Connections between Leavitt path algebras and graph C*-algebras
Is there a Rosetta Stone?

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Arizona State University C*-algebra seminar
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Overview

1. Leavitt path algebras
2. Connections to graph C^*-algebras
3. What we know: Similarities and Differences
4. What we don't know

General path algebras

K always denotes a field. *Any* field.

Let E be a directed graph. $E = (E^0, E^1, r, s)$

$$\bullet^{s(e)} \xrightarrow{e} \bullet^{r(e)}$$

The *path algebra* KE is the K-algebra with basis $\{p_i\}$ consisting of the directed paths in E. (View vertices as paths of length 0.)

$$p \cdot q = pq \ \text{ if } \ r(p) = s(q), \quad 0 \text{ otherwise.}$$

In particular, $s(e) \cdot e = e = e \cdot r(e)$.

Note: E^0 finite $\Leftrightarrow$ KE is unital; then $1_{KE} = \sum_{v \in E^0} v$.

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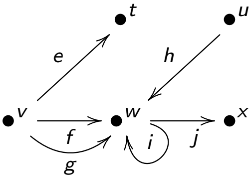

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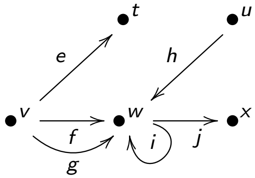

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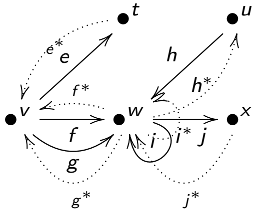

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Definition

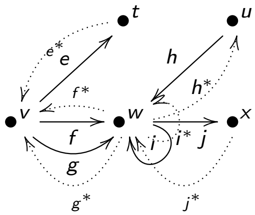
The Leavitt path algebra of E with coefficients in K

$$L_K(E) = K\widehat{E} \; / \; < (CK1), (CK2) >$$

Leavitt path algebras: Examples

Some sample computations in $L_{\mathbb{C}}(E)$ from the Example:

$\widehat{E} \;=$



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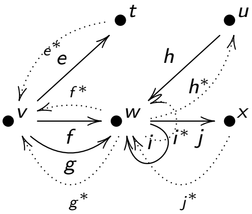

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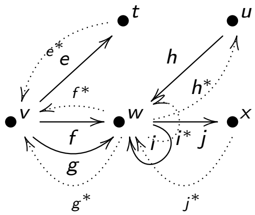

$$ee^* + ff^* + gg^* = v \quad g^*g = w \quad g^*f = 0$$

$$h^*h = w \quad hh^* = u \quad ff^* = \ldots \ \text{(no simplification)}$$

$$\text{But } (ff^*)^2 = f(f^*f)f^* = f \cdot w \cdot f^* = ff^*.$$

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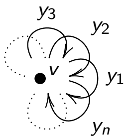
Then $L_K(E) \cong M_n(K)$.

$$E = \bullet^{v} \circlearrowright x$$

Then $L_K(E) \cong K[x, x^{-1}]$.

Leavitt path algebras: Examples

$E = R_n =$



Then $L_K(E) \cong L_K(1, n)$, the "Leavitt K-algebra of order n".

(W.G. Leavitt, Transactions. A.M.S. 1962).

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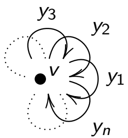

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$L_K(1, n)$ is the universal K-algebra R for which ${}_RR \cong {}_RR^n$.

$$L_K(1, n) = \langle x_1, ..., x_n, y_1, ..., y_n \mid x_i y_j = \delta_{i,j} 1_K, \sum_{i=1}^{n} y_i x_i = 1_K \rangle$$

Leavitt path algebras

Some general properties of Leavitt path algebras:

1. $L_K(E) = \mathrm{span}_K\{pq^* \mid p, q \text{ paths in } E\}$.
2. $L_K(E) \cong L_K(E)^{op}$.
3. $L_K(E)$ admits a natural $\mathbb{Z}$-grading: $\deg(pq^*) = \ell(p) - \ell(q)$.
4. $J(L_K(E)) = \{0\}$.

Graph C*-algebras

E any directed graph, $\mathcal{H}$ a Hilbert space.

Definition. A **Cuntz-Krieger** E**-family in** $B(\mathcal{H})$ is a collection of mutually orthogonal projections $\{P_v \mid v \in E^0\}$, and partial isometries $\{S_e \mid e \in E^1\}$ with mutually orthogonal ranges, for which:

(CK1) $S_e^* S_e = P_{r(e)}$ for all $e \in E^1$,

(CK2) $\sum_{\{e \mid s(e)=v\}} S_e S_e^* = P_v$ whenever v is a regular vertex, and

(CK3) $S_e S_e^* \leq P_{s(e)}$ for all $e \in E^1$.

The **graph C*-algebra** $C^*(E)$ of E is the universal C*-algebra generated by a Cuntz-Krieger E-family.

Graph C*-algebras

For $\mu = e_1 e_2 \cdots e_n$ a path in E,
let S_μ denote $S_{e_1} S_{e_2} \cdots S_{e_n} \in C^*(E)$.

The Key Connection: Consider

$$A = \mathrm{span}_{\mathbb{C}}\{P_v, S_\mu S_\nu^* \mid v \in E^0,\ \mu, \nu \text{ paths in } E\} \subseteq C^*(E).$$

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So it's probably not surprising that there are some close relationships between $L_{\mathbb{C}}(E)$ and $C^*(E)$.

Graph C*-algebras: Examples

Here are the graph C*-algebras which arise from the graphs of the previous examples.

$$E = \ \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

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Then $C^*(E) \cong M_n(\mathbb{C}) \cong L_{\mathbb{C}}(E)$.

$$E = \bullet^{v} \circlearrowright$$

Then $C^*(E) \cong C(\mathbb{T})$, the continuous functions on the unit circle.

Graph C*-algebras: Examples

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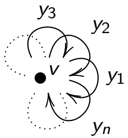

Then $C^*(E) \cong \mathcal{O}_n$, the Cuntz algebra of order n.

Brief History

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1962: Leavitt defines / investigates $L_K(1, n)$.

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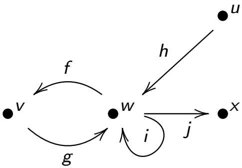
1980 - 2000: Various authors generalize Cuntz' construction; eventually, graph C*-algebras are defined / investigated.

2005: Leavitt path algebras are defined / investigated.

Some graph terminology

Example



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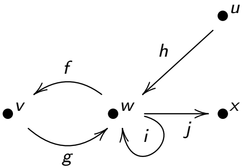

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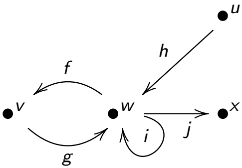

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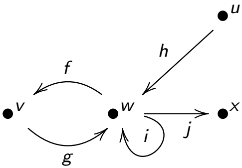

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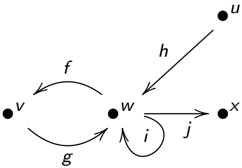

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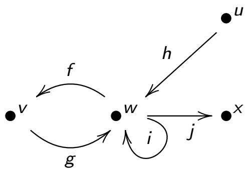

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Standing hypothesis: All graphs are finite (for now) ...

Similarities

We begin by looking at some similarities between the structure of $L_K(E)$ and the structure of $C^*(E)$.

Simplicity

Simplicity:

Algebraic: No nontrivial two-sided ideals.

Analytic: No nontrivial closed two-sided ideals.

Simplicity

Theorem: These are equivalent for any finite graph E:

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2. $L_K(E)$ is simple for any field K
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Big Question:

Can we go 'directly' between (1) or (2), and (3) or (4) ??

Purely infinite simplicity

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Same Big Question:

Can we go 'directly' between (1) or (2), and (3) or (4) ??

Rosetta Stone?

There are many additional examples of this sort of behavior:

For instance:

1. primitivity
2. exchange property
3. $\mathcal{V}$-monoid (in particular, $K_0(L_K(E)) \cong K_0(C^*(E))$)
4. possible values of stable rank

But there are no 'direct' proofs for any of them.

Is there some sort of Rosetta Stone ??

The Kirchberg Phillips Theorem

Kirchberg Phillips Theorem (2000): Classification result for a class of C*-algebras in terms of K-theoretic data.

In the context of graph C*-algebras for finite graphs, it looks like this:

Theorem: Suppose E and F are finite graphs for which $C^*(E)$ and $C^*(F)$ are purely infinite simple. Suppose

$$(K_0(C^*(E)), [1_{C^*(E)}]) \cong (K_0(C^*(F)), [1_{C^*(F)}]).$$

Then $C^*(E) \cong C^*(F)$ homeomorphically.

The KP Theorem plays an intriguing role in the Rosetta Stone question.

Matrices over Leavitt algebras

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W. Paschke and N. Salinas, Matrix algebras over $\mathcal{O}_n$, Michigan J. Math, 1979

For which $d \in \mathbb{N}$ is it the case that $\mathcal{O}_n \cong \mathrm{M}_d(\mathcal{O}_n)$?

The answer (in retrospect) follows from the Kirchberg Phillips Theorem: if and only of $\gcd(d, n-1) = 1$.

Matrices over Leavitt algebras

From the Leavitt path algebra side: Let $R = L_{\mathbb{C}}(1, n)$. So ${}_R R \cong {}_R R^n$.

So this gives in particular $R \cong \mathrm{M}_n(R)$ as rings.

Which then (for free) gives some additional isomorphisms, e.g.

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Also, ${}_RR \cong {}_RR^n \cong {}_RR^{2n-1} \cong {}_RR^{3n-2} \cong \ldots$, which also in turn yield ring isomorphisms

$$R \cong \mathrm{M}_n(R) \cong \mathrm{M}_{2n-1}(R) \cong \mathrm{M}_{3n-2}(R) \cong \ldots$$

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For instance, if $R = L(1,4)$, then it's not hard to show that $R \cong \mathrm{M}_2(R)$ as rings (even though $R \not\cong {}_R R^2$ as modules).
Idea: 2 and 4 are nicely related, so these eight matrices inside $\mathrm{M}_2(L(1,4))$ "work":

$$X_1 = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}, \ X_2 = \begin{pmatrix} x_3 & 0 \\ x_4 & 0 \end{pmatrix}, \ X_3 = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix}, \ X_4 = \begin{pmatrix} 0 & x_3 \\ 0 & x_4 \end{pmatrix}$$

together with their duals

$$Y_1 = \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix}, \ Y_2 = \begin{pmatrix} y_3 & y_4 \\ 0 & 0 \end{pmatrix}, \ Y_3 = \begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}, \ Y_4 = \begin{pmatrix} 0 & 0 \\ y_3 & y_4 \end{pmatrix}$$

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In general, using this same idea, we can show that:

if $d | n^t$ for some $t \in \mathbb{N}$, then $L(1, n) \cong \mathrm{M}_d(L(1, n))$.

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On the other hand ...

If $R = L(1,n)$, then the "type" of R is $n-1$. (Think: "smallest difference"). Bill Leavitt showed the following in his 1962 paper:

The type of $\mathrm{M}_d(L(1,n))$ is $\frac{n-1}{g.c.d.(d,n-1)}$.

In particular, if $g.c.d.(d,n-1) > 1$, then $L(1,n) \ncong \mathrm{M}_d(L(1,n))$.

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(Note: $d|n^t \ \Rightarrow \ g.c.d.(d,n-1) = 1$.)

Matrices over Leavitt algebras

Smallest interesting pair: Is $L(1,5) \cong \mathrm{M}_3(L(1,5))$?

We are led "naturally" to consider these five matrices (and their duals) in $\mathrm{M}_3(L(1,5))$:

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Matrices over Leavitt algebras

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Everything went along nicely... **except**, we couldn't see how to generate the matrix units $e_{1,3}$ and $e_{3,1}$ inside $\mathrm{M}_3(L(1,5))$ using these ten matrices.

Matrices over Leavitt algebras

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Instead, this set (together with duals) works:

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Matrices over Leavitt algebras

Theorem

(A-, Ánh, Pardo; Crelle's J. 2008) For any field K,

$$L_K(1,n) \cong \mathrm{M}_d(L_K(1,n)) \;\Leftrightarrow\; g.c.d.(d, n-1) = 1.$$

Indeed, more generally,

$$\mathrm{M}_d(L_K(1,n)) \cong \mathrm{M}_{d'}(L_K(1,n')) \;\Leftrightarrow$$
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Moreover, we can write down the isomorphisms explicitly.

Along the way, some elementary (but apparently new) number theory ideas come into play.

Given n, d with $g.c.d.(d, n-1) = 1$, there is a "natural" partition of $\{1, 2, \ldots, n\}$ into two disjoint subsets.

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Here's what made this second set of matrices work. Using this partition in the particular case $n = 5, d = 3$, then the partition of $\{1, 2, 3, 4, 5\}$ turns out to be the two sets

$$\{1, 4\} \quad \text{and} \quad \{2, 3, 5\}.$$

The matrices that "worked" are ones where we fill in the last columns with terms of the form $x_i x_1^j$ in such a way that i is in the same subset as the row number of that entry.

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The number theory underlying this partition in the general case where $\mathrm{g.c.d.}(d, n-1) = 1$ is elementary. But we are hoping to find some other 'context' in which this partition process arises.

Matrices over Leavitt algebras

Computations when $n = 5, d = 3$.

$\gcd(3, 5-1) = 1$. Now $5 = 1 \cdot 3 + 2$, so that $r = 2, r-1 = 1$, and define $s = d - (r-1) = 3 - 1 = 2$.

Consider the sequence starting at 1, and increasing by s each step, and interpret mod d $(1 \leq i \leq d)$. This will necessarily give all integers between 1 and d.

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Now break this set into two pieces: those integers up to and including $r - 1$, and those after. Since $r - 1 = 1$, here we get

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Now extend these two sets mod 3 to all integers up to 5.

$$\{1, 4\} \cup \{2, 3, 5\}$$

Matrices over Leavitt algebras

Does this look familiar?

Matrices over Leavitt algebras

Corollary. (Answer to the Paschke Salinas Question)

$$\mathcal{O}_n \cong \mathrm{M}_d(\mathcal{O}_n) \iff g.c.d.(d, n-1) = 1.$$

(And the isomorphisms are explicitly described.)

Proof. The explicitly constructed algebraic isomorphism between the matrices over Leavitt path algebras turns out to preserve the $*$ structure, and so (easily) can be shown to extend to the corresponding completions.

Matrices over Leavitt algebras

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$$G^+_{n,r} \cong G^+_{m,s} \quad \Leftrightarrow \quad m = n \text{ and } \mathrm{g.c.d.}(r, n-1) = \mathrm{g.c.d.}(s, n-1).$$

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Proof. Show that $G^+_{n,r}$ can be realized as an appropriate subgroup of the invertible elements of $M_r(L_{\mathbb{C}}(1,n))$, and then use the explicit isomorphisms provided in the A -, Ánh, Pardo result.

Differences

We now look at some differences between the structure of $L_K(E)$ and the structure of $C^*(E)$.

Primeness

Algebraic: R is a prime ring in case $\{0\}$ is a prime ideal of R; that is, in case for any two-sided ideals I, J of R, $I \cdot J = \{0\}$ if and only if $I = \{0\}$ or $J = \{0\}$.

Theorem. K any field, E any graph.
$L_K(E)$ is prime $\Leftrightarrow$ E is downward directed.

Primeness

Analytic: A is a prime C*-algebra in case $\{0\}$ is a prime ideal of A; that is, in case for any closed two-sided ideals I, J of R, $I \cdot J = \{0\}$ if and only if $I = \{0\}$ or $J = \{0\}$.

Theorem: $C^*(E)$ is prime $\Leftrightarrow$ E downward directed **and** satisfies Condition (L).

So for example $L_K(\bullet \circlearrowleft)$ is prime, but $C^*(\bullet \circlearrowleft)$ is not prime.

More Differences

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3. $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, but $L_{\mathbb{C}}(1,2) \otimes L_C(1,2) \ncong L_{\mathbb{C}}(1,2)$.

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2. There is no Bott periodicity for $L_K(E)$.

3. $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, but $L_{\mathbb{C}}(1,2) \otimes L_C(1,2) \ncong L_{\mathbb{C}}(1,2)$. Note: The fact that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ is used in an essential way in Phillips' proof of the KP Theorem.

1 Leavitt path algebras

2 Connections to graph C*-algebras

3 What we know: Similarities and Differences

4 What we don't know

What we don't know ...

We continue by looking at properties for which

we do not currently know

whether these give similarities or differences between the structure of $L_K(E)$ and the structure of $C^*(E)$.

The isomorphism question

Perhaps the most basic question ...

If $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$, does this imply $C^*(E) \cong C^*(F)$?

And conversely?

(Need to interpret "isomorphism" appropriately.)

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Partial answer: OK in case the graph algebras are simple.

But this result uses some heavy classification machinery, *including the Kirchberg Phillips Theorem*.

Answer not known in general.

Converse? It's not known whether $C^*(E) \cong C^*(F)$ implies $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$, even in the simple case.

An algebraic Kirchberg Phillips Theorem?

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We currently don't know whether there is an algebraic analog to the KP Theorem for purely infinite simple Leavitt path algebras. That is ....

Let K be a field. Suppose E and F are finite graphs for which $L_K(E)$ and $L_K(F)$ are purely infinite simple. Suppose

$$(K_0(L_K(E)), [1_{L_K(E)}]) \cong (K_0(L_K(F)), [1_{L_K(F)}]).$$

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$$(K_0(L_K(E)), [1_{L_K(E)}]) \cong (K_0(L_K(F)), [1_{L_K(F)}]).$$

Does this imply that $L_K(E) \cong L_K(F)$?

"Algebraic KP Question"

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Here's one approach which could possibly be used to answer the Algebraic KP Question. We try to re-prove or re-interpret the KP Theorem using techniques which might possibly be applicable in the algebraic setting. Here's a possible way to do that:

(Step 1) Use results from symbolic dynamics to show that the isomorphism $C^*(E) \cong C^*(F)$ follows in case one also assumes that $\det(I - A_E) = \det(I - A_F)$.

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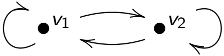
(Step 2) Use KK-theory to show that the graph C*-algebras $C^*(E_2)$ and $C^*(E_4)$ are isomorphic:

$E_2 =$ 

and

$E_4 =$ 

(These have identical K-theory, but different determinants.)

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(Step 4) Show such an isomorphism exists.

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For Leavitt path algebras we have:

"Restricted" Algebraic KP Theorem: In this situation, *if we also assume* $\det(I - A_E) = \det(I - A_F)$, then we get $L_K(E) \cong L_K(F)$. (The proof uses the same deep results from symbolic dynamics mentioned above.)

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Is there an explicit isomorphism from $C^*(E_2)$ to $C^*(E_4)$ that we can possibly exploit?

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If it turns out that $L_K(E_2) \cong L_K(E_4)$, it's not clear how one could use this to establish isomorphisms between Leavitt path algebras of different pairs of graphs for which the K-theory matches up but the signs of the determinants do not.

An algebraic Kirchberg / Phillips Theorem?

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There are three possibilities: Yes, No, and Sometimes. The answer will be interesting, no matter how things play out.

Thank you.

Thanks also to The Simons Foundation.