(i, 0, 0)] \, : \, i \in \mathbb{N}\} \cup \{[(0, 0, 1)]\}.$$

$[(0, 0, 1)]$ is an identity element for this Mad Vet semigroup.

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So W in this case is a *monoid*.

But W is not a group: e.g., there is no element $[x]$ in W for which

$$[(1, 0, 0)] + [x] = [(0, 0, 1)].$$

Mad Vet semigroups

The Big Question:

Given a Mad Vet scenario, when is the corresponding Mad Vet semigroup actually a group?

More Mad Vet scenarios ...

Mad Vet semigroups

Scenario #3.

M1: A $\longrightarrow$ B,C; M2: B $\longrightarrow$ A,C; M3: C $\longrightarrow$ A,B

Scenario #4.

M1: A $\longrightarrow$ 2A; M2: B $\longrightarrow$ 2B; M3: C $\longrightarrow$ 2C

Scenario #5.

M1: A $\longrightarrow$ B,C; M2: B $\longrightarrow$ A,B; M3: C $\longrightarrow$ A,C

Scenario #6.

M1: A $\longrightarrow$ B; M2: B $\longrightarrow$ C; M3: C $\longrightarrow$ C

Scenario #7.

M1: A $\longrightarrow$ A,B,C; M2: B $\longrightarrow$ A,C; M3: C $\longrightarrow$ A,B

Mad Vet semigroups

Subtle?

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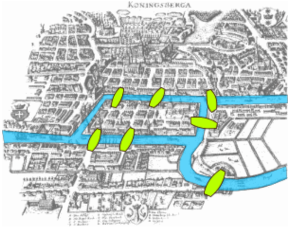
Among Scenarios #3-7, there are Mad Vet semigroups W for which:

1. W is an infinite group;
2. W is a finite noncyclic group;
3. W is a finite nonmonoid;
4. W is a finite cyclic group, not isomorphic to $\mathbb{Z}_3$; and
5. W is an infinite nonmonoid.

Some graph theory: context



Euler's "Bridges of Königsberg" problem.

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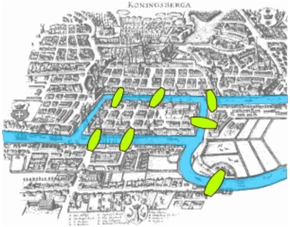

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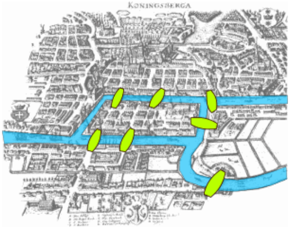

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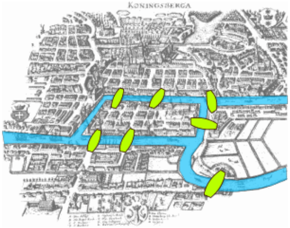

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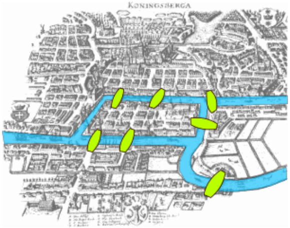

Euler's "Bridges of Königsberg" problem. Idea:

1. translate the problem to a question about graphs;
2. prove a theorem about graphs;
3. use the graph-theoretic result to answer original question.

Graph theory

Some graph theory terminology. (All graphs are *directed*.)

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1. A *sink* in a directed graph.
2. A *path* in a directed graph.
3. If v and w are vertices, v *connects to* w in case either $v = w$ or there is a path from v to w.
4. For a vertex v, a *cycle based at* v is a (nontrivial) path from v to v for which no vertices are repeated.

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6. If $C = f_1 f_2 \cdots f_m$ is a cycle in Γ, then an edge e is called an *exit for* C if the source vertex of e equals the source vertex for f_j (some j), but $e \neq f_j$.

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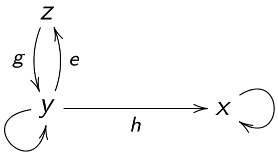
6. If $C = f_1 f_2 \cdots f_m$ is a cycle in Γ, then an edge e is called an *exit for* C if the source vertex of e equals the source vertex for f_j (some j), but $e \neq f_j$. (Intuitively, an exit for C is an edge e, not included in C, which provides a way to step off of C.)

Mad Vet graphs

Example.



The cycle eg based at y has two exits: h and the loop at y.

These same edges are also exits for the cycle ge based at z.
Similarly, the loop at y has exits e and h.

The loop at x has no exit.

This graph is not cofinal (e.g., x does not connect to eg).

Mad Vet Group Test

Theorem: Mad Vet Group Test. *The Mad Vet semigroup W of a Mad Vet scenario is a group if and only if the corresponding Mad Vet graph Γ has the following two properties.*

(1) Γ is cofinal; and

(2) Every cycle in Γ has an exit.

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Proof.: Long, but can be done using only basic graph-theoretic and group-theoretic ideas.

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(Actually, two proofs are known. More about that later.)

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Lemma. A commutative semigroup S is a group if and only if for each pair $x, z \in S$ there exists $y \in S$ for which $x + y = z$.

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www.maa.org → Publications → Periodicals → Mathematics Magazine → June 2010

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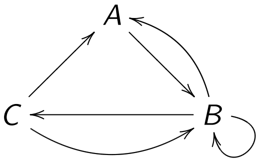
Here's the Mad Vet graph from Scenario #1 again:



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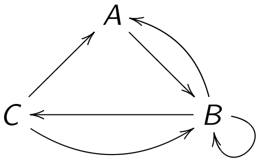

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Cofinal? YES.

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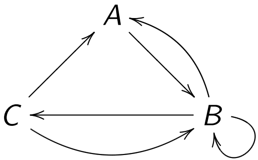

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Cofinal? YES. Every cycle has an exit? YES.

Mad Vet Group Test

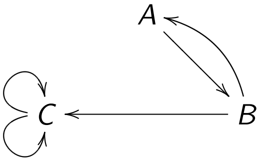
Here's the Mad Vet graph Θ of Scenario #2.

$$
\begin{tikzcd}
 & A \arrow[dr] & \\
C \arrow[loop above] \arrow[loop below] & & B \arrow[ul, bend right] \arrow[ll]
\end{tikzcd}
$$

(Recall: Machine 1: A → B Machine 2: B → A, C Machine 3: C → 2C)

Mad Vet Group Test

Here's the Mad Vet graph Θ of Scenario #2.



(Recall: Machine 1: A → B Machine 2: B → A, C Machine 3: C → 2C)

Cofinal? NO. (*C* does not connect to the cycle *ABA*.)

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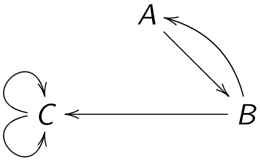

(Recall: Machine 1: A $\longrightarrow$ B Machine 2: B $\longrightarrow$ A, C Machine 3: C $\longrightarrow$ 2C)

Cofinal? NO. (C does not connect to the cycle ABA.)

(But every cycle does have an exit ...)

Mad Vet Group Test

Scenario #8. Let's analyze Mad Vet Bob's puzzle.

(Recall: Machine 1: A $\rightarrow$ 2B,5C Machine 2: B $\rightarrow$ 3A, 3C Machine 3: C $\rightarrow$ A,B)

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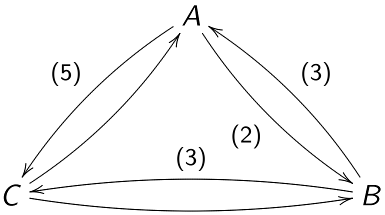

Mad Vet Group Test

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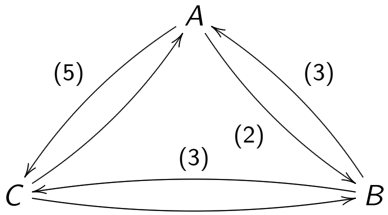

So Mad Vet Bob's semigroup is in fact a group.

Mad Vet Groups

Just exactly what group is it ?????

Mad Vet Groups

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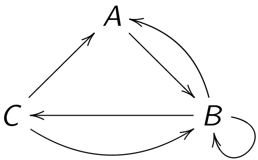
This question has a remarkably nice answer.

Any graph Γ has an associated *incidence matrix* A_Γ: if Γ has n vertices $v_1, v_2, \ldots, v_n$, then A_Γ is the $n \times n$ matrix (d_{ij}), where

$$d_{ij} \ = \# \text{ of edges starting at } v_i \text{ and ending at } v_j.$$

Mad Vet Groups

For example, if Δ is the graph of Scenario $\#1$,



then

$$A_\Delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Mad Vet Groups

Now form the matrix $I_n - A_\Gamma$.

For instance, using the above matrix A_Δ,

$$I_3 - A_\Delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Mad Vet Groups

Then put the (square) matrix $I_n - A_\Gamma$ in *Smith normal form*.

The Smith normal form of an $n \times n$ matrix having integer entries is a diagonal $n \times n$ matrix whose diagonal entries are nonnegative integers

$$\alpha_1, \alpha_2, \ldots, \alpha_q, 0, 0, \ldots, 0$$

such that α_i divides α_{i+1} for each $1 \leq i \leq q-1$.

Mad Vet Groups

The Smith normal form of a matrix A can be obtained by performing on A a combination of these matrix operations: interchanging rows or columns, or adding an integer multiple of a row [column] to another row [column]. The resulting Smith normal form of matrix A is thus of the form PAQ, where P and Q are integer-valued matrices with determinants equal to ± 1.

(Might need to tweak some signs at the end ...)

Mad Vet Groups

Here's an answer to the "just exactly what group is it?" question.

Mad Vet Group Identification Theorem. *Given a Mad Vet scenario with n species whose Mad Vet semigroup W is a group, let* Γ *be its associated Mad Vet graph. Let* $\alpha_1, \alpha_2, \ldots, \alpha_q$ *be the nonzero diagonal entries of the Smith normal form of the matrix* $I_n - A_\Gamma$.

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$$W \cong \mathbb{Z}_{\alpha_1} \oplus \mathbb{Z}_{\alpha_2} \oplus \cdots \oplus \mathbb{Z}_{\alpha_q} \oplus \mathbb{Z}^{n-q}.$$

(Notation: $\mathbb{Z}_1 = \{0\}$.)

Mad Vet Groups

Example. Letting Δ be the Mad Vet graph of Scenario #1, the Smith normal form of the matrix $I_3 - A_\Delta$ is the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Because we already know that Scenario #1's semigroup is a group, the Mad Vet Group Identification Theorem implies that it is isomorphic to $\mathbb{Z}_1 \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_3 \cong \{0\} \oplus \{0\} \oplus \mathbb{Z}_3 \cong \mathbb{Z}_3$, as expected.

Mad Vet Groups

Example. Let Φ be the Mad Vet graph of Scenario #8 (Mad Vet Bob's Puzzle). We've checked that Φ has the right properties, so that the corresponding Mad Vet semigroup is a group. Then I_Φ is the matrix

$$\begin{pmatrix} 0 & 2 & 5 \\ 3 & 0 & 3 \\ 1 & 1 & 0 \end{pmatrix}.$$

The Smith normal form of $I_3 - A_\Phi$ turns out to be matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 34 \end{pmatrix}.$$

So the corresponding group is isomorphic to $\mathbb{Z}_1 \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_{34} \cong \mathbb{Z}_{34}$.

Who cares?

Purely Infinite Simplicity Theorem. *For a finite directed sink-free graph* Γ*, the following are equivalent:*

(1) The Leavitt path algebra $L_{\mathbb{C}}(\Gamma)$ *is purely infinite and simple. (This is a statement about an algebraic structure.)*

(2) The graph C^**-algebra* $C^*(\Gamma)$ *is purely infinite and simple. (This is a statement about an analytic structure.)*

(3) Γ *is cofinal, and every cycle in* Γ *has an exit.*

(4) The graph semigroup W_Γ *is a group.*

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The equivalence of (1) and (2) remains a mystery.

We can get rid of the sink-free hypothesis in the general analysis.

Thanks!

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Acknowledgment The authors express their gratitude to Enrique Pardo for allowing them to use and modify his proof of the Graph Semigroup Group Test for this article; to Amelia Taylor and Brian Hopkins for carefully reading and offering helpful suggestions about the article; and to Ken Ross for his valuable comments, advice, and support. The first author was introduced to Mad Veterinarian puzzles at a June 2008 workshop on Math Teachers' Circles, sponsored by the American Institute of Mathematics, Palo Alto, CA. The author is grateful for AIM's support.

15 minutes of fame?

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Questions?