The graph menagerie:
Abstract algebra and The Mad Veterinarian

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(Joint work with Jessica Sklar, Pacific Lutheran University, Tacoma, WA)

Colorado College Fearless Friday Seminar, October 22, 2010
1. Introduction and brief history

2. Mad Vet scenarios

3. Mad Vet groups

4. Beyond the Mad Vet
Welcome to Bob’s Mad Veterinarian Puzzle Page

In September of 1998, after fiddling with this puzzle format for about a decade, I posted the first Mad Veterinarian puzzle to the rec.puzzles newsgroup:
A mad veterinarian has created three animal transmogrifying machines. Place a cat in the input bin of the first machine, press the button, and *whirr... bing!* Open the output bins to find two dogs and five mice.

The second machine can convert a dog into three cats and three mice, and the third machine can convert a mouse into a cat and a dog. Each machine can also operate in reverse (e.g. if you’ve got two dogs and five mice, you can convert them into a cat).

You have one cat.

1. Can you convert it into seven mice?
2. Can you convert it into a pack of dogs, with no mice or cats left over?
Puzzle solvers discovered that it was impossible to convert a single cat into seven mice, nor to a lonesome pack of dogs.

However, they posed and answered followup questions, such as *how many mice can be created from a single cat?* and *what’s the smallest number of cats that can be turned into just dogs?*
Below, I’ve set up several puzzles of this type, and a java applet that lets you solve them. Each applet deals with one set of machines and poses several conversions for you to try to solve.

How To Solve Mad Veterinarian Puzzles

Easy Three Animal Labratory Mar/17/2003
Original Three Animal Labratory Mar/17/2003
Hard Four Animal Labratory Mar/17/2003
Harder Four Animal Labratory Apr/1/2003
Schoolhouse Jelly Beans Apr/2/2003
Mad Vet puzzles and ...

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Leavitt path algebras !!
1. Introduction and brief history

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A *Mad Vet scenario* posits a Mad Veterinarian in possession of a finite number of transmogrifying machines, where

1. Each machine transmogrifies a single animal of a given species into a finite nonempty collection of animals from any number of species;
2. Each machine can also operate in reverse; and
3. There is one machine corresponding to each species in the menagerie.
Mat Vet Scenario #1

**Scenario #1.** Suppose a Mad Veterinarian has three machines with the following properties.

Machine 1 turns one ant into one beaver;
Machine 2 turns one beaver into one ant, one beaver and one cougar;
Machine 3 turns one cougar into one ant and one beaver.
Scenario #1. Suppose a Mad Veterinarian has three machines with the following properties.

Machine 1 turns one ant into one beaver;
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Let’s do some transmogrification!!
Given any Mad Vet scenario, its corresponding Mad Vet graph is the directed graph with

\[ V = \{A_1, A_2, \ldots, A_n\}, \]

and having, for each \( A_i, A_j \) in \( V \), exactly \( d_{i,j} \) edges with initial vertex \( A_i \) and terminal vertex \( A_j \), where the machine corresponding to species \( A_i \) produces \( d_{i,j} \) animals of species \( A_j \).
Example. Mad Vet scenario #1 has the following Mad Vet graph.

Recall:
Machine 1: Ant → Beaver
Machine 2: Beaver → Ant, Beaver, and Cougar
Machine 3: Cougar → Ant, Beaver
Key idea: Let’s say there are $n$ different species. Let

$$\mathbb{Z}^+$$ denote \{0, 1, 2, \ldots\}.

A \textit{menagerie} is an element of the set

$$S = (\mathbb{Z}^+)^n \setminus \{(0, 0, \ldots, 0)\}.$$
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There is a natural correspondence between menageries and nonempty collections of animals from species $A_1, A_2, \ldots, A_n$. 
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There is a natural correspondence between menageries and nonempty collections of animals from species \( A_1, A_2, \ldots, A_n \).

For instance, in Scenario #1 a collection of two beavers and five cougars would correspond to \( (0, 2, 5) \) in \( S \).
Mad Vet equivalence

There is a naturally arising relation $\sim$ on $S$:

Given $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in $S$, we write

$$a \sim b$$

if there is a sequence of Mad Vet machines that will transmogrify the collection of animals associated with menagerie $a$ into the collection of animals associated with menagerie $b$. 
There is a naturally arising relation \( \sim \) on \( S \):

Given \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \) in \( S \), we write

\[ a \sim b \]

if there is a sequence of Mad Vet machines that will transmogrify the collection of animals associated with menagerie \( a \) into the collection of animals associated with menagerie \( b \).

Using the three properties of a Mad Vet scenario, it is straightforward to show that \( \sim \) is an equivalence relation on \( S \).
Mad Vet equivalence

We focus on the set

\[ W = \{ [a] : a \in S \} \]

of equivalence classes of \( S \) under \( \sim \).

**Example.** Suppose that our Mad Vet of Scenario #1 starts with the menagerie \((1, 0, 0)\).

(Recall: Machine 1: A \( \rightarrow \) B Machine 2: B \( \rightarrow \) A, B, C Machine 3: C \( \rightarrow \) A,B)

Then, for example,

\[(1, 0, 0) \sim (0, 1, 0) \sim (1, 1, 1) \sim (2, 2, 0) \sim (4, 0, 0).\]

Rewritten,

\[ [(1, 0, 0)] = [(0, 1, 0)] = [(1, 1, 1)] = [(2, 2, 0)] = [(4, 0, 0)] \text{ in } W. \]
(Recall: Machine 1: $A \to B$ Machine 2: $B \to A, B, C$ Machine 3: $C \to A, B$)

**Claim.** $W$ is the 3-element set

$$\{(1, 0, 0), (2, 0, 0), (3, 0, 0)\}.$$
Mad Vet equivalence


Claim. $W$ is the 3-element set

$$\{(1,0,0), (2,0,0), (3,0,0)\}.$$

Reason. It’s not hard to see that any $(a, b, c)$ is equivalent to one of the menageries $(1,0,0)$, $(2,0,0)$, or $(3,0,0)$. 
Mad Vet equivalence


Why are these classes not equal to each other? Given a menagerie $m = (a, b, c)$, define the sum

$$s_m = a + b + 2c.$$  

(Intuitively: $s_m$ is the dollar value of menagerie $m$, where an Ant costs $1$, a Beaver $1$, and a Couger $2$.)
Mad Vet equivalence


Why are these classes not equal to each other? Given a menagerie \( m = (a, b, c) \), define the sum

\[
sm = a + b + 2c.
\]

(Intuitively: \( sm \) is the dollar value of menagerie \( m \), where an Ant costs $1, a Beaver $1, and a Couger $2.)

Then Machines 1 and 3 leave \( sm \) the same, while Machine 2 increases \( sm \) by 3 (and running Machine 2 in reverse decreases \( sm \) by 3). So any application of any machine to any menagerie leaves the total value of the menagerie invariant mod 3. So the three classes are distinct.
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Semigroups, monoids, and groups

Reminder / review of notation.

1. *semigroup*: associative operation.
Semigroups, monoids, and groups

Reminder / review of notation.

1 semigroup: associative operation.

   e.g. $\mathbb{N} = \{1, 2, 3, \ldots\}$ under addition.
Semigroups, monoids, and groups

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1. **Semigroup**: associative operation.
   e.g. $\mathbb{N} = \{1, 2, 3, \ldots\}$ under addition.

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3. **group**: monoid, for which each element has an inverse.
   
e.g. \( \mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, \ldots\} \) under addition.
Start with a Mad Vet scenario. Define addition on $W$ (the set of equivalence classes of menageries) by setting

$$[x] + [y] = [x + y].$$

Interpret as “unions” of menageries.

This operation is well defined.

“Mad Vet semigroup.”
Mad Vet semigroups

(Recall: Machine 1: $A \rightarrow B$ Machine 2: $B \rightarrow A, B, C$ Machine 3: $C \rightarrow A, B$)

Example.

$$\mathcal{W} = \{[(1, 0, 0)], [(2, 0, 0)], [(3, 0, 0)]\}.$$ 

We get, for instance,

$$[(1, 0, 0)] + [(1, 0, 0)] = [(1 + 1, 0, 0)] = [(2, 0, 0)],$$

as we’d expect.
Mad Vet semigroups

(Recall: Machine 1: $A \rightarrow B$  
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Mad Vet semigroups

(Recall: Machine 1: A \rightarrow B \quad Machine 2: B \rightarrow A, B, C \quad Machine 3: C \rightarrow A,B)

**Example.**

\[ W = \{ [(1, 0, 0)], [(2, 0, 0)], [(3, 0, 0)] \}. \]

We get, for instance,

\[ [(1, 0, 0)] + [(1, 0, 0)] = [(1 + 1, 0, 0)] = [(2, 0, 0)], \]

as we’d expect. But also

\[ [(1, 0, 0)] + [(3, 0, 0)] = [(4, 0, 0)] = [(1, 0, 0)]. \]

So \([(3, 0, 0)]\) behaves like an identity element with respect to the element \([(1, 0, 0)]\) in \(W\).
Mad Vet semigroups

Similarly

\[(2, 0, 0)] + [(3, 0, 0)] = [(2, 0, 0)], \text{ and } [(3, 0, 0)] + [(3, 0, 0)] = [(3, 0, 0)].\]
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\[(2, 0, 0)] + [(3, 0, 0)] = [(2, 0, 0)], \quad \text{and} \quad [(3, 0, 0)] + [(3, 0, 0)] = [(3, 0, 0)].\]

So for this Mad Vet scenario the Mad Vet semigroup $W$ is a monoid with identity $[(3, 0, 0)]$. 
Mad Vet semigroups

Actually, since

\[(1, 0, 0) + (2, 0, 0) = (3, 0, 0)\]

in \(W\), every element in \(W\) has an inverse.
Actually, since

\[(1, 0, 0)] + [(2, 0, 0)] = [(3, 0, 0)]

in \( W \), every element in \( W \) has an inverse.

So \( W \) is in fact a group, necessarily \( \mathbb{Z}_3 \).
Mad Vet semigroups

**Scenario #2.** Suppose the same Mad Vet has replaced two of her machines with new machines.

Machine 1 still turns one ant into one beaver; Machine 2 now turns one beaver into one ant and one cougar; Machine 3 now turns one cougar into two cougars.

In this situation $W$ is a monoid, but not a group.

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In this situation $W$ is a monoid, but not a group. In fact,  
$$W = \{(i, 0, 0) \mid i \in \mathbb{N}\} \cup \{(0, 0, 1)\}.$$  

$(0, 0, 1)$ is an identity element for this Mad Vet semigroup.  
So $W$ in this case is a *monoid*.  

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The graph menagerie
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In this situation $\mathcal{W}$ is a monoid, but not a group. In fact,

$$\mathcal{W} = \{(i, 0, 0) : i \in \mathbb{N}\} \cup \{(0, 0, 1)\}.$$  

$(0, 0, 1)$ is an identity element for this Mad Vet semigroup. So $\mathcal{W}$ in this case is a *monoid*.

But $\mathcal{W}$ is not a group: e.g., there is no element $[x]$ in $\mathcal{W}$ for which

$$[(1, 0, 0)] + [x] = [(0, 0, 1)].$$

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The Big Question:

Given a Mad Vet scenario, when is the corresponding Mad Vet semigroup actually a group?

More Mad Vet scenarios ...
Mad Vet semigroups

Scenario #3.

M1: A → B, C;  M2: B → A, C;  M3: C → A, B

Scenario #4.

M1: A → 2A;  M2: B → 2B;  M3: C → 2C

Scenario #5.

M1: A → B, C;  M2: B → A, B;  M3: C → A, C

Scenario #6.

M1: A → B;  M2: B → C;  M3: C → C

Scenario #7.

M1: A → A, B, C;  M2: B → A, C;  M3: C → A, B
Mad Vet semigroups

Subtle?
Subtle?

Among Scenarios #3-7, there are Mad Vet semigroups $W$ for which:

1. $W$ is an infinite group;
2. $W$ is a finite noncyclic group;
3. $W$ is a finite nonmonoid;
4. $W$ is a finite cyclic group, not isomorphic to $\mathbb{Z}_3$; and
5. $W$ is an infinite nonmonoid.
Euler’s “Bridges of Königsberg” problem.
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2. prove a theorem about graphs;
Euler’s “Bridges of Königsberg” problem. Idea:

1. translate the problem to a question about graphs;
2. prove a theorem about graphs;
3. use the graph-theoretic result to answer original question.
Some graph theory terminology. (All graphs are directed.)
Graph theory

Some graph theory terminology. (All graphs are directed.)

1. A sink in a directed graph.
2. A path in a directed graph.
3. If \( v \) and \( w \) are vertices, \( v \) connects to \( w \) in case either \( v = w \) or there is a path from \( v \) to \( w \).
4. For a vertex \( v \), a cycle based at \( v \) is a (nontrivial) path from \( v \) to \( v \) for which no vertices are repeated.
Some graph theory terminology. (All graphs are \textit{directed}.)

1. A \textit{sink} in a directed graph.

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3. If \(v\) and \(w\) are vertices, \(v\) \textit{connects to} \(w\) in case either \(v = w\) or there is a path from \(v\) to \(w\).

4. For a vertex \(v\), a \textit{cycle based at} \(v\) is a (nontrivial) path from \(v\) to \(v\) for which no vertices are repeated.

5. A finite graph \(\Gamma\) is \textit{cofinal} in case every vertex \(v\) of \(\Gamma\) connects to every cycle and to every sink in \(\Gamma\).
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4. For a vertex $v$, a cycle based at $v$ is a (nontrivial) path from $v$ to $v$ for which no vertices are repeated.
5. A finite graph $\Gamma$ is cofinal in case every vertex $v$ of $\Gamma$ connects to every cycle and to every sink in $\Gamma$.
6. If $C = f_1 f_2 \cdots f_m$ is a cycle in $\Gamma$, then an edge $e$ is called an exit for $C$ if the source vertex of $e$ equals the source vertex for $f_j$ (some $j$), but $e \neq f_j$. 

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5. A finite graph \( \Gamma \) is cofinal in case every vertex \( v \) of \( \Gamma \) connects to every cycle and to every sink in \( \Gamma \).
6. If \( C = f_1 f_2 \cdots f_m \) is a cycle in \( \Gamma \), then an edge \( e \) is called an exit for \( C \) if the source vertex of \( e \) equals the source vertex for \( f_j \) (some \( j \)), but \( e \neq f_j \). (Intuitively, an exit for \( C \) is an edge \( e \), not included in \( C \), which provides a way to step off of \( C \).)
**Example.**

The cycle $eg$ based at $y$ has two exits: $h$ and the loop at $y$. These same edges are also exits for the cycle $ge$ based at $z$. Similarly, the loop at $y$ has exits $e$ and $h$.

The loop at $x$ has no exit.

This graph is not cofinal (e.g., $x$ does not connect to $eg$).
Theorem: Mad Vet Group Test. The Mad Vet semigroup $W$ of a Mad Vet scenario is a group if and only if the corresponding Mad Vet graph $\Gamma$ has the following two properties.

1. $\Gamma$ is cofinal; and
2. Every cycle in $\Gamma$ has an exit.
Mad Vet Group Test

**Theorem: Mad Vet Group Test.** The Mad Vet semigroup $W$ of a Mad Vet scenario is a group if and only if the corresponding Mad Vet graph $\Gamma$ has the following two properties.

1. $\Gamma$ is cofinal; and
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**Proof.** Long, but can be done using only basic graph-theoretic and group-theoretic ideas.
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1. $\Gamma$ is cofinal; and
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Proof.: Long, but can be done using only basic graph-theoretic and group-theoretic ideas.

(Actually, two proofs are known. More about that later.)
An overview of one of the proofs.
Mad Vet Group Test

An overview of one of the proofs.

Lemma. A commutative semigroup $S$ is a group if and only if for each pair $x, z \in S$ there exists $y \in S$ for which $x + y = z$. 
An overview of one of the proofs.

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Proof: Good exercise for MA321 students.

(Converse to Theorem 25.1c in Anderson / Feil ...)
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**Lemma.** A commutative semigroup $S$ is a group if and only if for each pair $x, z \in S$ there exists $y \in S$ for which $x + y = z$.

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Here’s the Mad Vet graph from Scenario #1 again:

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Cofinal? YES.
Here’s the Mad Vet graph from Scenario #1 again:


Cofinal? YES. Every cycle has an exit? YES.
Here’s the Mad Vet graph $\Theta$ of Scenario #2.

(Recall: Machine 1: $A \rightarrow B$  
Machine 2: $B \rightarrow A, C$  
Machine 3: $C \rightarrow 2C$)
Here's the Mad Vet graph $\Theta$ of Scenario #2.

![Diagram of the Mad Vet graph]

(Recall: Machine 1: $A \rightarrow B$  Machine 2: $B \rightarrow A, C$  Machine 3: $C \rightarrow 2C$)

Cofinal? NO. ($C$ does not connect to the cycle $ABA$.)
Here’s the Mad Vet graph $\Theta$ of Scenario #2.

(Recall: Machine 1: $A \rightarrow B$  Machine 2: $B \rightarrow A, C$  Machine 3: $C \rightarrow 2C$)

Cofinal? NO. ($C$ does not connect to the cycle $ABA$.)

(But every cycle does have an exit ...)
**Scenario #8.** Let’s analyze Mad Vet Bob’s puzzle.

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(Recall: Machine 1: $A \rightarrow 2B, 5C$  Machine 2: $B \rightarrow 3A, 3C$  Machine 3: $C \rightarrow A, B$)
Scenario #8. Let’s analyze Mad Vet Bob’s puzzle.


So Mad Vet Bob’s semigroup is in fact a group.
Just exactly what group is it ?????
Mad Vet Groups

Just exactly what group is it ?????

This question has a remarkably nice answer.

Any graph $\Gamma$ has an associated incidence matrix $A_{\Gamma}$: if $\Gamma$ has $n$ vertices $v_1, v_2, \ldots, v_n$, then $A_{\Gamma}$ is the $n \times n$ matrix $(d_{ij})$, where

$$d_{ij} = \# \text{ of edges starting at } v_i \text{ and ending at } v_j.$$
Mad Vet Groups

For example, if $\Delta$ is the graph of Scenario #1, then

\[
A_{\Delta} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 1 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]
Now form the matrix $I_n - A_\Gamma$.

For instance, using the above matrix $A_\Delta$,

$$I_3 - A_\Delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$
Then put the (square) matrix $I_n - A_{\Gamma}$ in *Smith normal form*.

The Smith normal form of an $n \times n$ matrix having integer entries is a diagonal $n \times n$ matrix whose diagonal entries are nonnegative integers

$$\alpha_1, \alpha_2, \ldots, \alpha_q, 0, 0, \ldots, 0$$

such that $\alpha_i$ divides $\alpha_{i+1}$ for each $1 \leq i \leq q - 1$. 

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The graph menagerie
Mad Vet Groups

The Smith normal form of a matrix $A$ can be obtained by performing on $A$ a combination of these matrix operations: interchanging rows or columns, or adding an integer multiple of a row [column] to another row [column]. The resulting Smith normal form of matrix $A$ is thus of the form $PAQ$, where $P$ and $Q$ are integer-valued matrices with determinants equal to $\pm 1$. ( Might need to tweak some signs at the end ... )
Here’s an answer to the “just exactly what group is it?” question.

**Mad Vet Group Identification Theorem.** Given a Mad Vet scenario with $n$ species whose Mad Vet semigroup $W$ is a group, let $\Gamma$ be its associated Mad Vet graph. Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ be the nonzero diagonal entries of the Smith normal form of the matrix $I_n - A_\Gamma$. 
Here’s an answer to the “just exactly what group is it?” question.

**Mad Vet Group Identification Theorem.** *Given a Mad Vet scenario with n species whose Mad Vet semigroup \( W \) is a group, let \( \Gamma \) be its associated Mad Vet graph. Let \( \alpha_1, \alpha_2, \ldots, \alpha_q \) be the nonzero diagonal entries of the Smith normal form of the matrix \( I_n - A_\Gamma \). Then

\[
W \cong \mathbb{Z}_{\alpha_1} \oplus \mathbb{Z}_{\alpha_2} \oplus \cdots \oplus \mathbb{Z}_{\alpha_q} \oplus \mathbb{Z}^{n-q}.
\]

(Notation: \( \mathbb{Z}_1 = \{0\} \).)
Example. Letting $\Delta$ be the Mad Vet graph of Scenario #1, the Smith normal form of the matrix $I_3 - A_\Delta$ is the matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 3
\end{pmatrix}.
$$

Because we already know that Scenario #1’s semigroup is a group, the Mad Vet Group Identification Theorem implies that it is isomorphic to $\mathbb{Z}_1 \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_3 \cong \{0\} \oplus \{0\} \oplus \mathbb{Z}_3 \cong \mathbb{Z}_3$, as expected.
Example. Let $\Phi$ be the Mad Vet graph of Scenario #8 (Mad Vet Bob’s Puzzle). We’ve checked that $\Phi$ has the right properties, so that the corresponding Mad Vet semigroup is a group. Then $I_{\Phi}$ is the matrix

$$
\begin{pmatrix}
0 & 2 & 5 \\
3 & 0 & 3 \\
1 & 1 & 0
\end{pmatrix}.
$$

The Smith normal form of $I_3 - A_{\Phi}$ turns out to be matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 34
\end{pmatrix}.
$$

So the corresponding group is isomorphic to $\mathbb{Z}_1 \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_{34} \cong \mathbb{Z}_{34}$. 
1 Introduction and brief history

2 Mad Vet scenarios

3 Mad Vet groups

4 Beyond the Mad Vet
Purely Infinite Simplicity Theorem. For a finite directed sink-free graph $\Gamma$, the following are equivalent:

1. The Leavitt path algebra $L_\mathbb{C}(\Gamma)$ is purely infinite and simple. (This is a statement about an algebraic structure.)
2. The graph $C^*$-algebra $C^*(\Gamma)$ is purely infinite and simple. (This is a statement about an analytic structure.)
3. $\Gamma$ is cofinal, and every cycle in $\Gamma$ has an exit.
4. The graph semigroup $W_\Gamma$ is a group.
Who cares?

Notes.
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Until the recent Mad Vet work, the only proof we knew of $(3) \iff (4)$ was to show that each is equivalent to $(1)$. That proof ain’t easy.
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We can get rid of the sink-free hypothesis in the general analysis.
Thanks!

Gene Abrams

The graph menagerie
Acknowledgment  The authors express their gratitude to Enrique Pardo for allowing them to use and modify his proof of the Graph Semigroup Group Test for this article; to Amelia Taylor and Brian Hopkins for carefully reading and offering helpful suggestions about the article; and to Ken Ross for his valuable comments, advice, and support. The first author was introduced to Mad Veterinarian puzzles at a June 2008 workshop on Math Teachers’ Circles, sponsored by the American Institute of Mathematics, Palo Alto, CA. The author is grateful for AIM’s support.
15 minutes of fame?
15 minutes of fame?

The Mad Veterinarian (p. 168)

- A Remarkable Euler Square
- The Ergodic Theory Carnival
- Tower of Hanoi Graphs
- Drilling through a Sphere

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Questions?