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Leavitt path algebras

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Preface

The great challenge in writing a book about a topic of ongoing mathematical research interest lies in determining *who* and *what*. *Who* are the readers for whom the book is intended? *What* pieces of the research should be included?

The topic of **Leavitt path algebras** presents both of these challenges, in the extreme. Indeed, much of the beauty inherent in this topic stems from the fact that it may be approached from many different directions, and on many different levels.

The topic encompasses classical ring theory at its finest. While at first glance these Leavitt path algebras may seem somewhat exotic, in fact many standard, well-understood algebras arise in this context: matrix rings and Laurent polynomial rings, to name just two. Many of the fundamental, classical ring-theoretic concepts have been and continue to be explored here, including the ideal structure, $\mathbb{Z}$-grading, and structure of finitely generated projective modules, to name just a few.

The topic continues a long tradition of associating an algebra with an appropriate combinatorial structure (here, a directed graph), the subsequent goal being to establish relationships between the algebra and the associated structures. In this particular setting, the topic allows for (and is enhanced by) visual, pictorial representation via directed graphs. Many readers are no doubt familiar with the by-now classical way of associating an algebra over a field with a directed graph, the standard *path algebra*. The construction of the *Leavitt path algebra* provides another such connection. The path algebra and Leavitt path algebra constructions are indeed related, via algebras of quotients. However, one may understand Leavitt path algebras without any prior knowledge of the path algebra construction.

The topic has significant, deep connections with other branches of mathematics. For instance, many of the initial results in Leavitt path algebras were guided and motivated by results previously known about their analytic cousins, the *graph C*-algebras*. The study of Leavitt path algebras quickly matured to adolescence (when it became clear that the algebraic results are not implied by the C* results), and almost immediately thereafter to adulthood (when in fact some C* results, including some *new* C* results, were shown to follow from the algebraic results). Indeed, a number of longstanding questions in algebra have recently been resolved using Leavitt path algebras as a tool, thus further establishing the maturity of the subject.

The topic continues a deep tradition evident in many branches of mathematics in which K-theory plays an important role. Indeed, in retrospect, one can view Leavitt path algebras as precisely those algebras constructed to produce specified K-theoretic data in a universal way, data arising naturally from directed graphs. Much of the current work in the field is focused on better understanding just how large a role the K-theoretic data plays in determining the structure of these algebras.

Our goal in writing this book, the *Why?* of this book, simultaneously addresses both the *Who?* and *What?* questions. We provide here a self-contained presentation of the topic of Leavitt path algebras, a presentation which will allow readers having different backgrounds and different topical interests to understand and appreciate these structures. In particular, graduate students having only a first year course in ring theory should find most of the material in this book quite accessible. Similarly, researchers who don't self-identify as algebraists (e.g., people working in C*-algebras or symbolic dynamics) will be able to understand how these Leavitt path algebras stem from, or apply to, their own research interests. While most of the results contained here have appeared elsewhere in the literature, a few of the central results appear here for the first time. The style will be relatively informal. We will often provide historical motivation and overview, both to increase the reader's understanding of the subject and to play up the connections with other areas of mathematics. Although space considerations clearly require us to eliminate some otherwise interesting and important topics from inclusion, we provide an extensive bibliography for those readers who seek additional information about various topics which arise herein.

More candidly, our real *Why?* for writing this book is to share what we know about Leavitt path algebras in such a way that others might become prepared, and subsequently inspired, to join in the game.

GA
PA
MSM

Contents

Chapter 1
The basics of Leavitt path algebras: motivations, definitions and examples

{IntroductionChapter}

ABSTRACT: We introduce the central idea, that of a *Leavitt path algebra*. We start by describing the classical Leavitt algebras. We then proceed to give the definition of the Leavitt path algebra $L_K(E)$ for an arbitrary directed graph E and field K. After providing some basic examples, we show how Leavitt path algebras are related to the monoid realization algebras of Bergman, as well as to graph C*-algebras. We then introduce the more general construction of relative Cohn path algebras $C_K^X(E)$, and show how these are related to Leavitt path algebras. We finish by describing how any Cohn (specifically, Leavitt) path algebra may be constructed as a direct limit of Cohn (specifically, Leavitt) path algebras corresponding to finite graphs. We conclude the chapter with an historical overview of the subject.

In this the initial chapter of the book we introduce the *Leavitt path algebra* $L_K(E)$ which arises from a directed graph E and field K. We begin in Section 1.1 by reviewing a class of algebras defined and investigated in the early 1960's by W.G. Leavitt, the now-so-called *Leavitt algebra* $L_K(1,n)$ corresponding to any positive integer n and field K. The importance of these algebras is that they are the universal examples of algebras which fail to have the *Invariant Basis Number* property; to wit, if $R = L_K(1,n)$, then the free left R-modules R and R^n are isomorphic. Once the definition of $L_K(E)$ is given for any graph E, we will recover $L_K(1,n)$ as $L(R_n)$, where R_n is the graph having one vertex and n loops at that vertex.

With the general definition of a Leavitt path algebra presented in Section 1.2 in hand, we give in Section 1.3 the three fundamental examples of Leavitt path algebras: the Leavitt algebras; full matrix rings over K; and the Laurent polynomial algebra $K[x,x^{-1}]$. These three types of Leavitt path algebras will provide the motivation and intuition for many of the general results in the subject.

The subject did not arise in a vacuum. Indeed, there are intimate connections between Leavitt path algebras and a powerful monoid-realization result of Bergman. As well, there are strong and historically significant connections between Leavitt path algebras and *graph* C*-*algebras*. We describe both of the connections in Section 1.4.

As we will see, there are natural modifications to the definition of a Leavitt path algebra which provide the data to construct a (seemingly) more general class of algebras, the *relative Cohn path algebras* $C_K^X(E)$ corresponding to a graph E, a subset X of the vertices of E, and field K. Although the class of relative Cohn path algebras contain as specific examples the class of Leavitt path algebras, we will see in Section 1.5 that every relative Cohn path algebra $C_K^X(E)$ is in fact isomorphic to the Leavitt path algebra $L_K(E(X))$ for some germane graph $E(X)$.

Although the motivating examples of Leavitt path algebras arise from finite graphs, the definition of $L_K(E)$ allows for the construction even when E is infinite. Indeed, much of the interesting work and many of the applications-related results about Leavitt path algebras arise in the situation where E is infinite. We show in Section 1.6 that, perhaps surprisingly, every Leavitt path algebra may be viewed as a direct limit (in an appropriate category) of Leavitt path algebras associated to finite graphs.

We conclude the chapter by presenting in Section 1.7 a brief historical overview of the subject.

{LeavittalgebrasSec}

1.1 A motivating construction: the Leavitt algebras

A student's first exposure to the theory of rings more than likely involves a study of various "basic examples", typically including fields, $\mathbb{Z}$, matrix rings over fields, and polynomial rings with coefficients in a field. It is not hard to show that each of these rings R has the *Invariant Basis Number (IBN) property* :

IBN: If m and m' are positive integers with the property that
the free left modules R^m and $R^{m'}$ are isomorphic, then $m = m'$.

Less formally, a ring has the IBN property (more succinctly: *is IBN*) in case any two bases (i.e., linearly independent spanning sets) of any free left R-module have the same number of elements. It turns out that many general classes of rings have this property (e.g., noetherian rings and commutative rings), classes of rings which include all of the basic examples with which the student first made acquaintance. (Typically, the student would have encountered the fact that the field of real numbers has the IBN property in an undergraduate course on linear algebra.)

Unfortunately, since all of the examples the student first encounters have the IBN property, the student more than likely is left with the wrong impression, as there are many important classes of rings which are not IBN. Perhaps the most common such example is the ring $B = \mathrm{End}_K(V)$, where V is an infinite dimensional vector space over a field K. Then B is not IBN (with a vengeance!): it is not hard to show that the free left B-modules B^m and $B^{m'}$ are isomorphic for *all* positive integers m, m'.

{moduletypeDef}

Definition 1.1.1. Suppose R is not IBN. Let $m \in \mathbb{N}$ be minimal with the property that $R^m \cong R^{m'}$ as left R-modules for some $m' > m$. For this m, let n denote the minimal such m'. In this case we say that R has *module type* (m,n).

So, for example, $B = \mathrm{End}_K(V)$ has module type $(1,2)$. We note that in the definition of module type it is easy to show that the same m,n arise if one considers free right R-modules, rather than left.

As we shall see, there is a perhaps surprising amount of structure inherent in non-IBN rings. To start with, in the groundbreaking article [100], Leavitt proves the following fundamental result.

{LeavittalgebraThm}

Theorem 1.1.2. *For each pair of positive integers $n > m$ and field K there exists a unital K-algebra $L_K(m,n)$, unique up to K-algebra isomorphism, such that:*

(i) *$L_K(m,n)$ has module type (m,n), and*
(ii) *for each unital K-algebra A having module type (m,n) there exists a unit-preserving K-algebra homomorphism $\phi : L_K(m,n) \to A$ which satisfies certain (natural) compatibility conditions.*

Our motivational focus here is on non-IBN rings of module type $(1,n)$ for some $n > 1$. In particular, such a ring then has the property that there exist isomorphisms of free modules

$$\phi \in \mathrm{Hom}_R(R^1, R^n) \text{ and } \psi \in \mathrm{Hom}_R(R^n, R^1), \quad \text{for which} \quad \psi \circ \phi = \iota_R \text{ and } \phi \circ \psi = \iota_{R^n},$$

where ι denotes the identity map on the appropriate module. Using the usual interpretation of homomorphisms between free modules as matrix multiplications (a description which the student encounters for the real numbers in an undergraduate linear algebra course, and which is easily shown to be valid for any unital ring), we see that such isomorphisms exist if and only if there exist $1 \times n$ and $n \times 1$ R-vectors

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \text{ and } \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

$$\text{for which} \quad \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = (1_R) \text{ and } \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} = \begin{pmatrix} 1_R & 0 & \cdots & 0 \\ 0 & 1_R & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1_R \end{pmatrix}.$$

Rephrased,

$${}_R R^1 \cong {}_R R^n \text{ for some } n > 1$$

if and only if there exist $2n$ elements $x_1,\ldots,x_n,y_1,\ldots,y_n$ of R for which

{Leavitteq}

$$\sum_{i=1}^{n} x_i y_i = 1_R \quad \text{and} \quad y_i x_j = \delta_{ij} 1_R \ \text{(for all } 1 \le i,j \le n). \tag{1.1}$$

The relations displayed in (1.1) provide the key idea in constructing the Leavitt algebras, and will play a central role in motivating the subsequent more general construction of Leavitt path algebras. For example, in the ring $B = \mathrm{End}_K(V)$ having module type $(1,2)$, it is straightforward to describe a set x_1,x_2,y_1,y_2 of $2\cdot 2 = 4$ elements of R which behave in this way.

Indeed, given $n > 1$, it is relatively easy to construct an algebra A which contains $2n$ elements behaving as do those in (1.1). Specifically, let K be any field, let

$$S = K\langle X_1,\ldots,X_n,Y_1,\ldots,Y_n\rangle$$

be the free associative K-algebra in $2n$ non-commuting variables, let I denote the ideal of S generated by the relations

$$I \ = \ \langle \sum_{i=1}^{n} X_i Y_i - 1,\ Y_i X_j - \delta_{ij} 1 \ \mid \ 1 \le i,j \le n\rangle,$$

and let

$$A = S/I.$$

Then the set $\{x_i = \overline{X_i}, y_j = \overline{Y_j} \mid 1 \le i,j \le n\}$ behaves in the desired way (by construction), so that $A^1 \cong A^n$ as left A-modules.

At this point one must be careful: although we have just constructed a K-algebra A for which $A^1 \cong A^n$, we cannot conclude that the module type of A is $(1,n)$ until we can guarantee the minimality of n. (For instance, it's not immediately clear that the algebra $A = S/I$ is necessarily nonzero.) But this is precisely what Leavitt establishes in [100]. Indeed, the K-algebra $L_K(1,n)$ of Theorem 1.1.2 is exactly the algebra $A = S/I$ constructed here. We formalize this in the following.

{LeavittalgebraDef}

Definition 1.1.3. Let K be any field, and $n > 1$ any integer. Then the *Leavitt K-algebra of type* $(1,n)$, denoted $L_K(1,n)$, is the K-algebra

$$K\langle X_1,\ldots,X_n,Y_1,\ldots,Y_n\rangle \ / \ \langle \sum_{i=1}^{n} X_i Y_i - 1,\ Y_i X_j - \delta_{ij} 1 \ \mid \ 1 \le i,j \le n \rangle.$$

Notationally, it is often more convenient to view $R = L_K(1,n)$ as the free associative K-algebra on the $2n$ variables $x_1,\ldots,x_n,y_1,\ldots,y_n$, subject to the relations $\sum_{i=1}^{n} x_i y_i = 1_R$ and $y_i x_j = \delta_{ij} 1_R$ $(1 \le i,j \le n)$. Specifically, $L_K(1,n)$ is the universal K-algebra of type $(1,n)$.

We summarize our discussion thus far. Although non-IBN rings might seem exotic on first sight, they in fact occur naturally. Non-IBN rings having module type $(1,n)$ can be constructed with relative ease. The key ingredient to produce such rings is the existence of elements $x_1,\ldots,x_n,y_1,\ldots,y_n$ for which the relations displayed in (1.1) are satisfied.

For those readers curious about the previous "surprising amount of structure" comment, we conclude this section with the following morsel of supporting evidence, established by Leavitt in [101].

{LeavittalgebrasimpleThm}

Theorem 1.1.4. *For all $n \ge 2$, and for any field K, $L_K(1,n)$ is a simple K-algebra.*

This remarkable result will in fact follow as a corollary of the more general results presented in Chapter 2.

{LPADefinitionSection}

1.2 Leavitt path algebras

With the construction of the Leavitt algebras $L_K(1,n)$ as motivational backdrop, we are nearly in position to present the central idea of this book, the Leavitt path algebras. We start by setting some basic notation and definitions.

{ringtheorynotation}

Notation 1.2.1. If K is a field, then by $K^\times$ we denote the nonzero elements of K, i.e., the invertible elements. $\mathbb{Z}$ denotes the set of integers; $\mathbb{Z}^+ = \{0,1,2,\dots\}$; $\mathbb{N} = \{1,2,3,\dots\}$.

{graphDef}

Definitions 1.2.2 A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1 and two functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. We place no restriction on the cardinalities of E^0 and E^1, nor on properties of the functions r and s. Throughout, the word "graph" will always mean "directed graph".

If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. A vertex v for which $s^{-1}(v) = \emptyset$ is called a *sink*, while a vertex v for which $r^{-1}(v) = \emptyset$ is called a *source*. In other words, v is a sink (resp., source) if v is not the source (resp., range) of any edge of E. A vertex which is both a source and a sink is called *isolated*. A vertex v such that $|s^{-1}(v)|$ is infinite is called an *infinite emitter*. If v is either a sink or an infinite emitter, we call v a *singular vertex*; otherwise, v is called a *regular vertex*. The expressions $\mathrm{Sink}(E)$, $\mathrm{Source}(E)$, $\mathrm{Isol}(E)$, $\mathrm{Reg}(E)$, and $\mathrm{Inf}(E)$ will be used to denote, respectively, the sets of sinks, sources, isolated vertices, regular vertices, and infinite emitters of E.

A *path* μ in a graph E is a sequence of edges $\mu = e_1, e_2, \dots, e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) = s(e_1)$ is the *source* of μ, $r(\mu) = r(e_n)$ is the *range* of μ, and $n = \ell(\mu)$ (or $n = |\mu|$) is the *length* of μ. We typically denote μ by using the more efficient notation $e_1 e_2 \cdots e_n$. We view the vertices of E as paths of length 0; to streamline notation, we will sometimes extend the functions s and r to E^0 by defining $s(v) = r(v) = v$ for $v \in E^0$. If $\mu = e_1 e_2 \cdots e_n$ is a path then we denote by μ^0 the set of its vertices, that is, $\mu^0 = \{s(e_1), r(e_i) \mid 1 \le i \le n\}$. For $n \ge 2$ we define E^n to be the set of paths in E of length n, and define $\mathrm{Path}(E) = \bigcup_{n\ge 0} E^n$, the set of all paths in E.

Here now are the main objects of our desire.

{LPADef}

Definition 1.2.3. (**Leavitt path algebras**) Let E be an arbitrary (directed) graph and K any field. We define a set $(E^1)^*$ consisting of symbols of the form $\{e^* \mid e \in E^1\}$. The *Leavitt path algebra of E with coefficients in K*, denoted $L_K(E)$, is the free associative K-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$, subject to the following relations:

(V) $vv' = \delta_{v,v'} v$ for all $v, v' \in E^0$,
(E1) $s(e)e = er(e) = e$ for all $e \in E^1$,
(E2) $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$,
(CK1) $e^* e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$, and
(CK2) $v = \sum_{\{e \in E^1 \mid s(e) = v\}} ee^*$ for every regular vertex $v \in E^0$.

Phrased another way, $L_K(E)$ is the free associative K-algebra on the symbols $E^0 \cup E^1 \cup (E^1)^*$, modulo the ideal generated by the five types of relations indicated in the previous list.

{LPAextendedGraph}

Remark 1.2.4. There is a connection between the classical notion of *path algebras* and the notion of Leavitt path algebras, which we describe here. As a brief reminder, if K is a field and $G = (G^0, G^1)$ is a directed graph then the *path K-algebra of G*, denoted KG, is defined as the free associative K-algebra generated as an algebra by the set $G^0 \cup G^1$, with relations given by (V) and (E1) of Definition 1.2.3. Equivalently, KG is the K-algebra having $\mathrm{Path}(G)$ as basis, and in which multiplication is defined by the K-linear extension of path concatenation (i.e., $p \cdot q = pq$ if $r(p) = s(q)$, 0 otherwise).

Given a graph E, we define the *extended graph of E* (also sometimes called the *double graph of E*) as the new graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$, where $(E^1)^* = \{e^* \mid e \in E^1\}$, and the functions r' and s' are defined as

$$r'|_{E^1} = r,\ s'|_{E^1} = s,\ r'(e^*) = s(e),\ \text{and } s'(e^*) = r(e) \text{ for all } e \in E^1.$$

(In other words, each edge e^* in $(E^1)^*$ has orientation the reverse of that of its counterpart $e \in E^1$.) Then $L_K(E)$ is the quotient of the path K-algebra $K\widehat{E}$ by the ideal of $K\widehat{E}$ generated by relations given in (CK1) and (CK2) of Definition 1.2.3.

{universalhomomorphismpr

Remark 1.2.5. (**The Universal Property of** $L_K(E)$) Suppose E is a graph, and A is a K-algebra which contains a set of pairwise orthogonal idempotents $\{a_v \mid v \in E^0\}$, and two sets $\{a_e \mid e \in E^1\}$, $\{b_e \mid e \in E^1\}$ for which

(1) $a_{s(e)}a_e = a_e a_{r(e)} = a_e$ and $a_{r(e)}b_e = b_e a_{s(e)} = b_e$ for all $e \in E^1$,
(2) $b_f a_e = \delta_{e,f} a_{r(e)}$ for all $e, f \in E^1$, and
(3) $a_v = \sum_{\{e \in E^1 | s(e)=v\}} a_e b_e$ for every regular vertex $v \in E^0$.

We call such a family an *E-family* in A. By the relations defining the Leavitt path algebra, there exists a unique K-algebra homomorphism $\varphi : L_K(E) \to A$ such that $\varphi(v) = a_v$, $\varphi(e) = a_e$, and $\varphi(e^*) = b_e$ for all $v \in E^0$ and $e \in E^1$. We will often refer to this as the *Universal Property* of $L_K(E)$.

{ghostNotation}

Notation 1.2.6. We sometimes refer to the edges in the graph E as the *real edges*, and the additional edges of $\widehat{E}$ (i.e., the elements of $(E^1)^*$) as the *ghost edges*. If $\mu = e_1 e_2 \cdots e_n$ is a path in E, then the element $e_n^* \cdots e_2^* e_1^*$ of $L_K(E)$ is denoted by μ^*.

Remark 1.2.7. Less formally (but no less accurately), one may view the Leavitt path algebra $L_K(E)$ as follows. Consider the standard path algebra $K\widehat{E}$ of the extended graph. Then impose on $K\widehat{E}$ the following relations:

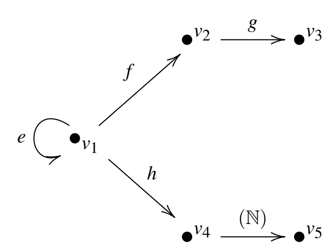
(1) If e is an edge of E, we replace any expression of the form e^*e in $K\widehat{E}$ by the vertex $r(e)$.
(2) If e and f are distinct edges in E, then we define $e^*f = 0$ in $K\widehat{E}$.
(3) If v is a regular vertex, then the sum over *all* terms of the form ee^* for which $s(e) = v$ is replaced by v in $K\widehat{E}$.

The resulting algebra is precisely $L_K(E)$.

{graphExample}

In the standard pictorial description of a directed graph E, we use the notation $\bullet^v \xrightarrow{(n)} \bullet^w$ to indicate that there are n distinct edges e_i in E for which $s(e_i) = v$ and $r(e_i) = w$; the value of n may be finite or infinite.

Example 1.2.8. An example will no doubt help clarify the definition of a Leavitt path algebra. Let E be the graph pictorially described by



Here are some representative computations in $L_K(E)$ (for any field K).

$$v_1 f = f = f v_2 \text{ by (E1), while } v_2 f^* = f^* = f^* v_1 \text{ by (E2)}$$

$$f^* f = v_2, \text{ while } f^* h = f^* e = 0 \text{ both by (CK1)}$$

$$v_1 = ee^* + ff^* + hh^* \text{ by (CK2)}$$

$$gg^* = v_2 \text{ by (CK2) (the sum contains only one term)}$$

We observe that there is no (CK2) relation at v_4 (as $v_4 \in \mathrm{Inf}(E)$); neither is there a (CK2) relation at the sinks v_3 and v_5.

Remark 1.2.9. We note that the construction of the Leavitt path algebra for a graph E over a field K can be extended in the obvious way to the construction of the Leavitt path *ring* for a graph E over an arbitrary unital ring R. (See for example [136], where the author studies Leavitt path algebras with coefficients in a commutative ring.)

The existence of a multiplicative identity in $L_K(E)$ depends on whether or not E^0 is finite (see Lemma 1.2.12 below). But even in nonunital situations, there is still much structure to be exploited.

{enoughidempotentsdef}

Definition 1.2.10. An associative ring R is said to have a *set of local units* F in case F is a set of idempotents in R having the property that, for each finite subset $r_1,\dots,r_n$ of R, there exists $f \in F$ for which $fr_if = r_i$ for all $1 \le i \le n$. Rephrased, a set of idempotents $F \subseteq R$ is a set of local units for R in case each finite subset of R is contained in a (unital) subring of the form fRf for some $f \in F$.

An associative ring R is said to have *enough idempotents* in case there exists a set of nonzero orthogonal idempotents E in R for which ${}_RR = \oplus_{e\in E} Re$ as left R-modules.

It is easy to show that any ring with enough idempotents E is necessarily a ring with local units, where the set F can be taken as the set of sums of distinct elements of E.

For a ring with local units, an abelian group M is a left R-module in case there is a (standard) module action of R on M, but with the added proviso that $RM = M$. (This is the appropriate generalization of the requirement that $1_R \cdot m = m$ for all m in a module M over a unital ring R.)

For a field K, a ring R with local units is said to be a *K-algebra* in case R is a K-vector space (with scalar action $\cdot$), and $(k\cdot r)s = k\cdot(rs)$ for all $k \in K$, $r,s \in R$.

{ringIdeal}

Remark 1.2.11. In any K-algebra R with local units, every (one-sided, resp., two-sided) ring ideal of R is a (one-sided, resp., two-sided) K-algebra ideal of R. This is easy to see: for instance, let I be a ring left ideal of R, let $k \in K$ and $y \in I$. Let $u \in R$ with $y = uy$. Then $ky = k(uy) = (ku)y \in RI \subseteq I$.

We give now some basic properties of the elements of $L_K(E)$.

{basicpropertiesLem}

Lemma 1.2.12. *Let E be an arbitrary graph and K any field. Let γ,λ,μ,ρ be elements of* $\mathrm{Path}(E)$.

(i) *Products of monomials in $L_K(E)$ are computed here:*

$$(\gamma\lambda^*)(\mu\rho^*) = \begin{cases} \gamma\kappa\rho^* & \textit{if} \quad \mu = \lambda\kappa \textit{ for some } \kappa \in \mathrm{Path}(E) \\ \gamma\sigma^*\rho^* & \textit{if} \quad \lambda = \mu\sigma \textit{ for some } \sigma \in \mathrm{Path}(E) \\ 0 & \textit{otherwise} \end{cases}$$

In particular, if $\ell(\lambda) = \ell(\mu)$, then $\lambda^*\mu \ne 0$ if and only if $\lambda = \mu$, in which case $\lambda^*\mu = r(\lambda)$.

(ii) *The K-action on the algebra $L_K(E)$ is trivial; that is,*

$$(k\gamma\lambda^*)(k'\mu\rho^*) = kk'(\gamma\lambda^*\mu\rho^*)$$

for $k,k' \in K$.

(iii) *The algebra $L_K(E)$ is spanned as a K-vector space by the set of monomials of the form*

$$\{\gamma\lambda^* \mid \gamma,\lambda \in \mathrm{Path}(E) \textit{ for which } r(\gamma) = r(\lambda)\}.$$

In other words, every nonzero element x of $L_K(E)$ may be expressed as

$$x = \sum_{i=1}^n k_i\gamma_i\lambda_i^*,$$

where $k_i \in K^\times$, and $\gamma_i,\lambda_i \in \mathrm{Path}(E)$ with $r(\gamma_i) = r(\lambda_i)$ for each $1 \le i \le n$. We note that this representation is not unique; i.e., the displayed monomials do not form a basis of $L_K(E)$.

(iv) *The algebra $L_K(E)$ is unital if and only if E^0 is finite. In this case,*

$$1_{L_K(E)} = \sum_{v\in E^0} v.$$

(v) *For each $\alpha \in L_K(E)$ there exists a finite set of distinct vertices $V(\alpha)$ for which $\alpha = f\alpha f$, where $f = \sum_{v\in V(\alpha)} v$. Moreover, the algebra $L_K(E)$ is a ring with enough idempotents (consisting of the vertices E^0), and thus a ring with local units (consisting of sums of distinct elements of E^0).*

Proof. (i) By (CK1), any expression of the form e^*f in $L_K(E)$ reduces either to 0 or to the vertex $r(e)$, from which the statement follows by a straightforward computation.

(ii) follows directly from the definition of $L_K(E)$ as the free K-algebra on various generators.

(iii) follows easily from (i).

(iv) For E^0 finite, the indicated element acts as the identity by the representation of elements of $L_K(E)$ given in (iii). If E^0 is infinite, then there is no element of $L_K(E)$ which acts as an identity on each element of the set $\{v \mid v \in E^0\}$.

(v) By the orthogonality given in Definition 1.2.3(V), it is clear that any sum of distinct vertices in $L_K(E)$ yields an idempotent. Now let $\alpha = \sum_{i=1}^{m} k_i \gamma_i \lambda_i^*$ be an arbitrary element of $L_K(E)$, and let $V(\alpha)$ denote the (finite) set of vertices which appear either as $s(\gamma_i)$ or as $s(\lambda_i)$ for some $1 \le i \le n$. If we define $f = \sum_{v \in V(\alpha)} v$, then an easy computation yields that $\alpha = f\alpha f$. The additional statements follow in the same manner. □

Definitions 1.2.13 We say that a graph E is *connected* if $\widehat{E}$ is a connected graph in the usual sense, that is, if given any two vertices $u, v \in E^0$ there exist $h_1, h_2, \ldots, h_m \in E^1 \cup (E^1)^*$ such that $\eta = h_1 h_2 \cdots h_m$ is a path in $\widehat{E}$ such that $s(\eta) = u$ and $r(\eta) = v$. The *connected components* of a graph E are the graphs $\{E_i\}_{i \in \Lambda}$ such that E is the disjoint union $E = \sqcup_{i \in \Lambda} E_i$, where every E_i is connected. {ConnectedComp}

We close the section by recording the following observation, which is easily verified utilizing the Universal Property of $L_K(E)$ (Remark 1.2.5).

Proposition 1.2.14. *Let E be an arbitrary graph and K any field. Suppose $E = \sqcup_{i \in \Lambda} E_i$ is a decomposition of E into its connected components. Then $L_K(E) \cong \oplus_{i \in \Lambda} L_K(E_i)$.*

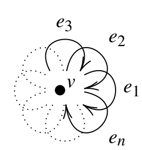
1.3 The three fundamental examples of Leavitt path algebras

{LPAExamplesSection}

Part of the beauty of the Leavitt path algebras is that they include many well-known, but seemingly disparate, classes of algebras. To make these connections clear, we introduce some notation which will be used throughout. {basicgraphsnotation}

Notation 1.3.1. We let R_n denote the *rose with n petals* graph having one vertex and n loops:

R_n =



In particular, a special role in the theory is played by the graph R_1:

R_1 =

For any $n \in \mathbb{N}$ we let A_n denote the *oriented n-line graph* having n vertices and $n-1$ edges:

$$A_n \;=\; \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

The examples presented in the following three propositions may be viewed as the three primary colors of Leavitt path algebras. Making good now on a promise offered earlier, we validate our claim that the Leavitt algebras $L_K(1,n)$ are truly motivating examples for the more general notion of Leavitt path algebra. {LeavittalgebraisL(Rn)}

Proposition 1.3.2. *Let $n \ge 2$ be any positive integer, and K any field. Let $L_K(1,n)$ be the Leavitt K-algebra of type $(1,\ n)$ presented in Definition 1.1.3, and let R_n be the rose with n petals. Then*

$$L_K(1,n) \cong L_K(R_n).$$

Proof. That these two algebras are isomorphic follows directly from the definition of $L_K(1,n)$ as a quotient of the free associative algebra on $2n$ variables, modulo the relations given in display (1.1). Specifically, we map $x_i \mapsto e_i$ and $y_i \mapsto e_i^*$. Then the relations given in (1.1) are precisely the relations provided by the (CK1) and (CK2) relations of Definition 1.2.3. $\square$

The rose with one petal produces a more-familiar (although less-exotic) algebra. Prior to the description of $L_K(R_1)$, the following remark is very much in order.

{eestarRem}

Remark 1.3.3. If E is a graph and $e \in E^1$, then the element ee^* of $L_K(E)$ is always an idempotent, since using (CK1) we have $(ee^*)(ee^*) = e(e^*e)e^* = er(e)e^* = ee^*$. However, ee^* does *not* equal $s(e)$ unless e is the only edge emitted by $s(e)$ (since in that case the (CK2) relation reduces to the equation $s(e) = ee^*$).

For any field K, the *Laurent polynomial K-algebra* is the associative K-algebra generated by the two symbols x and y, with relations $xy = yx = 1$. For obvious reasons this algebra is denoted by $K[x,x^{-1}]$. The elements of $K[x,x^{-1}]$ may be written as $\sum_{i=m}^{n} k_i x^i$ (where $k_i \in K$ and $m \le n \in \mathbb{Z}$); note in particular that the exponents are allowed to include negative integers. Viewed another way, $K[x,x^{-1}]$ is the group algebra of $\mathbb{Z}$ over K.

{LaurantalgebraisL(R1)}

Proposition 1.3.4. *Let K be any field. Then*

$$K[x,x^{-1}] \cong L_K(R_1).$$

Proof. By the (CK1) relation and Lemma 1.2.12(iv) we have $x^*x = v = 1$ in $L_K(R_1)$. But since v emits only the edge x, Remark 1.3.3 yields $xx^* = v = 1$ in $L_K(R_1)$ as well, and the result now follows. $\square$

The third of the three primary colors of Leavitt path algebras moves us from the less-exotic $K[x,x^{-1}]$ to the almost-mundane matrix algebras $\mathrm{M}_n(K)$.

{matrixalgebraisL(An)}

Proposition 1.3.5. *Let K be any field, and $n \ge 1$ any positive integer. Then*

$$\mathrm{M}_n(K) \cong L_K(A_n).$$

Proof. Let $\{f_{i,j} \mid 1 \le i,j \le n\}$ denote the standard matrix units in $\mathrm{M}_n(K)$. We define the map $\varphi : L_K(A_n) \to \mathrm{M}_n(K)$ by setting $\varphi(v_i) = f_{i,i}$, $\varphi(e_i) = f_{i,i+1}$, and $\varphi(e_i^*) = f_{i+1,i}$. Using Remark 1.3.3, it is then easy to check that φ is an isomorphism of K-algebras as desired. $\square$

The title of this section notwithstanding, we provide a fourth example of a well-known classical algebra which arises as a specific example of a Leavitt path algebra.

{toeplizexample}

Example 1.3.6. The *Toeplitz graph* is the graph

$$E_T \quad = \quad \begin{tikzcd} \bullet^u \arrow[loop left, "e"] \arrow[r, "f"] & \bullet^v \end{tikzcd} \quad .$$

Let K be any field. We denote by $\mathscr{T}_K$ the *algebraic Toeplitz K-algebra*

$$\mathscr{T}_K = L_K(E_T).$$

{ToeplitzisJacobsonProp}

Proposition 1.3.7. *For any field K, the Leavitt path algebra $L_K(E_T)$ is isomorphic to the free associative K-algebra $K\langle x,y\rangle$, modulo the single relation $xy = 1$. Rephrased, the algebraic Toeplitz K-algebra $\mathscr{T}_K$ is the K-algebra $K\langle U,V\rangle$ investigated by Jacobson in* [88].

Proof. We begin by noting that in $L_K(E_T)$ we have the relations $ee^* + ff^* = u$ and $u + v = 1$. We consider the elements $X = e^* + f^*$ and $Y = e + f$ of $L_K(E_T)$. Then by (CK1) we have $XY = u + v = 1$, while $YX = ee^* + ff^* = u \neq 1$ by (CK1) and (CK2). The subalgebra of $\mathscr{T}_K = L_K(E_T)$ generated by X and Y then contains $1 - u = v$, which in turn gives that this subalgebra contains $e = Yu$, $f = Yv$, $e^* = uX$, and $f^* = vX$. These observations establish that the map $\varphi : K\langle U,V\rangle \to L_K(E_T)$ given by the extension of $\varphi(U) = e^* + f^*, \varphi(V) = e + f$ is a surjective K-algebra homomorphism. The injectivity of φ will follow from results in Section 1.5; see specifically Example 1.5.20. $\square$

{ConnectionsSection}

1.4 Connections and motivations: the algebras of Bergman, and graph C*-algebras

In presenting a description of the Leavitt algebras $L_K(1,n)$ in the very first section of this book, our intent was to provide some sort of "natural" motivation for the relations which define the more general Leavitt path algebras. In this section we present two additional avenues which lead in a natural way to the description of Leavitt path algebras. The first such avenue takes us through a description of the finitely generated projective modules over a ring, while the second provides an expedition through the world of C*-algebras. These two topics will be explored much more extensively, and in more generality, in Chapters 3 and 5 respectively.

{V(R)Def}

Definition 1.4.1. Let R be any unital ring. We denote by $\mathscr{V}(R)$ the semigroup whose elements are the isomorphism classes of the finitely generated projective left R-modules, with operation given by $[P]+[Q]=[P\oplus Q]$.

Clearly $\mathscr{V}(R)$ is a commutative monoid for any ring R, with zero element $[\{0\}]$. In addition, it is apparent that $\mathscr{V}(R)$ has the property that

{BergmanCondition1}

$$x+y=[\{0\}] \text{ in } \mathscr{V}(R) \ \text{ if and only if } \ x=y=[\{0\}]. \tag{1.2}$$

Since R is assumed here to be unital (we will relax this requirement later), then each finitely generated projective left R-module is isomorphic to a direct summand of R^n for some integer n, so it is similarly apparent that the element $I=[R]$ of $\mathscr{V}(R)$ has the property that

{BergmanCondition2}

$$\forall\, x\in\mathscr{V}(R)\ \ \exists\, y\in\mathscr{V}(R) \text{ and } n\in\mathbb{N} \text{ for which } x+y=nI. \tag{1.3}$$

In a groundbreaking construction conceived and executed by Bergman in [51], it is shown that, in this context, anything that *can* happen in fact *does* happen. That is, if S is any finitely generated commutative monoid having the (necessary) properties described in displays (1.2) and (1.3), and K is any field, then there exists an explicitly constructed unital K-algebra R for which $\mathscr{V}(R)\cong S$. Moreover, this K-algebra is universal in the sense that, for any unital K-algebra T having $\mathscr{V}(T)\cong S$, then there exists a nonzero homomorphism $\varphi: R\to T$ which induces the identity on S.

We now define, for any graph E, an associated semigroup M_E; with the previous three sections in mind, the relations which describe M_E should seem familiar.

{MsubEDef}

Definition 1.4.2. Let E be an arbitrary graph. We denote by M_E the free abelian monoid on a set of generators $\{a_v \mid v\in E^0\}$, modulo relations given by

$$a_v=\sum_{\{e\in E^1 \mid s(e)=v\}} a_{r(e)} \tag{1.4}$$

for each $v\in \mathrm{Reg}(E)$.

So to any graph E we can associate the semigroup M_E, and to any graph E and field K we can associate the semigroup $\mathscr{V}(L_K(E))$. We will prove the following in Chapter 3; this result shows that these two semigroups are intimately related.

{MsubEisoV(L(E))Theorem}

Theorem 1.4.3. *Let E be any row-finite graph and K any field. Then, using the presentation of the semigroup M_E given in Definition 1.4.2, $L_K(E)$ is precisely the universal K-algebra corresponding to the semigroup M_E as guaranteed by Bergman's Theorem [51, Theorem 6.2]. In particular,*

$$\mathscr{V}(L_K(E))\cong M_E.$$

The upshot of this discussion is that, with the Leavitt algebras $L_K(1,n)$ having been presented as our first motivational offering, there is now a second motivating description of the Leavitt path algebras (arising from row-finite graphs): they are precisely the universal K-algebras which arise in [51, Theorem 6.2] for semigroups of the form M_E. This is no small conclusion, in the sense that for general commutative monoids which satisfy displayed conditions (1.2) and (1.3), it is rare that one can so explicitly describe the corresponding universal K-algebras.

In fact, the Leavitt algebras $L_K(1,n)$ play a basic role in Bergman's analysis. Specifically, let $\mathbb{Z}_{n-1}$ be the standard cyclic group of order $n-1$, and let S be the semigroup $\mathbb{Z}_{n-1} \cup \{z\}$ where $z+g = g = g+z$ for all $g \in S$. Then S is a commutative monoid satisfying (1.2) and (1.3) above, and $L_K(1,n)$ is the universal K-algebra corresponding to S. We will investigate this construction much more deeply in Chapter 3.

{graphCstarDef}

And now for something completely different. While the next few paragraphs (and various subsequent portions of this book) discuss the notion of a C*-algebra, readers may choose to skip these portions while still gaining an in-focus picture of Leavitt path algebras. In any event, it behooves us to remark that C*-algebras are always algebras in the usual ring-theoretic sense over the field of complex numbers $\mathbb{C}$.

Definitions 1.4.4 Let E be an arbitrary graph. (In the following context it is typically assumed that the sets E^0 and E^1 are at most countable, but we need not make those assumptions here.) A *Cuntz-Krieger E-family* in a C*-algebra B consists of a set of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and a set of partial isometries $\{s_e \mid e \in E^1\}$ satisfying

$$s_e^* s_e = p_{r(e)} \text{ for } e \in E^1, \ p_v = \sum_{\{e \ \mid \ s(e)=v\}} s_e s_e^* \ \text{ whenever } v \in \mathrm{Reg}(E), \text{ and } s_e s_e^* \leq p_{s(e)} \text{ for } e \in E^1.$$

It is shown in [93] that there is a C*-algebra $C^*(E)$, called the *graph C^*-algebra of E*, generated by a universal Cuntz-Krieger E-family $\{s_e, p_v\}$; in other words, for every Cuntz-Krieger E-family $\{t_e, q_v\}$ in a C*-algebra B, there is a homomorphism $\pi = \pi_{t,q} : C^*(E) \to B$ such that $\pi(s_e) = t_e$ and $\pi(p_v) = q_v$ for all $e \in E^1, v \in E^0$.

The relations presented in Definitions 1.4.4 clearly smack of those which generate the Leavitt path algebras, so it is probably not surprising that there is a strong connection between the structures $L_{\mathbb{C}}(E)$ and $C^*(E)$. In fact, we will show in Chapter 5 that $L_{\mathbb{C}}(E)$ embeds as a $\mathbb{C}$-algebra inside $C^*(E)$ in a natural way, and that $C^*(E)$ may be realized as the completion of $L_{\mathbb{C}}(E)$ in an appropriate topology.

The main point to be made here is that the Leavitt path $\mathbb{C}$-algebra $L_{\mathbb{C}}(E)$ can be realized and motivated as an algebraic foundation upon which $C^*(E)$ can be built. We will note often throughout the later chapters that while there are striking (indeed, *compellingly mysterious*) similarities amongst some of the results pertaining to the two structures $L_{\mathbb{C}}(E)$ and $C^*(E)$, there are other situations in which perhaps-anticipated parallels between this structures are indeed different. Further, while the Leavitt path $\mathbb{C}$-algebra $L_{\mathbb{C}}(E)$ is then naturally motivated by the $\mathbb{C}$-algebra $C^*(E)$ in this way, we shall see that the structural properties of $L_{\mathbb{C}}(E)$ typically pass to identical structural properties of $L_K(E)$ for any field K.

As of the writing of this book, there is no vehicle which allows one to easily establish results on the algebra side as direct consequences of results on the analytic side, or vice versa.

{Cohnpathalgsec}

1.5 The Cohn path algebras and connections to Leavitt path algebras

In the previous section we focused on two different constructions, both of which naturally led to the construction of Leavitt path algebras: the "realization algebras" of Bergman, and the graph C*-algebras. In this section we present a third construction, the *relative Cohn path algebras* $C_K^X(E)$, and specifically the Cohn path algebras $C_K(E)$, which also can be used to produce Leavitt path algebras.

The relative Cohn path algebras will serve two main purposes here. First, it will be trivial to show that every Leavitt path algebra is a quotient of a relative Cohn path algebra by an appropriately defined ideal. As will become apparent, the vector space structure of a Cohn path algebra is straightforward (e.g., a basis of $C_K(E)$ is easy to describe). This structure in turn will allow us to almost seamlessly achieve various results about Leavitt path algebras simply by appealing to quotient-preserving properties. Second, the relative Cohn path algebras will allow us to further showcase the ubiquity of the Leavitt path algebras. Specifically, for any graph E we will show that each relative Cohn path algebra $C_K^X(E)$ (including $C_K(E)$ itself) is isomorphic to the Leavitt path algebra $L_K(F)$ of some graph F.

The motivational information given in the previous section was presented almost as an advertising teaser ("stay tuned for further details!", the hard work to be confronted in subsequent chapters). In contrast, our description and use of the relative Cohn path algebras will require us to get our hands dirty right away. We start with the most important of these.

Definition 1.5.1. Let E be an arbitrary graph and K any field. We define a set $(E^1)^*$ consisting of symbols of the form $\{e^* \mid e \in E^1\}$. The *Cohn path algebra of E with coefficients in K*, denoted by $C_K(E)$, is the free associative K-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$, subject to the relations given in (V), (E1), (E2), and (CK1) of Definition 1.2.3.

In other words, $C_K(E)$ is the algebra generated by the same symbols as those which generate $L_K(E)$, but on which we do *not* impose the (CK2) relation. Since by (CK1) we have $e^*f = \delta_{e,f}r(e)$ in $C_K(E)$ for $e, f \in E^1$ (and the lack of the (CK2) relation in $C_K(E)$ notwithstanding), it is easy to show that there is still some information to be had about expressions of the form ee^* in $C_K(E)$: namely, that the family $\{ee^* \mid e \in E^1\}$ is a set of orthogonal idempotents in $C_K(E)$. What we do not impose in $C_K(E)$ is any relationship between this family and the set of vertices E^0 in $C_K(E)$.

Remark 1.5.2. In a manner similar to the explanation given in Remark 1.2.4, another way of looking at Cohn path algebras is the following: $C_K(E)$ is the quotient of the path K-algebra over the extended graph $K\widehat{E}$ by the ideal of $K\widehat{E}$ generated by the relations given in (CK1).

In [60], P.M. Cohn introduced and studied the collection of K-algebras $\{U_K(1,n) \mid n \in \mathbb{N}\}$ (for any field K); these have come to be known as the *Cohn algebras*, and as such we have come to use the notation $C_K(1,n)$ for these. It is clear that for each $n \in \mathbb{N}$ we have $C_K(R_n) \cong C_K(1,n)$. Thus the algebras $C_K(1,n) \cong C_K(R_n)$ stand in relation to the more general Cohn path algebras in precisely the same way that the Leavitt algebras $L_K(1,n) \cong L_K(R_n)$ stand in relation to the more general Leavitt path algebras.

Remark 1.5.3. As with Leavitt path algebras, we can define analogously the *Cohn path ring* $C_R(E)$ for any unital ring R and graph E.

Example 1.5.4. The algebra investigated by Jacobson which was presented in Proposition 1.3.7 is the quintessential example of a Cohn path algebra. Specifically, the free associative K-algebra $K\langle U,V\rangle$ modulo the single relation $UV = 1$ is precisely the Cohn path algebra $C_K(R_1)$, where R_1 is as usual the graph with one vertex and one loop.

The following result follows directly from the definition of the indicated algebras.

Proposition 1.5.5. *Let E be an arbitrary graph and K any field. Let I be the ideal of the Cohn path algebra $C_K(E)$ generated by the set*

$$\{v - \sum_{e \in s^{-1}(v)} ee^* \mid v \in \mathrm{Reg}(E)\}.$$

Then

$$L_K(E) \cong C_K(E)/I$$

as K-algebras.

Unlike the situation in the Leavitt path algebras, inside the Cohn path algebras every element can be expressed *in a unique way* as a linear combination of the terms $\lambda\nu^*$, with λ and ν paths in E for which $r(\lambda) = r(\nu)$.

Proposition 1.5.6. *Let E be an arbitrary graph and K any field. Then*

$$\mathscr{B} = \{\lambda\nu^* \mid \lambda, \nu \in \mathrm{Path}(E), r(\lambda) = r(\nu)\}$$

is a K-basis of $C_K(E)$.

Proof. Let A be the K-vector space with basis $\mathscr{B}$. We define a bilinear product on A by the formula

$$(\lambda_1\nu_1^*)(\lambda_2\nu_2^*) = \begin{cases} \lambda_1\lambda_2'\nu_2^* & \text{if} \quad \lambda_2 = \nu_1\lambda_2' \text{ for some } \lambda_2' \in \mathrm{Path}(E) \\ \lambda_1(\nu_1')^*\nu_2^* & \text{if} \quad \nu_1 = \lambda_2\nu_1' \text{ for some } \nu_1' \in \mathrm{Path}(E) \\ 0 & \text{otherwise.} \end{cases}$$

To see that this gives the structure of an associative K-algebra on A we only need to check that $x = y$, where $x = (\lambda_1\nu_1^*)((\lambda_2\nu_2^*)(\lambda_3\nu_3^*))$ and $y = ((\lambda_1\nu_1^*)(\lambda_2\nu_2^*))(\lambda_3\nu_3^*)$. A tedious computation shows that

$$x = y = \begin{cases} \lambda_1\lambda_2'\lambda_3'\nu_3^* & \text{if} \quad \lambda_3 = \nu_2\lambda_3' \quad \text{and} \quad \lambda_2 = \nu_1\lambda_2' \\ \lambda_1\lambda_3'\nu_3^* & \text{if} \quad \lambda_3 = \nu_2\lambda_3''\lambda_3' \quad \text{and} \quad \nu_1 = \lambda_2\lambda_3'' \\ \lambda_1(\nu_1')^*\nu_3^* & \text{if} \quad \lambda_3 = \nu_2\lambda_3' \quad \text{and} \quad \nu_1 = \lambda_2\lambda_3'\nu_1' \\ \lambda_1\lambda_2'(\nu_2')^*\nu_3^* & \text{if} \quad \nu_2 = \lambda_3\nu_2' \quad \text{and} \quad \lambda_2 = \nu_1\lambda_2' \\ \lambda_1(\nu_1')^*(\nu_2')^*\nu_3^* & \text{if} \quad \nu_2 = \lambda_3\nu_2' \quad \text{and} \quad \nu_1 = \lambda_2\nu_1' \\ 0 & \text{otherwise.} \end{cases}$$

as desired. This clearly yields the result. □

Corollary 1.5.7. *Let E be an arbitrary graph and K any field. The restriction of the canonical projection $K\hat{E} \to C_K(E)$ is injective on the subspace generated by the paths in E and the paths in E^*. In particular the maps $KE \to C_K(E)$ and $KE^* \to C_K(E)$ are injective.*

Now we construct certain natural quotient algebras of Cohn path algebras. For $v \in \mathrm{Reg}(E)$, consider the following element q_v of $C_K(E)$:

$$q_v = v - \sum_{e \in s^{-1}(v)} ee^*.$$

Proposition 1.5.8. *The elements q_v are idempotents of $C_K(E)$. Moreover, $q_vC_K(E)q_w = \delta_{v,w}q_vK$ for each pair $v, w \in \mathrm{Reg}(E)$.*

Proof. A simple computation shows that $\{q_v \mid v \in \mathrm{Reg}(E)\}$ is a family of pairwise orthogonal idempotents in $C_K(E)$. Now let $v \in E^0$ and $f \in E^1$. If $f \notin s^{-1}(v)$ then $e^*f = 0$ for all $e \in s^{-1}(v)$. On the other hand, if $f \in s^{-1}(v)$ then $ee^*f = 0$ for $e \neq f$, while $ff^*f = f$. Thus we see that $\sum_{e \in s^{-1}(v)} ee^*f = vf$, and in a similar way that $\sum_{e \in s^{-1}(v)} f^*ee^* = f^*v$, for all $f \in E^1$. So

$$f^*q_v = 0 = q_vf \tag{1.5}$$

for all $f \in E^1$ and $v \in \mathrm{Reg}(E)$. This yields that $q_vC_K(E)q_w = Kq_vq_w = \delta_{v,w}q_vK$, as desired. □

Definition 1.5.9. Let E be an arbitrary graph and K any field. Let X be any subset of $\mathrm{Reg}(E)$. We denote by I^X the K-algebra ideal of $C_K(E)$ generated by the idempotents $\{q_v \mid v \in X\}$. The *Cohn path algebra of E relative to X*, denoted $C_K^X(E)$, is defined to be the quotient K-algebra

$$C_K(E)/I^X.$$

Clearly this notion of the relative Cohn path algebra links the Cohn and Leavitt path algebra constructions, as we see immediately that

$$C_K(E) = C^{\emptyset}(E) \ \text{ and } \ L_K(E) = C_K^{\mathrm{Reg}(E)}(E).$$

Generalizing the Universal Property for Leavitt path algebras (Remark 1.2.5), we have the following.

Remark 1.5.10. Suppose E is a graph, X is a subset of $\mathrm{Reg}(E)$, and A is a K-algebra which contains a set of pairwise orthogonal idempotents $\{a_v \mid v \in E^0\}$, and two sets $\{a_e \mid e \in E^1\}$, $\{b_e \mid e \in E^1\}$ for which

(1) $a_{s(e)}a_e = a_ea_{r(e)} = a_e$ and $a_{r(e)}b_e = b_ea_{s(e)} = b_e$ for all $e \in E^1$,

(2) $b_f a_e = \delta_{e,f} a_{r(e)}$ for all $e, f \in E^1$, and
(3) $a_v = \sum_{\{e \in E^1 \mid s(e) = v\}} a_e b_e$ for every vertex $v \in X$.

By the relations defining the relative Cohn path algebra, there exists a unique K-algebra homomorphism $\varphi : C_K^X(E) \to A$ such that $\varphi(v) = a_v$, $\varphi(e) = a_e$, and $\varphi(e^*) = b_e$ for all $v \in E^0$ and $e \in E^1$. We will often refer to this as the *Universal Property* of $C_K^X(E)$.

{basisCXE}

Proposition 1.5.11. *Let E be an arbitrary graph and K any field. Let X be a subset of* $\mathrm{Reg}(E)$*. Then a K-basis of I^X is given by the family $\lambda q_v \mu^*$, where $v \in X$ and $\lambda, \mu \in \mathrm{Path}(E)$ with $r(\lambda) = r(\mu) = v$. For $v \in X$ let $\{e_1^v, \dots, e_{n_v}^v\}$ be an enumeration of the elements of $s^{-1}(v)$. Then a K-basis of $C_K^X(E)$ is given by the family*

$$\mathscr{B}'' = \mathscr{B} \setminus \{\lambda e_{n_v}^v (e_{n_v}^v)^* \nu^* \mid r(\lambda) = r(\nu) = v\},$$

where $\mathscr{B} = \{\lambda \nu^* \mid r(\lambda) = r(\nu)\}$ is the canonical basis of $C_K(E)$ given in Proposition 1.5.6.

Proof. By the displayed equation (1.5), we have that the elements $\lambda q_v \mu^*$, for $v \in X$ and $\lambda, \mu \in \mathrm{Path}(E)$ with $r(\lambda) = v = r(\mu)$, generate I^X. To show that they are linearly independent, assume that there is an equation

$$\sum k_{\gamma,\mu} \gamma q_v \mu^* = 0$$

with $k_{\gamma,\mu} \in K$. Expressing the left hand side as a linear combination of monomials $\lambda \nu^*$, and using the linear independence of these monomials (Proposition 1.5.6), we immediately get $k_{\gamma,\mu} = 0$ for all γ, μ.

Let $\mathscr{B}'$ be the basis of I^X just constructed. To show the second part of the proposition, it is enough to prove that $\mathscr{B}' \cup \mathscr{B}''$ is a basis of $C_K(E)$. Clearly every element $\lambda \nu^*$ of the basis $\mathscr{B}$ of $C_K(E)$ can be written as a linear combination of the elements in $\mathscr{B}' \cup \mathscr{B}''$. On the other hand, any nonzero linear combination of elements in $\mathscr{B}'$ must involve (with a nonzero coefficient) a monomial of the form $\lambda e_{n_v}^v (e_{n_v}^v)^* \nu^*$, and so it cannot be a linear combination of elements in $\mathscr{B}''$. This shows that $\mathscr{B}' \cup \mathscr{B}''$ is a basis of $C_K(E)$. □

{basisforL(E)Cor}

As $L_K(E) = C_K^{\mathrm{Reg}(E)}(E)$, Proposition 1.5.11 immediately yields the following.

Corollary 1.5.12. *Let E be an arbitrary graph and K any field. Let $\mathscr{B} = \{\lambda \nu^* \mid r(\lambda) = r(\nu)\}$ be the canonical basis of $C_K(E)$ given in Proposition 1.5.6. For each vertex $v \in \mathrm{Reg}(E)$, let $\{e_1^v, \dots, e_{n_v}^v\}$ be an enumeration of the elements of $s^{-1}(v)$. Then a basis of $L_K(E)$ is given by the family*

$$\mathscr{B}'' = \mathscr{B} \setminus \{\lambda e_{n_v}^v (e_{n_v}^v)^* \nu^* \mid r(\lambda) = r(\nu) = v \in \mathrm{Reg}(E)\}.$$

Proposition 1.5.11 easily yields the following three consequences as well.

{injKE}

Corollary 1.5.13. *Let E be an arbitrary graph and K any field. The restriction of the canonical projection $K\hat{E} \to L_K(E)$ is injective on the subspace generated by the paths in E and the paths in E^*. In particular the maps $KE \to L_K(E)$ and $KE^* \to L_K(E)$ are injective.*

{tensor=L}

Corollary 1.5.14. *Let R and S be unital rings, with R commutative, and suppose there exists a unital ring homomorphism $R \to Z(S)$ (where $Z(S)$ denotes the center of S). Let E be an arbitrary graph, and suppose $X \subseteq \mathrm{Reg}(E)$. Then there are ring isomorphisms*

$$C_R^X(E) \otimes_R S \cong C_S^X(E) \cong S \otimes_R C_R^X(E).$$

In particular,

$$L_R(E) \otimes_R S \cong L_S(E) \cong S \otimes_R L_R(E).$$

Proof. We see that the computations made in Propositions 1.5.6 and 1.5.11 are independent of the coefficient ring, so that we have, for instance, $C_R^X(E) \otimes_R S = (\bigoplus_{\mathfrak{b} \in \mathscr{B}''} \mathfrak{b}R) \otimes_R S \cong \bigoplus_{\mathfrak{b} \in \mathscr{B}''} \mathfrak{b}S = C_S^X(E)$. □

{IndependenceOfPaths}

Corollary 1.5.15. *Let E be an arbitrary graph and K any field. Then any set of distinct elements of* $\mathrm{Path}(E)$ *is linearly independent in the Cohn path algebra $C_K(E)$, as well as in the Leavitt path algebra $L_K(E)$.*

One of the nice things about Cohn path algebras is that they turn out, perhaps unexpectedly, to be Leavitt path algebras. In fact, we will show that any relative Cohn path algebra $C_K^X(E)$ is isomorphic to the Leavitt path algebra of a graph $E(X)$ which is obtained by adding various new vertices and edges to E.

{FgraphDef}

Definition 1.5.16. Let E be an arbitrary graph and K any field. Let X be a subset of $\operatorname{Reg}(E)$, and define $Y := \operatorname{Reg}(E) \setminus X$. Let $Y' = \{v' \mid v \in Y\}$ be a disjoint copy of Y. For $v \in Y$ and for each edge $e \in r_E^{-1}(v)$, we consider a new symbol e'. We define the graph $E(X)$, as follows:

$$E(X)^0 = E^0 \sqcup Y' \quad \text{and} \quad E(X)^1 = E^1 \sqcup \{e' \mid r_E(e) \in Y\}.$$

For $e \in E^1$ we define $r_{E(X)}(e) = r_E(e)$ and $s_{E(X)}(e) = s_E(e)$, and define $s_{E(X)}(e') = s_E(e)$ and $r_{E(X)}(e') = r_E(e)'$ for the new symbols e'.

{exam:Cohn=Lea}

Less formally, the graph $E(X)$ is built from E and X by adding a new vertex to E corresponding to each element of $Y = \operatorname{Reg}(E) \setminus X$, and then including new edges to each of these new vertices as appropriate. Observe in particular that each of the new vertices $v' \in Y'$ is a sink in $E(X)$, so that $\operatorname{Reg}(E) = \operatorname{Reg}(E(X))$. In case $X = \operatorname{Reg}(E)$, then $E = E(X)$.

Example 1.5.17. Let E be the following graph:

$$\begin{tikzcd} \bullet^{v} \arrow[r, "f"] & \bullet^{u} \arrow[loop right, "e"] \end{tikzcd}$$

Take $X = \emptyset$, so that $Y = \operatorname{Reg}(E) = \{u, v\}$. Then the graph $E(X)$ is the following:

$$\begin{tikzcd} \bullet^{v} \arrow[r, "f"] \arrow[dr, "f'"] & \bullet^{u} \arrow[loop right, "e"] \arrow[d, "e'"] \\ \bullet^{v'} & \bullet^{u'} \end{tikzcd}$$

For any ring R, if f and g are idempotents of R then it is standard in the literature to write $f \leq g$ in case $fg = gf = f$. (We note, however, that this notation is not consistent with the notation $v \leq w$ used in a situation where $v, w \in E^0$ and v, w are viewed as idempotent elements of $L_K(E)$; however, used in context, this should not cause confusion.)

{CohnisLea}

As noted previously, every Leavitt path algebra arises (easily) as a relative Cohn path algebra, to wit, $L_K(E) = C_K^{\operatorname{Reg}(E)}(E)$. Perhaps more surprising is the following (very useful) result, which shows the converse.

Theorem 1.5.18. *Let E be an arbitrary graph and K any field. Let X any subset of* $\operatorname{Reg}(E)$*, and let $E(X)$ be the graph constructed in Definition 1.5.16. Then*

$$C_K^X(E) \cong L_K(E(X)).$$

Proof. We define a K-algebra homomorphism $\phi\colon C_K^X(E) \to L_K(E(X))$ as follows. Write $Y = \operatorname{Reg}(E) \setminus X$. For a vertex v of E define $\phi(v) = v + v'$ if $v \in Y$, and $\phi(v) = v$ otherwise. Moreover, for $e \in E^1$, define $\phi(e) = e$ if $r_E(e) \notin Y$ and $\phi(e) = e + e'$ if $r_E(e) \in Y$, and define $\phi(e^*) = \phi(e)^*$. Clearly relation (V) is preserved by ϕ. To show that relation (E1) is preserved by ϕ, we consider first the case where $r_E(e) \notin Y$. Then $\phi(e) = e$, $\phi(r_E(e)) = r_E(e)$ and $s_{E(X)}(e) = s_E(e) \leq \phi(s_E(e))$, so

$$\phi(s_E(e))\phi(e) = s_E(e)e = e = er_E(e) = \phi(e)\phi(r_E(e)).$$

If $v := r_E(e) \in Y$ then $\phi(e) = e + e'$ and $\phi(v) = v + v'$, and $s_{E(X)}(e) = s_{E(X)}(e') \leq \phi(s_E(e))$, so that

$$\phi(s_E(e))\phi(e) = s_E(e)(e + e') = e + e' = \phi(e) = (e + e')(v + v') = \phi(e)\phi(r_E(e)),$$

as desired. Relations (E2) follow by applying $*$ to the above. Now we consider relation (CK1). If $e \neq f$ then clearly $\phi(e)^*\phi(f) = 0$. If $r_E(e) \notin Y$ then $\phi(e)^*\phi(e) = e^*e = r_E(e) = \phi(r_E(e))$. If $r_E(e) \in Y$ then

$$\phi(e)^*\phi(e) = (e^* + (e')^*)(e + e') = r_E(e) + r_E(e)^* = \phi(r_E(e)).$$

We must check that the (CK2) relation holds for the vertices in X. If $v\in X$ then $\phi(v)=v$ and $s_{E(X)}^{-1}(v)=s_E^{-1}(v)\sqcup\{e'\mid s_E(e)=v \text{ and } r_E(e)\in Y\}$, so that

$$\begin{aligned}\phi(v)-\sum_{e\in s_E^{-1}(v)}\phi(e)\phi(e)^* &= v-\sum_{r_E(e)\notin Y}ee^*+\sum_{r_E(e)\in Y}(e+e')(e^*+(e')^*)\\ &= v-\sum_{s_E(e)=v}ee^*-\sum_{s_E(e)=v,r_E(e)\in Y}e'(e')^*=0.\end{aligned}$$

So we have shown that ϕ is a well-defined homomorphism.

Assume that $v\in Y$. Then a similar computation to the one presented above, using this time that $\phi(v)=v+v'$, yields that $\phi(q_v)=v'$, where q_v is defined prior to Proposition 1.5.8. It follows that $v,v'\in \operatorname{Im}(\phi)$. Now we have, for $e\in E^1$ such that $r_E(e)=v\in Y$, that $\phi(e)v=(e+e')v=e$ and $\phi(e)v'=e'$, so that $e,e'\in\operatorname{Im}(\phi)$. It follows that ϕ is surjective.

Now we build the inverse homomorphism $\psi\colon L_K(E(X))\to C_K^X(E)$. This is dictated by the above computations, so that we necessarily must set $\psi(v)=v$ if $v\notin Y$, and $\psi(v)=v-q_v$, $\psi(v')=q_v$ if $v\in Y$. For $e\in E^1$, set $\psi(e)=e$ if $r_E(e)\notin Y$, and set $\psi(e)=e(v-q_v)$, $\psi(e')=eq_v$ if $r_E(e)=v\in Y$. It is straightforward to show that all the defining relations of $L_K(E(X))$ are preserved by ψ, so that we get a well-defined homomorphism from $L_K(E(X))$ to $C_K^X(E)$. We check here the preservation of the (CK2) relation, and leave the others to the reader. Since $\operatorname{Reg}(E(X))=\operatorname{Reg}(E)$ we need to consider only the regular vertices of E. Let $v\in\operatorname{Reg}(E)$. Relation (CK2) in $L_K(E(X))$ may be presented as

$$v=\sum_{s_E(e)=v,r_E(e)\notin Y}ee^*+\sum_{s_E(e)=v,r_E(e)\in Y}ee^*+\sum_{s_E(e)=v,r_E(e)\in Y}e'(e')^*.$$

If $v\in X$ then

$$\begin{aligned}&\sum_{s_E(e)=v,r_E(e)\notin Y}\psi(e)\psi(e)^*+\sum_{s_E(e)=v,r_E(e)\in Y}\psi(e)\psi(e)^*+\sum_{s_E(e)=v,r_E(e)\in Y}\psi(e')\psi(e')^*\\ &=\sum_{s_E(e)=v,r_E(e)\notin Y}ee^*+\sum_{s_E(e)=v,r_E(e)\in Y}e(r_E(e)-q_{r_E(e)})e^*+\sum_{s_E(e)=v,r_E(e)\in Y}eq_{r_E(e)}e^*\\ &=\sum_{s_E(e)=v}ee^*=v=\psi(v).\end{aligned}$$

On the other hand, if $v\in Y$ then the same computation as above gives

$$\sum_{s_E(e)=v,r_E(e)\notin Y}\psi(e)\psi(e)^*+\sum_{s_E(e)=v,r_E(e)\in Y}\psi(e)\psi(e)^*+\sum_{s_E(e)=v,r_E(e)\in Y}\psi(e')\psi(e')^*=v-q_v=\psi(v),$$

as desired.

It is now straightforward to show that both compositions $\psi\circ\phi$ and $\phi\circ\psi$ give the identity on the generators of the corresponding algebras, thus these maps are the identity on their respective domains. It follows that ϕ is an isomorphism. $\square$

Here are two specific consequences of Theorem 1.5.18.

{firstCohnisLeavittExample}

Example 1.5.19. Consider the graphs

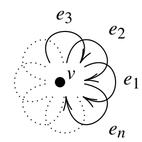
$$E \;=\; \begin{tikzcd} \bullet^v \arrow[r, "f"] & \bullet^u \arrow[loop right, "e"] \end{tikzcd} \qquad \text{and} \qquad F \;=\; \begin{tikzcd} \bullet^v \arrow[r, "f"] \arrow[dr, "f'"] & \bullet^u \arrow[loop right, "e"] \arrow[d, "e'"] \\ \bullet^{v'} & \bullet^{u'} \end{tikzcd}$$

Then $C_K(E)\cong L_K(F)$ since $C_K(E)=C_K^{\emptyset}(E)$ (this is true for any graph E), and, as observed in Example 1.5.17, $F=E(X)$ for $X=\emptyset$.

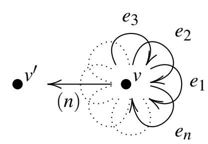
{CohnofR1isLeavittExample}

As with the Leavitt path algebras, the “rose with n petals” graphs R_n play an important role in the context of Cohn path algebras as well. We demonstrate now what the graph $R_n(X)$ looks like for $X = \emptyset$. This in particular will demonstrate how the Toeplitz algebra arises naturally from the Cohn path algebra point of view.

Example 1.5.20. If

$R_n \;=$ 

and $X = \emptyset$, then it is easy to show that

$R_n(X) \;=$  .

In particular, for $E = R_1 = \bullet^v \circlearrowleft e$, we get $R_1(X) \;=\; \bullet^{v'} \longleftarrow \bullet^v \circlearrowleft e \;= E_T$, the graph of Example 1.3.6. Specifically, Proposition 1.3.7 together with Theorem 1.5.18 give K-isomorphisms

$$K\langle U,V \mid UV = 1\rangle \;\cong\; C_K(R_1) \;\cong\; L_K(E_T) \;=\; \mathscr{T}_K.$$

{E(X)properties}

We finish the section by making some easily checked, eventually useful observations about the relationship between the graphs E and $E(X)$ for any $X \subseteq \mathrm{Reg}(E)$.

Proposition 1.5.21. *Let E be any graph, and X any subset of* $\mathrm{Reg}(E)$. *Let Y denote* $\mathrm{Reg}(E) \setminus X$.

(i) *E is acyclic if and only if $E(X)$ is acyclic.*
(ii) *E is finite if and only if $E(X)$ is finite.*
(iii) *E is row-finite if and only if $E(X)$ is row-finite.*
(iv) *The sinks of $E(X)$ are precisely the sinks of E together with the vertices $\{v' | v \in Y\}$.*
(v) *If v is a source in E, then v is also a source in $E(X)$. If moreover $v \in Y$, then v' is an isolated vertex in $E(X)$. Any isolated vertex of E is also isolated in $E(X)$.*

{directlimits}

1.6 Direct limits in the context of Leavitt path algebras

The Leavitt path algebras of finite graphs not only play an historically important role in the theory, they also quite often provide key information regarding the structure of Leavitt path algebras corresponding to arbitrary graphs. We show in this section how the Leavitt path algebra $L_K(E)$ of any graph E may be viewed as the direct limit of certain subalgebras of $L_K(E)$, where each of these subalgebras is isomorphic to the Leavitt path algebra of some finite graph.

We start by offering the following cautionary note. It may be tempting to think that if F is a subgraph of E, then, using the obvious identification, we should have $L_K(F)$ is a subalgebra of $L_K(E)$. However, this is not true in general, as a moment’s reflection reveals that the (CK2) relation at a vertex v viewed in $L_K(F)$ need not be compatible with the (CK2) relation at that same vertex v when viewed as an element of $L_K(E)$. For example, the obvious graph embedding of R_2 into R_3 does not extend to an algebra homomorphism from $L_K(R_2)$ to $L_K(R_3)$. However, in certain situations a subgraph F embeds in E in a way compatible with the (CK2) relations, or, more generally, with the (CK2) relations imposed at a given subset $Y \subseteq \mathrm{Reg}(F)$. This is the motivating idea behind the main concepts of this section. We start by reminding the reader of a basic idea in graphs, one which we will need to modify and expand upon in order to make it useful in our context.

{graphhomdef}

Definition 1.6.1. A *graph homomorphism* $\varphi\colon F=(F^0,F^1,r_F,s_F)\to E=(E^0,E^1,r_E,s_E)$ is a pair of maps $\varphi^0\colon F^0\to E^0$ and $\varphi^1\colon F^1\to E^1$ such that $r_E(\varphi^1(e))=\varphi^0(r_F(e))$ and $s_E(\varphi^1(e))=\varphi^0(s_F(e))$ for every $e\in F^1$.

As the observation made above about the embedding of R_2 into R_3 demonstrates, a graph homomorphism from F to E need not induce a homomorphism of algebras $L_K(F)\to L_K(E)$. However, the following additional conditions on a graph homomorphism will allow such an extension to the algebra level.

{categoryGdefinition}

Definition 1.6.2. We consider the category $\mathscr{G}$, defined as follows. The objects of $\mathscr{G}$ are pairs (E,X), where E is a graph and $X\subseteq \mathrm{Reg}(E)$. If $(F,Y),(E,X)\in \mathrm{Ob}(\mathscr{G})$, then $\psi=(\psi^0,\psi^1):(F,Y)\to(E,X)$ is a morphism in $\mathscr{G}$ in case

{homsfromcatGmorphisms}

(1) $\psi:F\to E$ is a graph homomorphism for which $\psi^0:F^0\to E^0$ and $\psi^1:F^1\to E^1$ are injective,
(2) $\psi^0(Y)\subseteq X$, and
(3) for all $v\in Y$, ψ^1 restricts to a bijection $\psi^1:s_F^{-1}(v)\to s_E^{-1}(\psi^0(v))$.

We note that a morphism $\psi:(F,Y)\to(E,X)$ in $\mathscr{G}$ depends not only on the underlying graphs F and E, but on the distinguished sets of vertices Y and X as well.

Lemma 1.6.3. *Suppose $\psi=(\psi^0,\psi^1):(F,Y)\to(E,X)$ is a morphism in $\mathscr{G}$. Then there exists a homomorphism of K-algebras $\overline{\psi}:C_K^Y(F)\to C_K^X(E)$.*

Proof. We define $\overline{\psi}:C_K^Y(F)\to C_K^X(E)$ as the extension of ψ on F^0 and F^1. We define $\overline{\psi}(f^*)=\psi(f)^*$ for all $f\in F^1$. As F^0, F^1, and $(F^1)^*$ generate $C_K^Y(F)$ as an algebra, this will yield a K-algebra homomorphism with domain $C_K^Y(F)$, once we show that the defining relations on $C_K^Y(F)$ are preserved.

The idempotent and orthogonality properties of relation (V) are preserved by $\overline{\psi}$ because ψ^0 is injective. (Note that if $v\neq w$ in F^0 then $\overline{\psi}(vw)=\overline{\psi}(0)$, while $\overline{\psi}(v)\overline{\psi}(w)=0$ using injectivity.) That relations (E1) and (E2) are preserved by $\overline{\psi}$ follows from the hypothesis that ψ is a graph homomorphism. That (CK1) is preserved by $\overline{\psi}$ follows because ψ^1 is injective (using an argument similar to the one given for relation (V)). Finally, the condition that ψ^1 restricts to a bijection from $s_F^{-1}(v)$ onto $s_E^{-1}(\psi^0(v))$ for every $v\in Y$ yields the preservation of (CK2) under $\overline{\psi}$ at the elements of Y. Thus, we get the desired extension of ψ to an algebra homomorphism $\overline{\psi}\colon C_K^Y(F)\to C_K^X(E)$. $\square$

{continuous-functor}

Proposition 1.6.4. *The category $\mathscr{G}$ has arbitrary direct limits. Moreover, for any field K, the assignment $(E,X)\mapsto C_K^X(E)$ extends to a continuous functor from the category $\mathscr{G}$ to the category K-alg of not-necessarily-unital K-algebras.*

Proof. We first show that $\mathscr{G}$ admits direct limits. Let I be an upward directed partially ordered set, and let $\{(E_i,X_i)_{i\in I},(\varphi_{ji})_{i,j\in I,j\geq i}\}$ be a directed system in $\mathscr{G}$. (So for each $j\geq i$ in I, $\varphi_{ji}:(E_i,X_i)\to(E_j,X_j)$ is a morphism in $\mathscr{G}$.) For $s=0,1$, set $E^s=\bigsqcup_{i\in I}E_i^s/\sim$, where $\sim$ is the equivalence relation on $\bigsqcup_{i\in I}E_i^s$ given by the following: For $\alpha\in E_i^s$ and $\beta\in E_j^s$, set $\alpha\sim\beta$ if and only if there is an index $k\in I$ such that $i\leq k$ and $j\leq k$ and $\varphi_{ki}^s(\alpha)=\varphi_{kj}^s(\beta)$. Observe that $E=(E^0,E^1)$ is a graph in a natural way, and there are injective graph homomorphisms $\psi_i=(\psi_i^0,\psi_i^1)\colon E_i\to E$ such that $E^s=\bigcup_{i\in I}\psi_i^s(E_i^s)$, $s=0,1$. Note that E^s is the direct limit of (E_i^s,φ_{ji}^s) in the category of sets. Now set $X=\bigcup_{i\in I}\psi_i^0(X_i)$. We see that ψ_i defines a graph homomorphism from E_i to E for all $i\in I$, such that $\psi_i=\psi_j\circ\varphi_{ji}$ for all $j\geq i$. Clearly ψ_i satisfies conditions (1) and (2) in Definition 1.6.2. To check condition (3), take any vertex v in X_i, for $i\in I$. Then $s_E^{-1}(\psi_i^0(v))=\bigcup_{j\geq i}\psi_j^1(s_{E_j}^{-1}(\varphi_{ji}^0(v)))$. But since for $j\geq i$ the map φ_{ji}^1 induces a bijection between $s_{E_i}^{-1}(v)$ and $s_{E_j}^{-1}(\varphi_{ji}^0(v))$, and $\psi_i^1=\psi_j^1\circ\varphi_{ji}^1$, it follows that

$$\psi_j^1(s_{E_j}^{-1}(\varphi_{ji}^0(v)))=\psi_j^1(\varphi_{ji}^1(s_{E_i}^{-1}(v)))=\psi_i^1(s_{E_i}^{-1}(v))\,,$$

so that ψ_i^1 induces a bijection from $s_{E_i}^{-1}(v)$ onto $s_E^{-1}(\psi_i^0(v))$. This gives (3) of Definition 1.6.2, and shows that each ψ_i is a morphism in the category $\mathscr{G}$.

We now check that $((E,X),\psi_i)$ is the direct limit of the directed system $((E_i,X_i),\varphi_{ji})$. Let $\{\gamma_i\colon(E_i,X_i)\to(G,Z)\mid i\in I\}$ be a compatible family of morphisms in $\mathscr{G}$. Define $\gamma\colon E\to G$ by the rule

$$\gamma^s(\psi_i(\alpha)) = \gamma_i^s(\alpha)\,,$$

for $\alpha \in E_i^s$, $s = 0,1$. It is obvious that γ is the unique graph homomorphism from E to G such that $\gamma_i = \gamma \circ \psi_i$ for all $i \in I$. Since, for $v \in E_i^0$, ψ_i^1 induces a bijection from $s_{E_i}^{-1}(v)$ onto $s_E^{-1}(\psi_i^0(v))$, and γ_i^1 induces a bijection from $s_{E_i}^{-1}(v)$ onto $s_G^{-1}(\gamma_i^0(v))$, it follows that γ^1 induces a bijection from $s_E^{-1}(\psi_i^0(v))$ onto $s_G^{-1}(\gamma_i^0(v)) = s_G^{-1}(\gamma^0(\psi_i^0(v)))$. This shows that γ defines a morphism in the category $\mathscr{G}$, and clearly γ is the unique object in the category $\mathscr{G}$ such that $\gamma_i = \gamma \circ \psi_i$ for all $i \in I$, showing that (E,X) is the direct limit of $((E_i,X_i),\varphi_{ji})$.

If $\psi\colon (F,Y) \to (E,X)$ is a morphism in $\mathscr{G}$, then there is an induced K-algebra homomorphism $\overline{\psi}\colon C_K^Y(F) \to C_K^X(E)$ by Lemma 1.6.3, and clearly the assignment $\psi \mapsto \overline{\psi}$ is functorial. Let

$$((E_i,X_i)_{i\in I},(\varphi_{ji})_{i,j\in I, j\geq i})$$

be a directed system in $\mathscr{G}$. Let $((E,X),\psi_i)$ be the direct limit in $\mathscr{G}$ of the directed system $((E_i,X_i),\varphi_{ji})$. We have to check that $(C_K^X(E),\overline{\psi_i})$ is the direct limit of the directed system $(C_K^{X_i}(E_i),\overline{\varphi_{ji}})$. Let $\gamma_i\colon C_K^{X_i}(E_i) \to A$ be a compatible family of K-algebra homomorphisms, where A is a K-algebra. Define $\gamma\colon C_K^X(E) \to A$ by the rule

$$\gamma(\psi_i^s(\alpha)) = \gamma_i(\alpha)\,, \quad \gamma(\psi_i^s(\alpha)^*) = \gamma_i(\alpha^*)\,,$$

for $\alpha \in E_i^s$, $i \in I$, $s = 0,1$. We have to check that relations (V), (E1), (E2), (CK1) are preserved by γ, and that relation (CK2) at all the vertices in X is also preserved by γ. It is straightforward to check (using appropriate injectivity hypotheses) that relations (V), (E1), (E2) and (CK1) are satisfied. Let $w \in X$. Then there is $v \in X_i$, for some $i \in I$, such that $w = \psi_i^0(v)$. Since ψ_i^1 induces a bijection from $s_{E_i}^{-1}(v)$ onto $s_E^{-1}(\psi_i^0(v)) = s_E^{-1}(w)$, we get

$$\gamma(w) = \gamma(\psi_i^0(v)) = \gamma_i(v) = \sum_{e\in s_{E_i}^{-1}(v)} \gamma_i(e)\gamma_i(e^*) = \sum_{e\in s_{E_i}^{-1}(v)} \gamma(\psi_i^1(e))\gamma(\psi_i^1(e)^*) = \sum_{f\in s_E^{-1}(w)} \gamma(f)\gamma(f^*)\,.$$

This shows that relation (CK2) at $w \in X$ is preserved by γ. It follows that γ is a well-defined K-algebra homomorphism. For $i \in I$, the maps γ_i and $\gamma \circ \overline{\psi_i}$ agree on the generators $E_i^0 \cup E_i^1 \cup (E_i^1)^*$ of $C_K^{X_i}(E_i)$, so we get $\gamma_i = \gamma \circ \overline{\psi_i}$. This shows that $(C_K^X(E),\overline{\psi_i})$ is the direct limit of the directed system $(C_K^{X_i}(E_i),\overline{\varphi_{ji}})$, as desired. $\square$

Although morphisms in $\mathscr{G}$ give rise to algebra homomorphisms between the associated relative Cohn path algebras as per the previous result, and although the morphisms in $\mathscr{G}$ are injective maps by definition, the induced algebra homomorphisms need not be injective. For instance, the identity map gives rise to a morphism $\iota : (R_n,\emptyset) \to (R_n,\{v\})$ in $\mathscr{G}$, where v is the unique vertex of the rose with n petals graph R_n. However, the corresponding induced map is the canonical surjection $C_K(1,n) \to L_K(1,n)$, which is not injective (as the nonzero element $v - \sum_{i=1}^n e_i e_i^*$ of $C_K(1,n)$ is mapped to zero in $L_K(1,n)$).

However, by adding an additional condition to morphisms in $\mathscr{G}$, we can ensure that the induced algebra homomorphisms are injective.

{completemorphismsdef}

Definition 1.6.5. Suppose $\psi = (\psi^0,\psi^1) : (F,Y) \to (E,X)$ is a morphism in $\mathscr{G}$. We say that ψ is *complete* in case, for every $v \in F^0$,

$$\text{if } \psi^0(v) \in X \text{ and } s_F^{-1}(v) \neq \emptyset, \text{ then } v \in Y.$$

That is, ψ is complete in case each of the vertices in X which are in $\mathrm{Im}(\psi^0)$, and which come from a non-sink in F, in fact come from Y. Note that a morphism ψ is complete if and only if $Y = (\psi^0)^{-1}(X) \cap \mathrm{Reg}(F)$.

We note that a complete morphism $\varphi : (F,\mathrm{Reg}(F)) \to (E,\mathrm{Reg}(E))$ is not in general the same as a *CK-morphism* as defined in [79], but the two ideas coincide when E is row-finite.

{completemorphismsinGgivemonos}

Lemma 1.6.6. *Suppose $\psi = (\psi^0,\psi^1) : (F,Y) \to (E,X)$ is a complete morphism in $\mathscr{G}$. Then the induced homomorphism $\overline{\psi} : C_K^Y(F) \to C_K^X(E)$ described in Lemma 1.6.3 is a monomorphism of K-algebras.*

Proof. Using Corollary 1.5.12 and the notation there, for every regular vertex $v \in F^0$, if $\{e_1^v,\dots,e_{n_v}^v\}$ is an enumeration of the elements of $s^{-1}(v)$, then a basis for $C_K^Y(F)$ is

$$\mathscr{B}''(F,Y) = \mathscr{B} \setminus \{\lambda e^v_{n_v}(e^v_{n_v})^* \nu^* | r(\lambda) = r(\nu) = v \in Y\}.$$

If $v \in Y$, then the map ψ^1 induces a bijection from $s_F^{-1}(v) = \{e_1^v, \dots, e_{n_v}^v\}$ onto $s_E^{-1}(\psi^0(v))$, so that $s_E^{-1}(\psi^0(v)) = \{\psi^1(e_1^v), \dots, \psi^1(e_{n_v}^v)\}$. We take a corresponding basis $\mathscr{B}''(E,X)$ of $C_K^X(E)$ such that, for $v \in Y$, the enumeration $\{e_1^{\psi^0(v)}, \dots, e_{n_v}^{\psi^0(v)}\}$ of the edges in $s_E^{-1}(\psi^0(v))$ is given by $e_i^{\psi^0(v)} = \psi^1(e_i^v)$, for $i = 1, \dots, n_v$.

The injectivity conditions on ψ^0 and ψ^1 give that ψ extends to an injective map from Path$(\widehat{F})$ to Path$(\widehat{E})$. It is now clear that $\overline{\psi}$ restricts to an injective map from the basis $\mathscr{B}''(F,Y)$ of $C_K^Y(F)$ into a subset of the basis $\mathscr{B}''(E,X)$ of $C_K^X(E)$. Indeed, the role here of the completeness condition is in assuring that the images of the basis elements $\lambda e_i^v(e_i^v)^* \nu^*$, $i = 1, \dots, n_v$, for v a regular vertex in F such that $v \notin Y$, belong to the basis $\mathscr{B}''(E,X)$ of $C_K^X(E)$ associated to (E,X). This is so because if $v \in \mathrm{Reg}(F) \setminus Y$, then $\psi^0(v) \notin X$ by completeness of ψ, and so the elements $\overline{\psi}(\lambda e_i^v(e_i^v)^* \nu^*)$ belong to the basis $\mathscr{B}''(E,X)$.

Therefore $\overline{\psi}$ is injective, as desired. □

{def:complete}

Definition 1.6.7. We say that a subgraph F of a graph E is *complete* in case the inclusion map

$$(F, \mathrm{Reg}(F) \cap \mathrm{Reg}(E)) \to (E, \mathrm{Reg}(E))$$

is a (complete) morphism in the category $\mathscr{G}$. Less formally, F is a *complete* subgraph of E in case for each $v \in F^0$, whenever $s_F^{-1}(v) \neq \emptyset$ and $0 < |s_E^{-1}(v)| < \infty$, then $s_F^{-1}(v) = s_E^{-1}(v)$. In words, a subgraph F of a graph E is complete in case, whenever v is a vertex in F which emits at least one edge in F and finitely many in E (and so also finitely many in F, because F is a subgraph of E), then the edges emitted at v in the subgraph F are precisely *all* of the edges emitted at v in the full graph E.

By Lemma 1.6.6, if F is a complete subgraph of E then we get an embedding

$$C_K^{\mathrm{Reg}(F)\cap\mathrm{Reg}(E)}(F) \hookrightarrow L_K(E) = C_K^{\mathrm{Reg}(E)}(E).$$

In case $\mathrm{Reg}(F) \cap \mathrm{Reg}(E) = \mathrm{Reg}(F)$ (for instance, in case E is row-finite), then a complete subgraph F of E yields that the canonical inclusion map $F \hookrightarrow E$ gives rise to an embedding of $L_K(F) \hookrightarrow L_K(E)$.

In the example given above, R_2 is not a complete subgraph of R_3. This is because $\mathrm{Reg}(R_3) = \{v\} = \mathrm{Reg}(R_2)$, so that $\mathrm{Reg}(R_2) \cap \mathrm{Reg}(R_3) = \{v\}$; and the inclusion map from $s_{R_2}^{-1}(v) \to s_{R_3}^{-1}(v)$ is not a bijection. In contrast, the inclusion morphism $(R_2, \emptyset) \hookrightarrow (R_3, \emptyset)$ *is* a complete morphism in $\mathscr{G}$. On the other hand, consider the infinite rose graph R_∞, and let R_n be any finite subgraph of R_∞. Then R_n is a complete subgraph of R_∞, since $\mathrm{Reg}(R_n) \cap \mathrm{Reg}(R_\infty) = \{v\} \cap \emptyset = \emptyset$, and the morphism $(R_n, \emptyset) \hookrightarrow (R_\infty, \emptyset)$ is complete.

The following definition generalizes Definition 1.6.7, and it will be useful later on.

{def:Scomplete}

Definition 1.6.8. Let E be a graph and let S be a subset of $\mathrm{Reg}(E)$. We say that a subgraph F of a graph E is *S-complete* in case the inclusion map

$$(F, \mathrm{Reg}(F) \cap S) \to (E, S)$$

is a (complete) morphism in the category $\mathscr{G}$. Thus, F is an *S-complete* subgraph of E in case for each $v \in S$, we have $s_F^{-1}(v) = s_E^{-1}(v)$ whenever $s_F^{-1}(v) \neq \emptyset$.

We note that the literature contains alternate definitions of the notion of a *complete* subgraph of a graph, see e.g. [14]. However, the notion of completeness is identical across all definitions whenever the given graph is row-finite.

The notion of a complete morphism in $\mathscr{G}$, and the attendant notion of a complete subgraph, will allow us to produce homomorphisms from various relative Cohn path algebras over appropriately chosen subgraphs F of E to the Leavitt path algebra $L_K(E)$. This will in turn, by an application of Theorem 1.5.18, allow us to realize any Leavitt path algebra $L_K(E)$ as a direct limit of algebras, each of which is itself the Leavitt path algebra of a finite graph built from E.

{injlim}

Lemma 1.6.9. *Every object (E,X) of $\mathscr{G}$ is a direct limit in the category $\mathscr{G}$ of a directed system of the form $\{(F_i, X_i) \mid i \in I\}$, for which each F_i is a finite graph and all the maps $(F_i, X_i) \to (E,X)$ are complete morphisms in $\mathscr{G}$.*

Proof. Clearly, E is the set theoretic union of its finite subgraphs. Let G be a finite subgraph of E. Define a finite subgraph F of E as follows:

$$F^0 = G^0 \cup \{r_E(e) \mid e \in E^1 \text{ and } s_E(e) \in G^0 \cap X\}$$

and

$$F^1 = \{e \in E^1 \mid s_E(e) \in G^0 \cap X\}.$$

Now notice that the set of vertices in $F^0 \cap X$ that emit edges in F is precisely the set $G^0 \cap X$, and if v is one of these vertices, then $s_E^{-1}(v) = s_F^{-1}(v)$. This shows that the inclusion map $(F, \mathrm{Reg}(F) \cap X) \hookrightarrow (E, X)$ is a complete morphism in $\mathscr{G}$. In particular, any finite subgraph G of E gives rise to a finite complete subobject $(F, \mathrm{Reg}(F) \cap X)$ of (E, X).

{algdirectlimitarbitrary}

Since the union of a finite number of finite complete subobjects of (E, X) is again a finite complete subobject of (E, X), it follows that (E, X) is the direct limit in the category $\mathscr{G}$ of the directed family of its finite complete subobjects $(F, \mathrm{Reg}(F) \cap X)$. $\square$

Now applying Lemma 1.6.9, Proposition 1.6.4 and Lemma 1.6.6, we have established the following useful result.

Theorem 1.6.10. *Let E be an arbitrary graph and K any field. Let X be any subset of* $\mathrm{Reg}(E)$. *Then as objects in the category K-*alg, *we have*

$$C_K^X(E) = \varinjlim_F \{C_K^{\mathrm{Reg}(F) \cap X}(F)\},$$

where $(F, \mathrm{Reg}(F) \cap X)$ ranges over all finite complete subobjects of (E, X) (i.e., F ranges over all X-complete subgraphs of E). Moreover, each of the homomorphisms $C_K^{\mathrm{Reg}(F) \cap X}(F) \to C_K^X(E)$ is injective. In particular,

$$L_K(E) = \varinjlim_F \{C_K^{\mathrm{Reg}(F) \cap \mathrm{Reg}(E)}(F)\},$$

where F ranges over all finite complete subgraphs of E, with all homomorphisms $C_K^{\mathrm{Reg}(F) \cap \mathrm{Reg}(E)}(F) \to L_K(E)$ being injective.

{L(E)isdirectlimit}

We are now in position to establish the aforementioned result regarding direct limits.

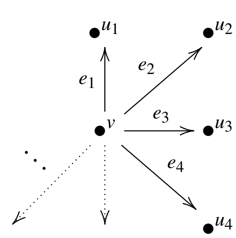
Corollary 1.6.11. *Let E be any graph and K any field. Let X be any subset of* $\mathrm{Reg}(E)$. *Then $C_K^X(E)$ is the direct limit in K-*alg *of subalgebras, each of which is isomorphic to the Leavitt path algebra of a finite graph. In particular, $L_K(E)$ is the direct limit of unital subalgebras (with not-necessarily-unital transition homomorphisms), each of which is isomorphic to the Leavitt path algebra of a finite graph.*

{BreakingVertex1}

Proof. This follows directly from Theorems 1.6.10 and 1.5.18. $\square$

To clarify the ideas of the previous two results, we present the following examples.

Example 1.6.12. Let $C_{\mathbb{N}}$ be the *infinite clock graph* pictured here



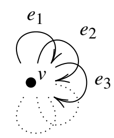
In this example, we have $L_K(C_{\mathbb{N}}) \cong \varinjlim_{n \in \mathbb{N}} C_K(C_n)$, where $C_K(C_n) = C_K^{\emptyset}(C_n) \cong L_K(C_n(\emptyset))$ is the Cohn path algebra of the n-edges clock C_n.

{BreakingVertex2}

Example 1.6.13. We let $R_{\mathbb{N}}$ denote the *rose with* $\mathbb{N}$ *petals* graph having one vertex and $\mathbb{N}$ loops:

{BreakingVertex3}

$R_{\mathbb{N}} \quad =$



In this example, we have $L_K(R_{\mathbb{N}}) \cong \varinjlim_{n\in\mathbb{N}} C_K(R_n)$, where $C_K(R_n) = C_K^{\emptyset}(R_n) \cong L_K(R_n(\emptyset))$ is the Cohn path algebra of the n-edges rose.

Example 1.6.14. Let $A_{\mathbb{N}}$ be the *infinite line graph*

$$A_{\mathbb{N}} \quad = \quad \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots\cdots\cdots$$

{acyclicisdirlimfiniteacyclic}

Here we have $L_K(A_{\mathbb{N}}) \cong \varinjlim_{n\in\mathbb{N}} L_K(A_n)$, because the graph $A_{\mathbb{N}}$ is row-finite (see Corollary 1.6.16 below). In this situation the transition homomorphisms $L_K(A_n) \to L_K(A_{n+1})$ can be identified with the maps $\mathrm{M}_n(K) \to \mathrm{M}_{n+1}(K)$ (cf. Proposition 1.3.5) sending x to $x \oplus 0$. This yields that $L_K(A_{\mathbb{N}}) \cong \mathrm{M}_{\mathbb{N}}(K)$, the K-algebra of $\mathbb{N} \times \mathbb{N}$ matrices consisting of those matrices having at most finitely many nonzero entries. (This isomorphism will also follow from Theorem 2.6.14 below.)

As a consequence of the results in this section which will prove to be quite useful later, we offer the following.

Proposition 1.6.15. *Let E be any acyclic graph. Then $L_K(E)$ is the direct limit, with injective transition homomorphisms, of algebras $\{L_K(F_i) \mid i \in I\}$, where each F_i is a finite acyclic graph.*

{rowfiniteL(E)isdirectlimit}

Proof. As subgraphs of E, the graphs F which arise in Theorem 1.6.10 are necessarily acyclic. But $C_K^{\mathrm{Reg}(F)\cap\mathrm{Reg}(E)}(F) \cong L_K(F(\mathrm{Reg}(F)\cap\mathrm{Reg}(E)))$ by Theorem 1.5.18, and $F(\mathrm{Reg}(F)\cap\mathrm{Reg}(E))$ is acyclic by Proposition 1.5.21(1). $\square$

We conclude this section by noting that the above direct limit construction may be streamlined in the row-finite case, for in that situation the regular vertices of E are precisely the non-sinks, and the set intersections $\mathrm{Reg}(F)\cap\mathrm{Reg}(E)$ are precisely the sets $\mathrm{Reg}(F)$. So by Theorem 1.6.10 we get

Corollary 1.6.16. *Let E be any row-finite graph. Then $L_K(E)$ is the directed union of unital subalgebras (with not-necessarily-unital transition homomorphisms), each of which is isomorphic to the Leavitt path algebra of a finite complete subgraph of E.*

{BriefHistorySection}

1.7 A brief restrospective on the history of Leavitt path algebras

A brief retrospective on the subject's genesis is in order here. (A much fuller account may be found in [2].) The accomplishments achieved during the initial investigation by Leavitt in the late 1950's and early 1960's into the structure of non-IBN rings were followed up by P.M. Cohn's work (see e.g. [60]) in the mid 1960's on the algebras $U_{1,n}$ (herein denoted $C_K(1,n)$), and by Bergman's work in the mid 1970's on the $\mathcal{V}$-monoid question. The algebras $L_K(1,n)$ and $C_K(1,n)$ were not again the subject of intense interest until more than a quarter century later, when they were dusted off and studied anew in [33], [24], and [29]. (Perhaps this hiatus of interest was due to Cohn's remark in [60] that these algebras " ... may be regarded as pathological rings"?) As noted previously, the algebras $C_K(1,n) \cong C_K(R_n)$ stand in relation to the more general Cohn path algebras in precisely the same way that the Leavitt algebras $L_K(1,n) \cong L_K(R_n)$ stand in relation to the more general Leavitt path algebras.

Working in a different corner of the mathematical universe, Cuntz in the late 1970's investigated a class of C*-algebras arising from a natural question in physics, the now-so-called *Cuntz algebras* $\mathcal{O}_n$ (see [63]). Subsequently, Cuntz and Krieger in [64] realized that the Cuntz algebras are specific cases of a more general

C*-algebra structure which could be associated with any finite $0/1$ matrix, the now-so-called *Cuntz-Krieger C*-algebras*. (The names Cuntz and Krieger give rise to the letters which comprise the notation (CK1) and (CK2); this notation is now standard in both the algebraic and analytic literature to describe the appropriate conditions on the algebras.) Subsequently, it was realized that the Cuntz-Krieger algebras were themselves specific cases of an even more general C*-algebra structure, the *graph C*-algebras* defined in [141] and then initially investigated in depth in [94].

Using the 20/20 vision provided by the passage of a few years' time, it is fair to say that there were two seminal papers which wound up serving as the launching pad for the study of Leavitt path algebras: [7] and [35]. The work for both of these articles was initiated in 2004, but the two groups of authors did not become aware of the others' efforts until Spring 2005, at which time it was immediately clear that the algebras under study in these two articles were identical. It is interesting to note that although the topic discussed in both [7] and [35] is the then-newly-described notion of Leavitt path algebras, the results in the two articles are in fact completely disjoint. Indeed, the former contains results for Leavitt path algebras which mimic some of the corresponding graph C*-algebra results (e.g., regarding simplicity of the algebras). In fact, the construction given in [7] was motivated directly by interpreting the C*-algebra equations displayed in Definitions 1.4.4 from a purely algebraic point of view. (The analogous interpretation relating $L_{\mathbb{C}}(1,n)$ and $\mathscr{O}_n$ had already been noted in [33].) On the other hand, [35] contains results describing Bergman's construction in the specific setting of graph monoids, as well as theretofore unknown information about the $\mathscr{V}$-monoid of the graph C*-algebras. The common, historically appropriate name "Leavitt path algebras" which now describes these structures was then agreed upon by the two groups of authors while [7] and [35] were in press.

The results presented in this opening chapter are meant to give the reader both an historical overview of the subject and a foundation for results which will be presented in subsequent chapters. The results described in Sections 1.1 through 1.4 have by now resided in the literature for a number of years, and are for the most part well-known. On the other hand, the main ideas of Sections 1.5 and 1.6 are contributions to the theory which either make their first appearance in the literature here, or made their appearance in literature motivated in part by pre-publication versions of this book.

Again donning our historical 20/20 lenses, it seems clear now that Cohn's aforementioned "pathological rings" observation rather significantly missed the mark. As we hope will become apparent to the reader throughout this book, in fact these rings are quite natural, structurally quite interesting, and really quite beautiful.

Chapter 2
Two-sided ideals

{chap2}

ABSTRACT: In this chapter we investigate the ideal structure of Leavitt path algebras. We start by describing the natural $\mathbb{Z}$-grading on $L_K(E)$. We then present the Reduction Theorem; this result describes how elements of $L_K(E)$ may be transformed in some specified way to either a vertex or a cycle without exits. Numerous consequences are discussed, including the Uniqueness Theorems. We then establish in the Structure Theorem for Graded Ideals a precise relationship between graded ideals and explicit sets of idempotents (arising from hereditary and saturated subsets of vertices, together with breaking vertices). With this description of the graded ideals having been achieved, we focus in the remainder of the chapter on the structure of all ideals. We achieve in the Structure Theorem for Ideals an explicit description of the entire ideal structure of $L_K(E)$ (including both the graded and non-graded ideals) for an arbitrary graph E and field K. This result utilizes the Structure Theorem for Graded Ideals together with the analysis of the ideal generated by vertices which lie on cycles having no exits. A number of ring-theoretic results follow almost immediately from the Structure Theorem for Ideals, including the Simplicity Theorem. Along the way, we describe the socle of a Leavitt path algebra, and we achieve a description of the finite dimensional Leavitt path algebras.

In this chapter we investigate the ideal structure of Leavitt path algebras. In the introductory paragraphs we present many of the graph-theoretic ideas which will be useful throughout the subject. There is a natural $\mathbb{Z}$-grading on $L_K(E)$, which we discuss in Section 2.1. With this grading so noted, we will see in subsequent sections that the graded ideals with respect to this grading play a fundamental structural role. In Section 2.2 we consider the Reduction Theorem. Important consequences of this result include the two Uniqueness Theorems (also presented in Section 2.2), as well as various structural results about Leavitt path algebras (which comprise Section 2.3). In Section 2.4 we show that the quotient of a Leavitt path algebra by a graded ideal is itself isomorphic to a Leavitt path algebra. In Section 2.5 we show that the graded ideals of a Leavitt path algebra arise as ideals generated from data given by prescribed subsets of the graph E. Specifically, in the Structure Theorem for Graded Ideals (Theorem 2.5.8), we establish a precise relationship between graded ideals and explicit sets of idempotents. In the row-finite case, these sets of idempotents consist of hereditary saturated sets of vertices, while in the more general case additional sets of idempotents (arising from breaking vertices) are necessary. As well, we show that a graded ideal viewed as an algebra in its own right is isomorphic to a Leavitt path algebra.

With a description of the graded ideals having been achieved, we focus in the remainder of the chapter on the structure of all ideals. We start in Section 2.6 by considering the socle of a Leavitt path algebra. Along the way, we achieve a description of the finite dimensional Leavitt path algebras. In Section 2.7 we identify the ideal generated by the set of those vertices which connect to a cycle having no exits. The denouement of Chapter 2 occurs in Section 2.8, in which we present the Structure Theorem for Ideals (Theorem 2.8.10), an explicit description of the entire ideal lattice of $L_K(E)$ (including both the graded and non-graded ideals) for an arbitrary graph E and field K. This key result weaves the Structure Theorem for Graded Ideals together with the analysis of the ideal investigated in the previous section. A number of ring-theoretic results follow almost immediately from the Structure Theorem for Ideals, including the Simplicity Theorem; we present those in Section 2.9.

{I(X)definition}

Notation 2.0.1. For a ring or algebra R and subset $X \subseteq R$, denote by $I(X)$ the ideal of R generated by X.

{moregraphdefinitions}

While only very basic graph-theoretic ideas and terminology were needed to define the Leavitt path algebras, additional graph-theoretic concepts will play a huge role in analyzing the structure of these algebras. We collect many of those in the following.

Definitions 2.0.2. Let $E = (E^0, E^1, r, s)$ be an arbitrary graph.

(i) Let $\mu = e_1 e_2 \cdots e_n \in \text{Path}(E)$. If $n = \ell(\mu) \geq 1$, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at* v.
(ii) A *closed simple path based at* v is a closed path $\mu = e_1 e_2 \cdots e_n$ based at v, such that $s(e_j) \neq v$ for every $j > 1$. We denote by $CSP(v)$ the set of all such paths.
(iii) If $\mu = e_1 e_2 \cdots e_n$ is a closed path based at v and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle based at* v. Note that a cycle is a closed simple path based at any of its vertices, but not every closed simple path based at v is a cycle, because a closed simple path may visit some of its vertices (other than v) more than once.
(iv) Suppose $\mu = e_1 e_2 \cdots e_n$ is a cycle based at the vertex v. Then for each $1 \leq i \leq n$, the path $\mu_i = e_i e_{i+1} \cdots e_n e_1 \cdots e_{i-1}$ is a cycle based at the vertex $s(e_i)$. (In particular, $\mu_1 = \mu$.) The *cycle of* μ is the collection of cycles $\{\mu_i\}$ based at $s(e_i)$.
(v) A *cycle* c is a set of paths consisting of the cycle of μ for μ some cycle based at a vertex v.
(vi) The *length of a cycle* c is the length of any of the paths in c. In particular, a cycle of length 1 is called a *loop*. (We note that the definition of the word *cycle* is somewhat non-standard, but will serve our purposes well here.)
(vii) A (directed) graph E is said to be *acyclic* in case it does not have any closed paths based at any vertex of E, equivalently if it does not have any cycles based at any vertex of E.

{DefCondK}

Definition 2.0.3. A graph E satisfies *Condition* (K) if for each $v \in E^0$ which lies on a closed simple path, there exist at least two distinct closed simple paths α, β based at v.

{preorderdef}

Definition 2.0.4. Let $E = (E^0, E^1, r, s)$ be a graph. We define a preorder $\leq$ on E^0 given by:

$$w \leq v \text{ in case there is a path } \mu \in \text{Path}(E) \text{ such that } s(\mu) = v \text{ and } r(\mu) = w.$$

(We will sometimes equivalently write $v \geq w$ in this situation.) If $v \in E^0$ then the *tree of* v, denoted $T(v)$, is the set $\{w \mid w \in E^0, v \geq w\}$. (This notation is standard in the context of Leavitt path algebras; note, however, that $T(v)$ need not be a "tree" in the sense of undirected graphs, as $T(v)$ may indeed contain closed paths.) If $X \subseteq E^0$, we define $T(X) := \bigcup_{v \in X} T(v)$. Note that $T(X)$ is the smallest hereditary subset of E^0 containing X.

editaryandsaturateddefinitions}

Definitions 2.0.5. Let E be a graph, and $H \subseteq E^0$.

(i) We say H is *hereditary* if whenever $v \in H$ and $w \in E^0$ for which $v \geq w$, then $w \in H$.
(ii) We say H is *saturated* if whenever $v \in \text{Reg}(E)$ has the property that $\{r(e) \mid s(e) = v\} \subseteq H$, then $v \in H$. (In other words, H is saturated if, for any non-sink vertex v which emits a finite number of edges in E, if *all* of the range vertices $r(e)$ for those edges e having $s(e) = v$ are in H, then v must be in H as well.)

We denote by $\mathscr{H}_E$ (or simply by $\mathscr{H}$ when the graph E is clear) the set of those subsets of E^0 which are both hereditary and saturated.

We refer back to the graph E given in Example 1.2.8. We see that the set $S_1 = \{v_3\}$ is hereditary (trivially), but not saturated, since the vertex v_2 emits all of its edges (there is only one) into S_2, but v_2 itself is not in S_2. However, the set $S_2 = \{v_2, v_3\}$ is both hereditary and saturated: while v_1 emits edges into S_2, not *all* of the edges emitted from v_1 have ranges in S_2.

{heredsatclosuredef}

Definition 2.0.6. If X is a subset of E^0, then the *hereditary saturated closure of* X, denoted $\overline{X}$, is the smallest hereditary and saturated subset of E^0 containing X. (Since the intersection of hereditary (resp., saturated) subsets of E^0 is again hereditary (resp., saturated), $\overline{X}$ is well defined.)

We denote by $S(X)$ the set of all vertices obtained by applying the saturated condition among the elements of X, that is,

$$S(X) := \{v \in \mathrm{Reg}(E) \mid \{r(e) \mid s(e) = v\} \subseteq X\} \cup X.$$

For $X \subseteq E^0$, the hereditary saturated closure of X may be inductively constructed as follows.

{HerSatGen}

Lemma 2.0.7. *Let X be a nonempty subset of vertices of a graph E. We define $X_0 := T(X)$, and for $n \geq 0$ we define inductively $X_{n+1} := S(X_n)$. Then $\overline{X} = \bigcup_{n\geq 0} X_n$.*

Proof. It is immediate to see that every hereditary and saturated subset of E^0 containing X must contain $\bigcup_{n\geq 0} X_n$. Note that every X_n is hereditary (it is easy to show that if $Y \subseteq E^0$ is hereditary, then so is $S(Y)$), which implies that $\bigcup_{n\geq 0} X_n$ is hereditary as well. We now show that $\bigcup_{n\geq 0} X_n$ is saturated. Take $v \in \mathrm{Reg}(E)$ such that $r(s^{-1}(v)) \subseteq \bigcup_{n\geq 0} X_n$; since $X_n \subseteq X_{n+1}$ and $r(s^{-1}(v))$ is a finite subset, there exists $N \in \mathbb{N}$ such that $r(s^{-1}(v)) \subseteq X_N$, hence $v \in X_{N+1}$ as required. $\square$

{involution}

We finish the introduction to this chapter by describing how the path algebra $K\hat{E}$ of K over the extended graph $\hat{E}$ can be endowed with an involution, as follows.

Lemma 2.0.8. *Let E be an arbitrary graph and K any field. Let $^- : K \to K$ be an involution on K. Then the following map can be extended to a unique involution $^* : K\hat{E} \to K\hat{E}$:*

(1) $(kv)^* = \overline{k}v$ *for every $k \in K$ and $v \in E^0$.*
(2) $(k\gamma)^* = \overline{k}\gamma^*$ *for every $k \in K$ and $\gamma \in \mathrm{Path}(E)$.*
(3) $(k\gamma^*)^* = \overline{k}\gamma$ *for every $k \in K$ and $\gamma \in \mathrm{Path}(E)$.*

In particular, $(KE)^* = KE^*$.

{involutiononrelCohnpathalg}

Proof. Define the map $\rho : E^0 \cup E^1 \cup (E^1)^* \to (K\hat{E})^{op}$ by setting $\rho(v) = v$, $\rho(e) = e^*$, and $\rho(e^*) = e$ for $v \in E^0$ and $e \in E^1$. It is easy to see that ρ is compatible with the relations (V), (E1) and (E2) in $K\hat{E}$, and hence ρ can be extended in a unique way to a homomorphism of K-algebras $\rho : K\hat{E} \to (K\hat{E})^{op}$. This homomorphism ρ is precisely the involution in the statement. $\square$

Corollary 2.0.9. *Let E be an arbitrary graph, let $X \subseteq \mathrm{Reg}(E)$, and let K be any field. Let $^- : K \to K$ be an involution on K. Then there is a unique involution $^* : C_K^X(E) \to C_K^X(E)$ satisfying the three properties of Lemma 2.0.8.*

Consequently, taking the involution to be the identity map, we have that $C_K^X(E)$ is isomorphic to its opposite ring $C_K^X(E)^{op}$. In particular, $L_K(E) \cong L_K(E)^{op}$.

{gradingsection}

2.1 The $\mathbb{Z}$-grading

One of the most important properties of the class of Leavitt path algebras is that each $L_K(E)$ is a $\mathbb{Z}$-graded K-algebra. As we shall see, this grading provides the key ingredient which allows us to achieve many structural results about Leavitt path algebras, as well as to streamline proofs of additional results.

In this section we will explore the natural $\mathbb{Z}$-grading on $L_K(E)$ (the one induced by the length of paths). Of particular importance will be the structure of the zero component of any Leavitt path algebra relative to this grading.

Definitions 2.1.1 . Let G be a group and A an algebra over a field K. We say that A is *G-graded* if there exists a family $\{A_\sigma\}_{\sigma\in G}$ of K-subspaces of A such that

$$A = \bigoplus_{\sigma\in G} A_\sigma \ \text{ as } K\text{-spaces, and } \ A_\sigma \cdot A_\tau \subseteq A_{\sigma\tau} \text{ for each } \sigma,\tau \in G.$$

{idealgenbydegreezeroelements}

An element x of A_σ is called a *homogeneous element* σ. An ideal I of a G-graded K-algebra A is said to be a *graded ideal* if $I \subseteq \sum_{\sigma\in G}(I \cap A_\sigma)$, or, equivalently, if

$$y = \textstyle\sum_{\sigma\in G} y_\sigma \in I \ \text{ implies } \ y_\sigma \in I \ \text{ for every } \ \sigma \in G.$$

Remark 2.1.2. Let e denote the identity element of the group G. It is straightforward to show that if A is a G-graded ring, and X is a subset of A_e, then the ideal $I(X)$ of A generated by X is a graded ideal.

It is easy to prove that the quotient of a G-graded algebra $A = \sum_{\sigma\in G} A_\sigma$ by a graded ideal I is a G-graded algebra, with the natural grading induced by that of A. Specifically, consider the projection map $A \to A/I$ via $a \mapsto \overline{a}$, and denote A/I by $\overline{A}$. Then, using the graded property of I, for any $\sigma \in G$ the homogenous component $\overline{A}_\sigma$ of $\overline{A}$ of degree σ is $\overline{A}_\sigma := \overline{A_\sigma}$. Hence

$$\overline{A} = \bigoplus_{\sigma\in G} \overline{A_\sigma}\ .$$

In general, not every ideal in a Leavitt path algebra is graded (see, e.g., Examples 2.1.7). It will be shown in Section 2.4 that graded ideals can be obtained from specified subsets of vertices. Concretely, Leavitt path algebras whose ideals are all graded will be shown to coincide with the exchange Leavitt path algebras; equivalently, to coincide with those Leavitt path algebras whose associated graph satisfies Condition (K).

We recall here that for an arbitrary graph E and field K the Leavitt path algebra $L_K(E)$ can be obtained as a quotient of the Cohn path algebra $C_K(E)$ by the ideal I generated by $\{v - \sum_{e\in s^{-1}(v)} ee^* \mid v \in \mathrm{Reg}(E)\}$ (Proposition 1.5.5). We establish that the Cohn path algebra has a natural $\mathbb{Z}$-grading given by the length of the monomials, which thereby will induce a $\mathbb{Z}$-grading on $L_K(E)$. (Although we derive the grading on $L_K(E)$ from the grading on $C_K(E)$, a more direct proof may also be produced.)

{gradingDef}

{gradingExtGraph}

Definition 2.1.3. Let E be an arbitrary graph and K any field. For any $v \in E^0$ and $e \in E^1$, define $\deg(v) = 0$, $\deg(e) = 1$ and $\deg(e^*) = -1$. For any monomial $kx_1\cdots x_m$, with $k \in K$ and $x_i \in E^0 \cup (E^1 \cup E^1)^*$, define $\deg(kx_1\cdots x_m) = \sum_{i=1}^m \deg(x_i)$. Finally, for any $n \in \mathbb{Z}$ define

$$A_n := \mathrm{span}_K(\{x_1\cdots x_m \mid x_i \in E^0 \cup E^1 \cup (E^1)^* \text{ with } \deg(x_1\cdots x_m) = n\}).$$

Proposition 2.1.4. *With the notation of Definition 2.1.3, $K\hat{E} = \bigoplus_{n\in\mathbb{Z}} A_n$ as K-subspaces, and this decomposition defines a $\mathbb{Z}$-grading on the path algebra $K\widehat{E}$.*

Proof. By Remark 2.1.2, the ideal I generated by the relations (V), (E1) and (E2) is graded, hence $K\widehat{E}$, which is isomorphic to $K\langle\{E^0 \cup E^1 \cup (E^1)^*\}\rangle/I$, is graded as in the indicated decomposition. $\square$

{gradingCE}

Corollary 2.1.5. *Let E be an arbitrary graph and K any field.*

(i) *For any subset X of* $\mathrm{Reg}(E)$*, the Cohn path algebra $C_K^X(E)$ of E relative to X is a $\mathbb{Z}$-graded K-algebra with the grading induced by the length of paths.*

(ii) $C_K(E) = \bigoplus_{n\in\mathbb{Z}} C_n$, *where*

$$C_n := \mathrm{span}_K(\{\gamma\lambda^* \mid \gamma,\lambda \in \mathrm{Path}(E) \ \textit{and}\ \ell(\gamma) - \ell(\lambda) = n\}),$$

defines a $\mathbb{Z}$-grading on the Cohn path algebra $C_K(E)$.

(iii) $L_K(E) = \bigoplus_{n\in\mathbb{Z}} L_n$, *where*

$$L_n := \mathrm{span}_K(\{\gamma\lambda^* \mid \gamma,\lambda \in \mathrm{Path}(E) \ \textit{and}\ \ell(\gamma) - \ell(\lambda) = n\}),$$

defines a $\mathbb{Z}$-grading on the Leavitt path algebra $L_K(E)$.

Proof. Items (ii) and (iii) are particular cases of (i), hence we will prove only this case. By definition (see Definition 1.5.9), the relative Cohn path algebra $C_K^X(E) = K\hat{E}/I$, where I is the K-algebra ideal of $K\hat{E}$ generated by relations of the forms (V), (E1), (E2), (CK1) and by the idempotents $\{q_v \mid v \in X\}$, where $q_v = v - \sum_{e\in s^{-1}(v)} ee^*$. Proposition 2.1.4 establishes that the path algebra $K\hat{E}$ is $\mathbb{Z}$-graded. But I is generated by homogeneous elements of degree 0, hence it is a graded ideal by Remark 2.1.2; consequently, the quotient $K\hat{E}/I$ gives a $\mathbb{Z}$-graded algebra. $\square$

Remark 2.1.6. {idealsinKxxinverse} This remark will turn out to be quite useful in understanding the ideal structure of general Leavitt path algebras. There is a natural $\mathbb{Z}$-grading on the Laurent polynomial algebra $A = K[x,x^{-1}]$, given by setting $A_i = Kx^i$ for all $i \in \mathbb{Z}$. Furthermore, it is well-known (and easy to prove) that the set of units in $K[x,x^{-1}]$ consists of the set $\{kx^i \mid k \in K^\times, i \in \mathbb{Z}\}$. Consequently, the only graded ideals of $K[x,x^{-1}]$ are the two ideals $\{0\}$ and $K[x,x^{-1}]$ itself.

Moreover, there are infinitely many non-graded ideals in $K[x,x^{-1}]$, since every nontrivial ideal of $K[x,x^{-1}]$ is generated by a unique element of the form $1 + k_1x + \cdots + k_nx^n$ with $k_n \neq 0$.

Consider a field K and a group G. Given two G-graded K-algebras $A = \oplus_{\sigma \in G} A_\sigma$ and $B = \oplus_{\sigma \in G} B_\sigma$, a K-algebra homomorphism f from A into B is said to be a *graded homomorphism* if $f(A_\sigma) \subseteq B_\sigma$ for every $\sigma \in G$. It is easy to show that $\operatorname{Ker}(f)$ is a graded ideal of A in this case. If there exists a K-algebra isomorphism $f : A \to B$ for which both f and f^{-1} are graded homomorphisms, then we say that A and B are *graded isomorphic*.

Examples 2.1.7. {gradingexamples} We demonstrate how the $\mathbb{Z}$-grading on $L_K(E)$ manifests in two fundamental cases.

First, let A_n be the oriented n-line graph $\bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet$ of Notation 1.3.1. In Proposition 1.3.5 we established that $L_K(A_n) \cong \mathrm{M}_n(K)$, by writing down an explicit isomorphism φ between these two algebras. For each integer t with $-(n-1) \leq t \leq n-1$ we consider the K-subspace A_t of $A = \mathrm{M}_n(K)$ consisting of those elements $(a_{i,j})$ for which $a_{i,j} = 0$ for each pair i,j having $i - j \neq t$. (Less formally, A_t consists of the elements of the t^{th}-superdiagonal of A.) For $|t| \geq n$ we set $A_t = \{0\}$. Then it is easy to see (and well-known) that $\oplus_{t \in \mathbb{Z}} A_t$ is a $\mathbb{Z}$-grading of $\mathrm{M}_n(K)$. Furthermore, $\varphi : L_K(A_n) \to \mathrm{M}_n(K)$ is a graded isomorphism with respect to this grading.

Now let R_1 be the graph $\bullet^v \circlearrowright e$, also of Notation 1.3.1. In Proposition 1.3.4 we showed that $L_K(R_1) \cong K[x,x^{-1}]$, via an isomorphism which takes v to 1 and e to x. With the usual grading on $K[x,x^{-1}]$ (described in Remark 2.1.6), this isomorphism is clearly graded. This immediately implies that there are infinitely many non-graded ideals in $L_K(R_1)$, to wit, any ideal generated by a non-monomial expression in e and/or e^*. For instance, $I(v+e)$ is such an ideal. The only graded ideals of $L_K(R_1)$ are $L_K(R_1)$ itself, and $\{0\}$.

We showed in Chapter 1 that the path K-algebra KE over a graph E and the path K-algebra KE^* over the graph E^* can be seen as subalgebras of the Cohn path algebra $C_K(E)$ (Corollary 1.5.7) and of the Leavitt path algebra $L_K(E)$ (Corollary 1.5.13). In fact, both KE and KE^* are graded subalgebras of both $C_K(E)$ and $L_K(E)$.

Lemma 2.1.8. {TheMonomorphismIsGraded} *Let E be an arbitrary graph and K any field.*

(i) *The canonical map $K\hat{E} \to C_K(E)$ is a $\mathbb{Z}$-graded algebra homomorphism. The restrictions $KE \to C_K(E)$ and $KE^* \to C_K(E)$ are $\mathbb{Z}$-graded algebra monomorphisms.*
(ii) *The canonical map $K\hat{E} \to L_K(E)$ is a $\mathbb{Z}$-graded algebra homomorphism. The restrictions $KE \to L_K(E)$ and $KE^* \to L_K(E)$ are $\mathbb{Z}$-graded algebra monomorphisms.*

Proof. The canonical projections given in Corollary 1.5.7 and in Corollary 1.5.13 are K-algebra monomorphisms sending homogeneous elements of degree n into elements of the same degree. □

The proof of the following result is easy, so we omit it.

Lemma 2.1.9. {gradediso} *Let E be an arbitrary graph and K any field. Let I be the ideal of the Cohn path algebra generated by the set $\{v - \sum_{e \in s^{-1}(v)} ee^* \mid v \in \mathrm{Reg}(E)\}$. Then $L_K(E)$ and $C_K(E)/I$ are $\mathbb{Z}$-graded isomorphic K-algebras.*

Lemma 2.1.9 is a particular case of

Proposition 2.1.10. {CohngradedisotoLeavitt} *Let E be an arbitrary graph and K any field. Let X be any subset of* $\mathrm{Reg}(X)$. *Then $C_K^X(E)$ and $L_K(E(X))$ are $\mathbb{Z}$-graded isomorphic K-algebras.*

Proof. By reconsidering the proof of Theorem 1.5.18, it is clear that the given isomorphism indeed respects the grading. □

For the remainder of this section we will focus on the structure of the *zero components* $(C_K(E))_0$ of $C_K(E)$ and $(L_K(E))_0$ of $L_K(E)$ with respect to the grading described above. As we shall see, these subrings will play important roles in the sequel. Let S be a subset of $\mathrm{Reg}(E)$. Given $k \in \mathbb{Z}^+$, let X be a finite set of paths of E of length $\leq k$. For $0 \leq i \leq k$, let X_i be the set of initial paths of elements of X of length i, and let Y_i be the set of edges which appear in position i in a path of an element of X. That is,

{CompleteSubset}

$$X_i = \{\lambda \in \mathrm{Path}(E) \mid |\lambda| = i \text{ and there exists } \lambda' \text{ such that } \lambda\lambda' \in X\}, \text{ and}$$

$$Y_i = \{e \in E^1 \mid \text{there exists } \lambda, \gamma \in \mathrm{Path}(E) \text{ such that } |\lambda| = i-1, \text{ and } \lambda e \gamma \in X\}.$$

Note that X_0 is the set of source vertices of paths in X. For a path λ of length $\geq i$, denote by λ_i the initial segment of λ of length i, so that $\lambda = \lambda_i \lambda'$, with $|\lambda_i| = i$.

Definitions 2.1.11. Let S, X, X_i, Y_i, and k be as above. We say that X is an *S-complete subset* of $\mathrm{Path}(E)$ if the following conditions are satisfied:

(i) All the paths in X of length $< k$ end in a sink.
(ii) For every $\lambda \in X$, every $i < |\lambda|$ such that $r(\lambda_i) \in S$ and every $e \in s^{-1}(r(\lambda_i))$, we have that $\lambda_i e = \gamma_{i+1}$ for some $\gamma \in X$.
(iii) For any $\lambda \in X_i$ $(1 \leq i < k)$ and any $e \in Y_{i+1}$ such that $r(\lambda) = s(e)$, we have $\lambda e \in X_{i+1}$.

{S-completesubsets}

Recall that we defined the notion of an S-complete subgraph in Chapter 1 (see Definition 1.6.8). This notion should not be confused with the just defined concept of S-complete subset of paths of a graph.

There is a natural way to build S-complete finite subsets of $\mathrm{Path}(E)$ from S-complete finite subgraphs of E, as follows. The goal is to extend the paths in the S-complete finite subgraph to either paths of length k, or to paths of length less than k which end in a sink, in a specifically described way.

Proposition 2.1.12. *Let F be a finite S-complete subgraph of E and $k \geq 1$. Then there exists an S-complete subset of* $\mathrm{Path}(E)$ *of paths of length $\leq k$ which contains all the paths of length k of F, as well as all the paths of length $< k$ of F which end in a sink of E. More precisely, there is a finite S-complete subgraph F' of E containing F such that X is the set of all paths of F' of length k starting at a vertex of F together with the set of all paths of F' of length $< k$ starting at a vertex of F and ending in a sink of E.*

Proof. For a vertex v of E with $v \in (E^0 \setminus (\mathrm{Sink}(E) \cup S)) \cap (\mathrm{Sink}(F) \cup (E^0 \setminus F^0))$, we choose and fix some $e_v \in s_E^{-1}(v)$.

For each $v \in E^0$ and each $t \geq 1$, we denote by $\Gamma(v,t)$ the set of all paths of length $\leq t$ which satisfy the following conditions:

1. All paths in $\Gamma(v,t)$ start at v.
2. The paths in $\Gamma(v,t)$ either have length t, or have length $< t$ and end in a sink of E.
3. If $\alpha_1 \alpha_2 \cdots \alpha_s \in \mathrm{Path}(E)$ (where $\alpha_i \in E^1$) belongs to $\Gamma(v,t)$, then for each i such that $s(\alpha_i) \in (E^0 \setminus S) \cap (\mathrm{Sink}(F) \cup (E^0 \setminus F^0))$ we have $\alpha_i = e_{s(\alpha_i)}$. Moreover, for each i such that $s(\alpha_i) \in F^0 \setminus \mathrm{Sink}(F)$, we have $\alpha_i \in F^1$.

The idea here is that we extend paths of length less than k arbitrarily in vertices of S, by using edges in F whenever we can; while we extend such paths by a predetermined edge if the vertex does not belong to S, is not a sink in E, and we cannot extend it by using edges in F. Observe that $\Gamma(v,t)$ is finite. Now note the following:

(a) Every path λ in $\Gamma(v,s)$, with $s < t$, can be extended to a path τ in $\Gamma(v,t)$, i.e., there is a path λ' such that $\lambda\lambda' \in \Gamma(v,t)$.

(b) If $\gamma \in \Gamma(v,t)$ and γ' is an initial segment of γ of positive length s, then $\gamma' \in \Gamma(v,s)$.

(c) If $\gamma \in \Gamma(v,t)$ and γ' is a final segment of γ of positive length s, then $\gamma' \in \Gamma(s(\gamma'),s)$.

Let $\Gamma^{(1)}$ denote the set of paths of F of length k together with the paths of F of length $< k$ which end in a sink of E.

Let $\Gamma^{(2)}$ denote the set of paths of length $\leq k$ consisting of all paths of the form $\lambda\mu$, where λ is a path of F of length $< k$ which ends in a sink of F which is not a sink in E, and $\mu \in \Gamma(r(\lambda), k - |\lambda|)$.

Let X be the (disjoint) union of $\Gamma^{(1)}$ and $\Gamma^{(2)}$. To complete the proof, we need to check that X is an S-complete subset of $\mathrm{Path}(E)$.

Observe that

$$X = \bigcup_{v \in F^0} \Gamma(v,k).$$

Condition (i) in the definition of S-complete subset is obviously satisfied. For condition (ii), let $\lambda \in X$, $i < |\lambda|$ such that $r(\lambda_i) \in S$, and $e \in s^{-1}(r(\lambda_i))$. Note that $\lambda_i e \in \Gamma(s(\lambda_i e), i+1)$, so by (a) $\lambda_i e$ can be extended to a path in $\gamma \in \Gamma(s(\lambda),k)$. If γ is a path of F then $\gamma \in \Gamma^{(1)}$. Otherwise we have $\gamma \in \Gamma^{(2)}$.

Finally we check (iii). Let $\lambda \in X$, $1 \le i < k$, and $e \in Y_{i+1}$ such that $r(\lambda_i) = s(e)$. Then $\lambda_i \in \Gamma(s(\lambda), i)$, and $e\mu \in \Gamma(s(e), k-i)$ for a certain path μ (because $e \in Y_{i+1}$). Therefore $\lambda_i e \mu \in X$, so that $\lambda_i e \in X_{i+1}$, as desired.

The last statement is shown as follows. Let v be a vertex of E which appears as a non-final vertex of a path from X. If $v \in F^0 \setminus \mathrm{Sink}(F)$, then we set $s_{F'}^{-1}(v) = s_F^{-1}(v)$. If $v \in S$, then we set $s_{F'}^{-1}(v) = s_E^{-1}(v)$. If $v \in (E^0 \setminus (\mathrm{Sink}(E) \cup S)) \cap (\mathrm{Sink}(F) \cup (E^0 \setminus F^0))$, then we set $s_{F'}^{-1}(v) = \{e_v\}$. The graph F' is the smallest subgraph of E containing F and all these edges. □

{matricialalgebradef}

Definition 2.1.13. A *matricial K-algebra* is a finite direct product of full matrix algebras (of finite size) over a field K.

Let S be a subset of $\mathrm{Reg}(E)$, and let X be an S-complete finite subset of $\mathrm{Path}(E)$ consisting of paths of length $\le k$. We define

$$\mathscr{G}(X) = \mathrm{span}_K(\lambda\mu^* \mid \lambda, \mu \in X, |\lambda| = |\mu|).$$

{lem:0component}

Proposition 2.1.14. *Let E be an arbitrary graph and K any field. Let S be a subset of* $\mathrm{Reg}(E)$. *Let X be an S-complete finite subset of* $\mathrm{Path}(E)$ *consisting of paths of length $\le k$. For $1 \le i \le k$, we consider the following K-subspaces $\mathscr{F}_i(X)$ of $C_K^S(E)$:*

$$\mathscr{F}_i(X) \textit{ is the } K-\textit{linear span in } C_K^S(E) \textit{ of the elements } \lambda\Big(v - \sum_{e \in Y_i, s(e)=v} ee^*\Big)\mu^*,$$

where $\lambda, \mu \in X_{i-1}$, $r(\lambda) = r(\mu) = v \notin S$, and $Y_i \cap s^{-1}(v) \neq \emptyset$. We set

$$\mathscr{F}(X) = \mathscr{G}(X) + \sum_{i=1}^{k} \mathscr{F}_i(X).$$

Then $\mathscr{F}(X)$ is a matricial K-algebra. Moreover, $(C_K^S(E))_0$ is the direct limit of the different subalgebras $\mathscr{F}(X)$, where X ranges over all the S-complete finite subsets of $\mathrm{Path}(E)$.

Proof. We will show:

(1) for every $1 \le i \le k$, $\mathscr{F}_i(X)$ is a matricial K-algebra, and

(2) for $i \neq j$ we have $\mathscr{F}_i(X) \cdot \mathscr{F}_j(X) = 0$. In particular, the sum $\mathscr{F}(X) = \sum_{i=1}^{k} \mathscr{F}_i(X)$ is a direct sum.

To establish these two statements, write an element $\lambda(v - \sum_{e \in Y_i, s(e)=v} ee^*)\mu^*$ in $\mathscr{F}(X)$ as $\lambda\tau_i(v)\mu^*$, where $\tau_i(v) = v - \sum_{e \in Y_i, s(e)=v} ee^*$. To show (1) for $1 \le i \le k$, observe that if $\lambda\tau_i(v)\mu^*$ and $\gamma\tau_i(w)\eta^*$ belong to $\mathscr{F}_i(X)$, and $v \neq w$ then we have

$$\lambda\tau_i(v)\mu^* \cdot \gamma\tau_i(w)\eta^* = 0.$$

If $v = w$ then

$$\lambda\tau_i(v)\mu^* \cdot \gamma\tau_i(v)\eta^* = \delta_{\mu,\gamma}\lambda\tau_i(v)\eta^*.$$

It follows that $\mathscr{F}_i(X) = \bigoplus_v \mathscr{F}_{i,v}(X)$, where $\mathscr{F}_{i,v}(X)$ is the linear span of the set of elements of the form $\lambda\tau_i(v)\mu^*$. Moreover $\mathscr{F}_{i,v}(X)$ is a matrix algebra over K of size $|X_{i-1}|$. This shows (1).

Now assume that $i \neq j$ and that $\alpha = \lambda\tau_i(v)\mu^*$ and $\beta = \gamma\tau_j(w)\eta^*$ belong to $\mathscr{F}_i(X)$ and $\mathscr{F}_j(X)$ respectively. Assume for convenience that $j > i$. Then $\alpha\beta = 0$ unless $\gamma = \mu\gamma'$, with $|\gamma'| = j - i > 0$, in which case

$$\alpha \cdot \beta = \lambda \tau_i(v)\gamma' \tau_j(w)\eta^*.$$

Write $\gamma' = f\gamma''$. Then $f \in Y_i$ and $s(f) = r(\mu) = v$ and thus

$$\tau_i(v)\gamma' = (v - \sum_{e\in Y_i, s(e)=v} ee^*)f\gamma'' = (f-f)\gamma'' = 0.$$

It follows that $\alpha\beta = 0$. This shows that $\sum_{i=1}^k \mathscr{F}_i(X)$ is a direct sum.

The space $\mathscr{G}(X)$ is also a matricial K-algebra, indeed

$$\mathscr{G}(X) = \Big[\bigoplus_{i=0}^{k-1} \bigoplus_{v\in \mathrm{Sink}(E)} \mathscr{G}_{i,v}(X)\Big] \bigoplus \Big[\bigoplus_{v\in E^0} \mathscr{G}_{k,v}(X)\Big],$$

where $\mathscr{G}_{i,v}(X)$ is the K-linear span of the set of elements of the form $\lambda\mu^*$, where $\lambda, \mu \in X$, $|\lambda| = |\mu| = i$ and $r(\lambda) = r(\mu) = v$. (This property relies on condition (i) in the definition of an S-complete subset of Path(E).) It is easy to show that the above sum is direct and also that each $\mathscr{G}_{i,v}(X)$ is a finite matrix K-algebra of size the number of elements of X with the prescribed conditions on length and range.

The proof that $\mathscr{G}(X) \cdot \mathscr{F}_i(X) = 0$ for all i is similar to the above. Hence we get the direct sum

$$\mathscr{F}(X) = \mathscr{G}(X) \bigoplus (\bigoplus_{i=1}^k \mathscr{F}_i(X)).$$

We now describe the transition homomorphisms $\mathscr{F}(X) \to \mathscr{F}(X')$, for appropriate pairs of S-complete finite subsets X, X' of Path(E). Suppose that X is an S-complete finite subset of paths of length $\le k$ and that X' is an S-complete finite subset of paths of length $\le \ell$. Then we write $X \le X'$ in case $k \le \ell$ and every path in X can be extended to a path in X', that is, for each λ in X there is a path λ' such that $\lambda\lambda'$ belongs to X'. Observe that only paths of length k can be properly extended. The condition $X \le X'$ implies that $X_i \subseteq X_i'$ for $1 \le i \le k$. Also $X < X'$ implies $k < \ell$.

To describe the transition homomorphism $\mathscr{F}(X) \to \mathscr{F}(X')$ for $X < X'$, we need to specify a rule that allows us eventually to write any of the generators of $\mathscr{F}(X)$ as a linear combination of the generators in $\mathscr{F}(X')$. Let us write $\tau_i(v)$ and $\tau_i'(v)$ for the corresponding elements $v - \sum_{e\in Y_i, s(e)=v} ee^*$ and $v - \sum_{e\in Y_i', s(e)=v} ee^*$ respectively.

We first describe the map on $\mathscr{G}(X)$. Let v be a vertex in E, and suppose that $\lambda, \mu \in X_i$ and $r(\lambda) = r(\mu) = v$. If v is a sink then $\lambda\mu^*$ belongs to $\mathscr{F}_\ell(X')$, so the map is the identity in this case. If $v \in S$ then $i = k$ and

$$\lambda\mu^* = \lambda(\sum_{e\in s^{-1}(v)} ee^*)\mu^* = \sum_{e\in s^{-1}(v)} (\lambda e)(\mu e)^*.$$

Note that, for $e \in s^{-1}(v)$, λe and μe can be enlarged to a path in X' by S-completeness of X' (condition (ii)). If $v \notin S$ then

$$\lambda\mu^* = \lambda(\sum_{e\in Y_{k+1}'} ee^*)\mu^* + \lambda\tau_{k+1}'(v)\mu^*.$$

Note that $\lambda\tau_{k+1}'(v)\mu^* \in \mathscr{F}_{k+1}(X')$ and that the paths $\lambda e, \mu e$, with $e \in Y_{k+1}'$, can be enlarged to paths in X', again by S-completeness of X' (condition (iii)). In this way, an inductive procedure gives the description of the transition mapping $\mathscr{G}(X) \to \mathscr{F}(X')$.

Now let $\lambda\tau_i(v)\mu^*$ be a generating element of $\mathscr{F}_i(X)$, for $1 \le i \le k$. Then

$$\lambda\tau_i(v)\mu^* = \lambda\tau_i'(v)\mu^* + \sum_{f\in Y_i'\setminus Y_i} (\lambda f)(\mu f)^*,$$

and $\lambda\tau_i'(v)\mu^* \in \mathscr{F}_i(X')$, whilst $\lambda f, \mu f$ can be enlarged to paths in X' for all $f \in Y_i' \setminus Y_i$ so that we can proceed as above in order to obtain the image of $\lambda f f^*\mu^*$ in $\mathscr{F}(X')$. This allows us to describe the transition homomorphism $\mathscr{F}_i(X) \to \mathscr{F}(X')$.

Finally, let $a = \sum_{\lambda,\mu\in T, |\lambda|=|\mu|} k_{\lambda,\mu}\lambda\mu^*$ be an arbitrary element in $(C_K^S(E))_0$, where T is a finite set of paths in E. There is a finite S-complete subgraph F of E such that all the paths in T have all their edges in F. Let k be an upper bound for the length of the paths in T. By using Proposition 2.1.12, we can find an S-complete finite subset of $\mathrm{Path}(E)$ consisting of paths of length $\le k$ such that all paths in T can be enlarged to paths in X. Now the above procedure enables us to write a as an element of $\mathscr{F}(X)$. This shows that $(C_K^S(E))_0$ is the direct limit of the different subalgebras $\mathscr{F}(X)$, where X ranges over all the S-complete finite subsets of $\mathrm{Path}(E)$, and completes the proof. □

A foundational reference for the material in the remainder of this section is [81, Section 2.3]. Every injective K-algebra homomorphism

$$\phi\colon A = \mathrm{M}_{n_1}(K)\times\cdots\times\mathrm{M}_{n_r}(K) \longrightarrow B = \mathrm{M}_{m_1}(K)\times\cdots\times\mathrm{M}_{m_s}(K)$$

is conjugate to a block diagonal one, and so it is completely determined by its multiplicity matrix $M = (m_{ji}) \in \mathrm{M}_{s\times r}(\mathbb{Z}^+)$, which has the property that $\sum_{i=1}^r m_{ji}n_i \le m_j$ for $j = 1,\dots,s$. If ϕ is unital, then the above inequality is an equality. Note that the injectivity hypothesis is equivalent to the statement that there is no zero column in the matrix M. For $i \in \{1,\dots,r\}$, the numbers m_{ji} can be computed as follows. Take a minimal idempotent e_i in the component $\mathrm{M}_{n_i}(K)$ of A. Then $\phi(e_i)$ can be written as $\phi(e_i) = \sum_{j=1}^s\sum_{m=1}^{m_{ji}} g_{j,m}^{(i)}$, where, for each j, $\{g_{j,1}^{(i)},\dots,g_{j,m_{ji}}^{(i)}\}$ are pairwise orthogonal minimal idempotents in the factor $\mathrm{M}_{m_j}(K)$ of B.

{Asubnsdefinition}

Definition 2.1.15. Let E be a finite graph and denote by $A_E = (a_{i,j}) \in \mathrm{M}_{E^0\times E^0}(\mathbb{Z}^+)$ the *incidence* or *adjacency* matrix of E, where $a_{v,w} = |\{e\in E^1 \mid s(e)=v, r(e)=w\}|$. We let A_{ns} denote the matrix A with the zero-rows removed; that is, A_{ns} is the (not necessarily square) matrix gotten from A by removing the rows corresponding to the sinks of E.

We are now in position to give an explicit description of the zero component of the Leavitt path algebra of a finite graph.

{cor:finitegraph0-component}

Corollary 2.1.16. *Let E be a finite graph and K any field. For each $n\in\mathbb{Z}^+$ let $L_{0,n}\subseteq L_K(E)$ denote the K-linear span of elements of the form $\gamma\eta^*$ where $\gamma,\eta\in\mathrm{Path}(E)$ for which $|\gamma|=|\eta|=n$ and $r(\gamma)=r(\eta)$, together with elements of the form $\gamma\eta^*$ where $\gamma,\eta\in\mathrm{Path}(E)$ for which $|\gamma|=|\eta|<n$ and $r(\gamma)=r(\eta)$ is a sink in E. Then we have*

$$(L_K(E))_0 \;=\; \bigcup_{n=0}^{\infty} L_{0,n}.$$

For each v in E^0, and each $n\in\mathbb{Z}^+$, we denote by $P(n,v)$ the set of paths γ in E such that $|\gamma|=n$ and $r(\gamma)=v$. Then

$$L_{0,n} \cong \Big[\prod_{m=0}^{n-1}\Big(\prod_{v\in\mathrm{Sink}(E)}\mathrm{M}_{|P(m,v)|}(K)\Big)\Big]\times\Big[\prod_{v\in E^0}\mathrm{M}_{|P(n,v)|}(K)\Big].$$

The transition homomorphism $L_{0,n}\to L_{0,n+1}$ is the identity on the factors $\prod_{v\in\mathrm{Sink}(E)}\mathrm{M}_{|P(m,v)|}(K)$, for $0\le m\le n-1$, and also on the factor $\prod_{v\in\mathrm{Sink}(E)}\mathrm{M}_{|P(n,v)|}(K)$ of the last term of the displayed formula. The transition homomorphism

$$\prod_{v\in E^0\setminus\mathrm{Sink}(E)}\mathrm{M}_{|P(n,v)|}(K)\to\prod_{v\in E^0}\mathrm{M}_{|P(n+1,v)|}(K)$$

has multiplicity matrix equal to A_{ns}^t.

Proof. All these facts follow directly from the proof of Proposition 2.1.14. For instance, observe that for $v\in E^0\setminus\mathrm{Sink}(E)$ and $\lambda\in P(n,v)$, we have that $\lambda\lambda^*$ is a minimal idempotent in the factor $\mathrm{M}_{|P(n,v)|}(K)$ of $L_{0,n}$ and that by the (CK2) relation

$$\lambda\lambda^* = \sum_{e\in s^{-1}(v)}(\lambda e)(\lambda e)^*,$$

so that, for $w \in E^0$, the multiplicity $m_{w,v}$ of the inclusion map

$$\prod_{v \in E^0 \setminus \mathrm{Sink}(E)} \mathrm{M}_{|P(n,v)|}(K) \to \prod_{v \in E^0} \mathrm{M}_{|P(n+1,v)|}(K)$$

is precisely $a_{v,w}$, which shows that $M = A_{ns}^t$. $\square$

We note that the K-subspaces $L_{0,n}$ described in the previous result form a filtration of $(L_K(E))_0$, given by the K-linear span of the paths $\gamma\nu^*$ such that $|\gamma| = |\nu| \leq n$ and $r(\gamma) = r(\nu)$.

{zerocomponentR2}

Example 2.1.17. Let $E = R_2$, with vertex v and edges e, f. Then for each $n \in \mathbb{Z}^+$ we have $|P(n,v)| = 2^n$. There are no sinks in E, so that $A_{ns}^t = A = (2)$. Thus $L_{0,n} \cong \mathrm{M}_{2^n}(K)$ for each $n \in \mathbb{Z}^+$, and the transition homomorphism from $L_{0,n}$ to $L_{0,n+1}$ takes an element $(m_{i,j})$ of $\mathrm{M}_{2^n}(K)$ to the element $(m_{i,j}I_2)$ of $\mathrm{M}_{2^{n+1}}(K)$, where I_2 is the 2×2 identity matrix. Thus $(L_K(R_2))_0 \cong \varinjlim_{n \in \mathbb{Z}^+} \mathrm{M}_{2^n}(K)$. (See also [3, Section 2] for further analysis of this direct limit.)

{zerocomponentToeplitz}

Example 2.1.18. Let E_T be the Toeplitz graph of Example 1.3.6, and let $\mathscr{T}$ denote the algebraic Toeplitz K-algebra $L_K(E_T)$. Then easily we see that $|P(n,u)| = |P(n,v)| = 1$ for all $n \in \mathbb{Z}^+$. In particular $\mathscr{T}_{0,0} \cong K \times K$. By Corollary 2.1.16 we have that

$$\mathscr{T}_{0,n} \cong [\prod_{m=0}^{n-1} K] \times [K \times K] \cong K^{n+2}$$

for each $n \in \mathbb{N}$. The transition homomorphism from $\mathscr{T}_{0,n}$ to $\mathscr{T}_{0,n+1}$ takes $(r_0, \ldots, r_{n-1}, r_n, r_{n+1}) \in K^{n+2}$ to $(r_0, \ldots, r_{n-1}, r_n, r_{n+1}, r_{n+1}) \in K^{n+3}$. Thus $\mathscr{T}_0$ is isomorphic to the subring of the direct product $\prod_{m=0}^{\infty} K$ consisting of those elements which are eventually constant.

{redUniqueness}

2.2 The Reduction Theorem and the Uniqueness Theorems

The name of this section derives in part from the name given to Theorem 2.2.11, a result which will prove to be an extremely useful tool in a variety of contexts. For instance, we will see how it yields both Theorems 2.2.15 and 2.2.16 with only a modicum of additional effort. The Reduction Theorem 2.2.11 will also be key to establishing various ring-theoretic properties of an arbitrary Leavitt path algebra, among other uses.

{cminusn}

Notation 2.2.1. For a cycle c based at the vertex v, we will use the following notation:

$$c^0 := v, \quad \text{and} \quad c^{-n} := (c^*)^n \text{ for all } n \in \mathbb{N}.$$

{exitDef}

Definitions 2.2.2. Let E be a graph, let $\mu = e_1 e_2 \cdots e_n$ be a path in E, and let $e \in E^1$.

(i) We say that e is an *exit* for μ if there exists i ($1 \leq i \leq n$) such that $s(e) = s(e_i)$ and $e \neq e_i$.
(ii) We say that E satisfies *Condition* (L) if every cycle in E has an exit.

{exitsExample}

Examples 2.2.3. Here is how Condition (L) manifests in the fundamental graphs of the theory.

(i) Let E be the graph R_n ($n \geq 2$), with edges $\{e_1, e_2, \ldots, e_n\}$. Each e_i is a path of length 1 in E, and each e_j ($j \neq i$) is an exit for e_i (since $s(e_i) = s(e_j)$ for all i, j). In particular, E satisfies Condition (L).
(ii) On the other hand, the path e consisting of the unique loop in the graph $E = R_1$ has no exit (and thus R_1 does not satisfy Condition (L)).
(iii) In the oriented n-line graph A_n, no element of $\mathrm{Path}(A_n)$ has an exit. However, A_n does satisfy Condtion (L) vacuously, as A_n is acyclic.
(iv) In the Toeplitz graph E_T of Example 1.3.6, the edge f is an exit for the loop e (which is the unique cycle in E_T). So E_T satisfies Condition (L).

{Pc(E)def}

Notation 2.2.4. Let E be an arbitrary graph. We denote by $P_c(E)$ the set of all vertices v of E which are in cycles without exits; i.e., $v \in c^0$ for some cycle c having no exits.

{noexitsremark}

Remark 2.2.5. If e is an edge of a path without exits, then $s^{-1}(s(e))$ is a singleton (necessarily e itself), so that the (CK2) relation at $s(e)$ reduces to the equation $s(e) = ee^*$.

We start by exploring the structure of a corner of a Leavitt path algebra at a vertex which lies in a cycle without exits.

{p(c)definition}

Definition 2.2.6. Let E be an arbitrary graph and K any field. For every cycle c based at a vertex v in E, and every polynomial $p(x) = \sum_{i=m}^{n} k_i x^i \in K[x,x^{-1}]$ $(m \leq n;\, m,n \in \mathbb{Z})$, we denote by $p(c)$ the element

$$p(c) := \sum_{i=m}^{n} k_i c^i \in L_K(E)$$

(using Notation 2.2.1).

{CicloSinSalidas}

Lemma 2.2.7. *Let E be an arbitrary graph and K any field. If c is a cycle without exits based at a vertex v, then*

$$vL_K(E)v = \left\{ \sum_{i=m}^{n} k_i c^i \mid k_i \in K,\ m \leq n, m, n \in \mathbb{Z} \right\} \cong K[x,x^{-1}],$$

via an isomorphism that sends v to 1, c to x and c^* to x^{-1}.

Proof. Write $c = e_1 \cdots e_n$, where $e_i \in E^1$. We establish first that any $\gamma \in \mathrm{Path}(E)$ such that $s(\gamma) = v$ is of the form $c^m \tau_q$, where $m \in \mathbb{Z}^+$, $\tau_q = e_1 \cdots e_q$ for $1 \leq q < n$, $\tau_0 = v$, and $\deg(\gamma) = mn + q$. We proceed by induction on $\deg(\gamma)$. If $\deg(\gamma) = 1$ and $s(\gamma) = s(e_1)$ then $\gamma = e_1$. Suppose now that the result holds for any $\lambda \in \mathrm{Path}(E)$ with $s(\lambda) = v$, $\deg(\lambda) \leq sn + t$, and consider any $\gamma \in \mathrm{Path}(E)$ with $s(\gamma) = v$ and $\deg(\gamma) = sn + t + 1$. We can write $\gamma = \gamma' f$ with $\gamma' \in \mathrm{Path}(E)$, $s(\gamma') = v$, $f \in E^1$ and $\deg(\gamma') = sn + t$, so by the induction hypothesis $\gamma' = c^s e_1 \cdots e_t$. Since c has no exits, $s(f) = r(e_t) = s(e_{t+1})$ implies $f = e_{t+1}$. Thus $\gamma = \gamma' e_{t+1} = c^s e_1 \cdots e_{t+1}$.

Now let $\gamma\lambda^*$ be such that $\gamma, \lambda \in \mathrm{Path}(E)$, $s(\gamma) = s(\lambda) = v$. If $\deg(\gamma) = \deg(\lambda)$ and $\gamma\lambda^* \neq 0$, we have $\gamma\lambda^* = c^p e_1 \cdots e_k e_k^* \cdots e_1^* c^{-p} = v$ (using the result of the previous paragraph together with Remark 2.2.5). On the other hand $\deg(\gamma) > \deg(\lambda)$ and $\gamma\lambda^* \neq 0$ imply $\gamma\lambda^* = c^{d+q} e_1 \cdots e_k e_k^* \cdots e_1^* c^{-q} = c^d$, $d \in \mathbb{N}$. In a similar way, from $\deg(\gamma) < \deg(\lambda)$ and $\gamma\lambda^* \neq 0$ follows $\gamma\lambda^* = c^q e_1 \cdots e_k e_k^* \cdots e_1^* c^{-q-d} = c^{-d}$, $d \in \mathbb{N}$.

For any $\alpha \in vL_K(E)v$, write $\alpha = \sum_{i=1}^{p} k_i \gamma_i \beta_i^* + kv$, with $k_i, k \in K$ and $\gamma_i, \beta_i \in \mathrm{Path}(E)$ such that $s(\gamma) = s(\lambda) = v$. Then, using what has been established in the previous paragraphs, we get $\alpha = \sum_{i=0}^{p} k_i c^{m_i}$, where $\deg(\gamma_i \beta_i^*) = m_i n$ for some $m_i \in \mathbb{Z}$.

Define $\varphi \colon K[x,x^{-1}] \to L_K(E)$ by $\varphi(1) = v$, $\varphi(x) = c$ and $\varphi(x^{-1}) = c^*$. It is a straightforward routine to check that φ is a monomorphism of K-algebras with image $vL_K(E)v$, so that $vL_K(E)v$ is isomorphic to the K-algebra $K[x,x^{-1}]$. $\square$

We note that the isomorphism φ of the previous result is a graded isomorphism precisely when the cycle c is a loop. Also, we note that Lemma 2.2.7 allows us to easily reestablish Proposition 1.3.4, namely, that $L_K(R_1)$ is isomorphic to $K[x,x^{-1}]$.

The following result provides a significant portion of the Reduction Theorem; effectively, it will allow us to "reduce" various elements of $L_K(E)$ to a nonzero scalar multiple of a vertex.

{previo}

Lemma 2.2.8. *Let E be an arbitrary graph and K any field. Suppose that v is a vertex of E for which $T(v) \cap P_c(E) = \emptyset$; in other words, for every $w \in E^0$ for which $v \geq w$, w does not lie on a cycle without exits. Let $\alpha := kv + \sum_{i=1}^{n} k_i \tau_i \in KE$, where $n \in \mathbb{N}$, $k, k_i \in K^\times$ and $\tau_i \in \mathrm{Path(E)} \setminus \{v\}$ with $s(\tau_i) = r(\tau_i) = v$, for which $\tau_i \neq \tau_j$. Then there exists $\gamma \in \mathrm{Path}(E)$, with $s(\gamma) = v$, such that $\gamma^* \alpha \gamma = kr(\gamma)$.*

Proof. We may suppose that $0 < \deg(\tau_1) \leq \ldots \leq \deg(\tau_n)$. Since the τ_i's are paths starting and ending at v, and $T(v) \cap P_c(E) = \emptyset$, there exists $\gamma \in \mathrm{Path}(E)$ such that $\tau_1 = \gamma\tau'$ (with $\tau' \in \mathrm{Path}(E)$), $s(\gamma) = v$ and $|s^{-1}(r(\gamma))| > 1$. For those values of i for which there exists τ_i' such that $\tau_i = \gamma\tau_i'$ we have $\gamma^* \tau_i \gamma = \tau_i' \gamma$; otherwise $\gamma^* \tau_i \gamma = 0$. After reordering the subindices we get $\gamma^* \alpha \gamma = kr(\gamma) + \sum_{i=1}^{m} k_i \tau_i' \gamma$, with $m \leq n$. Let e be the initial edge of $\tau_1' \gamma$. Observe that $s(\tau_1') = r(\gamma)$, and $|s^{-1}(r(\gamma))| > 1$. So there exists $f \in s^{-1}(r(\gamma))$ such that $f \neq e$. We have

$$f^*\gamma^*\alpha\gamma f = kr(f) + \sum_{i=2}^{m} k_i f^*\tau_i'\gamma f,$$

and $f^*\tau_i'\gamma f$ is either a path in real edges, or zero. Moreover, $T(r(f)) \cap P_c(E) = \emptyset$ as $r(f) \leq v$ and $T(v) \cap P_c(E) = \emptyset$. Hence we have reached the same initial conditions, but using fewer summands. So continuing in this way we eventually produce a nonzero multiple of a vertex. □

{gdeg}

Definitions 2.2.9 . A monomial $e_1 \cdots e_m f_1^* \cdots f_n^*$ in a path algebra $K\hat{E}$ over an extended graph $\hat{E}$, where $e_i, f_j \in E^1$ and $m, n \in \mathbb{Z}^+$, is said to have *degree in ghost edges* (or simply *ghost degree*) equal to n. Monomials in KE are said to have degree in ghost edges equal to 0. The *degree in ghost edges* of an expression of the form $\sum_{i=1}^n k_i\gamma_i\lambda_i^*$, with $k_i \in K^\times$, denoted $\mathrm{gdeg}(\sum_{i=1}^n k_i\gamma_i\lambda_i^*)$, is defined to be the maximum of the degree in ghost edges of the monomials $\gamma_i\lambda_i^*$.

Because the representation of an element $\alpha \in L_K(E)$ as an element of the form $\sum_{i=1}^n k_i\gamma_i\lambda_i^*$ is not uniquely determined, the direct extension of the notion of "degree in ghost edges" to elements of $L_K(E)$ is not well-defined. However, we define the degree in ghost edges of an element $\alpha \in L_K(E)$, also denoted $\mathrm{greg}(\alpha)$, to be the minimum of the degrees in ghost edges among all the representations of α as an expression $\sum_{i=1}^n k_i\gamma_i\lambda_i^*$ in $L_K(E)$ as above.

{lem:reducing-the-gdg}

Lemma 2.2.10. *Let E be an arbitrary graph and K any field. Let α be an element of $L_K(E)$ with positive degree in ghost edges and let $e \in E^1$. Then* $\mathrm{gdeg}(\alpha e) < \mathrm{gdeg}(\alpha)$.

Proof. Let $\alpha = \sum_{i=1}^n k_i\gamma_i\lambda_i^*$, $k_i \in K^\times$, be an expression of α with smallest degree in ghost edges. Note that if the degree in ghost edges of a monomial $\gamma_j\lambda_j^*$ is positive, then $\mathrm{gdeg}(\gamma_j\lambda_j^* e) < \mathrm{gdeg}(\gamma_j\lambda_j^*)$. The result follows. □

{trespuntouno}

We now come to the key result of this section. Roughly speaking, this theorem says that any nonzero element of a Leavitt path algebra may be "reduced", via multiplication on the left and right by appropriate paths, to either a vertex or to a cycle without exits (or perhaps both).

Theorem 2.2.11. (Reduction Theorem) *Let E be an arbitrary graph and K any field. For any nonzero element $\alpha \in L_K(E)$ there exist $\mu, \eta \in \mathrm{Path}(E)$ such that either:*

{3.1vertex}
{3.1cycle}

(1) $0 \neq \mu^*\alpha\eta = kv$, *for some $k \in K^\times$ and $v \in E^0$, or*
(2) $0 \neq \mu^*\alpha\eta = p(c)$, *where c is a cycle without exits and $p(x)$ is a nonzero polynomial in $K[x, x^{-1}]$.*

Proof. The first step will be to show that for $0 \neq \alpha \in L_K(E)$ there exists $\eta \in \mathrm{Path}(E)$ such that $0 \neq \alpha\eta \in KE$. Let $v \in E^0$ be such that $\alpha v \neq 0$ (such a vertex v exists by Lemma 1.2.12(v)). Write $\alpha v = \sum_{i=1}^r \alpha_i e_i^* + \alpha'$, where $\alpha_i \in L_K(E)r(e_i)$, $\alpha' \in (KE)v$, $e_i \in E^1$, $e_i \neq e_j$ for every $i \neq j$, and $s(e_i) = v$ for all $1 \leq i \leq r$.

Note that if $\mathrm{gdeg}(\alpha v) = 0$, then we are done.

Suppose otherwise that $\mathrm{gdeg}(\alpha v) > 0$. If $\alpha v e_j = 0$ for every $j \in \{1, \ldots, r\}$, then multiplying the equation $\alpha v = \sum_{i=1}^r \alpha_i e_i^* + \alpha'$ on the right by e_j gives $0 = \alpha v e_j = \alpha_j + \alpha' e_j$, so $\alpha_j = -\alpha' e_j$, and $\alpha v = \sum_{i=1}^r (-\alpha' e_i e_i^*) + \alpha' = \alpha'((\sum_{i=1}^r -e_i e_i^*) + v) \neq 0$. In particular, $0 \neq (\sum_{i=1}^r -e_i e_i^*) + v$ and $\alpha' \neq 0$. So by (CK2) there exists $f \in s^{-1}(v) \setminus \{e_1, \ldots, e_r\}$. Now, by the structure of KE, $\alpha v f = \alpha' f \in KE \setminus \{0\}$, and we have finished the proof of the first step.

On the other hand, suppose that there exists $j \in \{1, \ldots, r\}$ such that $\alpha v e_j \neq 0$. There is no loss of generality if we consider $j = 1$. Then $0 \neq \alpha v e_1 = \alpha_1 + \alpha' e_1 = (\alpha_1 + \alpha' e_1)r(e_1)$, where $\mathrm{gdeg}(\alpha_1 + \alpha' e_1) < \mathrm{gdeg}(\alpha v)$ by Lemma 2.2.10. Repeating this argument a finite number of times, we reach $\eta \in \mathrm{Path}(E)$ with $\alpha\eta \in KE \setminus \{0\}$.

Now pick $0 \neq \alpha \in L_K(E)$. By the previous paragraph, we know that there exists $\eta \in \mathrm{Path}(E)$ such that $\beta := \alpha\eta \in KE \setminus \{0\}$. Write $\beta = \sum_{i=1}^s k_i\gamma_i$, with $k_i \in K^\times$, $\gamma_i \in \mathrm{Path}(E)$, and with $r(\gamma_i) = r(\eta) =: v$ for every i. We will prove the result by induction on s.

Suppose $s = 1$. If $\deg(\gamma_1) = 0$, then there is nothing to prove. If $\deg(\gamma_1) > 0$, then $\gamma_1^*\alpha\eta = \gamma_1^*\beta = k_1\gamma_1^*\gamma_1 = k_1 r(\gamma_1) \neq 0$.

Now suppose the result is true for any element having at most $s-1$ summands. Write again $\beta = \sum_{i=1}^s k_i\gamma_i$, where $k_i \in K^\times$, $\gamma_i \in \mathrm{Path}(E)$, $\gamma_i \neq \gamma_j$ if $i \neq j$ and $\deg(\gamma_i) \leq \deg(\gamma_{i+1})$ for every $i \in \{1, \ldots, s-1\}$. Then $0 \neq \gamma_1^*\beta = k_1 v + \sum_{i=2}^s k_i\gamma_1^*\gamma_i$.

If $\gamma_1^*\gamma_i = 0$ for some $i \in \{2,\dots,s\}$, then apply the induction hypothesis to get the result. Otherwise, $0 \neq \mu := \gamma_1^*\beta = k_1 v + \sum_{i=2}^{s} k_i\mu_i$, where the μ_i are paths starting and ending at v and satisfying $0 < \deg(\mu_2) \leq \dots \leq \deg(\mu_s)$. If $T(v) \cap P_c(E) = \emptyset$, then by Lemma 2.2.8 there exists a path τ such that $\tau^*\gamma_1^*\alpha\eta\tau = \tau^*\mu\tau = k_1 r(\tau)$, and we are done. If $T(v) \cap P_c(E) \neq \emptyset$, then there is a path ρ starting at v such that $w := r(\rho)$ is a vertex in a cycle c without exits. In this case, $0 \neq \rho^*\gamma_1^*\alpha\eta\rho = \rho^*\mu\rho \in wL_K(E)w$, and by Lemma 2.2.7 the proof is complete. □

We note that both cases in The Reduction Theorem 2.2.11 can occur simultaneously: for instance, in $L_K(R_1)$ we have $e^*e = v$, which is simultaneously a vertex as well as the base of a cycle without exits.

The conclusion we obtained in the first step of the proof of the Reduction Theorem, and a consequence of it, will be of great use later on, so we note them in the following two results.

{path} {pathuno} {pathdos} {vertices}

Corollary 2.2.12. *Let E be an arbitrary graph and K any field. Let α be a nonzero element in $L_K(E)$.*

(i) *There exists $\eta \in \mathrm{Path}(E)$ such that $0 \neq \alpha\eta \in KE$.*
(ii) *If α is a homogeneous element of $L_K(E)$, then there exists $\eta \in \mathrm{Path}(E)$ such that $0 \neq \alpha\eta$ is a homogeneous element of KE.*

Corollary 2.2.13. *Let E be an arbitrary graph and K any field. Let α be a nonzero homogeneous element of $L_K(E)$. Then there exist $\mu,\eta \in \mathrm{Path}(E)$, $k \in K^\times$, and $v \in E^0$ such that $0 \neq \mu^*\alpha\eta = kv$.*

In particular, every nonzero graded ideal of $L_K(E)$ contains a vertex.

Proof. By Corollary 2.2.12(ii) there exists $\eta \in \mathrm{Path}(E)$ for which $0 \neq \alpha\eta$ is a homogeneous element in KE. So we may write $\alpha\eta = \sum_{i=1}^{n} k_i\beta_i$ where the β_i are distinct paths in E, and the lengths of the β_i are equal. But then $\beta_1^*\beta_1 = r(\beta_1)$, while $\beta_1^*\beta_i = 0$ for all $2 \leq i \leq n$ by Lemma 1.2.12(i). Thus $\beta_1^*\alpha\eta = k_1 r(\beta_1)$, as desired.

The particular statement follows immediately. □

{CondLimplVertices}

We noted in Examples 2.1.7 that the Leavitt path algebra $L_K(R_1)$ contains infinitely many nontrivial non-graded ideals. Since the single vertex of R_1 acts as the identity element of $L_K(R_1)$, none of these ideals contains a vertex. The following result shows that the existence of ideals in $L_K(R_1)$ which do not contain any vertices is a consequence of the fact that the graph R_1 contains a cycle without exits.

Proposition 2.2.14. *Let E be a graph satisfying Condition* (L) *and K any field. Then every nonzero ideal of $L_K(E)$ contains a vertex.*

Proof. Let I be a nonzero ideal of $L_K(E)$, and let α be a nonzero element in I. Since E satisfies Condition (L) then by the Reduction Theorem there exist $\mu,\eta \in \mathrm{Path}(E)$ such that $0 \neq \mu^*\alpha\eta = kv$ with $v \in E^0$ and $k \in K^\times$. This implies $0 \neq v = k^{-1}\mu^*\alpha\eta \in L_K(E)IL_K(E) \subseteq I$. □

The converse of Proposition 2.2.14 is also true, as will be proved in Proposition 2.9.13.

{GR-uniqueness} {CK-uniqueness}

Two results of fundamental importance which are direct consequences of the Reduction Theorem are the following Uniqueness Theorems. These results can be considered as the analogs of the Gauge-Invariant Uniqueness Theorem ([117, Theorem 2.2]) and the Cuntz-Krieger Uniqueness Theorem ([117, Theorem 2.4]) for graph C*-algebras.

Theorem 2.2.15. (Graded Uniqueness Theorem) *Let E be an arbitrary graph and K any field. If A is a $\mathbb{Z}$-graded ring, and $\pi : L_K(E) \to A$ is a graded ring homomorphism with $\pi(v) \neq 0$ for every vertex $v \in E^0$, then π is injective.*

Theorem 2.2.16. (Cuntz-Krieger Uniqueness Theorem) *Let E be an arbitrary graph which satisfies Condition (L), let K be any field, and let A be any K-algebra. If $\pi : L_K(E) \to A$ is a ring homomorphism with $\pi(v) \neq 0$ for every vertex $v \in E^0$, then π is injective.*

Proof of Theorems 2.2.15 and 2.2.16. We use the basic fact that the kernel of any ring homomorphism is an ideal of the domain. For the Graded Uniqueness Theorem, as π is a graded homomorphism we have that $\mathrm{Ker}(\pi)$ is a graded ideal of $L_K(E)$. Thus $\mathrm{Ker}(\pi)$ is either $\{0\}$ or contains a vertex, by Corollary 2.2.13. For the Cuntz-Krieger Uniqueness Theorem, we use Proposition 2.2.14 to conclude that $\mathrm{Ker}(\pi)$ is either $\{0\}$ or contains a vertex in this situation as well. Since the hypotheses of both statements presumes that π sends vertices to nonzero elements, the only option is $\mathrm{Ker}(\pi) = 0$ in both cases. $\square$

{variante}

The next result is similar in flavor to the two Uniqueness Theorems.

Proposition 2.2.17. *Let E be an arbitrary graph and K any field. Let A be a $\mathbb{Z}$-graded K-algebra and let $\pi : L_K(E) \to A$ be a (not necessarily graded) K-algebra homomorphism for which $\pi(v) \neq 0$ for every vertex $v \in E^0$, and for which π maps each cycle without exits in E to a nonzero homogeneous element of nonzero degree in A. Then π is injective.*

Proof. By hypothesis, $\mathrm{Ker}(\pi)$ is an algebra ideal of $L_K(E)$ which does not contain vertices. If $\mathrm{Ker}(\pi)$ is nonzero, then by the Reduction Theorem $\mathrm{Ker}(\pi)$ contains a nonzero element $p(c)$, where $p(x) = \sum_{i=-m}^{n} k_i x^i \in K[x, x^{-1}]$ and c is a cycle without exits. Let $q(x) = x^m p(x) \in K[x]$; then $q(c) = c^m p(c) = \sum_{i=0}^{n+m} k_{i-m} c^i \in \mathrm{Ker}(\pi)$. So $0 = \pi(q(c)) = q(\pi(c)) = \sum_{i=0}^{n+m} k_{i-m} \pi(c)^i$. But this is impossible since $\pi(c)$ is a nonzero homogeneous element of nonzero degree in A. $\square$

We finish out the section by giving a direct application of the Graded Uniqueness Theorem, in which we demonstrate an embedding of Leavitt path algebras corresponding to naturally arising subgraphs F of a given graph E.

Definition 2.2.18. (The restriction graph) Let E be an arbitrary graph, and let H be a hereditary subset of E^0. We denote by E_H the *restriction graph*:

$$E_H^0 := H, \quad E_H^1 := \{e \in E^1 \mid s(e) \in H\},$$

{restuno}

and the source and range functions in E_H are simply the source and range functions in E, restricted to H.

Proposition 2.2.19. *Let E be an arbitrary graph and K any field. Let H be a hereditary subset of E^0.*

(i) Consider the assignment

$$v \mapsto v, \quad e \mapsto e, \quad \text{and } e^* \mapsto e^*$$

{restrdos}

(for $v \in E_H^0$ and $e \in E_H^1$), which maps elements of $L_K(E_H)$ to elements of $L_K(E)$. Then this assignment extends to a $\mathbb{Z}$-graded monomorphism of Leavitt path algebras $\varphi : L_K(E_H) \to L_K(E)$.

(ii) If H is finite, then $\varphi(L_K(E_H)) = p_H L_K(E) p_H$, where $p_H = \sum_{v \in H} v \in L_K(E)$.

Proof. (i) Consider these elements of $L_K(E)$: $a_v = v$, $a_e = e$, and $b_e = e^*$ for $v \in E_H^0, e \in E_H^1$. Then by definition we have that the set $\{a_v, a_e, b_e\}$ is an E_H-family in $L_K(E)$, so the indicated assignment extends to a K-algebra homomorphism $\varphi : L_K(E_H) \to L_K(E)$ by the Universal Property 1.2.5. That φ is a graded homomorphism is clear from the definition of the grading on $L_K(E_H)$ and $L_K(E)$. That φ is a monomorphism then follows from an application of the Graded Uniqueness Theorem 2.2.15.

(ii) We show that (ii) follows from (i). Since every element in $L_K(E)$ is a K-linear combination of elements of the form $\gamma\lambda^*$ with $\gamma, \lambda \in \mathrm{Path}(E)$, then every element in $p_H L_K(E) p_H$ is a K-linear combination of elements $\gamma\lambda^*$, with $\gamma, \lambda \in \mathrm{Path}(E)$ having $s(\gamma), s(\lambda) \in H$. Thus $\gamma\lambda^* \in \mathrm{Im}(\varphi)$. The containment $\mathrm{Im}(\varphi) \subseteq p_H L_K(E) p_H$ is immediate using that p_H is the multiplicative identity of $L_K(E_H)$. $\square$

{FirstConsequences}

2.3 Additional consequences of the Reduction Theorem

As part of the power of the Reduction Theorem 2.2.11 we will see that every Leavitt path algebra is semiprime, semiprimitive, and nonsingular. Numerous additional applications of the Reduction Theorem will be presented throughout the sequel.

Recall that a ring R is said to be *semiprime* if, for every ideal I of R, $I^2 = 0$ implies $I = 0$. A ring R is said to be *semiprimitive* in case the Jacobson radical $J(R)$ of R is zero.

{semiprimenessLPA}

Proposition 2.3.1. *Let E be an arbitrary graph and K any field. Then the Leavitt path algebra $L_K(E)$ is semiprime.*

{semiprimitive}

Proof. Let I be a nonzero ideal of $L_K(E)$, and consider a nonzero element $\alpha \in I$. By the Reduction Theorem 2.2.11, there exist $\gamma, \lambda \in \mathrm{Path}(E)$ such that $\gamma^*\alpha\lambda = kv$ or $\gamma^*\alpha\lambda = p(c) \in wL_K(E)w$, where $k \in K^\times$, $v, w \in E^0$, $c \in P_c(E)$ and $w \in c^0$. Then $kv \in I$ or $p(c) \in I$. Observe that since $(kv)^2 = k^2 v \neq 0$ and $(p(c))^2 \neq 0$ (use that $wL_K(E)w$ has no nonzero zero divisors, by Lemma 2.2.7), $I^2 \neq 0$ and hence $L_K(E)$ is semiprime. $\square$

Proposition 2.3.2. *Let E be an arbitrary graph and K any field. Then the Leavitt path algebra $L_K(E)$ is semiprimitive.*

Proof. Denote by J the Jacobson radical of $L_K(E)$, and suppose there is a nonzero element $\alpha \in J$. By the Reduction Theorem 2.2.11, there exist $\mu, \eta \in \mathrm{Path}(E)$ such that $0 \neq \mu^*\alpha\eta = kv$ or $\mu^*\alpha\eta = p(c) \in wL_K(E)w$, where $k \in K^\times$, $v, w \in E^0$, $c \in P_c(E)$ and $w \in c^0$. In the first case we would have $v \in J$, but this is not possible, as the Jacobson radical contains no nonzero idempotents. In the second case, let u denote $s(c)$. Then $\mu^*\alpha\eta$ is a nonzero element in $J \cap uL_K(E)u$, which coincides with the Jacobson radical of $uL_K(E)u$ by [89, §III.7, Proposition 1]. But by Lemma 2.2.7 $uL_K(E)u \cong K[x, x^{-1}]$ which has zero Jacobson radical. In both cases we get a contradiction, hence $J = 0$. $\square$

We note that Proposition 2.3.2 indeed directly implies Proposition 2.3.1, as it is well known that any semiprimitive ring is semiprime. We have included Proposition 2.3.1 simply to provide an additional example of the power of the Reduction Theorem.

{RadGraded}

We present here a second approach to establishing that every Leavitt path algebra is semiprimitive. This approach makes use of an extension of an unpublished result of Bergman about the Jacobson radical of unital $\mathbb{Z}$-graded rings; this extension (Lemma 2.3.3) may be of interest in its own right.

Lemma 2.3.3. *Let R be a $\mathbb{Z}$-graded ring that contains a set of local units consisting of homogeneous elements. Then the Jacobson radical $J(R)$ of R is a graded ideal of R.*

Proof. Given $x \in J(R)$, decompose x into its homogeneous components: $x = x_{-n} + \cdots + x_{-1} + x_0 + x_1 + \cdots + x_n$, where $n \in \mathbb{N}$ (and x_i can be zero). By Lemma 1.2.12(v) there exists a sum of distinct vertices $u \in L_K(E)$ for which $x = uxu$. Since u is a homogeneous element of degree 0, and we get that

$$x = uxu = ux_{-n}u + \cdots + ux_{-1}u + ux_0u + ux_1u + \cdots + ux_nu$$

{nonsingdefinitions}

is also a decomposition of x into its homogeneous components inside the unital ring uRu, so that $x_i = ux_iu$ for every $i \in \{-n, \ldots, -1, 0, 1, \ldots n\}$. As the corner uRu is also a $\mathbb{Z}$-graded ring, and as $J(uRu) = uJ(R)u$, the displayed equation yields a decomposition of the element x in the Jacobson radical of uRu, which is a graded ideal of the $\mathbb{Z}$-graded unital ring uRu (see [108, 2.9.3 Corollary], or the aforementioned unpublished result of Bergman). Therefore every x_i is in $J(uRu)$, and, consequently, in $J(R)$. $\square$

A second proof of Proposition 2.3.2. By Lemma 2.3.3 and Corollary 2.2.13, if the Jacobson radical of $L_K(E)$ were nonzero, then it would contain a vertex, hence a nonzero idempotent, which is impossible. $\square$

Definitions 2.3.4. Let R be a ring and $x \in R$. The *left annihilator of x in R*, denoted by $\mathrm{lan}_R(x)$ (or more simply by $\mathrm{lan}(x)$ if the ring R is understood), is the set $\{r \in R \mid rx = 0\}$. A left ideal I of R is said to be *essential* if $I \cap I' \neq 0$ for every nonzero left ideal I' of R. In this situation we write $I \triangleleft_e^l R$. The set

$$Z_l(R) = \{x \in R \mid \mathrm{lan}(x) \ \triangleleft_e^l R\},$$

which is an ideal of R (see [96, Corollary 7.4]), is called the *left singular ideal* of R. The ring R is called *left nonsingular* if $Z_l(R) = \{0\}$. *Right nonsingular* rings are defined similarly, while *nonsingular* means that R is both left and right nonsingular.

A very useful tool to overcome the lack of a unit element in a ring or algebra, and to translate problems from a nonunital context to a unital one, are local rings at elements. This notion was first introduced in the context of associative algebras in [72]. We refer the reader to [76] for a fuller account of the transfer of various properties between rings and their local rings at elements.

Definition 2.3.5. Let R be a ring and let $a \in R$. The *local ring* of R *at* a is defined as $R_a = aRa$, with sum inherited from R, and product given by $axa \cdot aya = axaya$.

Notice that if e is an idempotent in the ring R, then the local ring of R at e is just the corner eRe. The following result can be found in [76].

Lemma 2.3.6. *Let R be a semiprime ring. Then:*

(i) *If $a \in Z_l(R)$, then $Z_l(R_a) = R_a$.*
(ii) *$Z_l(R_a) \subseteq Z_l(R)$.*
(iii) *R is left nonsingular if and only if R_a is left nonsingular for every $a \in R$.*

Proposition 2.3.7. *Let E be an arbitrary graph and K any field. Then $L_K(E)$ is nonsingular.*

Proof. Suppose that the left singular ideal $Z_l(L_K(E))$ contains a nonzero element α. By the Reduction Theorem there exist $\gamma, \mu \in \mathrm{Path}(E)$ such that $0 \neq \gamma^* \alpha \mu \in Kv$ for some vertex $v \in E^0$, or $0 \neq \gamma^* \alpha \mu \in uL_K(E)u \cong K[x, x^{-1}]$ (by Lemma 2.2.7), where u is a vertex in a cycle without exits. Since, for any ring R, $Z_l(R)$ is an ideal of R and does not contain nonzero idempotents, the first case cannot happen.

In the second case, denote by β the nonzero element $\gamma^* \alpha \mu \in Z_l(L_K(E))$, and for notational convenience denote the Leavitt path algebra $L_K(E)$ by L. Then, by Lemma 2.3.6(i) (which can be applied due to Proposition 2.3.1), $Z_l(L_\beta) = L_\beta$. It is not difficult to see that $L_\beta = (L_u)_\beta$, and therefore, $Z_l((L_u)_\beta) = (L_u)_\beta$. Note that $L_u \cong K[x,\, x^{-1}]$, which is a nonsingular ring. This implies, by Lemma 2.3.6 (iii), that every local algebra of L_u at an element is left nonsingular; in particular, $L_\beta = Z_l(L_\beta) = 0$. Now the semiprimeness of L yields $\beta = 0$, a contradiction.

The right nonsingularity of $L_K(E)$ follows from Corollary 2.0.9. □

2.4 Graded ideals: basic properties and quotient graphs

In this section we present a description of the graded ideals of a Leavitt path algebra. The main goal here (Theorem 2.4.8) is to show that every graded ideal can be constructed from a hereditary saturated subset of E^0, possibly augmented by a set of *breaking vertices* (cf. Definition 2.4.4). With this information in hand, we then proceed to analyze the quotient algebra $L_K(E)/I$ for a graded ideal I. Specifically, we show in Theorem 2.4.15 that there exists a graph F for which $L_K(E)/I \cong L_K(F)$ as $\mathbb{Z}$-graded K-algebras.

This introductory analysis of the graded ideal structure will provide a foundation for the remaining results of Chapter 2. Looking forward, we will use the ideas of this section to explicitly describe the lattice of graded ideals of $L_K(E)$ in terms of graph-theoretic properties; to show how graded ideals of $L_K(E)$ are themselves Leavitt path algebras in their own right; and how the graded ideals, together with various sets of cycles in E and polynomials in $K[x]$, provide complete information about the lattice of *all* ideals of $L_K(E)$.

We start by presenting a description of the elements in the ideal generated by a hereditary subset of vertices.

Lemma 2.4.1. *Let E be an arbitrary graph and K any field. Let H be a hereditary subset of E^0. Then the ideal $I(H)$ of $L_K(E)$ consists of elements of $L_K(E)$ of the form*

$$I(H) = \left\{ \sum_{i=1}^{n} k_i \gamma_i \lambda_i^* \mid n \geq 1,\ k_i \in K^\times,\ \gamma_i, \lambda_i \in \mathrm{Path}(E) \text{ such that } r(\gamma_i) = r(\lambda_i) \in H \right\}.$$

Moreover, if $\overline{H}$ denotes the saturated closure of H, then $I(H) = I(\overline{H})$.

Proof. Let J denote the set presented in the display. To see that J is an ideal of $L_K(E)$ we need to show that for every element of the form $\alpha\beta^*$, where $r(\alpha) = r(\beta) = u \in H$, and for every $a, b \in L_K(E)$, we have $a\alpha u \beta^* b \in J$. Taking into account statements (i) and (iii) of Lemma 1.2.12, it is enough to prove that $\gamma\lambda^* u \mu \eta^* \in J$ for every $\gamma, \lambda, \mu, \eta \in \mathrm{Path}(E)$ and $u \in H$.

If $\gamma\lambda^* u\mu\eta^* = 0$ we are done. Suppose otherwise that $\gamma\lambda^* u\mu\eta^* \neq 0$. By Lemma 1.2.12(i), $\gamma\lambda^* u\mu\eta^* = \gamma\mu'\eta^*$ if $\mu = \lambda\mu'$, or $\gamma\lambda^* u\mu\eta^* = \gamma(\lambda')^*\eta^*$ if $\lambda = \mu\lambda'$. Note that $u = s(\mu)$ and H hereditary imply $r(\mu) \in H$, therefore, $r(\mu') = r(\mu) \in H$ in the first case, and $r(\lambda') = r(\mu) \in H$ in the second case, which imply $\gamma\lambda^* u\mu\eta^* \in J$ in both cases. This shows that J is an ideal of $L_K(E)$; as it contains H and must be contained in every ideal containing H, it must coincide with $I(H)$.

Now we prove $I(H) = I(\overline{H})$. Clearly $I(H) \subseteq I(\overline{H})$. Conversely, we will show by induction that $H_n \subseteq I(H)$ for every $n \in \mathbb{Z}^+$ (where the notation H_n is as in Lemma 2.0.7). For $n = 0$ there is nothing to prove, as $H_0 = T(H) = H \subseteq I(H)$. Suppose $H_{n-1} \subseteq I(H)$ and take $u \in H_n$. Then $s^{-1}(u) = \{e_1, \ldots, e_m\}$, and so $\{r(e_i) | 1 \leq i \leq m\} = r(s^{-1}(u)) \subseteq H_{n-1}$, which is contained in $I(H)$ by the induction hypothesis. This means $u = \sum_{i=1}^m e_i e_i^* = \sum_{i=1}^m e_i r(e_i) e_i^* \in I(H)$ and the proof is complete. □

{vertH}

Corollary 2.4.2. *Let E be an arbitrary graph and K any field. Let H be a nonempty hereditary subset of E^0. Then for every nonzero homogeneous $x \in I(H)$ there exist $\alpha, \beta \in \mathrm{Path}(E)$ such that $\alpha^* x \beta = ku$ for some $k \in K^\times$ and $u \in H$.*

Proof. Given the nonzero homogeneous element $x \in I(H)$, apply Corollary 2.2.13 to choose $\lambda, \mu \in \mathrm{Path}(E)$ such that $k^{-1}\lambda^* x \mu = v$ for some $k \in K^\times$ and $v \in E^0$. Since $x \in I(H)$ this equation gives that $v \in I(H)$. So by Lemma 2.4.1 we may write $v = \sum_{i=1}^m k_i' \lambda_i \mu_i^*$ with $k_i' \in K^\times$ and $\lambda_i, \mu_i \in \mathrm{Path}(E)$ with $r(\lambda_i) = r(\mu_i) \in H$. Then $0 \neq r(\mu_1) = \mu_1^* \mu_1 = \mu_1^* v \mu_1 = k^{-1} \mu_1^* \lambda^* x \mu \mu_1 \in H$, so that $r(\mu_1) = u$, $\mu_1^* \lambda^* = \alpha^*$ and $\mu\mu_1 = \beta$ satisfy the assertion. □

The following result demonstrates the natural, fundamental connection between the (CK1) and (CK2) condition on the elements of $L_K(E)$ on the one hand, and the ideal structure of $L_K(E)$ on the other. Recall the definition of the set $\mathscr{H}_E$ of hereditary saturated subsets of E given in Definitions 2.0.5.

{Idhersat}

Lemma 2.4.3. *Let E be an arbitrary graph and K any field. Let I be an ideal of $L_K(E)$, Then $I \cap E^0 \in \mathscr{H}_E$.*

Proof. Let $v, w \in E^0$ be such that $v \geq w$, and $v \in I$. So there exists a path $p \in \mathrm{Path}(E)$ with $v = s(p)$ and $w = r(p)$. Then Lemma 1.2.12(i) implies that $w = p^* p = p^* v p \in I$. This shows that $I \cap E^0$ is hereditary.

Now let $u \in \mathrm{Reg}(E)$, and suppose $r(e) \in I$ for every $e \in s^{-1}(u)$. By (CK2), $u = \sum_{e \in s^{-1}(u)} ee^* = \sum_{e \in s^{-1}(u)} e r(e) e^* \in I$. Thus $I \cap E^0$ is saturated. □

{AdmissibleVertices}

One eventual goal in our study of the graded ideals in a Leavitt path algebra is the Structure Theorem for Graded Ideals, Theorem 2.5.8. The idea is to associate with each graded ideal of $L_K(E)$ some data inherent in the underlying graph. The previous lemma establishes a first connection of this type. The following graph-theoretic idea will provide a key ingredient in this association.

Definitions 2.4.4. Let E be an arbitrary graph and K any field. Let H be a hereditary subset of E^0, and let $v \in E^0$. We say that v is a *breaking vertex of H* if v belongs to the set

$$B_H := \{v \in E^0 \setminus H \mid v \in \mathrm{Inf}(E) \text{ and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

In words, B_H consists of those vertices of E which are infinite emitters, which do not belong to H, and for which the ranges of the edges they emit are all, except for a finite (but nonzero) number, inside H. For $v \in B_H$, we define the element v^H of $L_K(E)$ by setting

$$v^H := v - \sum_{e \in s^{-1}(v) \cap r^{-1}(E^0 \setminus H)} ee^*.$$

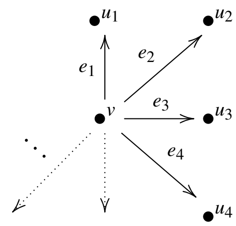
We note that any such v^H is homogeneous of degree 0 in the standard $\mathbb{Z}$-grading on $L_K(E)$. For any subset $S \subseteq B_H$, we define $S^H \subseteq L_K(E)$ by setting $S^H = \{v^H \mid v \in S\}$.

{bvE0andemptyremark}

Of course a row-finite graph contains no breaking vertices, so that this concept does not play a role in the study of the Leavitt path algebras arising from such graphs.

Remark 2.4.5. Let E be an arbitrary graph. It is easy to show both that $B_\emptyset = \emptyset$, and that $B_{E^0} = \emptyset$. The latter is trivial, while the former follows by noting that $|s^{-1}(v) \cap r^{-1}(E^0 \setminus \emptyset)| = \infty$ for any $v \in \mathrm{Inf}(E)$.

The clarify the concept of a breaking vertex, we revisit the infinite clock graph $C_{\mathbb{N}}$ of Example 1.6.12.



Let U denote the set $\{u_i \mid i \in \mathbb{N}\} = C^0_{\mathbb{N}} \setminus \{v\}$. Let H be a subset of U. Since the elements of H are sinks in E, H is clearly hereditary. If $U \setminus H$ is infinite, or if $H = U$, then $B_H = \emptyset$. On the other hand, if $U \setminus H$ is finite, then $B_H = \{v\}$, and in this situation, $v^H = v - \sum_{\{i \mid r(e_i) \in U \setminus H\}} e_i e_i^*$.

{IdGenerated}

For any hereditary subset H of a graph E, and for any $S \subseteq B_H$, the ideal $I(H \cup S^H)$ of $L_K(E)$ is graded, as it is generated by elements of $L_K(E)$ of degree zero (see Remark 2.1.2). We describe more explicitly this ideal in the following result.

Lemma 2.4.6. *Let E be an arbitrary graph and K any field. Let H be a hereditary subset of vertices of E, and S a subset of B_H. Then*

$$I(H \cup S^H) = \operatorname{span}_K(\{\gamma\lambda^* \mid \gamma, \lambda \in \operatorname{Path}(E) \text{ such that } r(\gamma) = r(\lambda) \in H\}) \\ + \operatorname{span}_K(\{\alpha v^H \beta^* \mid \alpha, \beta \in \operatorname{Path}(E) \text{ and } v \in S\}).$$

Moreover, the first summand equals $I(H)$, while the second summand (call it J) is a subalgebra of $L_K(E)$ for which $I(S^H) \subseteq I(H) + J$.

Proof. Clearly $I(H \cup S^H) = I(H) + I(S^H)$. Moreover, by virtue of Lemma 2.4.1, the first summand in the displayed formula of the statement coincides with $I(H)$. Now we study $I(S^H)$. Take $v \in S$, and denote the set $s^{-1}(v) \cap r^{-1}(E^0 \setminus H)$ by $\{f_1, \ldots, f_n\}$, where $n \in \mathbb{N}$. For any $e \in E^1$, compute $e^* v^H$ and $v^H e$. If $s(e) \neq v$, then $e^* v^H = v^H e = 0$. Otherwise, if $s(e) = v$, we distinguish two cases. If $e = f_j$ for some j, then $e^* v^H = e^*(v - \sum_{i=1}^n f_i f_i^*) = f_j^* v - f_j^* f_j f_j^* = f_j^* - f_j^* = 0$, and, as well, $v^H e = (v - \sum_{i=1}^n f_i f_i^*)e = v f_j - f_j f_j^* f_j = f_j - f_j = 0$. If on the other hand $e \notin \{f_1, \ldots, f_m\}$, then $e \in s^{-1}(v) \cap r^{-1}(H)$, so that $r(e) \in H$, and $e^* v^H = e^* = r(e)e^* \in I(H)$. Similarly, $v^H e = e = er(e) \in I(H)$. This means that for $\alpha, \beta \in \operatorname{Path}(E)$ we have $\alpha^* v^H = 0$, or $\alpha^* v^H = \alpha^* \in I(H)$; similarly, $v^H \beta = 0$ or $v^H \beta = \beta \in I(H)$. In either case the resulting product is in $I(H)$, and so $I(S^H) \subseteq I(H) + J$. To see that J is a subalgebra, apply the previous calculation and use that $H v^H = v^H H = 0$ for every $v \in S$. This finishes our proof because $I(H) + J \subseteq I(H) + I(S^H)$. $\square$

{sumher}

Here is a useful application of Lemma 2.4.6.

Proposition 2.4.7. *Let E be an arbitrary graph and K any field. Let $\{H_i\}_{i \in \Lambda}$ be a family of hereditary pairwise disjoint subsets of a graph E. Then*

$$I\left(\overline{\bigsqcup_{i \in \Lambda} H_i}\right) = I\left(\bigsqcup_{i \in \Lambda} H_i\right) = \bigoplus_{i \in \Lambda} I(H_i) = \bigoplus_{i \in \Lambda} I\left(\overline{H_i}\right).$$

Proof. The final equality follows from Lemma 2.4.1. It is easy to see that the union of any family of hereditary subsets is again hereditary, hence $H := \bigcup_{i \in \Lambda} H_i$ is a hereditary subset of E^0. Thus the first equality also follows from Lemma 2.4.1.

By Lemma 2.4.1 every element x in $I(H)$ can be written as $x = \sum_{l=1}^n k_l \alpha_l \beta_l^*$, where $k_l \in K^\times$, $\alpha_l, \beta_l \in \operatorname{Path}(E)$ and $r(\alpha_l) = r(\beta_l) \in H$. Separate the vertices appearing as ranges of the α_l's depending on the H_i's they belong to, and apply again Lemma 2.4.1. This gives $x \in \sum_{i \in \Lambda} I(H_i)$, so that $I(H) \subseteq \sum_{i \in \Lambda} I(H_i)$. The containment $\sum_{i \in \Lambda} I(H_i) \subseteq I(H)$ is clear.

So all that remains is to show that the sum $\sum_{i \in \Lambda} I(H_i)$ is direct. If this is not the case, there exists $j \in \Lambda$ such that $I(H_j) \cap \sum_{j \neq i \in \Lambda} I(H_i) \neq 0$. Since for every l, $I(H_l)$ is a graded ideal, we get that $I(H_j) \cap \sum_{j \neq i \in \Lambda} I(H_i)$ is a graded ideal as well, so there exists a nonzero homogeneous element $y \in I(H_j) \cap \sum_{j \neq i \in \Lambda} I(H_i)$. By Corollary 2.4.2 there exist $\alpha, \beta \in \mathrm{Path}(E)$ and $k \in K^\times$ such that $0 \neq k^{-1}\alpha^* y \beta = w \in H_j$. Observe that w also belongs to $I(\bigcup_{j \neq i \in \Lambda} H_i)$. Write $w = \sum_{l=1}^n k_l \alpha_l \beta_l^*$, with $k_l \in K^\times$, $\alpha_l, \beta_l \in \mathrm{Path}(E)$, and $r(\alpha_l) = r(\beta_l) \in \bigcup_{j \neq i \in \Lambda} H_i$. Then $0 \neq r(\beta_1) = \beta_1^* \beta_1 = \beta_1^* w \beta_1 \in \bigcup_{j \neq i \in \Lambda} H_i$. On the other hand, $s(\alpha_1) = w \in H_j$ implies (since H_j is a hereditary set) $r(\alpha_1) \in H_j$; therefore, $r(\alpha_1) = r(\beta_1) \in H_j \cap \left(\bigcup_{j \neq i \in \Lambda} H_i \right)$, a contradiction. □

{StructureGradedIdeals}

We now deepen the connection between graded ideals of $L_K(E)$ and various subsets of E^0.

Theorem 2.4.8. *Let E be an arbitrary graph and K any field. Then every graded ideal I of the Leavitt path algebra $L_K(E)$ is generated by $H \cup S^H$, where $H = I \cap E^0 \in \mathscr{H}_E$, and $S = \{v \in B_H \mid v^H \in I\}$.*

In particular, every graded ideal of $L_K(E)$ is generated by a set of homogeneous idempotents.

Proof. It is immediate to see that $I(H \cup S^H) \subseteq I$. Now we show $I \subseteq I(H \cup S^H)$. As I is a graded ideal, it is enough to consider nonzero homogeneous elements of the form $\alpha = \alpha v$ of I, where $v \in E^0$.

We will prove $\alpha v \in I(H \cup S^H)$ by induction on the degree in ghost edges of the elements in I. Suppose first $\mathrm{gdeg}(\alpha) = 0$. Then, $\alpha = \sum_{i=1}^m k_i \gamma_i$, with $k_i \in K^\times$, $m \in \mathbb{N}$, and $\gamma_i \in \mathrm{Path}(E)$ with $r(\gamma_i) = v$. As α is a homogeneous element, we may consider those γ_i's having the same degree (i.e., length) as that of α. Moreover, we may suppose all the γ_i's are distinct, hence $\gamma_i^* \gamma_j = 0$ for $i \neq j$ by Lemma 1.2.12(i). Then for every j, $k_j^{-1} \gamma_j^* \alpha v = k_j^{-1} \gamma_j^* (\sum_{i=1}^m k_i \gamma_i) = k_j^{-1} k_j \gamma_j^* \gamma_j = r(\gamma_j) = v \in I \cap E^0 = H$. This means $\alpha v \in I(H) \subseteq I(H \cup S^H)$.

We now suppose the result is true for appropriate elements of $L_K(E)$ having degree in ghost edges strictly less than $n \in \mathbb{N}$, and prove the result for $\mathrm{gdeg}(\alpha v) = n$. Write $\alpha v = \sum_{i=1}^m \mu_i e_i^* + \lambda$, with $\mu_i \in L_K(E)$, $e_i \in E^1$ and $\lambda \in KE$, in such a way that this is a representation of αv of minimal degree in ghost edges.

If $\lambda = 0$ then for every i we have $\alpha v e_i = \mu_i$, which is in $I(H \cup S^H)$ by the induction hypothesis, and we have finished. Hence, we may assume that $\lambda \neq 0$.

As α is homogeneous, we may choose μ_i and λ to be homogeneous as well. Write $\lambda = \sum_{l=1}^n k_l \lambda_l$ for some $k_l \in K^\times$ and λ_l distinct paths of the same length. We first observe that v cannot be a sink because $e_i^* = e_i^* v$ implies $v = s(e_i)$ for every i; in particular, $s^{-1}(v) \neq \emptyset$. Choose $f \in s^{-1}(v)$. If $e_i^* f = 0$ for every i, then $\alpha v f = \lambda f$, which is in $I(H \cup S^H)$ by the previous case. Otherwise, suppose $e_j^* f \neq 0$ for some j. By (CK1) this happens precisely when $f = e_j$, and hence $\alpha v f = (\sum_{i=1}^m \mu_i e_i^* + \lambda) f = \mu_j e_j^* f + \lambda f = \mu_j + \lambda f$, which lies in $I(H \cup S^H)$ by the induction hypothesis. (Note that the induction hypothesis can be applied because $\mathrm{gdeg}(\mu_j + \lambda f) < \mathrm{gdeg}(\alpha v)$.) In any case, $\alpha v f \in I(H \cup S^H)$. Now, if v is not an infinite emitter then $\alpha v = \alpha \sum_{f \in s^{-1}(v)} f f^* \in I(H \cup S^H)$. If v is an infinite emitter, then either $v \in H$, in which case $\alpha v \in I(H \cup S^H)$, or $v \notin H$, in which case $v \in B_H$, as follows. For any $f \in s^{-1}(v) \cap r^{-1}(E^0 \setminus H)$, observe that f must coincide with some e_i because otherwise $\alpha f = \sum_{i=1}^m \mu_i e_i^* f + \lambda f = \lambda f \in I$ would imply $r(f) = f^* f = f^* k_1^{-1} \lambda_1^* \lambda f \in I \cap E^0 = H$, a contradiction. Thus $s^{-1}(v) \cap r^{-1}(E^0 \setminus H) \subseteq \{e_i \mid 1 \leq i \leq m\}$, and so $v \in B_H$.

Now write $\alpha v = \alpha v^H + \alpha \sum_{\{f \in s^{-1}(v) \cap r^{-1}(E^0 \setminus H)\}} f f^*$. Since $\alpha f \in I(H \cup S^H)$ for all $f \in s^{-1}(v) \cap r^{-1}(E^0 \setminus H)$, to show that $\alpha v \in I(H \cup S^H)$, it is enough to show that $v \in S$. We compute

$$e_i^* v^H = e_i^* (v - \sum_{f \in s^{-1}(v) \cap r^{-1}(E^0 \setminus H)} f f^*) = \begin{cases} 0 & \text{if} \quad e_i \in s^{-1}(v) \cap r^{-1}(E^0 \setminus H) \\ e_i^* v & \text{if} \quad e_i \in s^{-1}(v) \cap r^{-1}(H). \end{cases}$$

In the second of these two cases, $s(e_i^*) = r(e_i) \in H$. In either case $e_i^* v^H \in I(H)$.

But $\alpha v^H \in I$ and $e_i^* v^H \in I$ imply $\lambda v^H \in I$, hence $k_1 v^H = \lambda_1^* (\lambda v^H) \in I$, therefore $v^H \in I$ and so $v \in S$ as desired. □

Proposition 2.4.9 is an immediate consequence of Theorem 2.4.8.

{rowfinitegradedideals}

Proposition 2.4.9. *Let E be a row-finite graph and K any field. Then every graded ideal I of $L_K(E)$ is generated by a hereditary and saturated subset of E^0, specifically, $I = I(I \cap E^0)$.*

{gradedidealsgenbyselfadj}

Let $(A,*)$ be an algebra with involution. An ideal I of A is said to be *self adjoint* if $y^* \in I$ whenever $y \in I$. Not every ideal in a Leavitt path algebra is self-adjoint. For instance, consider an arbitrary field K and let E be the graph R_1. Then the ideal I of $L_K(E)$ generated by $v+e+e^3$ is not self-adjoint, as follows. Identify $L_K(R_1)$ and $K[x,x^{-1}]$ via the isomorphism given in Proposition 1.3.4. Our statement rephrased says that $I(1+x+x^3)$ is not a self-adjoint ideal, which is clear as otherwise we would have $1+x^{-1}+x^{-3} \in I(1+x+x^3)$, which is impossible by an observation made in Remark 2.1.6.

By observing that any ideal in an arbitrary graded ring with involution which is generated by a set of self-adjoint elements is necessarily self-adjoint, we record this consequence of Theorem 2.4.8.

Corollary 2.4.10. *Let E be an arbitrary graph and K any field. If I is a graded ideal of $L_K(E)$, then $I = I(X)$ for some set X of homogeneous self-adjoint idempotents in $L_K(E)$. Specifically, every graded ideal of a Leavitt path algebra is self adjoint.*

The converse to Corollary 2.4.10 does not hold. For instance, the ideal $I = I(v+e)$ of $L_K(R_1)$ is self adjoint, as $v+e^* = e^*(v+e) \in I$. However, I is not graded, as noted in Examples 2.1.7. Indeed, this same behavior is exhibited by any ideal of $L_K(R_1)$ of the form $I(p(e))$, where $p(x) \in K[x,x^{-1}]$ is not homogeneous and has the property that $p(x)^* = x^n p(x)$ for some integer n.

{firstquotgraphdef}

In the next section we will strengthen Theorem 2.4.8 to show that in fact there is a bijection between the graded ideals of $L_K(E)$ and pairs of the form (H,S^H). In order to establish that distinct pairs of this form correspond to distinct graded ideals, we analyze the K-algebras which arise as quotients of a Leavitt path algebra by graded ideals. As we shall see, such quotients turn out to be Leavitt path algebras in their own right.

Definition 2.4.11. (The quotient graph by a hereditary subset) Let E be an arbitrary graph, and let H be a hereditary subset of E^0. We denote by E/H the *quotient graph of E by H*, defined as follows:

$$(E/H)^0 = E^0 \setminus H, \quad \text{and} \quad (E/H)^1 = \{e \in E^1 \mid r(e) \notin H\}.$$

The range and source functions for E/H are defined by restricting the range and source functions of E to $(E/H)^1$.

{quotientgeneral}

We anticipate the following result with a brief discussion. We will show that the quotient algebra $L_K(E)/I(H \cup S^H)$ is isomorphic to a relative Cohn path algebra for the quotient graph E/H (with respect to an appropriate subset of vertices), and then subsequently apply Proposition 2.1.10. The intuitive idea underlying Theorem 2.4.12 is as follows. Let H be a hereditary saturated subset of E^0. Then the breaking vertices B_H of H are precisely the infinite emitters in E which become regular vertices in E/H. If $S \subseteq B_H$, and we consider the ideal $I(H \cup S^H)$ of $L_K(E)$, then we are only imposing relation (CK2) on the vertices corresponding to S in the quotient ring $L_K(E)/I(H \cup S^H)$. So it is natural to expect that the quotient $L_K(E)/I(H \cup S^H)$ will be a relative Cohn path algebra with respect to the set $X = (\mathrm{Reg}(E) \setminus H) \cup S$.

Theorem 2.4.12. *Let E be an arbitrary graph and K any field. Let $H \in \mathscr{H}_E$, $S \subseteq B_H$, and $X = (\mathrm{Reg}(E) \setminus H) \cup S$. Then there exists a $\mathbb{Z}$-graded isomorphism of K-algebras*

$$\overline{\Psi} : L_K(E)/I(H \cup S^H) \to C_K^X(E/H).$$

Proof. We consider the assignment (which we denote by Ψ) of elements of the set $E^0 \cup E^1 \cup (E^1)^*$ with specific elements of $C_K^X(E/H)$ given as follows: for each $v \in E^0$ and $e \in E^1$,

$$\Psi(v) = \begin{cases} v & \text{if } v \notin H \\ 0 & \text{otherwise,} \end{cases} \qquad \Psi(e) = \begin{cases} e & \text{if } r(e) \notin H \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \Psi(e^*) = \begin{cases} e^* & \text{if } s(e^*) \notin H \\ 0 & \text{otherwise.} \end{cases}$$

Using this assignment, a set of straightforward computations yields that the collection

$$\{\Psi(v), \Psi(e), \Psi(e^*) \mid v \in E^0, e \in E^1\}$$

is an E-family in $C_K^X(E/H)$. So by the Universal Property of $L_K(E)$ (Remark 1.2.5) there is a unique extension of Ψ to a K-algebra homomorphism

$$\Psi : L_K(E) \to C_K^X(E/H).$$

We note that Ψ is indeed a $\mathbb{Z}$-graded homomorphism, as clearly Ψ preserves the grading of each of the generators of $L_K(E)$. By the definition of E/H, it is immediate that Ψ is surjective. As well, Ψ is clearly 0 on $I(H)$. But we also have that $\Psi(v^H) = 0$ for $v \in S$, because $S \subseteq X$. Consequently, there is an induced map

$$\overline{\Psi} : L_K(E)/I(H \cup S^H) \to C_K^X(E/H).$$

We now define an inverse map for $\overline{\Psi}$. The map Φ is defined as follows: for $v \in (E/H)^0$ and $e \in (E/H)^1$, set

$$\Phi(v) = v + I(H \cup S^H), \qquad \Phi(e) = e + I(H \cup S^H), \qquad \text{and} \qquad \Phi(e^*) = e^* + I(H \cup S^H).$$

By the Universal Property of $C_K^X(E/H)$ (Remark 1.5.10), Φ extends to a K-algebra homomorphism $\Phi : C_K^X(E/H) \to L_K(E)/I(H \cup S^H)$. It is then straightforward to verify that the compositions $\Phi \circ \overline{\Psi}$ and $\overline{\Psi} \circ \Phi$ give the identity on the canonical generators, and therefore give the identity on the corresponding algebras. $\square$

Here are two specific consequences of Theorem 2.4.12.

{quotientremark}

Corollary 2.4.13. *Let K be any field.*

(i) *Suppose E is a row-finite graph, and $H \in \mathscr{H}_E$. Then $L_K(E)/I(H) \cong L_K(E/H)$ as $\mathbb{Z}$-graded K-algebras.*
(ii) *If E is an arbitrary graph, $H \in \mathscr{H}_E$, and $S = B_H$, then*

$$L_K(E)/I(H \cup B_H^H) \;\cong\; C_K^{\mathrm{Reg}(E/H)}(E/H) \;=\; L_K(E/H).$$

Proof. (i) In this case $S = \emptyset$, so that $X = \mathrm{Reg}(E) \setminus H$, and thus $C_K^X(E/H) = L_K(E/H)$. Now apply Theorem 2.4.12.

(ii) In a similar manner, we have in the more general case that $X = (\mathrm{Reg}(E) \setminus H)) \cup B_H = \mathrm{Reg}(E/H)$, so that we may again apply Theorem 2.4.12. $\square$

Theorem 2.4.12 gives a description of the quotient of a Leavitt path algebra by a graded ideal as a relative Cohn path algebra. But by defining a new type of quotient graph, we can in fact describe the quotient of a Leavitt path algebra by a graded ideal as the Leavitt path algebra over this new graph.

{quotientgeneraldef}

Definition 2.4.14. (The quotient graph incorporating breaking vertices) Let E be an arbitrary graph, $H \in \mathscr{H}_E$, and $S \subseteq B_H$. We denote by $E/(H,S)$ the *quotient graph of E by (H,S)*, defined as follows:

$$(E/(H,S))^0 = (E^0 \setminus H) \cup \{v' \mid v \in B_H \setminus S\},$$

$$(E/(H,S))^1 = \{e \in E^1 \mid r(e) \notin H\} \cup \{e' \mid e \in E^1 \text{ and } r(e) \in B_H \setminus S\},$$

and range and source maps in $E/(H,S)$ are defined by extending the range and source maps in E when appropriate, and by in addition setting $s(e') = s(e)$ and $r(e') = r(e)'$.

We note that the quotient graph E/H given in Definition 2.4.11 is precisely the graph $E/(H,B_H)$ in the context of this broader definition. (In particular, we point out that E/H is *not* the same $E/(H,\emptyset)$.)

With this definition, and using Theorem 2.4.12 and Theorem 1.5.18, we get the following.

{quotientisLpaarbitrarytheorem}

Theorem 2.4.15. *Let E be an arbitrary graph and K any field. Then the quotient of $L_K(E)$ by a graded ideal of $L_K(E)$ is $\mathbb{Z}$-graded isomorphic to a Leavitt path algebra. Specifically, there is a $\mathbb{Z}$-graded K-algebra isomorphism*

$$\overline{\Psi} : L_K(E)/I(H \cup S^H) \;\to\; L_K(E/(H,S)),$$

where $\overline{\Psi}$ is defined as in Theorem 2.4.12.

Proof. By Theorem 2.4.12, we have $L_K(E)/I(H \cup S^H) \cong C_K^X(E/H)$, where $X = (\mathrm{Reg}(E) \setminus H) \cup S$. But then $\mathrm{Reg}(E/H) \setminus X = B_H \setminus S$. Therefore, the graph $(E/H)(X)$ from Definition 1.5.16 coincides with the quotient graph $E/(H,S)$, and Theorem 1.5.18 gives that $C_K^X(E/H) \cong L_K(E/(H,S))$ naturally, thus yielding the result. $\square$

{recoveringHandsS}

We close this section with another consequence of Theorem 2.4.12.

Corollary 2.4.16. *Let E be an arbitrary graph and K any field. Suppose $H \in \mathscr{H}_E$ and let $S \subseteq B_H$.*

(i) $I(H \cup S^H) \cap E^0 = H$. *In particular,* $I(H) \cap E^0 = H$.
(ii) $S = \{v \in B_H \mid v^H \in I(H \cup S^H)\}$.

Proof. (i). The containment $H \subseteq I(H \cup S^H)$ is clear. Conversely, for $v \in E^0 \setminus H$, we observe that $\Psi(v)$ is a nonzero element in $C_K^X(E/H)$, where Ψ is the isomorphism given in Theorem 2.4.12. Thus $v \notin I(H \cup S^H)$.

(ii). The containment $S \subseteq \{v \in B_H \mid v^H \in I(H \cup S^H)\}$ is clear. For the reverse containment, observe that in a manner analogous to that used in the proof of (i) we have $\Psi(v^H) \neq 0$ for any $v \in B_H \setminus S$. This shows that $v^H \notin I(H \cup S^H)$, as required. □

{latticeLgrSection} {gradedidealsnotation}

2.5 The Structure Theorem for Graded Ideals, and the internal structure of graded ideals

In the previous section we have developed much of the machinery which will allow us to achieve the main goal of the current section, the Structure Theorem for Graded Ideals (Theorem 2.5.8), which gives a complete description of the lattice of graded ideals of a Leavitt path algebra in terms of specified subsets of E^0.

{orderonHsubE}

Definition 2.5.1. Let E be an arbitrary graph and K any field. Denote by $\mathscr{L}_{gr}(L_K(E))$ the lattice of graded ideals of $L_K(E)$, with order given by inclusion, and supremum and infimum given by the usual operations of ideal sum and intersection.

{reticuloH}

Remark 2.5.2. Let E be an arbitrary graph. We define in $\mathscr{H}_E$ a partial order by setting $H \leq H'$ in case $H \subseteq H'$. Using this ordering, $\mathscr{H}_E$ is a complete lattice, with supremum $\vee$ and infimum $\wedge$ in $\mathscr{H}_E$ given by setting $\vee_{i\in\Gamma} H_i := \overline{\cup_{i\in\Gamma} H_i}$ and $\wedge_{i\in\Gamma} H_i := \cap_{i\in\Gamma} H_i$ respectively.

Definition 2.5.3. Let E be an arbitrary graph. We set

$$\mathscr{S} = \bigcup_{H\in\mathscr{H}_E} \mathscr{P}(B_H),$$

where $\mathscr{P}(B_H)$ denotes the set of all subsets of B_H. We denote by $\mathscr{T}_E$ the subset of $\mathscr{H}_E \times \mathscr{S}$ consisting of pairs of the form (H,S), where $S \in \mathscr{P}(B_H)$. We define in $\mathscr{T}_E$ the following relation:

$$(H_1,S_1) \leq (H_2,S_2) \quad \text{if and only if} \quad H_1 \subseteq H_2 \text{ and } S_1 \subseteq H_2 \cup S_2.$$

The following comments, which explain why the order in $\mathscr{T}_E$ has been defined as in the previous display, will help clarify the proof of the upcoming proposition. For a graph E, a hereditary saturated subset H of E^0, and a breaking vertex $v \in B_H$, define

$$A(v,H) := s^{-1}(v) \cap r^{-1}(E^0 \setminus H).$$

Note that $A(v,H)$ is a finite nonempty subset of E^1.

Now suppose that H_1 and H_2 are hereditary saturated subsets of vertices in E, with $H_1 \subseteq H_2$. Let $v \in B_{H_1}$. Since $H_1 \subseteq H_2$ then $v \in B_{H_2}$, unless it happens to be the case that $r(s^{-1}(v)) \subseteq H_2$ (since by definition a breaking vertex for a set must emit at least one edge whose range is outside the set). If $v \in B_{H_2}$, then write

$$A(v,H_1) = A(v,H_2) \sqcup B,$$

where $B = \{e \in A(v,H_1) \mid r(e) \in H_2\}$. In this case we have

{eqDesc}

$$v^{H_1} = v^{H_2} - \sum_{e \in B} ee^*. \tag{2.1}$$

{neworder}

Proposition 2.5.4. *Let E be an arbitrary graph. For $(H_1,S_1),(H_2,S_2) \in \mathscr{T}_E$, we have*

$$(H_1,S_1) \leq (H_2,S_2) \iff I(H_1 \cup S_1{}^{H_1}) \subseteq I(H_2 \cup S_2^{H_2}).$$

In particular, $\leq$ is a partial order on $\mathscr{T}_E$.

Proof. For notational convenience, set $I(H_i,S_i) := I(H_i \cup S_i^{H_i})$ for $i = 1,2$.

Suppose that $I(H_1,S_1) \subseteq I(H_2,S_2)$. Then $H_1 \subseteq H_2$ by Corollary 2.4.16(i). Now let $v \in S_1$. We will show that $v \in H_2 \cup S_2$. If on the one hand $r(s^{-1}(v)) \subseteq H_2$ then we have

$$v = v^{H_1} + \sum_{e \in A(v,H_1)} ee^* \in I(H_1,S_1) + I(H_2) \subseteq I(H_2,S_2),$$

so that $v \in H_2$ (by again invoking Corollary 2.4.16(i)). If on the other hand there is some $e \in s^{-1}(v)$ such that $r(e) \notin H_2$, then necessarily $v \notin H_2$ (since H_2 is hereditary). So, since we already know that $H_1 \subseteq H_2$, we see that $v \in B_{H_2}$. Moreover, we have, by (2.1),

$$v^{H_2} = v^{H_1} + \sum_{e \in B} ee^* \in I(H_1,S_1) + I(H_2) \subseteq I(H_2,S_2).$$

Hence $v \in S_2$ by Corollary 2.4.16(ii). So we have shown $S_1 \subseteq H_2 \cup S_2$, which yields $(H_1,S_1) \leq (H_2,S_2)$ by definition.

Conversely, suppose that $(H_1,S_1) \leq (H_2,S_2)$. This gives in particular that $I(H_1) \subseteq I(H_2)$, so we only need to check that $v^{H_1} \in I(H_2,S_2)$ for $v \in S_1$. So let $v \in S_1$. If on the one hand $r(s^{-1}(v)) \subseteq H_2$, then $v \in H_2$ because $S_1 \subseteq H_2 \cup S_2$ and $v \notin S_2$ (since $v \notin B_{H_2}$). If on the other hand there is some $e \in s^{-1}(v)$ such that $r(e) \notin H_2$, then $v \in B_{H_2}$ and, by (2.1) we have

$$v^{H_1} = v^{H_2} - \sum_{e \in B} ee^* \in I(H_2,S_2) + I(H_2) \subseteq I(H_2,S_2),$$

showing that $v^{H_1} \in I(H_2,S_2)$. Thus we obtain that $I(H_1,S_1) \subseteq I(H_2,S_2)$. □

For the proof of Proposition 2.5.6 we need to introduce a refinement of the definition of saturation which allows us to consider breaking vertices.

{defSatExt}

Definition 2.5.5. Let E be an arbitrary graph. Let H be a hereditary subset of E^0, and consider a subset $S \subseteq H \cup B_H$. The *S-saturation* of H is defined as the smallest hereditary subset H' of E^0 satisfying the following properties:

(i) $H \subseteq H'$.
(ii) H' is saturated.
(iii) If $v \in S$ and $r(s^{-1}(v)) \subseteq H'$, then $v \in H'$.

We denote by $\overline{H}^S$ the S-saturation of H.

To build the S-saturation of H we proceed as in Lemma 2.0.7. Concretely, for every $n \in \mathbb{Z}^+$ we define inductively the hereditary subsets $\Lambda_n^S(H)$ as follows. Let $\Lambda_0^S(H) := H$. For $n \geq 1$, we put

$$\Lambda_n^S(H) = \Lambda_{n-1}^S(H) \cup \{v \in E^0 \setminus \Lambda_{n-1}^S(H) \mid v \in \mathrm{Reg}(E) \cup S \text{ and } r(s^{-1}(v)) \subseteq \Lambda_{n-1}^S(H)\}.$$

It can be easily shown that $\overline{H}^S = \cup_{n \geq 0} \Lambda_n^S(H)$.

{supANDinf}

Proposition 2.5.6. *Let E be an arbitrary graph. Then with the partial order $\leq$ on $\mathscr{T}_E$ given in Definition 2.5.3, $(\mathscr{T}_E, \leq)$ is a complete lattice, with supremum $\vee$ and infimum $\wedge$ in $\mathscr{T}_E$ given by:*

$$(H_1,S_1)\vee(H_2,S_2)=(\overline{H_1\cup H_2}^{S_1\cup S_2},\ (S_1\cup S_2)\setminus\overline{H_1\cup H_2}^{S_1\cup S_2})\qquad \textit{and}$$

$$(H_1,S_1)\wedge(H_2,S_2)=(H_1\cap H_2,\ (S_1\cap S_2)\cup((S_1\cup S_2)\cap(H_1\cup H_2))).$$

Proof. The fact that $\leq$ is a partial order is established in Proposition 2.5.4.

We first verify the displayed formula for the supremum. Observe that $(\overline{H_1\cup H_2}^{S_1\cup S_2},\ (S_1\cup S_2)\setminus\overline{H_1\cup H_2}^{S_1\cup S_2})\in\mathscr{T}_E$, and that it contains (H_i,S_i) for $i=1,2$.

To show minimality, let $(H,S)\in\mathscr{T}_E$ be such that $(H_i,S_i)\leq(H,S)$ for $i=1,2$. In order to show that $\overline{H_1\cup H_2}^{S_1\cup S_2}\subseteq H$, it suffices, by Definition 2.5.5, to prove, inductively, that $\Lambda_n^{S_1\cup S_2}(H_1\cup H_2)\subseteq H$. For $n=0$ this is clear by assumption. Now, assume $n\geq 1$ and that $\Lambda_{n-1}^{S_1\cup S_2}(H_1\cup H_2)\subseteq H$. Pick $v\in\Lambda_n^{S_1\cup S_2}(H_1\cup H_2)$. If $v\in\mathrm{Reg}(E)$, then v belongs to H because H is saturated. Now suppose $v\in S_1\cup S_2$. By definition and the induction hypothesis, we have

$$r(s^{-1}(v))\subseteq\Lambda_{n-1}^{S_1\cup S_2}(H_1\cup H_2)\subseteq H.$$

In particular, this implies $v\notin S$. Since $v\in S_1\cup S_2\subseteq H\cup S$, we conclude that $v\in H$, completing the induction step. The inclusion $(S_1\cup S_2)\setminus\overline{H_1\cup H_2}^{S_1\cup S_2}\subseteq H\cup S$ is immediate.

Now we verify the indicated expression for the infimum, i.e., we will show that $(H_1\cap H_2,\ (S_1\cap S_2)\cup((S_1\cup S_2)\cap(H_1\cup H_2)))$ is a lower bound for the pair $(H_1,S_1),(H_2,S_2)$, and is the maximal such. First, note that $(H_1\cap H_2,\ S_1\cap S_2\cup((S_1\cup S_2)\cap(H_1\cup H_2)))\leq(H_i,S_i)$ for $i=1,2$; to see this, use $H_i\cap S_i=\emptyset$ for $i=1,2$, so that

$$(S_1\cap S_2)\cup((S_1\cup S_2)\cap(H_1\cup H_2))=(S_1\cap S_2)\cup(S_1\cap H_2)\cup(S_2\cap H_1).$$

Now, suppose $(H,S)\leq(H_i,S_i)$. Then $H\subseteq H_1\cap H_2$ and $S\subseteq H_i\cup S_i$ and so

$$S\subseteq(H_1\cup S_1)\cap(H_2\cup S_2)=(H_1\cap H_2)\cup(S_1\cap S_2)\cup(S_1\cap H_2)\cup(S_2\cap H_1),$$

which by the formula above shows $(H,S)\leq(H_1\cap H_2,\ (S_1\cap S_2)\cup((S_1\cup S_2)\cap(H_1\cup H_2)))$. $\square$

The following example shows the need of considering the notion of S-saturation.

{S-saturationexample}

Examples 2.5.7 (1) Let E be the following graph:

$$\begin{tikzcd}
v_1 \bullet \arrow[r, "(\infty)"] \arrow[d] & v_2 \bullet \\
v_3 &
\end{tikzcd}$$

Let $H_1=\{v_2\}$, $S_1=\{v_1\}$; $H_2=\{v_3\}$, $S_2=\emptyset$. Note that $\overline{H_1\cup H_2}$ does not contain the vertex v_1, which is not a breaking vertex for $H_1\cup H_2$ as $r(s^{-1}(v_1))\subseteq H_1\cup H_2$. This is the reason why we have to consider the S-saturation, which is:

$$\Lambda_1^{S_1}(H_1\cup H_2)=\{v_1,v_2,v_3\}$$

and, consequently, the formula in Proposition 2.5.6 gives that $(H_1,S_1)\vee(H_2,S_2)=(E^0,\emptyset)$.

(2) Let G be the following graph:

$$\begin{tikzcd}
v_1 \bullet \arrow[r, "(\infty)"] \arrow[d] & v_2 \bullet & \\
v_3 \bullet & w_2 \bullet \arrow[l] & \bullet w_1 \arrow[l] \arrow[ul, "(\infty)"']
\end{tikzcd}$$

Let $H_1=\{v_2\}$, $S_1=\{v_1\}$; $H_2=\{v_3\}$, $S_2=\emptyset$. Then

$$\Lambda_1^{S_1}(H_1\cup H_2)=\{v_1,v_2,v_3,w_2\}$$

and

$$\Lambda_2^{S_1}(H_1 \cup H_2) = \{v_1, v_2, v_3, w_2, w_1\}.$$

In this case, again the formula in Proposition 2.5.6 gives that $(H_1,S_1) \vee (H_2,S_2) = (G^0, \emptyset)$.

We now have all the pieces in place to achieve our previously stated goal, in which we give a precise description of the graded ideals of $L_K(E)$ in terms of specified subsets of E^0.

{GradIdgen}

Theorem 2.5.8. (Structure Theorem for Graded Ideals) *Let E be an arbitrary graph and K any field. Then the map φ given here provides a lattice isomorphism:*

$$\varphi : \mathscr{L}_{gr}(L_K(E)) \to \mathscr{T}_E \qquad via \qquad I \mapsto (I \cap E^0, S)$$

where $S = \{v \in B_H \mid v^H \in I\}$ for $H = I \cap E^0$. The inverse φ' of φ is given by:

$$\varphi' : \mathscr{T}_E \to \mathscr{L}_{gr}(L_K(E)) \qquad via \qquad (H,S) \mapsto I(H \cup S^H).$$

Proof. By Lemma 2.4.3 and the definition of S, the map φ is well defined. The map φ' is clearly well defined. By Theorem 2.4.8 we get that $\varphi'\varphi = \mathrm{Id}_{\mathscr{L}_{gr}(L_K(E))}$. On the other hand, Corollary 2.4.16 yields that $\varphi\varphi' = \mathrm{Id}_{\mathscr{T}_E}$.

Now we prove that φ' preserves the order. Suppose that $(H_1,S_1),(H_2,S_2) \in \mathscr{T}_E$ are such that $(H_1,S_1) \leq (H_2,S_2)$. Then $H_1 \subseteq H_2$ and $S_1 \subseteq H_2 \cup S_2$. It is easy to see that $H_1 \subseteq I(H_2 \cup S^{H_2})$. Now we prove $S^{H_1} \subseteq I(H_2 \cup S^{H_2})$. Take $v^{H_1} \in S^{H_1}$. Then $v^{H_1} = v - \sum_{s(e)=v, r(e)\notin H_1} ee^*$ for some infinite emitter $v \in E^0$. We must distinguish two cases: first, if $v \in H_2$, then $v^{H_1} \in I(H_2) \subseteq I(H_2 \cup S^{H_2})$, while second, if $v \in S_2$, then

$$v^{H_1} = v^{H_2} - \sum_{s(e)=v, r(e)\in H_2 \setminus H_1} ee^* \in I(H_2 \cup S^{H_2}).$$

The final step is to show that φ preserves the order. To this end, consider two graded ideals I_1 and I_2 such that $I_1 \subseteq I_2$. Then $H_1 := I_1 \cap E^0 \subseteq H_2 := I_2 \cap E^0$. Now we show $S_1 \subseteq S_2$, where $S_i := \{v \in B_{H_i} \mid v^{H_i} \in I_i\}$, for $i = 1,2$. Take $v \in S_1$. We again must distinguish two cases. Suppose first that for every $e \in E^1$ such that $s(e) = v$ we have $r(e) \in H_2$. Then

$$v = v^{H_1} + \sum_{s(e)=v, r(e)\in H_2 \setminus H_1} ee^* \in I_1 + I_2 = I_2,$$

and thus $v \in I_2 \cap E^0 = H_2$. On the other hand, suppose that there exists $e \in E^1$ such that $s(e) = v$ and $r(e) \notin H_2$. Then $v \in B_{H_2}$ and

$$v^{H_2} = v^{H_1} + \sum_{s(e)=v, r(e)\in H_2 \setminus H_1} ee^* \in I_1 + I_2 = I_2.$$

This implies $v \in S_2$. We obtain that $S_1 \subseteq H_2 \cup S_2$ and hence that $(H_1,S_1) \leq (H_2,S_2)$. □

{GradId}

We record the Structure Theorem for Graded Ideals in the situation where the graph is row-finite.

Theorem 2.5.9. *Let E be a row-finite graph and K any field. The following map φ provides a lattice isomorphism:*

$$\varphi : \mathscr{L}_{gr}(L_K(E)) \to \mathscr{H}_E \qquad via \qquad \varphi(I) = I \cap E^0,$$

with inverse given by

$$\varphi' : \mathscr{H}_E \to \mathscr{L}_{gr}(L_K(E)) \qquad via \qquad \varphi'(H) = I(H).$$

{idealsofclockexample}

Example 2.5.10. The following is a description of all graded ideals of the Leavitt path algebra of the infinite clock graph $C_{\mathbb{N}}$ of Example 1.6.12. Recall that U denotes the set $\{u_i \mid i \in \mathbb{N}\}$ of all "non-center" vertices of $C_{\mathbb{N}}$. It is clear that the hereditary saturated subsets of $C_{\mathbb{N}}$ are $\emptyset$, $C_{\mathbb{N}}^0$, and subsets H of U. (Note that if v is in a hereditary subset H of $C_{\mathbb{N}}$, then necessarily $H = C_{\mathbb{N}}^0$.) For a subset H of U, there is a breaking vertex (namely, v) for H precisely when $U \setminus H$ is nonempty and finite. With this information in hand, we use Theorem 2.5.8 to conclude that a complete irredundant set of graded ideals of $L_K(C_{\mathbb{N}})$ is:

$$\{0\}, \quad L_K(C_{\mathbb{N}}), \quad I(H) \text{ for } H \subseteq U, \quad \text{and} \quad I(H \cup \{v - \sum_{e \in r^{-1}(U \setminus H)} ee^*\}) \text{ for } H \subsetneqq U \text{ having } U \setminus H \text{ finite.}$$

Of interest are the following consequences of the Structure Theorem for Graded Ideals.

{intersectionequalsproduct}

Corollary 2.5.11. *Let E be an arbitrary graph and K any field. Let J_1 and J_2 be graded ideals of $L_K(E)$. Then $J_1 \cdot J_2 = J_1 \cap J_2$.*

Proof. The containment $J_1 \cdot J_2 \subseteq J_1 \cap J_2$ holds for any two-sided ideals in any ring. For the reverse containment, we use Theorem 2.5.8 to guarantee that we can write the graded ideal $J_1 \cap J_2$ as $I(H \cup S^H)$ for some $(H,S) \in \mathscr{T}_E$. So it suffices to show that each of the elements in the generating set $H \cup S^H$ of $J_1 \cap J_2$ is in $J_1 \cdot J_2$. But this follows immediately, as each of these elements is idempotent. □

Recall that a graded algebra A is said to be *graded artinian* (resp., *graded noetherian*) in case A satisfies the descending chain condition (resp., ascending chain condition) on graded two-sided ideals. We need an observation which will be used more than once in the sequel.

{...sonTequivtochaincondsonHLemma}

Lemma 2.5.12. *Let E be an arbitrary graph. Then the following are equivalent.*

(1) *The lattice $\mathscr{T}_E$ satisfies the ascending (resp., descending) chain condition with respect to the partial order given in Definition 2.5.3.*
(2) *The lattice $\mathscr{H}_E$ satisfies the ascending (resp., descending) chain condition (under set inclusion), and, for each $H \in \mathscr{H}_E$, the corresponding set B_H of breaking vertices is finite.*

Proof. We prove the ascending chain condition statement; the proof for the descending chain condition is essentially identical. So suppose the a.c.c. holds in $\mathscr{T}_E$. Let $H_1 \subseteq H_2 \subseteq \dots$ be an ascending chain of hereditary saturated subsets of vertices in E. Then we get an ascending chain $(H_1,\emptyset) \le (H_2,\emptyset) \le \dots$ in $\mathscr{T}_E$. By hypothesis, there is an integer n such that $(H_n,\emptyset) = (H_{n+1},\emptyset) = \dots$. This implies that $H_n = H_{n+1} = \dots$, showing that the a.c.c holds in $\mathscr{H}_E$. Let $H \in \mathscr{H}_E$. Then the corresponding set B_H of breaking vertices of H must be finite, since otherwise B_H would contain an infinite ascending chain of subsets $S_1 \subsetneqq S_2 \subsetneqq \dots$, and this would then give rise to a proper ascending chain $(H,S_1) \lneqq (H,S_2) \lneqq \dots$ in $\mathscr{T}_E$, contradicting the hypothesis that a.c.c. holds in $\mathscr{T}_E$.

Conversely, suppose the a.c.c. holds in $\mathscr{H}_E$, and that B_H is a finite set for each $H \in \mathscr{H}_E$. Consider an ascending chain $(H_1,S_1) \le (H_2,S_2) \le \dots$ in $\mathscr{T}_E$. This gives rise to an ascending chain $H_1 \subseteq H_2 \subseteq \dots$ in $\mathscr{H}_E$, and so there is an integer n such that $H_i = H_n = H$ for all $i \ge n$. So from the n^{th} term onwards, the given chain in $\mathscr{T}_E$ is of the form $(H,S_n) \le (H,S_{n+1}) \le \dots$, where $S_n, S_{n+1}, \dots$ are subsets of B_H. Observe that since $B_H \cap H = \emptyset$, it follows from the definition of $\le$ on $\mathscr{T}_E$ that we have an ascending chain $S_n \subseteq S_{n+1} \subseteq \dots$. Since B_H is a finite set, there is a positive integer m such that $S_{n+m} = S_{n+m+i}$ for all $i \ge 0$. This establishes the a.c.c. in $\mathscr{T}_E$. □

Now combining the Structure Theorem for Graded Ideals with Lemma 2.5.12, we get

{gradedchainconditions}

Proposition 2.5.13. *Let E be an arbitrary graph and K any field. Consider the standard $\mathbb{Z}$-grading on $L_K(E)$.*

(i) *$L_K(E)$ is graded artinian if and only if the set $\mathscr{H}_E$ satisfies the descending chain condition with respect to inclusion, and, for each $H \in \mathscr{H}_E$, the set B_H of breaking vertices is finite.*
(ii) *$L_K(E)$ is graded noetherian if and only if the set $\mathscr{H}_E$ satisfies the ascending chain condition with respect to inclusion, and, for each $H \in \mathscr{H}_E$, the set B_H of breaking vertices is finite.*

{gradedchainconditionsEfinite}

Corollary 2.5.14. *Let E be a finite graph and K any field. Then $L_K(E)$ is both graded artinian and graded noetherian.*

For another direct consequence of the Structure Theorem for Graded Ideals, recall that a graded algebra A is said to be *graded simple* if $A^2 \neq 0$, and A has no graded ideals other than 0 and A. Since $L_K(E)$ is a ring with local units for any graph E and field K, we have $L_K(E)^2 \neq 0$. Thus Theorem 2.5.8 immediately yields

{grsimplicity}

Corollary 2.5.15. *Let E be an arbitrary graph and K any field. Then $L_K(E)$ is graded simple if and only if the only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*

We conclude our discussion of the graded ideals in a Leavitt path algebra by establishing that every graded ideal in a Leavitt path algebra is itself, up to isomorphism, the Leavitt path algebra of an explicitly-described graph. Since dealing with breaking vertices makes the proof of the result for arbitrary graded ideals less "visual", and because a number of our results in the sequel will rely only on this more specific setting, we start our analysis by considering graded ideals of the form $I(H)$ for $H \in \mathscr{H}_E$.

{def1.3DHSz}

Definition 2.5.16. (The hedgehog graph for a hereditary subset) Let E be an arbitrary graph. Let H be a nonempty hereditary subset of E^0. We denote by $F_E(H)$ the set

$$F_E(H) = \{\alpha \in \mathrm{Path}(E) \mid \alpha = e_1 \cdots e_n, \text{ with } s(e_1) \in E^0 \setminus H,\ r(e_i) \in E^0 \setminus H \text{ for all } 1 \le i < n, \text{ and } r(e_n) \in H\}.$$

We denote by $\overline{F}_E(H)$ another copy of $F_E(H)$. If $\alpha \in F_E(H)$, we will write $\overline{\alpha}$ to refer to a copy of α in $\overline{F}_E(H)$. We define the graph ${}_HE = ({}_HE^0, {}_HE^1, s', r')$ as follows:

$${}_HE^0 = H \cup F_E(H), \qquad \text{and} \qquad {}_HE^1 = \{e \in E^1 \mid s(e) \in H\} \cup \overline{F}_E(H).$$

The source and range functions s' and r' are defined by setting $s'(e) = s(e)$ and $r'(e) = r(e)$ for every $e \in E^1$ such that $s(e) \in H$; and by setting $s'(\overline{\alpha}) = \alpha$ and $r'(\overline{\alpha}) = r(\alpha)$ for all $\overline{\alpha} \in \overline{F}_E(H)$.

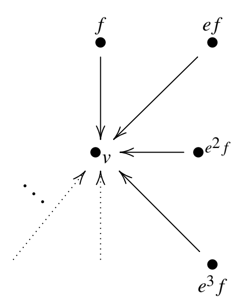
Intuitively, $F_E(H)$ can be viewed as H, together with a new vertex corresponding to each path in E which ends at a vertex in H, but for which none of the previous edges in the path ends at a vertex in H. For every such new vertex, a new edge is added going into H. So the net effect is that in $F_E(H)$, the only paths entering the subgraph H have common length 1; pictorially, the situation evokes an image of the quills (edges into H) on the body of a hedgehog or porcupine (H itself), whence the name.

{cyclesinhedgehog}

Remark 2.5.17. We note that, by construction, the cycles in the hedgehog graph ${}_HE$ are precisely the cycles in H. In particular, as H is hereditary, every cycle without exits in ${}_HE$ arises from a cycle without exits in H.

{HEexample}

Example 2.5.18. Let E_T be the Toeplitz graph $e \circlearrowleft \bullet^u \xrightarrow{f} \bullet^v$, and let H denote the hereditary subset $\{v\}$. Then $F_{E_T}(H) = \{e^n f \mid n \in \mathbb{Z}^+\}$, and ${}_HE_T$ is the graph



If I is an ideal of a ring R, then I itself may be viewed as a ring in its own right. (Of course I need not be unital, nor need it contain a set of local units, e.g., the ideal $2\mathbb{Z}$ of $\mathbb{Z}$.) Similarly, if I is an algebra ideal of a K-algebra A, then I may be viewed as a K-algebra in its own right. We note in this regard that the K-ideal $I(1+x)$ of $K[x,x^{-1}]$ does not contain any nonzero idempotents, hence $I(1+x)$ when viewed as a K-algebra cannot contain a set of local units. Using the identification established between $L_K(R_1)$ and $K[x,x^{-1}]$, this implies in particular that the ideal $I(v+e)$ of $L_K(R_1)$ cannot be isomorphic to the Leavitt path algebra of any graph, as any Leavitt path algebra is an algebra with local units. These comments provide context for the following result.

{LPAideal}

Theorem 2.5.19. *Let E be an arbitrary graph and K any field. Let H be a nonempty hereditary subset of E. Then $I(H)$, when viewed as a K-algebra, is K-algebra isomorphic to $L_K({}_HE)$.*

Proof. We define a map $\varphi : \{u \mid u \in {}_HE^0\} \cup \{e \mid e \in {}_HE^1\} \cup \{e^* \mid e \in {}_HE^1\} \to I(H)$ by the following rule:

$$\varphi(v) = \begin{cases} v & \text{if } v \in H \\ \alpha\alpha^* & \text{if } v = \alpha \in F_E(H), \end{cases} \quad \varphi(e) = \begin{cases} e & \text{if } e \in E^1 \\ \alpha & \text{if } e = \overline{\alpha} \in \overline{F}_E(H), \end{cases} \quad \text{and } \varphi(e^*) = \begin{cases} e^* & \text{if } e \in E^1 \\ \alpha^* & \text{if } e = \overline{\alpha} \in \overline{F}_E(H). \end{cases}$$

Note that for α, β distinct elements in $F_E(H)$ we have $\alpha^*\beta = 0$, so $\{\varphi(u) \mid u \in {}_HE^0\}$ is a set of orthogonal idempotents in $I(H)$. Moreover, it is not difficult to establish that this set, jointly with $\{\varphi(e) \mid e \in {}_HE^1\}$ and $\{\varphi(e^*) \mid e \in {}_HE^1\}$, is an ${}_HE$-family in $I(H)$. So by the Universal Property, φ extends to a K-algebra homomorphism from $L_K({}_HE)$ into $I(H)$.

To see that φ is onto, by Lemma 2.4.1 it is enough to show that every vertex of H and every finite path α of E with $r(\alpha) \in H$ are in the image of φ. For any $v \in H$, $\varphi(v) = v$, so that this case is clear. Now, let $\alpha = \alpha_1 \cdots \alpha_n$ with $\alpha_i \in E^1$. If $s(\alpha_1) \in H$, then $\alpha = \varphi(\alpha_1) \cdots \varphi(\alpha_n)$. Suppose that $s(\alpha_1) \in E^0 \setminus H$ and $r(\alpha_n) \in H$. Then, there exists $1 \leq j \leq n-1$ such that $r(\alpha_j) \in E^0 \setminus H$ and $r(\alpha_{j+1}) \in H$. Thus, $\alpha = \alpha_1 \cdots \alpha_{j+1} \cdot \alpha_{j+2} \cdots \alpha_n$, where $\beta = \alpha_1 \cdots \alpha_{j+1} \in F_E(H)$. Hence, $\alpha = \varphi(\overline{\beta})\varphi(\alpha_{j+2}) \cdots \varphi(\alpha_n)$.

To show injectivity, by Remark 2.5.17 we have that any cycle without exits in ${}_HE$ comes from a cycle without exits in E, where the vertices of the cycle are in H. So every cycle without exits in ${}_HE$ is mapped to a homogeneous nonzero element of nonzero degree in $I(H)$. The injectivity thereby follows by Proposition 2.2.17. □

{newgraph}

In what follows, we will generalize Theorem 2.5.19 in Theorem 2.5.22 by showing that in fact every graded ideal in a Leavitt path algebra is isomorphic to a Leavitt path algebra.

Definition 2.5.20. (The generalized hedgehog graph construction, incorporating breaking vertices) Let E be an arbitrary graph, H a nonempty hereditary subset of E, and $S \subseteq B_H$. We define:

$$F_1(H,S) := \{\alpha \in \text{Path}(E) \mid \alpha = e_1 \cdots e_n,\ r(e_n) \in H \text{ and } s(e_n) \notin H \cup S\}, \text{ and}$$

$$F_2(H,S) := \{\alpha \in \text{Path}(E) \mid |\alpha| \geq 1 \text{ and } r(\alpha) \in S\}.$$

For $i = 1,2$ we denote a copy of $F_i(H,S)$ by $\overline{F}_i(H,S)$. We define the graph ${}_{(H,S)}E$ as follows:

$${}_{(H,S)}E^0 := H \cup S \cup F_1(H,S) \cup F_2(H,S), \text{ and}$$

{hedgehogrk}

$${}_{(H,S)}E^1 := \{e \in E^1 \mid s(e) \in H\} \cup \{e \in E^1 \mid s(e) \in S \text{ and } r(e) \in H\} \cup \overline{F}_1(H,S) \cup \overline{F}_2(H,S).$$

The range and source maps for ${}_{(H,S)}E$ are described by extending r and s to ${}_{(H,S)}E^1$, by defining $r(\overline{\alpha}) = \alpha$ and $s(\overline{\alpha}) = \alpha$ for all $\overline{\alpha} \in \overline{F}_1(H,S) \cup \overline{F}_2(H,S)$.

Remark 2.5.21. Here are some observations about the construction of the generalized hedgehog graph ${}_{(H,S)}E$.

(i) $F_1(H,S) \cap F_2(H,S) = \emptyset$.

(ii) Every cycle in E produces a cycle in ${}_{(H,S)}E$; moreover, cycles in ${}_{(H,S)}E$ come from cycles in E. Thus there is a bijection between the set of cycles in E and the set of cycles in ${}_{(H,S)}E$.

{grIdealsAndLPAs}

(iii) In the particular case $S = \emptyset$, we get:

$$F_1(H,\emptyset) = \{\alpha = e_1 \cdots e_n \in \text{Path}(E) \mid r(e_n) \in H \text{ and } s(e_n) \notin H\};\ \ F_2(H,\emptyset) = \emptyset;\ \ \text{and } {}_{(H,\emptyset)}E = {}_HE.$$

Thus Definition 2.5.20 indeed generalizes the construction of the graph ${}_HE$ given in Definition 2.5.16.

Theorem 2.5.22. *Let E be an arbitrary graph and K any field. Suppose $H \in \mathscr{H}_E$ and $S \subseteq B_H$. Then the graded ideal $I(H \cup S^H)$ of the Leavitt path algebra $L_K(E)$ is isomorphic as K-algebras to the Leavitt path algebra $L_K({}_{(H,S)}E)$.*

Proof. Let $\varphi : \{v \mid v \in {}_{(H,S)}E^0\} \cup \{e \mid e \in {}_{(H,S)}E^1\} \cup \{e^* \mid e \in {}_{(H,S)}E^1\} \to I(H \cup S^H)$ be the map such that:

$$\varphi(v) = \begin{cases} v & \text{if } v \in H \\ v^H & \text{if } v \in S \\ \alpha\alpha^* & \text{if } v = \alpha \in F_1(H,S) \\ \alpha r(\alpha)^H \alpha^* & \text{if } v = \alpha \in F_2(H,S)\,, \end{cases}$$

$$\varphi(e) = \begin{cases} e & \text{if } e \in E^1 \\ \alpha & \text{if } e = \overline{\alpha} \in \overline{F}_1(H,S) \\ \alpha r(\alpha)^H & \text{if } e = \overline{\alpha} \in \overline{F}_2(H,S)\,, \end{cases} \quad \text{and} \quad \varphi(e^*) = \begin{cases} e^* & \text{if } e \in E^1 \\ \alpha^* & \text{if } e = \overline{\alpha} \in \overline{F}_1(H,S) \\ r(\alpha)^H \alpha^* & \text{if } e = \overline{\alpha} \in \overline{F}_2(H,S) \end{cases}$$

for every $v \in {}_{(H,S)}E^0$ and every $e \in {}_{(H,S)}E^1$. It is not difficult to see that each of the elements $\varphi(v)$, $\varphi(e)$, and $\varphi(e^*)$ is an element of $I(H \cup S^H)$. In a manner similar to the proof of Theorem 2.5.19, one can show that the set

$$\{\varphi(v) \mid v \in {}_{(H,S)}E^0\} \cup \{\varphi(e) \mid e \in {}_{(H,S)}E^1\} \cup \{\varphi(e^*) \mid e \in {}_{(H,S)}E^1\}$$

is an ${}_{(H,S)}E$-family in $I(H \cup S^H)$. Consequently, by the Universal Property of $L_K({}_{(H,S)}E)$, the map φ can be uniquely extended to a K-algebra homomorphism from $L_K({}_{(H,S)}E)$ to $I(H \cup S^H)$.

The injectivity of φ follows from Proposition 2.2.17. To show surjectivity, recall the description of the generators of $I(H \cup S^H)$ given in Lemma 2.4.6. Using this, the only two things we must show are that $\alpha \in \mathrm{Im}(\varphi)$ for every $\alpha \in \mathrm{Path}(E)$ such that $r(\alpha) \in H$, and that $\alpha v^H \in \mathrm{Im}(\varphi)$ for every $\alpha \in \mathrm{Path}(E)$ such that $r(\alpha) = v \in S$.

To show the first statement, take $\alpha = e_1 \cdots e_n$ as indicated. There are four cases to analyze. First, if $s(e_1) \in H$ then $s(e_i) \in H$ for all i and $e_i \in {}_{(H,S)}E^1$. Hence, $\varphi(\alpha) = \varphi(e_1)\cdots\varphi(e_n) = e_1 \cdots e_n = \alpha$, which proves $\alpha \in \mathrm{Im}(\varphi)$. Second, suppose $\alpha = f e_1 \cdots e_n$ with $r(f) = s(e_1) \in H$ and $s(f) \in S$. Then $f \in {}_{(H,S)}E^1$, $s(e_i) \in H$ and $e_i \in {}_{(H,S)}E^1$ for all i. Therefore, $\varphi(\alpha) = \varphi(f)\varphi(e_1)\cdots\varphi(e_n) = f e_1 \cdots e_n = \alpha$ and so $\alpha \in \mathrm{Im}(\varphi)$. In the third case, if $\alpha = f_1 \cdots f_m e_1 \cdots e_n$ with $r(f_m) = s(e_1) \in H$ and $s(f_m) \notin H \cup S$, then $\beta := f_1 \cdots f_m \in F_1(H,S)$ and $e_i \in {}_{(H,S)}E^1$ for all i, so $\varphi(\overline{\beta} e_1 \cdots e_n) = \varphi(\overline{\beta})\varphi(e_1)\cdots\varphi(e_n) = \beta e_1 \cdots e_n = \alpha$ and so $\alpha \in \mathrm{Im}(\varphi)$. Finally, if $\alpha = f_1 \cdots f_m g e_1 \cdots e_n$ with $r(g) = s(e) \in H$, $s(g) \in S$ and $m \geq 1$, then $\beta := f_1 \cdots f_m \in F_2(H,S)$, $g \in {}_{(H,S)}E^1$ and $e_i \in {}_{(H,S)}E^1$ for all i; therefore, $\varphi(\overline{\beta} g e_1 \cdots e_n) = \varphi(\overline{\beta})\varphi(g)\varphi(e_1)\cdots\varphi(e_n) = \beta g e_e \cdots e_n = \alpha$, which shows again $\alpha \in \mathrm{Im}(\varphi)$.

Now we verify that $\alpha v^H \in \mathrm{Im}(\varphi)$ for every $\alpha \in \mathrm{Path}(E)$ such that $r(\alpha) = v \in S$. If $|\alpha| = 0$ then $v := \alpha$ is a vertex in S and $\varphi(v) = v^H = \alpha v^H$ so that $\alpha v^H \in \mathrm{Im}(\varphi)$. If $|\alpha| \geq 1$ then $\alpha \in F_2(H,S)$ and $\varphi(\alpha) = \alpha r(\alpha)^H$. This shows $\alpha r(\alpha)^H \in \mathrm{Im}(\varphi)$, and the proof is complete. □

Corollary 2.5.23. *Let E be an arbitrary graph and K any field. Then every graded ideal of $L_K(E)$ is K-algebra isomorphic to a Leavitt path algebra.*

{IdGrLPA}

Proof. Apply the Structure Theorem for Graded Ideals with Theorem 2.5.22. □

{I(H)notgrisototheother}

Remark 2.5.24. We note that, except for the obvious trivial cases, the isomorphism established in Theorem 2.5.22 between the graded ideal $I(H \cup S^H)$ of $L_K(E)$ and the Leavitt path algebra $L_K(E/(H,S))$ is not a graded isomorphism with respect to the induced grading on $I(H \cup S^H)$ coming from $L_K(E)$. This is because if α is a path in $F_E(H)$ having $|\alpha| \geq 2$, then the equation $\varphi(\overline{\alpha}) = \alpha$ reveals that φ does not preserve the grading.

In summary, we have now shown that the graded ideals of $L_K(E)$ are "natural" in the context of Leavitt path algebras: by Theorem 2.4.15 every quotient of a Leavitt path algebra by a graded ideal is again a Leavitt path algebra, and by Theorem 2.5.22 every graded ideal of a Leavitt path algebra is itself a Leavitt path algebra. In contrast, the quotient of a graded algebra by a non-graded ideal is not a graded algebra with respect to an induced grading; see the comments subsequent to Remark 2.1.2. Moreover, once we develop a description of the structure of all ideals in a Leavitt path algebra, we will be able to prove that non-graded ideals are necessarily not isomorphic to Leavitt path algebras (see Corollary 2.9.11).

We close this section by establishing yet another consequence of the Structure Theorem for Graded Ideals.

{interseccionhereditario

Proposition 2.5.25. *Let $\{H_i\}_{i\in\Lambda}$ be a family of hereditary subsets of an arbitrary graph E and K any field. Then as ideals of $L_K(E)$ we have:*

(i) $I(\cap_{i\in\Lambda}\overline{H_i}) = \cap_{i\in\Lambda} I(H_i)$.
(ii) *If Λ is finite, then* $I(\cap_{i\in\Lambda} H_i) = \cap_{i\in\Lambda} I(H_i)$.

Proof. (i) The containment $I(\cap_{i\in\Lambda}\overline{H_i}) \subseteq \cap_{i\in\Lambda} I(H_i)$ is clear because $I(H_i) = I\left(\overline{H_i}\right)$. Now we show the other one. Observe first that since the intersection of graded ideals is a graded ideal, by the Structure Theorem for Graded Ideals we get $\cap_{i\in\Lambda} I(\overline{H_i}) = I(H \cup S^H)$, where $H = \left(\cap_{i\in\Lambda} I(\overline{H_i})\right) \cap E^0 = \cap_{i\in\Lambda}\left(I(\overline{H_i}) \cap E^0\right) = \cap_{i\in\Lambda}\overline{H_i}$. Now, consider $v \in B_H$; we will see that $v^H \notin \cap_{i\in\Lambda} I(\overline{H_i})$. Since $v \notin H$, there is an $i \in \Lambda$ such that $v \notin \overline{H_i}$, hence $v \in B_{\overline{H_i}}$.

Write $\tilde{v}$ to denote either $v^{\overline{H_i}}$ (in case $v \in B_{\overline{H_i}}$), or v (in case $r(s^{-1}(v)) \subseteq \overline{H_i}$). Then we may write

$$\tilde{v} = v^H - \sum_{\substack{s(e)=v \\ r(e)\in\overline{H_i}\setminus H}} ee^*.$$

Since $\sum_{\{s(e)=v, r(e)\in\overline{H_i}\setminus H\}} ee^* \in I(\overline{H_i})$ and $v^H \in I(\overline{H_i})$, then $\tilde{v} \in I(\overline{H_i}) \cap E^0 = \overline{H_i}$, a contradiction. This implies $S = \emptyset$, giving the desired result.

(ii) When Λ is finite, then $\cap_{i\in\Lambda}\overline{H_i} = \overline{\cap_{i\in\Lambda} H_i}$, and consequently

$$I(\cap_{i\in\Lambda} H_i) = I(\overline{\cap_{i\in\Lambda} H_i}) = \cap_{i\in\Lambda} I(\overline{H_i}) = \cap_{i\in\Lambda} I(H_i). \qquad \square$$

{sectionsocle} {linepointsdefinition}

2.6 The socle

Because of its importance in the general theory, we present now a description of the socle of a Leavitt path algebra. Along the way, we will investigate various minimal left ideals of $L_K(E)$. This in turn will provide us with, among other things, an explicit description of the finite dimensional Leavitt path algebras.

Definitions 2.6.1. Let E be an arbitrary graph. Recall that for $v \in E^0$, we say that *there exists a cycle at v* if v is a vertex lying on some cycle in E. Also, recall that for $v \in E^0$, $T(v)$ denotes the set $\{w \in E^0 \mid v \geq w\}$.

A vertex $v \in E^0$ is called a *bifurcation* vertex (or it is said that *there is a bifurcation at v*) if $s_E^{-1}(v)$ contains at least two edges of E.

A vertex $u \in E^0$ is called a *line point* if there are neither bifurcations nor cycles at any vertex of $T(u)$.

The set of line points of the graph E will be denoted by $P_l(E)$.

{linepointsremarks}

Remark 2.6.2. Vacuously, any sink in E is a line point. The set of line points $P_l(E)$ is always a hereditary subset of E^0, although it is not necessarily saturated.

If $u \in P_l(E)$, then $T(u)$ is a sequence $T(u) = \{u_1, u_2, u_3, \dots\}$, where $u = u_1$, and where, for all $i \in \mathbb{N}$, there exists a unique edge $e_i \in E^1$ with $s(e_i) = u_i, r(e_i) = u_{i+1}$. This sequence is finite precisely when there exists a sink w of E in $T(u)$, in which case w is the last element of the sequence. Intuitively, $T(u)$ is then essentially just a "directed line starting at u", from which the name "line point" derives.

Consequently, for each pair $u_i, u_j \in T(u)$ with $i < j$, there exists a unique path $p_{i,j}$ in E for which $s(p_{i,j}) = u_i$ and $r(p_{i,j}) = u_j$. In particular, the lack of bifurcations at any vertex in $T(u)$ together with the (CK2) relation yields that $p_{i,j}p_{i,j}^* = u_i$ for any pair $u_i, u_j \in T(u)$ for which $i \leq j$.

A key role in the theory is played by rings of the following form.

{infinitematrixnotation}

Notation 2.6.3. Let Γ be an infinite set, and let S be any unital ring. We denote by

$$\mathrm{M}_\Gamma(S)$$

the ring consisting of those square matrices M, with rows and columns indexed by Γ, with entries from S, for which there are at most finitely many nonzero entries in M.

Clearly any such ring $\mathrm{M}_\Gamma(S)$ contains a set of enough idempotents, consisting of (finite) sums of distinct matrix units $e_{i,i}$.

A subset $\{\varepsilon_{\alpha,\beta} \mid \alpha,\beta \in \Gamma\}$ of an ideal T of a K-algebra R is called a *set of matrix units for* T in case $\varepsilon_{\alpha,\beta}\varepsilon_{\gamma,\kappa} = \delta_{\beta,\gamma}\varepsilon_{\alpha,\kappa}$ for all $\alpha,\beta,\gamma,\kappa \in \Gamma$, and $T = \mathrm{span}_K(\{\varepsilon_{\alpha,\beta}\})$. In this case, $T \cong \mathrm{M}_\Gamma(K)$ as K-algebras, via an isomorphism sending $\varepsilon_{\alpha,\beta}$ to the standard matrix element $e_{\alpha,\beta}$ (which is 1_K in row α, column β, and 0 elsewhere). The following result (which generalizes Proposition 1.3.5) allows us to explicitly describe the structure of the ideal $I(v)$ generated by a line point v. As a consequence of this description, we will be able to describe the structure of the socle of any Leavitt path algebra.

{idGenLinePoint}

Lemma 2.6.4. *Let E be an arbitrary graph and K any field. Let v be a line point in E. Let Λ_v denote the set $F_E(T(v))$; that is, Λ_v is the set of paths $\alpha \in \mathrm{Path}(E)$ for which $r(\alpha)$ meets $T(v)$ for the first time at $r(\alpha)$. Then*

$$I(v) \cong \mathrm{M}_{\Lambda_v}(K).$$

Proof. We construct a set of matrix units in $I(v)$, indexed by Λ_v, as follows. Write $T(v) = \{v_1, v_2, \dots\}$ as in Remark 2.6.2. By Lemma 2.4.1 and the observations offered in Remark 2.6.2, each element in $I(v)$ is a K-linear combination of elements of the form $\alpha x_{i,j}\lambda^*$, where $\alpha,\lambda \in F_E(T(v))$, and $x_{i,j} = p_{i,j}$ if $i \leq j$, or $x_{i,j} = p^*_{j,i}$ if $j \leq i$. We denote such $\alpha x_{i,j}\lambda^*$ by $e_{\alpha,\lambda}$.

Again using Remark 2.6.2, we see that the set $\{x_{i,j} \mid i,j \in \mathbb{N}\}$ has the multiplicative property $x_{i,j}x_{k,\ell} = \delta_{j,k}x_{i,\ell}$ for all $i,j,k,\ell \in \mathbb{N}$. Using this, it is then straightforward to establish that the set $\{e_{\alpha,\lambda} \mid \alpha,\lambda \in \Lambda_v\}$ is a set of matrix units for $I(v)$. $\square$

{I(v)incasevissink}
{fullmatrixringisLeavitt}

Corollary 2.6.5. *Let E be an arbitrary graph and K any field. Let v be a sink in E. Then $I(v) \cong \mathrm{M}_{\Lambda_v}(K)$, where Λ_v is the set of paths in E ending at v.*

Corollary 2.6.6. *Let K be any field. For any set Λ let E_Λ denote the graph with*

$$E^0_\Lambda = \{v\} \cup \{u_\lambda \mid \lambda \in \Lambda\} \quad \text{and} \quad E^1_\Lambda = \{f_\lambda \mid \lambda \in \Lambda\},$$

where $s(f_\lambda) = u_\lambda$ and $r(f_\lambda) = v$ for all $\lambda \in \Lambda$. Then $L_K(E_\Lambda) \cong \mathrm{M}_\Lambda(K)$. In particular, by taking disjoint unions of graphs of this form, any direct sum of full matrix rings over K arises as the Leavitt path algebra of a graph. (With Example 1.6.12 in mind, we sometimes refer to $E_\mathbb{N}$ as the infinite co-clock graph.*) We note that $E_{\mathbb{Z}^+}$ is precisely the graph arising in Example 2.5.18.*

{DefSocle}

Definitions 2.6.7. Let R be a ring. We say that a left ideal I of R is a *minimal left ideal* if $I \neq 0$ and I does not contain any left ideals of R other than 0 and I. (This is equivalent to saying that ${}_RI$ is a simple left R-module.) An idempotent $e \in R$ is called *left minimal* in case Re is a minimal left ideal of R. The *left socle* of R is defined to be the sum of all the minimal left ideals of R (or is defined to be $\{0\}$ in case R contains no minimal left ideals). The corresponding notions of *right minimal* and *right socle* are defined analogously.

{SemipSocle}

Remark 2.6.8. It is well known that for any ring R, both the left socle and the right socle of R are two-sided ideals of R. For a semiprime ring R the left and right socles of R coincide; in this case, either of these is called the *socle* of R, and is denoted by $\mathrm{Soc}(R)$. In particular, for E an arbitrary graph and K any field, the two-sided ideal $\mathrm{Soc}(L_K(E))$ denotes the sum of the minimal left (or right) ideals of $L_K(E)$ (when such exist), or denotes $\{0\}$ (when $L_K(E)$ contains no minimal one-sided left ideals).

The following result is standard (see e.g. [97, Section 3.4]).

{IdempMin}

Lemma 2.6.9. *Let R be a semiprime ring, and $e^2 = e \in R$. Then Re is a minimal left ideal of R if and only if eRe is a division ring.*

{nobifurcationlemma}

The structure of left ideals generated by vertices lies at the heart of the description of the socle of a Leavitt path algebra. Here is a fundamental observation in that regard.

Lemma 2.6.10. *Let E be an arbitrary graph and K any field. Let $w \in E^0$. If there exists a bifurcation at w (i.e., if $|s^{-1}(w)| \geq 2$), then the left ideal $L_K(E)w$ is not minimal.*

{1pvertices}

Proof. Suppose $e \neq f \in s^{-1}(w)$. Then ee^* and ff^* are nonzero elements of $L_K(E)w$. Since $ee^* \neq 0$, $L_K(E)ee^*$ is nonzero submodule of $L_K(E)w$. But $ff^* \notin L_K(E)ee^*$, since otherwise we would have $ff^* = ree^*$ for some $r \in L_K(E)$, which upon multiplication on the right by ff^* and using (CK1) would give $ff^* = 0$, a contradiction. $\square$

Proposition 2.6.11. *Let E be an arbitrary graph and K any field. A vertex v of E is a line point if and only if $L_K(E)v$ is a minimal left ideal of $L_K(E)$.*

Proof. Suppose first that v is a line point. Since $L_K(E)$ is semiprime (Proposition 2.3.1), in order to show that $L_K(E)v$ is a minimal left ideal it suffices to show (by Lemma 2.6.9) that $vL_K(E)v$ is a division ring. To that end, consider an arbitrary nonzero element $a \in vL_K(E)v$. Then a will be of the form $a = v(\sum_{i=1}^n k_i\lambda_i\mu_i^*)v = \sum_{i=1}^n k_i(v\lambda_i\mu_i^*v)$, for $\lambda_i, \mu_i \in \mathrm{Path}(E)$ such that $s(\lambda_i) = r(\mu_i^*) = v$ (so that $s(\mu) = v$), and $r(\lambda_i) = s(\mu_i^*) = v$ (so that $r(\mu) = v$). But then necessarily $\lambda_i = \mu_i$, because v is a line point and λ_i and μ_i start and end at the same vertex. So we get $\lambda_i\mu_i^* = \lambda_i\lambda_i^* = v$ (using Remark 2.6.2), yielding $a = \sum_{i=1}^n k_i \cdot v \in Kv$. This shows that $vL_K(E)v = Kv \cong K$.

Conversely, suppose $L_K(E)v$ is a minimal left ideal. We will see that no vertex in $T(v)$ has bifurcations, nor is any vertex in $T(v)$ the base of a cycle. We start by noting the following. For any $u \in T(v)$, let μ be a path such that $s(\mu) = v$ and $r(\mu) = u$. Then the map

$$\rho_\mu : L_K(E)v \to L_K(E)u \qquad av \mapsto av\mu = a\mu$$

is a nonzero epimorphism of left $L_K(E)$-modules, as for $\beta u \in L_K(E)u$ we have $\beta\mu^* \in L_K(E)v$, and $\rho_\mu(\beta\mu^*) = \beta\mu^*\mu = \beta u$. The minimality of $L_K(E)v$ implies that ρ_u is an isomorphism, so that $L_K(E)u$ must be minimal as well. In particular, by Lemma 2.6.9 $uL_K(E)u$ is a division ring.

{minimalvertexdef}

With these observations, we conclude first (by Lemma 2.6.10) that there are no bifurcations at w for every $w \in T(v)$, and second (by Lemma 2.2.7) that w is not the base of a cycle without exits in E for every $w \in T(v)$. Thus v is a line point. $\square$

{decompher}

Definition 2.6.12. For an arbitrary graph E and field K, we call a vertex $w \in E^0$ a minimal vertex in case $L_K(E)w$ is a minimal left ideal of $L_K(E)$.

Lemma 2.6.13. *Let E be an arbitrary graph and K any field. Then there exists a family $\{H_i\}_{i\in\Lambda}$ of hereditary subsets of E^0 such that $P_l(E) = \bigsqcup_{i\in\Lambda} H_i$, and $I(H_i) = I(v_i)$ as ideals of $L_K(E)$ for every $v_i \in H_i$ and $i \in \Lambda$.*

Proof. Define on $P_l(E)$ the following equivalence relation: for $u, v \in P_l(E)$, we say $u \equiv v$ if $I(u) = I(v)$. Let $\{H_i\}_{i\in\Lambda}$ be the set of all $\equiv$ equivalence classes.

{estructSocle}

We claim that each H_i is a hereditary subset of E^0. Indeed, suppose $u \in H_i$ and $v \in E^0$ such that $v = r(e)$ for some $e \in s^{-1}(u)$. Then $v \in P_l(E)$, as $P_l(E)$ is hereditary, and by hypothesis, $s^{-1}(u) = \{e\}$. This implies, by (CK1) and (CK2), $u = ee^* = eve^* \in I(v)$ and $v = e^*e = e^*ue \in I(u)$, hence $I(u) = I(v)$, and so $v \in H_i$. The rest of the conditions in the statement are obviously fulfilled. $\square$

We are now in position to describe the socle of a Leavitt path algebra.

Theorem 2.6.14. *Let E be an arbitrary graph and K any field. Decompose $P_l(E) = \bigsqcup_{i\in\Gamma} H_i$ as in Lemma 2.6.13. Then*

$$\mathrm{Soc}(L_K(E)) = I(P_l(E)) \cong \bigoplus_{i\in\Gamma} \mathrm{M}_{\Lambda_{v_i}}(K),$$

where for every $i \in \Gamma$, if v_i is an arbitrary element of H_i then $I(v_i) \cong \mathrm{M}_{\Lambda_{v_i}}(K)$ (with notation as in Lemma 2.6.4).

Proof. We begin by showing $I(P_l(E)) = \mathrm{Soc}(L_K(E))$. Proposition 2.6.11 gives that $I(P_l(E)) \subseteq \mathrm{Soc}(L_K(E))$. To establish the reverse inclusion note that, since $\mathrm{Soc}(L_K(E))$ is generated by the minimal left ideals of $L_K(E)$, it suffices to show that $a \in I(P_l(E))$ for every a for which $L_K(E)a$ is a minimal left ideal of $L_K(E)$.

Use the Reduction Theorem 2.2.11 to find $\mu, \eta \in \mathrm{Path}(E)$ such that either $0 \neq \mu^*a\eta = kv$ for some $k \in K^\times$ and $v \in E^0$, or $0 \neq \mu^*a\eta \in wL_K(E)w$, where w is a vertex in a cycle without exits. The second option is not possible, since $wL_K(E)w$ is isomorphic as an algebra to $K[x,x^{-1}]$ by Lemma 2.2.7, and so if the second option holds we have

$$\{0\} \neq w\mathrm{Soc}(L_K(E))w = \mathrm{Soc}(wL_K(E)w) \cong \mathrm{Soc}(K[x,x^{-1}]) = \{0\},$$

a contradiction.

Hence for some $v \in E^0$ and $k \in K^\times$ we have $\mu^* a\eta = kv$. By minimality of $L_K(E)a$, we get $L_K(E)\mu^* a = L_K(E)a$. Again by minimality, the nonzero surjection $\rho_\eta : L_K(E)\mu^* a \to L_K(E)v$ is an isomorphism. Thus $L_K(E)v \cong L_K(E)a$. In particular, $L_K(E)v$ is minimal, so that v is a line point by Proposition 2.6.11. But the isomorphism $L_K(E)v \cong L_K(E)a$ implies that $a = svt$ for some $s,t \in L_K(E)$, so that $a \in I(P_l(E))$.

In order to finish the proof of the theorem, we proceed as follows: $I(P_l(E)) = I(\bigsqcup_{i\in\Gamma} H_i) = \bigoplus_{i\in\Gamma} I(H_i)$ by Proposition 2.4.7. Now, use $I(H_i) = I(v_i)$ for any $v_i \in H_i$ (by construction), and apply Lemma 2.6.4. $\square$

As a consequence of Theorem 2.6.14, we get

{equivnolinepoints}

Corollary 2.6.15. *Let E be an arbitrary graph and K any field. The following are equivalent.*

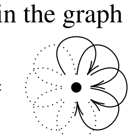
(1) *E contains no line points.*
(2) *$L_K(E)$ has no minimal idempotents.*

Proof. (1) implies (2) follows from the fact that if $L_K(E)$ has minimal idempotents, then $\mathrm{Soc}(L_K(E)) \neq \{0\}$, so that $P_l(E) \neq \emptyset$ by Theorem 2.6.14. That (2) implies (1) follows from Proposition 2.6.11. $\square$

{socleexamples}

Examples 2.6.16. In general, the relative size of $\mathrm{Soc}(L_K(E))$ within $L_K(E)$ can run the gamut, even among the fundamental examples of Leavitt path algebras. For instance:

(i) Since for each $n \in \mathbb{N}$ there are no line points in the graph

R_n = 

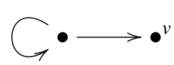
we conclude by Theorem 2.6.14 that $\mathrm{Soc}(L_K(R_n)) = \{0\}$. In particular, $\mathrm{Soc}(L_K(1,n)) = \{0\}$ for each of the Leavitt K-algebras $L_K(1,n)$. (We also recover the well-known fact that $\mathrm{Soc}(K[x,x^{-1}]) = \mathrm{Soc}(L_K(R_1)) = \{0\}$.)

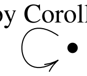
(ii) Since in the graph

$$A_n \quad = \quad \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

we have that $I(v_n) = L_K(A_n)$ for the line point v_n, we conclude by Theorem 2.6.14 that $\mathrm{Soc}(L_K(A_n)) = L_K(A_n)$. (Of course this result is easy to see from first principles, since $L_K(A_n) \cong \mathrm{M}_n(K)$.)

(iii) Since in the Toeplitz graph

E_T = 

the only line point is the vertex v, we conclude by Theorem 2.6.14 that $\mathrm{Soc}(L_K(E_T))$ is the ideal $I(v)$ of $L_K(E_T)$ generated by v. We see immediately that $\{0\} \subsetneqq \mathrm{Soc}(L_K(E_T)) \subsetneqq L_K(E_T)$.
Indeed, by Theorem 2.5.19, the ideal $I(v)$ is isomorphic to the Leavitt path algebra of the graph in Example 2.5.18, which in turn is isomorphic to $\mathrm{M}_{\mathbb{Z}^+}(K)$ by Corollary 2.6.6. Moreover, by Corollary 2.4.13(i) the quotient of $L_K(E_T)$ by the socle $I(v)$ is isomorphic to $L_K(E/\{v\}) \cong L_K($  $) \cong K[x,x^{-1}]$.

{structLPAs}

We finish the section by giving the aforementioned key consequence of our newly developed tools, in which we describe the structure of all finite dimensional Leavitt path algebras.

Theorem 2.6.17. (The Finite Dimension Theorem) *Let E be an arbitrary graph and K any field. The following conditions are equivalent.*

(1) *$L_K(E)$ is a finite dimensional Leavitt path K-algebra.*

(2) *E is a finite and acyclic graph.*
(3) *$L_K(E)$ is K-algebra isomorphic to $\oplus_{i=1}^m \mathrm{M}_{n_i}(K)$, where $m = |\mathrm{Sink}(E)|$ and n_i is the number of different paths ending at the sink v_i, for $1 \le i \le m$.*

Proof. (1) $\Rightarrow$ (2). Since $E^0 \cup E^1$ is a linearly independent set in $L_K(E)$ (apply Corollary 1.5.15), (1) implies that E must be finite. On the other hand, if c were a cycle in E, then applying Corollary 1.5.15 again would yield that $\{c^n\}_{n\in\mathbb{N}}$ is an independent set, contrary to the finite dimensionality of $L_K(E)$.

(2) $\Rightarrow$ (3). We show that $L_K(E) = \oplus_{i=1}^m I(v_i)$, where $\{v_1,\dots,v_m\} = \mathrm{Sink}(E)$. We note that in a finite acyclic graph E, there is a positive integer $b(E)$ for which every path in E has length at most $b(E)$. In addition, such a graph must contain at least one sink. Observe first that $\{\{v_i\}\}_{i=1}^m$ is a family of pairwise disjoint hereditary subsets of E. This implies, by Proposition 2.4.7, that $\sum_{i=1}^m I(v_i) = \oplus_{i=1}^m I(v_i)$.

Now consider an element $\alpha\beta^* \in L_K(E)$, with $\alpha, \beta \in \mathrm{Path}(E)$. If $r(\alpha) \in \mathrm{Sink}(E)$, then $\alpha\beta^* \in I(r(\alpha))$, which is one the $I(v_i)$'s. If this is not the case, then apply the (CK2) relation at $r(\alpha)$ to get

$$\alpha\beta^* = \alpha r(\alpha)\beta^* = \sum_{\{e\in s^{-1}(r(\alpha))\}} \alpha e e^* \beta^*.$$

If for every $e \in s^{-1}(r(\alpha))$ we have $r(e) \in \mathrm{Sink}(E)$, then we are done. Otherwise, rewrite every $r(e)$ which is not a sink as before, using (CK2). Since the graph is finite and acyclic, after at most $b(E)$ steps we have finished.

Finally, we note that m is exactly the cardinality of $\mathrm{Sink}(E)$, while by Corollary 2.6.5, n_i is the number of distinct paths ending in v_i.

(3) $\Rightarrow$ (1) is clear. □

{matricialLPAs}

We recall that a *matricial* K-algebra is a finite direct sum of full finite dimensional matrix algebras over the field K.

{acyclicgiveslocmat}
{locmatdef}

Remark 2.6.18 The Finite Dimension Theorem 2.6.17 yields that the matricial Leavitt path K-algebras (Definition 2.1.13) coincide precisely with the finite dimensional Leavitt path K-algebras. By Corollary 2.6.6, we see that every matricial K-algebra indeed arises as a Leavitt path K-algebra.

Definition 2.6.19. A *locally matricial K-algebra* is a direct limit of matricial K-algebras.

Proposition 2.6.20. *Let E be an acyclic graph and K any field. Then $L_K(E)$ is locally matricial.*

Proof. Write $L_K(E) = \varinjlim L_K(F_i)$, as in Proposition 1.6.15, where every F_i is a finite and acyclic graph. The result then follows, as each $L_K(F_i)$ is a matricial algebra by Theorem 2.6.17. □

Remark 2.6.21. The Finite Dimension Theorem 2.6.17 will play a central role in the theory of Leavitt path algebras. One immediate consequence is instructive. We see from Theorem 2.6.17 that the only information required to understand $L_K(E)$ up to K-algebra isomorphism when E is a finite acyclic graph is the number of sinks in E, and the number of paths ending in each of those sinks. In particular, this allows us to construct isomorphic Leavitt path algebras from non-isomorphic graphs. For example, let

$$E = \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad \text{and} \qquad F = \bullet \longrightarrow \bullet \longleftarrow \bullet \quad .$$

Then E and F are clearly not isomorphic as directed graphs (for instance, F has a vertex of invalence 2, while E does not). However, by Theorem 2.6.17 we get

$$L_K(E) \cong L_K(F) \cong \mathrm{M}_3(K),$$

since both E and F contain exactly one sink, and in both E and F there are exactly three paths ending at that sink.

{VerticesInCycles}

2.7 The ideal generated by the vertices in cycles without exits

For an arbitrary ring R, there are a number of ideals within R which merit special attention: the Jacobson radical of R, the socle of R, and the left singular ideal of R, to mention just a few. We have already identified these ideals (and others) in the context of Leavitt path algebras. However, there is one specific ideal within a Leavitt path algebra $L_K(E)$ which plays a central role in the description of the lattice $\mathscr{L}_{id}(L_K(E))$ of all two-sided ideals of $L_K(E)$: the ideal $I(P_c(E))$ generated by those vertices which lie on a cycle without exits. We describe $I(P_c(E))$ in this section.

Just as the ideal generated by the line points has importance (as it coincides with the socle of the corresponding Leavitt path algebra), the ideal generated by the vertices which lie on cycles without exits will also have an important place in the theory. In this case, the cycles without exits will play a role similar to that of the line points. In addition, we will be able to view this ideal as the ideal generated by the primitive non-minimal idempotents in $L_K(E)$ (such idempotents are discussed further in Section 3.5). Recall from Notation 2.2.4 that

$$P_c(E) := \{v \in E^0 \mid v \text{ is the base of some cycle } c \text{ for which } c \text{ has no exits}\}.$$

Indeed, $P_c(E)$ may be viewed as the disjoint union $P_c(E) = \sqcup_{i\in\Upsilon}\{c_i^0\}$, where $\{c_i\}_{i\in\Upsilon}$ is the set of distinct cycles without exits in E (i.e., for which $c_i^0 \neq c_j^0$ for $i \neq j$). Note that although $P_c(E)$ is clearly hereditary, it is not necessarily saturated. For instance, in the graph

$$\begin{tikzcd} \bullet_u \arrow[r] & \bullet_v \arrow[loop right] \end{tikzcd}$$

{idGenByOneCycle}

we have $P_c(E) = \{v\}$, which is a hereditary but not saturated subset of E^0. Note, however, that $I(P_c(E)) = I(\overline{P_c(E)})$, by Lemma 2.4.1.

Lemma 2.7.1. *Let E be an arbitrary graph and K any field. Let $v \in P_c(E)$, and let c be the cycle without exits such that $s(c) = v$. Let Λ_v denote the (possibly infinite) set of paths in E which end at v, but which do not contain all the edges of c. Then*

$$I(c^0) = I(v) \cong \mathrm{M}_{\Lambda_v}(K[x,x^{-1}]).$$

Proof. That $I(c^0) = I(v)$ is clear because, using the hypotheses that c has no exits, we have $\overline{\{v\}} = c^0$.

Consider the family

$$\mathscr{B} := \{\mu c^k \eta^* \mid \mu,\eta \in \Lambda_v, k \in \mathbb{Z}\},$$

where as usual c^0 denotes v and c^k denotes $(c^*)^{-k}$ for $k < 0$. By Corollary 1.5.12, $\mathscr{B}$ is a K-linearly independent set.

By Lemma 2.4.1 we have that every element in $I(v)$ is a K-linear combination of elements of the form $\alpha\beta^*$, where $r(\alpha) = r(\beta) \in T(v)$. But $T(v)$ consists precisely of the vertices in c, as c has no exits. So $\alpha = \mu c^\ell$ and $\beta = \eta c^m$ for some $\mu,\eta \in \Lambda_v$, and $\ell, m \geq 0$. This shows that $\mathscr{B}$ generates $I(v)$, so that $\mathscr{B}$ is a K-basis for $I(v)$.

We define $\varphi : I(v) \to \mathrm{M}_{\Lambda_v}(K[x,x^{-1}])$ by setting $\varphi(\mu c^k \eta^*) = x^k e_{\mu,\eta}$ for each $\mu c^k \eta^* \in \mathscr{B}$ (where $x^k e_{\mu,\eta}$ denotes the element of $\mathrm{M}_{\Lambda_v}(K[x,x^{-1}])$ which is x^k in the (μ,η) entry, and zero otherwise). Then one easily checks that φ is a K-algebra isomorphism. □

{fullmatrixringoverLaurentisLeavit

We record a consequence of Lemma 2.7.1 which is analogous to a previously noted consequence of Lemma 2.6.4.

Corollary 2.7.2. *Let K be any field. For any set Λ let E^c_Λ denote the graph with*

$$(E^c_\Lambda)^0 = \{v\} \cup \{u_\lambda \mid \lambda \in \Lambda\} \quad \textit{and} \quad (E^c_\Lambda)^1 = \{f_c\} \cup \{f_\lambda \mid \lambda \in \Lambda\},$$

where $s(f_\lambda) = u_\lambda$ and $r(f_\lambda) = v$ for all $\lambda \in \Lambda$, and f_c is a loop based at v. Then $L_K(E^c_\Lambda) \cong \mathrm{M}_\Lambda(K[x,x^{-1}])$. In particular, by taking disjoint unions of graphs of this form, any direct sum of full matrix rings over $K[x,x^{-1}]$ arises as the Leavitt path algebra of a graph.

Now using Proposition 2.4.7 together with Lemma 2.7.1, we have achieved the following.

Theorem 2.7.3. *Let E be an arbitrary graph and K any field. Then*

$$I(P_c(E)) \cong \oplus_{i\in\Upsilon} \mathrm{M}_{\Lambda_{v_i}}(K[x,x^{-1}]),$$

where $\{c_i\}_{i\in\Upsilon}$ is the set of distinct cycles without exits in E (i.e., for which $c_i^0 \neq c_j^0$ for $i\neq j$), and Λ_{v_i} is the set of paths in E which end at the base v_i of the cycle c_i, but do not contain all the edges of c_i.

For the following corollary, we will need to consider vertices for which its tree does not contain infinite bifurcations. We give a name to this set which will be useful, as we will see, to describe the center of a Leavitt path algebra.

Definition 2.7.4. Let E be an arbitrary graph. Denote by $P_{b^\infty}(E)$ the set of all vertices v in E^0 such that $T(v)$ has infinite bifurcations and by $P_{ne}(E)$ the set of vertices whose tree does not contain cycles with exits.

Corollary 2.7.5. *Let E be an arbitrary graph and K any field. Denote by H the set $P_l(E)\cup P_c(E)\subseteq E^0$.*

(i) *There is a graded isomorphism of graded K-algebras*

$$I(H) \cong \left(\oplus_{i\in\Upsilon_1}\mathrm{M}_{\Lambda_{v_i}}(K)\right)\oplus\left(\oplus_{i\in\Upsilon_2}\mathrm{M}_{\Lambda_{v_i}}(K[x,x^{-1}])\right).$$

(ii) *For every $v\in P_{ne}(E)\setminus P_{b^\infty}$ for which every path starting at v connects to H, there is a graded isomorphism of graded K-algebras*

$$I(H) \cong \left(\oplus_{i\in\Upsilon_1'}\mathrm{M}_{\Lambda_{v_i}}(K)\right)\oplus\left(\oplus_{i\in\Upsilon_2'}\mathrm{M}_{\Lambda_{v_i}}(K[x,x^{-1}])\right),$$

where $\Upsilon_j'\subseteq\Upsilon_j$ for $j=1,2$.

Proof. (i). It is clear that $P_l(E)$ and $P_c(E)$ are hereditary saturated disjoint subsets of E^0. By Proposition 2.4.7 we have that $I(H)=I(P_l(E))\oplus I(P_c(E))$. Now apply Theorems 2.6.14 and 2.7.3 to establish the result.

(ii). Using (CK2) we may write $v=\sum_l k_l\gamma_l\lambda_l^*$ for some $k_l\in K^\times$, $\gamma_l,\lambda_l\in \mathrm{Path}(E)$ and $r(\gamma_l)\in H$. Indeed, start by writing $v=\sum_{e\in s^{-1}(v)}ee^*$ (note that as there are not infinitely many bifurcations in vertices belonging to $T(v)$, in particular no vertex here is an infinite emitter). If $s(e)\in H$ for every such e, we have finished. If this is not the case then, for those e not having this property, use (CK2) to write $r(e)=\sum_{f\in s^{-1}(r(e))}ff^*$. If for every $f\in s^{-1}(r(e))$ we have $r(f)\in H$, the proof is finished. Otherwise we proceed in the same way as before. This process must eventually terminate, as $v\notin P_{b^\infty}$.

Now we have $v=\sum_l k_l\gamma_l\lambda_l^*\in I(H)$. This implies $I(v)\subseteq I(H)$. By (i) this last ideal is isomorphic to $\left(\oplus_{i\in\Upsilon_1}\mathrm{M}_{\Lambda_{v_i}}(K)\right)\oplus\left(\oplus_{i\in\Upsilon_2}\mathrm{M}_{\Lambda_{v_i}}(K[x,x^{-1}])\right)$. Now we appeal to the structure of the graded ideals of this ring and that $I(v)$ is a graded ideal (as it is generated by an element of zero degree) to get the result. □

Corollary 2.7.6. *Let E be a finite graph and K any field. Let $v\in P_{ne}(E)$. Then there exists positive integers m,n,r_i, and n_i for which*

$$I(v)\cong\left(\oplus_{i=1}^m\mathrm{M}_{r_i}(K)\right)\oplus\left(\oplus_{i=1}^n\mathrm{M}_{n_i}(K[x,x^{-1}])\right).$$

In particular, $I(v)$ is a noetherian K-subalgebra of $L_K(E)$.

Proof. Use Corollary 2.7.5(ii) with the fact that E finite implies $L_K(E)$ is unital to get that all Υ_j and Λ_{v_j} must be finite, for $j=1,2$. Once we then know the form of $I(v)$, the second statement follows immediately. □

The ideal we have described in Theorem 2.7.3 will play an important role in a Leavitt path algebra because as we now show, it captures all those ideals in the Leavitt path algebra which do not contain vertices.

{casoKE}

Lemma 2.7.7. *Let E be an arbitrary graph and K any field. Let J be a nonzero ideal of $L_K(E)$ such that $J \cap E^0 = \emptyset$. Then $\{0\} \neq J \cap KE \subseteq I(P_c(E))$.*

Proof. We first show that $\{0\} \neq J \cap KE$. Let y be a nonzero element in J. By the Reduction Theorem, either there exist $\alpha, \beta \in \mathrm{Path}(E)$ such that $\alpha^* y \beta = ku$ for some $u \in E^0$ and $k \in K^\times$, or $\alpha^* y \beta$ is a nonzero polynomial in a cycle without exits. Since J does not contain vertices, the first case cannot happen, and by multiplying by a power of the cycle without exits (if necessary), we produce a nonzero element in $J \cap KE$.

For such a nonzero element $x \in J \cap KE$, write $x = \sum_{u \in U} xu$, where $U = U(x)$ is the finite family of vertices of E such that $xu \neq 0$. Fix $u \in U$, and write $xu = \sum_{i=1}^{r} k_i \alpha_i$, with $k_i \in K^\times$, $\alpha_i = \alpha_i u \in \mathrm{Path}(E)$ for every i and $\alpha_i \neq \alpha_j$ for every $i \neq j$, and in such a way that $\deg(\alpha_i) \leq \deg(\alpha_{i+1})$ for every $i = 1, \ldots, r-1$.

We will prove that $xu \in I(P_c(E))$ by induction on the number r of summands. Note that $r \neq 1$ as otherwise we would have $xu = k_1 \alpha_1$, so $k_1^{-1} \alpha_1^* xu = u \in J$, a contradiction to the hypothesis. So the base case for the induction is $r = 2$.

Suppose first that $\deg(\alpha_1) = \deg(\alpha_2)$. In this case, since $\alpha_1 \neq \alpha_2$, we get $\alpha_1^* \alpha_2 = 0$ so that $k_1^{-1} \alpha_1^* xu = u \in J$, a contradiction again. This gives $\deg(\alpha_1) < \deg(\alpha_2)$, and then $\alpha_1^* xu = k_1 u + k_2 e_1 \cdots e_t$ for some $e_1, \cdots, e_t \in E^1$. By multiplying on the left and right hand sides by u we get

$$y_1 := u \alpha_1^* xu = k_1 u + k_2 u e_1 \cdots e_t u \in J \cap KE.$$

Observe that u and $e_1 \cdots e_t$ have different degrees, so since $k_1 u \neq 0$ we obtain that $y_1 \neq 0$. Moreover, as J does not contain vertices we have that $c := u e_1 \cdots e_t u \neq 0$, and thus c is a closed path based at u. We will prove that c does not have exits. Suppose on the contrary that there exist $w \in T(u)$ and $e, f \in E^1$ such that $e \neq f$, $s(e) = s(f) = w$, $c = aweb = aeb$ for some $a, b \in \mathrm{Path}(E)$. Then $\tau = af$ satisfies $\tau^* c = f^* a^* aeb = f^* eb = 0$ so that $\tau^* y_1 \tau = k_1 r(\tau) \in J$, again a contradiction. Thus by definition $u \in P_c(E)$, so that, in particular, $xu \in I(P_c(E))$. So the base case $r = 2$ for the induction has been established.

We now assume the result holds for $r \geq 2$ and prove it for $r+1$. Assume then that $xu = \sum_{i=1}^{r+1} k_i \alpha_i$; we distinguish two situations.

For the first case, suppose $\deg(\alpha_j) = \deg(\alpha_{j+1})$ for some $1 \leq j \leq r$. The element $\alpha_j^* xu \alpha_j = \alpha_j^* xu \alpha_j u \in J$ is nonzero, as follows: clearly each monomial remains with positive degree as $\deg(\alpha_j^* \alpha_i \alpha_j) = \deg(\alpha_i) \geq 0$. Moreover, at least $\alpha_j = \alpha_j^* \alpha_j \alpha_j$ appears in the expression for $\alpha_j^* xu \alpha_j$ because if we had $\alpha_j = \alpha_j^* \alpha_i \alpha_j$ for some $i \neq j$, then $\deg(\alpha_i) = \deg(\alpha_j)$, which implies $\alpha_j^* \alpha_i = 0$ and therefore $\alpha_j = 0$, a contradiction. This shows that $\alpha_j^* xu \alpha_j$ has at least one nonzero monomial summand, and because distinct paths of E are linearly independent (see Corollary 1.5.15), then $\alpha_j^* xu \alpha_j \neq 0$. Now, this element has at most r summands because $\alpha_j^* \alpha_{j+1} \alpha_j = 0$ and it satisfies the induction hypothesis, so that $u \in P_c(E)$.

The second case is when $\deg(\alpha_i) < \deg(\alpha_{i+1})$ for every $i = 1, \ldots, r$. Then $0 \neq \alpha_1^* xu = k_1 u + \sum_{i=2}^{r+1} k_i \beta_i$ with $\beta_i u = \beta_i \in \mathrm{Path}(E)$. Multiply again as follows:

$$y_2 := u \beta_{r+1}^* u \alpha_1^* xu \beta_{r+1} u = k_1 u + \sum_{i=2}^{r+1} k_i u \beta_{r+1}^* u \beta_i u \beta_{r+1} u \in J.$$

A similar argument to the one used above shows that y_2 is nonzero so that, in case some monomial summand of y_2 becomes zero, then y_2 satisfies the induction hypothesis, therefore $u \in P_c(E)$. If this is not the case, since β_{r+1} has maximum degree among the β_i, then

$$y_2 = k_1 u + k_2 \gamma_1 + k_3 \gamma_1 \gamma_2 + \ldots + k_{r+1} \gamma_1 \cdots \gamma_r,$$

where γ_i are closed paths based at u. We focus on γ_1. Proceeding in a similar fashion as before, we can conclude that γ_1 cannot have exits, as otherwise there would exist a path δ with $s(\delta) = u$ and $\delta^* \gamma_1 = 0$, which in turn would give $0 \neq \delta^* y_2 \delta = k_1 r(\delta) \in J$, a contradiction. Thus γ_1 is a closed path without exits, so that $r(\gamma) = u \in P_c(E)$, and finally $x = xu \in I(P_c(E))$.

Since this holds for every $u \in U$ we get $x = \sum_{u \in U} xu \in I(P_c(E))$. □

{nuj1}

Prior to achieving our main result about $I(P_c(E))$, we need a general result about path algebras.

Lemma 2.7.8. *Let E be an arbitrary graph and K any field.*

(i) *Let* $w \in E^0$*, let* $\mu \in \mathrm{Path}(E)$ *with* $r(\mu) = w$*, and let* $x \in KE$ *for which* $wx = x$*. If* $\mu x = 0$ *in* KE*, then* $x = 0$.

(ii) *Let* $v \in E^0$*, let* $\gamma \in \mathrm{Path}(E)$ *with* $s(\gamma) = v$*, and let* $y \in KE$ *for which* $yv = y$*. If* $y\gamma = 0$ *in* KE*, then* $y = 0$.

{idealsinvertices}

Proof. (i) Write $x = \sum_{i=1}^n k_i\mu_i \in KE$, where $k_i \in K^\times$, and the μ_i are distinct. Since $wx = x$, we may assume that $s(\mu_i) = w$ for all $1 \le i \le n$. In particular, each expression $\mu\mu_i$ is a path in E. Then from $\mu x = 0$ we get $\sum_{i=1}^n k_i\mu\mu_i = 0$, and since all the paths in the set $\{\mu\mu_i\}_{i=1}^n$ are distinct, they are K-linearly independent in KE (see Remark 1.2.4). Therefore $k_i = 0$ for all $1 \le i \le n$, and so $x = 0$.

Statement (ii) can be established analogously. □

Proposition 2.7.9. *Let* E *be an arbitrary graph and* K *any field. Let* J *be an ideal of* $L_K(E)$ *such that* $J \cap E^0 = \emptyset$*. Then* $J \subseteq I(P_c(E))$.

Proof. We may assume that $J \neq 0$. Let $0 \neq x \in J$, and write $x = \sum_{i=1}^n xu_i$ for the finite set of vertices $\{u_i \mid 1 \le i \le n\}$ for which $0 \neq xu_i$. As J is an ideal, $0 \neq xu_i \in J$, so that we can assume without loss of generality that $0 \neq x = xu$ for some $u \in E^0$.

We will show, by induction on the degree in ghost edges (recall Definitions 2.2.9), that if $xu \in J$, with $u \in E^0$, then $xu \in I(P_c(E))$. If $\mathrm{gdeg}(xu) = 0$, the result follows by Lemma 2.7.7. Suppose the result is true for elements having degree in ghost edges strictly less than $\mathrm{gdeg}(xu)$, and show it for $\mathrm{gdeg}(xu)$.

Write $x = \sum_{i=1}^r \beta_i e_i^* + \beta$, with $\beta_i \in L_K(E)$, $\beta = \beta u \in KE$ and $e_i \in E^1$, with $e_i \neq e_j$ for every $i \neq j$. Then $xue_i = \beta_i + \beta e_i \in J$; since $\mathrm{gdeg}(xue_i) < \mathrm{gdeg}(xu)$, by the induction hypothesis $\beta_i + \beta e_i \in I(P_c(E))$, for every $i \in \{1, \dots, r\}$.

Suppose first that u is a finite emitter. If $u = \sum_{i=1}^r e_ie_i^*$, then $xu = \sum_{i=1}^r \beta_ie_i^* + \sum_{i=1}^r \beta e_ie_i^* = \sum_{i=1}^r (\beta_i + \beta e_i)e_i^* \in I(P_c(E))$, and we have finished. If $u = \sum_{i=1}^r e_ie_i^* + \sum_{j=1}^s f_jf_j^*$ (where $f_j \in E^1$), then $xuf_j = \beta f_j \in J \cap KE$. By Lemma 2.7.7, $\beta f_j \in I(P_c(E))$ for every $j \in \{1, \dots, s\}$, hence $xu = \sum_{i=1}^r (\beta_i + \beta e_i)e_i^* + \sum_{j=1}^s \beta f_jf_j^* \in I(P_c(E))$.

Now, suppose that u is an infinite emitter. If $\beta = 0$ then for every j we have $xue_j = \beta_j \in I(P_c(E))$, by the induction hypothesis, and so $xu \in I(P_c(E))$. Now we are going to show by contradiction that the case $\beta \neq 0$ cannot happen, and thereby will complete the proof.

So suppose $\beta \neq 0$, and write $\beta = \sum_{i=1}^s k_i\beta_i'$, with $k_i \in K^\times$, and $\beta_i' \in \mathrm{Path}(E)$ distinct paths such that $|\beta_1'| \le \cdots \le |\beta_s'|$. Note that, as u is an infinite emitter, u is not in $I(\overline{P_c(E)})$. Since $\beta_i' = \beta_i'u$ then β_i' is not in $I(P_c(E))$ for any i. (Because $P_c(E)$ contains no infinite emitters (by definition), then neither does $\overline{P_c(E)}$, and so neither does $I(\overline{P_c(E)})$.) Let $f \in s^{-1}(u)$ such that $f \neq e_j$ for every j. By Lemma 2.7.8(ii) we have $\beta f \neq 0$; since $\beta f = xf$, by the induction hypothesis $\beta f \in I(P_c(E))$, therefore $0 \neq xf = \beta f \in I(P_c(E))$.

We shall see that $r(f) \in \overline{P_c(E)}$. Consider the algebra $L_K(E)/I(\overline{P_c(E)})$ and denote by $\overline{x}$ the class of an element x of $L_K(E)$ in this quotient. Note that $0 = \overline{\beta f} = \sum_{i=1}^s k_i\overline{\beta_i' f}$, hence, by Theorem 2.4.15 we have $\overline{\beta_i' f} = 0$, i.e., $\beta_i' f \in I(\overline{P_c(E)})$ for every i and so $r(f) = f^*(\beta_i')^*\beta_i' f \in I(\overline{P_c(E)}) \cap E^0 = \overline{P_c(E)}$ by Corollary 2.4.16(i). Then $f^*\beta_1'\beta f = k_1r(f) + \sum_{i=2}^s k_if^*(\beta_1')^*\beta_i' f$. Note that the second summand must be zero because otherwise for some $j \in \{2, \cdots, s\}$ we would have $\beta_j' = \beta_1' f\gamma$ for some $\gamma \in \mathrm{Path}(E)$, which is not possible because we know $\beta_j' \notin I(\overline{P_c(E)})$. Therefore $0 \neq k_1r(f) \in J$, a contradiction again. Thus $\beta = 0$, which completes the proof of the result. □

{densityproposition}

We finish the section by utilizing Lemma 2.7.8 to give a graph-theoretic description of when an ideal $I(H)$ is an essential ideal of $L_K(E)$.

Proposition 2.7.10. *Let* E *be an arbitrary graph and* K *any field. Let* H *be a hereditary subset of* E*. Then* $I(H)$ *is an essential (left / right / two-sided) ideal of* $L_K(E)$ *if and only if every vertex of* E *connects to a vertex in* H*.*

Proof. Since $L_K(E)$ is semiprime (Proposition 2.3.1), we may invoke [96, (14.1) Proposition] to conclude that $I(H)$ is essential as a left or right ideal if and only if it is essential as an ideal. Moreover, as $I(H)$ is a graded ideal, by [107, 2.3.5 Proposition] we have that essentiality and graded-essentiality (i.e., essentiality with respect to graded ideals) of $I(H)$ are equivalent. Hence, it suffices to show that $I(H)$ is a graded-essential ideal if and only if every vertex of E^0 connects to a vertex in H.

Suppose first that $I(H)$ is a graded essential ideal of $L_K(E)$. Let $v \in E^0$. If $H \cap T(v) = \emptyset$, then Proposition 2.5.25 would imply $I(H) \cap I(T(v)) = 0$, but this cannot happen as $I(H)$ is a graded essential ideal. Hence $H \cap T(v) \neq \emptyset$. This implies that v connects to a vertex in H.

Conversely, suppose $H \cap T(v) \neq \emptyset$ for each $v \in E^0$. Let J be a nonzero graded ideal and pick a nonzero homogeneous element $x = uxv \in J$, where $u, v \in E^0$. By Corollary 2.2.12(ii), there exists $\mu \in \mathrm{Path}(E)$ such that $0 \neq x\mu \in KE$. Denote $r(\mu)$ by w. By hypothesis w connects to a vertex in H, hence there exists $\lambda \in \mathrm{Path}(E)$ such that $w = s(\lambda)$ and $r(\lambda) \in H$. If $x\mu\lambda = 0$ then $x\mu \in uL_K(E)w \cap KE$ would satisfy $\lambda \in \mathrm{Path}(E) \cap ran(x\mu) = \emptyset$, by Lemma 2.7.8, a contradiction. Hence $0 \neq x\mu\lambda \in I(H) \cap J$, which establishes the result. $\square$

2.8 The Structure Theorem for Ideals, and the internal structure of ideals

{IdealStructureRF}

Now that we have in hand an explicit description of the lattice of graded ideals of a Leavitt path algebra (the Structure Theorem for Graded Ideals, Theorem 2.5.8), we turn our attention to explicitly describing the lattice of *all* ideals in a Leavitt path algebra. Although the structure of the field K played no role in the description of the graded ideals, the field will indeed play a pivotal role in this more general setting. The intuition which lies at the heart of this description is as follows. The prototypical example of a Leavitt path algebra which contains non-graded ideals is $L_K(R_1) \cong K[x, x^{-1}]$. The only graded ideals of $L_K(R_1)$, namely, $\{0\}$ and $L_K(R_1)$ itself, correspond to the two distinct hereditary saturated subsets of R_1. On the other hand, the non-graded ideals correspond to various polynomial expressions in the cycle c of R_1, specifically, are in bijective correspondence with polynomials of the form $1 + k_1x + \cdots + k_nx^n \in K[x]$, for $n > 0$ and $k_n \neq 0$. We will show in the main result of this section (the Structure Theorem for Ideals, Theorem 2.8.10) that such a bijection, one which associates hereditary saturated subsets of E^0 (possibly also with breaking vertices of such subsets) together with various cycles in E and polynomials in $K[x]$ on the one hand, with ideals of $L_K(E)$ on the other, may be established for arbitrary graphs E and fields K as well. To achieve this general result we will rely heavily on our previously completed analysis of the graded ideal structure of $L_K(E)$, together with the structure of the ideal $I(P_c)$ investigated in Section 2.7. It is not coincidental in this context that the loop in R_1 is the only closed simple path based at the vertex of R_1. Indeed, in general $L_K(E)$ will contain non-graded ideals only when E fails to satisfy Condition (K).

We remind the reader that when we talk about a *cycle based at a vertex* (say, v), then we mean a specific path $c = e_1 \cdots e_n$ in E (one for which $s(c) = r(c) = v$); on the other hand, when we speak about a *cycle*, we mean a collection of paths based at the different vertices of the path c (see Definitions 1.2.2).

Notation 2.8.1. Let E be an arbitrary graph. We define

$$C_u(E) = \{c \mid c \text{ is a cycle in } E \text{ for which } |CSP(v)| = 1 \text{ for every } v \in c^0\}, \quad \text{and}$$

$$C_{ne}(E) = \{c \mid c \text{ is a cycle in } E \text{ for which } c \text{ has no exits in } E\}.$$

Observe that $C_{ne}(E) \subseteq C_u(E)$ for any graph E, but not necessarily conversely: in the Toeplitz graph E_T, the unique cycle has an exit, but there is exactly one closed simple path at the vertex of that cycle.

{cehache}

Notation 2.8.2. Let E be an arbitrary graph. Let $H \in \mathscr{H}_E$. Denote by C_H the set

{CsubHinCsubu}

$$C_H = \{c \mid c \text{ is a cycle in } E \text{ such that } c^0 \cap H = \emptyset, \text{ and for which } r(e) \in H \text{ for every exit } e \text{ of } c\}.$$

We note that C_H corresponds precisely to the set of cycles without exits in the quotient graph E/H.

Lemma 2.8.3. *Let E be an arbitrary graph, and $H \in \mathscr{H}_E$. Then $C_H(E) \subseteq C_u(E)$.*

Proof. Let c be a cycle in C_H. We must show that $c \in C_u(E)$, that is, $|CSP(v)| = 1$ for every $v \in c^0$. But this holds because for every exit e of c the vertices in $T(r(e))$ are in H (because H is hereditary), and because $c^0 \cap H = \emptyset$. $\square$

Recall the preorder $\leq$ in E^0: given $v,w\in E^0$, $v\leq w$ if and only if there is a path $\mu\in \mathrm{Path}(E)$ such that $s(\mu)=w$ and $r(\mu)=v$.

{idealSinVerticesGral}

Notation 2.8.4. Let E be an arbitrary graph. For $u,v\in E^0$ we write $u<<v$ in case $u\leq v$ but $v\not\leq u$. For a cycle c in E, we define:

$$c^{<<}:=\{w\in E^0 \mid w<<v \text{ for every } v\in c^0\}.$$

Roughly speaking, $c^{<<}$ is the tree of the set of vertices which are ranges of exits for the cycle c, but for which there are no paths from such vertices which return back to the cycle c. For instance, for the Toeplitz graph E_T of Example 1.3.6, we have $c^{<<}=\{v\}$.

Proposition 2.8.5. *Let E be an arbitrary graph and K any field. Let I be an ideal of E. Denote by $H:=I\cap E^0$ and $S:=\{v\in B_H \mid v^H\in I\}$. Let J denote $I/I(H\cup S^H)$; using Theorem 2.4.15, we view J as an ideal of the Leavitt path algebra of the quotient graph $L_K(E/(H,S))$. Then:*

(i) $J\subseteq I(P_c(E/(H,S)))$.
(ii) *There exists a set $C\subseteq C_H$ and a set $P=\{p_c(x)\in K[x]\mid c\in C\}$ such that each $p_c(x)$ is a polynomial of the form $1+k_1x+\ldots+k_nx^n$, with $n>0$ and $k_n\neq 0$, in such a way that $J=\oplus_{c\in C}I(p_c(c))$. (Note that C is empty precisely when I is graded, which happens precisely when $J=\{0\}$.)*
(iii) *The sets C and P are uniquely determined by I.*

Proof. (i). Consider the ideal $J=I/I(H\cup S^H)$ of $L_K(E/(H,S))$. Recall that the vertices in $E/(H,S)$ are $(E^0\setminus H)\cup\{v'\mid v\in B_H\setminus S\}$, and observe that vertices v' with $v\in B_H\setminus S$ correspond to the classes of the elements v^H through the isomorphism $L_K(E/(H,S))\cong L_K(E)/I(H\cup S^H)$. It is clear from this that J does not contain vertices in the graph $E/(H,S)$. Now (i) follows by Proposition 2.7.9.

(ii) and (iii). By Theorem 2.7.3 we have an isomorphism

$$I(P_c(E/(H,S)))\cong\bigoplus_{i\in\Upsilon}\mathrm{M}_{\Lambda_i}(K[x,x^{-1}]),$$

where Υ is the set of cycles without exits in $E/(H,S)$. As observed previously, we may identify this set with C_H. We recall now these two well-known facts: first, that the ideals of a direct sum of matrix rings are direct sums of matrix rings over ideals of the base rings, and, second, that the Laurent polynomial ring $K[x,x^{-1}]$ is a principal ideal domain. Applying these two facts, along with (i) and the displayed isomorphism, we get that there exists a subset C of C_H and a set of polynomials P as in the statement, uniquely determined by J, for which

$$J\cong\bigoplus_{c\in C}\mathrm{M}_{\Lambda_c}(p_c(x)K[x,x^{-1}])\cong\bigoplus_{c\in C}I(p_c(c)),$$

as desired. $\square$

{HSCPdefinition}

The main result of this section is Theorem 2.8.10, which shows that there is a lattice isomorphism between ideals in the Leavitt path algebra $L_K(E)$ on the one hand, and triples consisting of elements in $\mathscr{T}_E$ (see Definition 2.5.3), certain subsets of cycles in E, and families of polynomials in $K[x]$ on the other. We now describe such triples.

Definition 2.8.6. Let E be an arbitrary graph and K any field. For every pair $(H,S)\in\mathscr{T}_E$, consider a subset C of C_H; for every element $c\in C$, take an arbitrary polynomial $p_c(x)=1+k_1x+\cdots+k_nx^n\in K[x]$, where $n>0$ and $k_n\neq 0$, and write $P=\{p_c(x)\mid c\in C\}$. We define $\mathscr{Q}_E$ as the set of triples:

$$\mathscr{Q}_E=\{((H,S),C,P)\}.$$

{relationinLHSCP}

To show that there is a bijection between $\mathscr{L}_{id}(L_K(E))$ and $\mathscr{Q}_E$ we will assign to every triple $((H,S),C,P)$ the ideal generated by $H\cup S^H\cup P_C$, where for $P=\{p_c(x)\mid c\in C\}$, P_C denotes the subset $\{p_c(c)\mid c\in C\}$ of $L_K(E)$.

Definition 2.8.7. Let E be an arbitrary graph and K any field. We define a relation $\leq$ on $\mathscr{Q}_E$ as follows. For elements $((H_1,S_1),C_1,P_1)$ and $((H_2,S_2),C_2,P_2)$ of $\mathscr{Q}_E$, we set $P_i:=\{p_c^{(i)}\mid c\in C_i\}$ for $i=1,2$. We then define

{latticeHSCPGral}

$$((H_1,S_1),C_1,P_1) \ \leq \ ((H_2,S_2),C_2,P_2) \quad \text{in case:}$$
$$(H_1,S_1) \leq (H_2,S_2), \quad C_1^0 \subseteq H_2 \cup C_2^0, \ \text{ and } \ p_c^{(2)} \mid p_c^{(1)} \text{ in } K[x] \text{ for every } c \in C_1 \cap C_2.$$

Proposition 2.8.8. *Let E be an arbitrary graph and K any field. Then the relation $\leq$ defined on $\mathscr{Q}_E$ in Definition 2.8.7 is a partial order. Furthermore, using this relation, $\mathscr{Q}_E$ is a lattice, in which the supremum and infimum operators are described as follows.*

For the supremum $\vee$ of two elements, we have

$$\begin{aligned} &((H_1,S_1),C_1,P_1) \ \vee \ ((H_2,S_2),C_2,P_2) \\ = \ &((\overline{H_1 \cup H_2 \cup C^0}^{S_1\cup S_2}, (S_1\cup S_2) \setminus \overline{H_1 \cup H_2 \cup C^0}^{S_1\cup S_2}), \ C_1 \vee C_2, \ \{\text{g.c.d.}(p_c^{(1)}, p_c^{(2)})\}_{c\in C_1\vee C_2}),\end{aligned}$$

where

$$C = \{c \in C_1 \cap C_2 \mid \text{g.c.d.}(p_c^{(1)}, p_c^{(2)}) = 1\}, \quad \textit{and}$$
$$C_1 \vee C_2 = C_1 \cup C_2 \setminus \{c \in C_1 \cup C_2 \mid c^0 \subseteq \overline{H_1 \cup H_2 \cup C^0}^{S_1\cup S_2}\}.$$

(We interpret $p_c^{(i)}$ as 0 if $c \notin C_i$ for $i = 1$ or 2.)

For the infimum $\wedge$ of two elements, we have

$$\begin{aligned} &((H_1,S_1),C_1,P_1) \ \wedge \ ((H_2,S_2),C_2,P_2) \\ = \ &((H_1,S_1)\wedge(H_2,S_2), \ C_1 \wedge C_2, \ \{\text{l.c.m.}(p_c^{(1)}, p_c^{(2)})\}_{c\in C_1\wedge C_2}),\end{aligned}$$

where

$$C_1 \wedge C_2 = (C_1 \cap C_2) \cup C_1^{H_2} \cup C_2^{H_1},$$
$$\textit{with } \ C_1^{H_2} := \{c \in C_1 \mid c^0 \subseteq H_2\} \ \textit{ and } \ C_2^{H_1} := \{c \in C_2 \mid c^0 \subseteq H_1\}.$$

(We interpret $p_c^{(i)}$ as 1 if $c \notin C_i$ for $i = 1$ or 2.)

Proof. It is immediate to see that $\leq$ is reflexive. To show the antisymmetric property we use the antisymmetric property of $\leq$ on $\mathscr{T}_E$ (see Proposition 2.5.6) and the fact that for $((H,S),C,P) \in \mathscr{Q}_E$ we have $C^0 \cap H = \emptyset$ (because $C \subseteq C_H$).

To prove the transitivity, take three triples in $\mathscr{Q}_E$ such that $((H_1,S_1),C_1,P_1) \leq ((H_2,S_2),C_2,P_2)$ and $((H_2,S_2),C_2,P_2) \leq ((H_3,S_3),C_3,P_3)$. Since $(H_1,S_1) \leq (H_2,S_2)$ and $(H_2,S_2) \leq (H_3,S_3)$, it follows that $(H_1,S_1) \leq (H_3,S_3)$. In addition, $C_1^0 \subseteq H_2 \cup C_2^0$ and $C_2^0 \subseteq H_3 \cup C_3^0$ implies $C_1^0 \subseteq H_2 \cup C_2^0 \subseteq H_3 \cup C_3^0$. Finally, let $c \in C_1 \cap C_3$. Note that $c \in C_3$ implies $c^0 \cap H_3 = \emptyset$, hence $c \in C_2$ because otherwise $c^0 \subseteq H_2 \cup C_2^0$ would imply $c^0 \subseteq H_2 \subseteq H_3$, a contradiction. Therefore $c \in C_1 \cap C_2 \cap C_3$, and from the relations $p_c^{(2)} \mid p_c^{(1)}$ and $p_c^{(3)} \mid p_c^{(2)}$ in $K[x]$ we get $p_c^{(3)} \mid p_c^{(1)}$ in $K[x]$. Hence $((H_1,S_1),C_1,P_1) \leq ((H_3,S_3),C_3,P_3)$.

Now we check that the formula given in the statement corresponds to the supremum. To this end, let $((H_1,S_1),C_1,P_1), ((H_2,S_2),C_2,P_2) \in \mathscr{Q}_E$. Denote the element

$$((\overline{H_1 \cup H_2 \cup C^0}^{S_1\cup S_2}, (S_1\cup S_2) \setminus \overline{H_1 \cup H_2 \cup C^0}^{S_1\cup S_2}), \ C_1 \vee C_2, \ \{\text{g.c.d.}(p_c^{(1)}, p_c^{(2)})\}_{c\in C_1\vee C_2})$$

by $((\widetilde{H},\widetilde{S}),\widetilde{C},\widetilde{P})$. It is not difficult to show that $((H_i,S_i),C_i,P_i) \leq ((\widetilde{H},\widetilde{S}),\widetilde{C},\widetilde{P})$ for $i = 1,2$.

Now take $((H',S'),C',P') \in \mathscr{Q}_E$ such that $((H_i,S_i),C_i,P_i) \leq ((H',S'),C',P')$ for $i = 1,2$. First we prove $(\widetilde{H},\widetilde{S}) \leq (H',S')$. Note that $H_1 \cup H_2 \subseteq H'$. Now we want to show that $C^0 \subseteq H'$. We start by showing that $C \cap C' = \emptyset$. Assume $c \in C \cap C'$. Then $c \in C_1 \cap C_2$ and $\text{g.c.d.}(p_c^{(1)}, p_c^{(2)}) = 1$ (recall the definition of C). Since $((H_i,S_i),C_i,P_i) \leq ((H',S'),C',P')$ and $c \in C_i \cap C'$ we have $p'_c | p_c^{(i)}$, for $i = 1,2$, where $P' = \{p'_c \mid c \in C'\}$. Hence $p'_c = 1$, contradicting the choice of p'_c (which, by definition, is a non invertible polynomial in $K[x,x^{-1}]$). Using that $C^0 \subseteq C_1^0 \subseteq H' \cup C'^0$, and taking into account that $C^0 \cap C'^0 = \emptyset$, we get $C^0 \subseteq H'$. This shows $H_1 \cup H_2 \cup C^0 \subseteq H'$. Since $S_1 \cup S_2 \subseteq H' \cup S'$ the same argument as in Proposition 2.5.6 shows

$$\overline{H_1 \cup H_2 \cup C^0}^{S_1\cup S_2} \subseteq H'.$$

{BiggestIdeal}

It is immediate that $S_1 \cup S_2 \setminus \overline{H_1 \cup H_2 \cup C^0}^{S_1 \cup S_2} \subseteq H' \cup S'$, and that $(C_1 \vee C_2)^0 \subseteq C_1^0 \cup C_2^0 \subseteq H' \cup (C')^0$.

Finally, note that for $c \in (C_1 \vee C_2) \cap C'$ we have that $p'_c | p_c^{(i)}$ for $i = 1,2$. Hence $p'_c \mid \text{g.c.d.}(p_c^{(1)}, p_c^{(2)})$. This concludes the proof of the formula for the supremum.

We leave to the reader the verification of the formula for the infimum. □

Lemma 2.8.9. *Let E be an arbitrary graph and K any field. For any ideal I of $L_K(E)$, let $H = I \cap E^0$, and $S^H = \{v \in B_H \mid v^H \in I\}$ (see Definitions 2.4.4). Then the largest graded ideal of $L_K(E)$ contained in I is precisely $I(H \cup S^H)$.*

Proof. Clearly $I(H \cup S^H) \subseteq I$. Now let J be any other graded ideal contained in I. Then by the Structure Theorem for Graded Ideals, $J = I(H' \cup S^{H'})$ for $H' = J \cap E^0 \subseteq I \cap E^0 = H$, and $S^{H'} = \{v \in B_{H'} \mid v^{H'} \in J\} \subseteq S^H$. □

{descripIdGral}

We now have all the tools in place to achieve the main result of this section, namely, a description of the collection of all two-sided ideals of $L_K(E)$. Recall that $\mathscr{L}_{id}(L_K(E))$ denotes the lattice of two-sided ideals of $L_K(E)$, under the usual order given by inclusion, and usual lattice operations given by $+$ and $\cap$.

Theorem 2.8.10. (Structure Theorem for Ideals) *Let E be an arbitrary graph and K any field. Then the following map is a lattice isomorphism:*

$$\begin{array}{rccc} \varphi : & \mathscr{Q}_E & \longrightarrow & \mathscr{L}_{id}(L_K(E)) \\ & ((H,S),C,P) & \mapsto & I(H \cup S^H \cup P_C) \end{array}$$

with inverse given by

$$\begin{array}{rccc} \varphi' : & \mathscr{L}_{id}(L_K(E)) & \longrightarrow & \mathscr{Q}_E \\ & I & \mapsto & ((H,S),C,P) \end{array}$$

where $H = I \cap E^0$, $S = \{v \in B_H \mid v^H \in I\}$, and C and P are as described in Proposition 2.8.5.

Proof. We start by showing that $\varphi'\varphi$ is the identity on $\mathscr{Q}_E$. Take $((H,S),C,P) \in \mathscr{Q}_E$, and denote by I its image under φ, that is, $I = I(H \cup S^H \cup P_C)$. We show that $I \cap E^0 = H$.

Clearly, $H \subseteq I \cap E^0 \subseteq I$. To see the reverse containment, consider $I/I(H \cup S^H) = I(\overline{P_C})$, where for any subset $X \subseteq L_K(E)$, $\overline{X}$ denotes the image of X under the epimorphism $\overline{\Psi} : L_K(E) \to L_K(E/(H,S))$ described in Theorems 2.4.12 and 2.4.15. Observe that for all $c \in C$ we have $\overline{c} \in C_{ne}(E/(H,S))$ and that $I/I(H \cup S^H)$ is an ideal of $L_K(E)/I(H \cup S^H)$ contained in $I(P_c(E/(H,S)))$. Concretely, we have

$$I/I(H,S) \cong \bigoplus_{\overline{c} \in \overline{C}} M_{\Lambda_{\overline{c}}}(p_c(x)K[x,x^{-1}]),$$

using the notation of Theorem 2.7.3. We want to see that there are no nonzero idempotents in $I/I(H \cup S^H)$. If e is an idempotent in $I/I(H \cup S^H)$, then the ideal J of $L_K(E)/I(H \cup S^H)$ generated by e is an idempotent ideal, contained in $I/I(H \cup S^H)$. However, by the structure of the ideal generated by $P_c(E/(H,S))$ (see Theorem 2.7.3), the only idempotent ideals of $I(P_c(E/(H,S)))$ are the direct sums of some subset of the ideals $\mathrm{M}_{\Lambda_i}(K[x,x^{-1}])$ appearing in the decomposition of $I(P_c(E/(H,S)))$ given by Theorem 2.7.3. Since all the polynomials p_c, for $c \in C$, are not invertible in $K[x,x^{-1}]$, we conclude that $J = 0$ and so that $e = 0$. Hence $I \cap E^0 \subseteq H$ by Corollary 2.4.16(i), and we have shown our claim.

We denote the set $\{v \in B_H \mid v^H \in I\}$ by S'. Then for $v \in S'$ we have that $\overline{v}$ is an idempotent in $I/I(H \cup S^H)$; apply again that this ideal has no nonzero idempotents to get $v^H \in I(H \cup S^H)$. Now, apply Corollary 2.4.16 (ii) to obtain that $v \in S$.

By the proof of Proposition 2.8.5 we see that the sets of cycles and of polynomials associated to the ideal $I = I(H \cup S^H) + I(P_C)$ are precisely the sets C and P. Therefore $\varphi'\varphi(\,((H,S),C,P)\,) = ((H,S),C,P)$.

Now we establish that the composition $\varphi\varphi'$ is the identity on $\mathscr{L}_{id}(L_K(E))$. To this end, consider $I \in \mathscr{L}_{id}(L_K(E))$. Recall from Proposition 2.8.5 that $\varphi'(I) = ((H,S),C,P)$, where $H = I \cap E^0$, $S = \{v \in B_H \mid v^H \in I\}$, and $C \subseteq C_H$ and $P = \{p_c\}_{c \in C}$ satisfy

$$I/I(H\cup S^H) = \bigoplus_{c\in C} I(p_c(\overline{c})).$$

Write $J=\varphi(\varphi'(I)) = I(H\cup S^H)+I(P_C)$ (where $P_C=\{p_c(c)\mid c\in C\}$). Since $I/I(H\cup S^H) = J/I(H\cup S^H)$, we get $I=J$ as desired. By Lemma 2.8.9, $I(H\cup S^H)$ is the largest graded ideal of $L_K(E)$ contained in I.

To finish the proof we check that both isomorphisms preserve the partial orders. First, assume that $((H_1,S_1),C_1,P_1)\leq((H_2,S_2),C_2,P_2)$. Since $(H_1,S_1)\leq(H_2,S_2)$, we get that $I(H_1\cup S_1^{H_1})\subseteq I(H_2\cup S_2^{H_2})$ by Theorem 2.5.8.

Now we want to show $I((P_1)_{C_1})\subseteq I(H_2\cup S_2^{H_2}\cup(P_2)_{C_2})$. Take $c\in C_1$. If $c\in C_2$ then $p_c^{(2)}|p_c^{(1)}$ and so $p_c^{(1)}(c)\in I((P_2)_{C_2})$. If $c\notin C_2$, then since $C_1^0\subseteq H_2\cup C_2^0$ we have $c^0\subseteq H_2$ and so $p_c^1(c)\in I(H_2)$. This shows that φ preserves the order.

In what follows we will prove that the map φ' also preserves the order. To this end, let I and J be in $\mathscr{L}_{id}(L_K(E))$ such that $I\subseteq J$. Again using Proposition 2.8.5 we have that $\varphi'(I)=((H_1,S_1),C_1,P_1)$ and $\varphi'(J)=((H_2,S_2),C_2,P_2)$, where H_i,S_i,C_i,P_i, for $i=1,2$, are as defined before. Again using Lemma 2.8.9, we have that the largest graded ideal $I(H_1\cup S_1^{H_1})$ of I is contained in the largest graded ideal $I(H_1\cup S_2^{H_2})$ of J. Hence, by Theorem 2.5.8, $(H_1,S_1)\leq(H_2,S_2)$.

To finish, we must prove $C_1^0\subseteq C_2^0\cup H_2$ and $p_c^{(2)}|p_c^{(1)}$ for every $c\in C_1\cap C_2$. First, we claim $C_{H_1}\subseteq C_{H_2}\cup H_2$. Consider $c\in C_{H_1}$. By definition, $c^0\cap H_1=\emptyset$ and $r(e)\in H_1$ for every exit e of c. If $c^0\cap H_2\neq\emptyset$, then we have finished. If $c^0\cap H_2=\emptyset$, we get $c\in C_{H_2}$ as $r(e)\in H_2$. Note that $I/I(H_1\cup S_1^{H_1})=\bigoplus_{c\in C_1} I(P_c^{(1)}(\overline{c}))$.

Denote by π the canonical homomorphism: $\pi: L_K(E)/I(H_1\cup S_1^{H_1})\longrightarrow L_K(E)/I(H_2\cup S_2^{H_2})$. Recall that

$$I(P_c(E/(H_1,S_1))) = \bigoplus_{c\in C_{H_1}} \mathrm{M}_{\Lambda_c}(K[\overline{c},\overline{c}^{-1}])\cong\bigoplus_{c\in C_{H_1}} \mathrm{M}_{\Lambda_c}(K[x,x^{-1}])$$

by Theorem 2.7.3 (where $\overline{c}$ denotes the class of c in $L_K(E)/I(H_1\cup S_1^{H_1})$), and thus

$$\mathrm{Ker}(\pi)\cap I(P_c(E/(H_1,S_1))) = \bigoplus_{\{c\in C_{H_1}\ \mid\ c^0\subseteq H_2\}} \mathrm{M}_{\Lambda_c}(K[\overline{c},\overline{c}^{-1}]).$$

Let $\tilde{c}$ denote the class of c in $L_K(E)/I(H_2\cup S_2^{H_2})$. Then, by the above,

$$\pi\left(I/I(H_1\cup S_1^{H_1})\right) = \bigoplus_{\{c\in C_1\ \mid\ c^0\cap H_2=\emptyset\}} I(p_c^{(1)}(\tilde{c}))\subseteq\pi\left(J/I(H_1\cup S_1^{H_1})\right) = J/I(H_2\cup S_2^{H_2}) = \bigoplus_{c\in C_2} I(p_c^{(2)}(\tilde{c})).$$

Therefore we have $\{c\in C_1\mid c^0\cap H_2=\emptyset\}\subseteq C_2$ and thus $c_1^0\subseteq H_2\cup C_2^0$. Finally we observe that for every $c\in C_1\cap C_2$ we have $p_c^{(2)}|p_c^{(1)}$ since $I(p_c^{(1)}(\tilde{c}))\subseteq I(p_c^{(2)}(\tilde{c}))$. This implies $((H_1,S_1),C_1,P_1)\leq((H_2,S_2),C_2,P_2)$, and thereby establishes the result. $\square$

As done with the Structure Theorem for Graded Ideals, we now record the Structure Theorem for Ideals in the case that E is row-finite.

{rowfiniteideals}

Proposition 2.8.11. *Let E be a row-finite graph and K any field. Then every ideal I of $L_K(E)$ is of the form $I(H\cup P_C)$, where $H=I\cap E^0$, and C and P are as described in Proposition 2.8.5.*

Here is an example of how Theorem 2.8.10 allows us to explicitly describe all the ideals of an important Leavitt path algebra.

{idealsinToeplitzalgebra}

Example 2.8.12. Let K be any field, and let E_T be the Toeplitz graph $c \circlearrowleft \bullet^u \xrightarrow{f} \bullet^v$ Easily we see that

$$\mathscr{H}_{E_T} = \{\emptyset,\{v\},\{u,v\}\}\quad\text{and}\quad C_u(E_T)=\{c\}.$$

Clearly there are no sets of breaking vertices in E_T. So by the Structure Theorem for Ideals, the complete set of ideals of $L_K(E_T)$ is given by:

{containment of lattices}

$$I(\emptyset) = \{0\}, \quad I(\{v\}), \quad I(\{u,v\}) = L_K(E_T), \quad \text{and}$$

$$\{I(\{v\} \cup \{p(c)\}) \mid p(x) = 1 + k_1 x + \ldots + k_n x^n \in K[x], \text{ with } k_n \neq 0 \text{ and } n \geq 1\}.$$

Remark 2.8.13. Let E be an arbitrary graph and K any field. Then there exist natural embeddings of lattices:

$$\begin{array}{ccccc} \mathscr{H}_E & \longrightarrow & \mathscr{T}_E & \longrightarrow & \mathscr{Q}_E \\ H & \mapsto & (H,\emptyset) & & \\ & & (H,S) & \mapsto & ((H,S),\emptyset,\emptyset)\ . \end{array}$$

{ProductOfQuadruples}

We conclude the section by presenting just one result which follows directly from the explicit description of the lattice of all two-sided ideals of $L_K(E)$ given in the Structure Theorem for Ideals 2.8.10. We will present numerous additional such results in Section 2.9. First, we introduce a binary operation $\cdot$ on $\mathscr{Q}_E$, under which $\mathscr{Q}_E$ becomes a commutative monoid.

Definition 2.8.14. Let E be an arbitrary graph and K any field. We define a binary operation $\cdot$ on $\mathscr{Q}_E$ as follows. For any $q_1 = ((H_1,S_1),C_1,P_1)$ and $q_2 = ((H_2,S_2),C_2,P_2) \in \mathscr{Q}_E$, set

$$q_1 \cdot q_2 = ((H_1,S_1) \wedge (H_2,S_2),\ C_1 \wedge C_2,\ \{p_c^{(1)} p_c^{(2)}\}_{c \in C_1 \wedge C_2}),$$

where

$$C_1 \wedge C_2 = (C_1 \cap C_2) \cup C_1^{H_2} \cup C_2^{H_1},$$

$$\text{with} \quad C_1^{H_2} = \{c \in C_1 \mid c^0 \subseteq H_2\} \quad \text{and} \quad C_2^{H_1} = \{c \in C_2 \mid c^0 \subseteq H_1\}.$$

{idempotentelementsinQ}

(We interpret $p_c^{(i)}$ as 1 if $c \notin C_i$ for $i = 1$ or 2.)

Clearly this operation is associative and commutative, and the neutral element is $((E^0,\emptyset),\emptyset,\emptyset)$.

{productsofidealsprop}

Remark 2.8.15. We note that the set of idempotent elements of $\mathscr{Q}_E$ is precisely $\mathscr{T}_E$.

{CommutativityProductOfIdeals}

Using the explicit description of the lattice isomorphism φ given in the proof of the Structure Theorem for Ideals, we get

Proposition 2.8.16. *Let $\varphi : \mathscr{L}_{id}(L_K(E)) \to \mathscr{Q}_E$ be the isomorphism of Theorem 2.8.10, and let I and J be elements of $\mathscr{L}_{id}(L_K(E))$. Then $\varphi(IJ) = \varphi(I) \cdot \varphi(J)$.*

Using Proposition 2.8.16, the fact that the map φ therein is a lattice isomorphism, and the obvious commutativity of the operation $\cdot$ on $\mathscr{Q}_E$, we achieve the following result, which is perhaps-surprising, in that $L_K(E)$ is in general far from commutative,

{AdditionalConseqSTfI}

Corollary 2.8.17. *Let E be an arbitrary graph and K any field. If I and J are arbitrary ideals of $L_K(E)$, then $IJ = JI$.*

2.9 Additional consequences of the Structure Theorem for Ideals. The Simplicity Theorem

The Structure Theorem for Ideals (Theorem 2.8.10) allows us great insight into various ring-theoretic properties of Leavitt path algebras. We record a number of those results in this section.

Consistent with our presentation of various consequences of the Structure Theorem for Graded Ideals, we begin by presenting the (non-graded) versions of results analogous to Proposition 2.5.13 and Corollary 2.5.15, namely, results about the simplicity and two-sided chain conditions of Leavitt path algebras.

Recall that an algebra A is said to be *simple* if $A^2 \neq 0$ and the only two-sided ideals of A are $\{0\}$ and A.

{simplicity}

Theorem 2.9.1. *(***The Simplicity Theorem***) Let E be an arbitrary graph and K any field. Then the Leavitt path algebra $L_K(E)$ is simple if and only if E satisfies the following conditions:*

(1) $\mathscr{H}_E = \{\emptyset, E^0\}$ *(i.e., the only hereditary saturated subsets of E^0 are $\emptyset$ and E^0), and*
(2) *E satisfies Condition (L) (i.e., every cycle in E has an exit).*

Proof. The Structure Theorem for Ideals 2.8.10 provides a lattice isomorphism φ from the lattice $\mathscr{Q}_E$ to the lattice of all two-sided ideals of $L_K(E)$. In particular, we see immediately that if H is a hereditary saturated subset of E^0 not equal to $\emptyset$ or E^0, then $\varphi(((H,\emptyset),\emptyset,\emptyset))$ is a nontrivial ideal of $L_K(E)$. Similarly, if c is a cycle in E without an exit, then $c \in C_\emptyset$ (see Notation 2.8.3), and then $\varphi(((\emptyset,\emptyset),\{c\},1+x))$ gives a nontrivial ideal of $L_K(E)$. Thus the two conditions on E are necessary for the simplicity of $L_K(E)$. {Leavittalgebraissimple}

Conversely, suppose E satisfies the two properties. First, as noted subsequent to Definition 2.4.4, we have that both $B_\emptyset = \emptyset$ and $B_{E^0} = \emptyset$. Additionally, $C_{E^0} = \emptyset$, and the hypothesis that every cycle in E has an exit yields that $C_\emptyset = \emptyset$ as well. Thus $\mathscr{Q}_E$ consists precisely of the two elements $((E^0,\emptyset),\emptyset,\emptyset)$ and $((\emptyset,\emptyset),\emptyset,\emptyset)\}$. The simplicity of $L_K(E)$ now follows from the Structure Theorem for Ideals. □

Example 2.9.2. Consider once again the graphs R_n consisting of one vertex and n loops. Obviously Condition (1) of the Simplicity Theorem is satisfied for R_n. When $n \geq 2$, Condition (2) is satisfied for R_n as well. Thus $L_K(R_n)$ is simple for $n \geq 2$; i.e., the Leavitt algebra $L_K(1,n)$ is simple for $n \geq 2$. We note that Condition (2) is not satisfied for the graph R_1, which implies that $L_K(R_1) \cong K[x,x^{-1}]$ is not simple. (Of course this last statement is well known.)

Remark 2.9.3. Note that graphs having infinite emitters may give rise to simple Leavitt path algebras: for example, the graph $R_\mathbb{N}$ having one vertex and countably many loops at that vertex satisfies the conditions of the Simplicity Theorem 2.9.1.

Due to its importance in the general theory of Leavitt path algebras, due to the importance that these attendant ideas and definitions will play later, and due to its historical significance, we offer now a second proof of the Simplicity Theorem. {infinitepathdef}

Definitions 2.9.4. Let E be an arbitrary graph. By an *infinite path* in E we mean a sequence $\gamma = e_1, e_2, \ldots$ for which $r(e_i) = s(e_{i+1})$ for all $i \in \mathbb{N}$. We often denote such γ by $e_1 e_2 \cdots$. (We note that the terminology *infinite path* is perhaps misleading, but standard: despite its name, an infinite path in E is not an element of $\mathrm{Path}(E)$.) By a *vertex in* an infinite path $\gamma = e_1, e_2, \ldots$ we mean a vertex of the form $s(e_i)$ for some $i \in \mathbb{N}$.

We denote by E^∞ the set of all infinite paths of E, and by $E^{\leq\infty}$ the set E^∞ together with the set of finite paths in E whose range vertex is a singular vertex.

We say that a vertex $v \in E^0$ is *cofinal* if for every $\gamma \in E^{\leq\infty}$ there is a vertex w in the path γ such that $v \geq w$. We say that a graph E is *cofinal* if every vertex in E is cofinal.

If c is a closed path in E, then c gives rise to the infinite path $ccc\cdots$ of E. Thus if E is cofinal, then in particular every vertex of E connects every cycle in E, and to every sink in E. {ojito}

Lemma 2.9.5. *Let E be a cofinal graph, and let $v \in E^0$ be a sink.*

(i) The only sink of E is v.
(ii) For every $w \in E^0$, $v \in T(w)$.
(iii) E contains no infinite paths. In particular, E is acyclic.

Proof.

(i) is obvious.
(ii) Since $T(v) = \{v\}$, the result follows from the definition of $T(v)$ by considering the path $\gamma = v \in E^{\leq\infty}$.
(iii) If $\alpha \in E^\infty$, then there exists $w \in \alpha^0$ such that $v \geq w$, which is impossible. Thus, in particular, E contains no closed paths. {cofinohersat}

Lemma 2.9.6. *A graph E is cofinal if and only if $\mathscr{H} = \{\emptyset, E^0\}$.*

Proof. Suppose E is cofinal. Let $H \in \mathscr{H}$ with $\emptyset \neq H \neq E^0$. We choose and fix $v \in E^0 \setminus H$, and subsequently build a path $\gamma \in E^{\leq\infty}$ such that $\gamma^0 \cap H = \emptyset$, as follows. If $v \in \mathrm{Sing}(E)$, take $\gamma = v$, and we are done. If not, then $v \in \mathrm{Reg}(E)$, so $0 < |s^{-1}(v)| < \infty$ and $r(s^{-1}(v)) \nsubseteq H$ (otherwise, H saturated implies $v \in H$). Hence, there exists $e_1 \in s^{-1}(v)$ such that $r(e_1) \notin H$. Let $\gamma_1 = e_1$ and repeat this process with $r(e_1)$. By recurrence either we reach a singular vertex, or we have an infinite path γ whose vertices are not in H, as desired. Now consider $w \in H$ (such exists as $\emptyset \neq H$ by hypothesis). By cofinality, there exists $z \in \gamma^0$ such that $w \geq z$, and by the hereditariness of H we get $z \in H$, contradicting the construction of γ.

Conversely, suppose that $\mathscr{H} = \{\emptyset, E^0\}$. Take $v \in E^0$ and $\gamma \in E^{\leq\infty}$, with $v \notin \gamma^0$ (the case $v \in \gamma^0$ is obvious). By hypothesis the hereditary saturated subset generated by v is E^0, i.e., $E^0 = \bigcup_{n\geq 0} \Lambda_n(v)$ as described in Lemma 2.0.7. Consider m, the minimum n such that $\Lambda_n(v) \cap \gamma^0 \neq \emptyset$, and let $w \in \Lambda_m(v) \cap \gamma^0$. If $m > 0$, then by minimality of m it must be that w is a regular vertex and that $r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$. Since w is a regular vertex and $\gamma = (\gamma_n) \in E^{\leq\infty}$, there exists $i \geq 1$ such that $s(\gamma_i) = w$ and $r(\gamma_i) = w' \in \gamma^0$, the latter meaning that $w' \in r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$, contradicting the minimality of m. Therefore $m = 0$ and then $w \in \Lambda_0(v) = T(v)$, as we needed. □

The previous discussion allows us to re-establish the Simplicity Theorem without the need to invoke the full power of the Structure Theorem for Ideals 2.8.10.

{simplicitytheoremsecondform}

Theorem 2.9.7. *(***The Simplicity Theorem, revisited***) Let E be an arbitrary graph and K any field. Then the Leavitt path algebra $L_K(E)$ is simple if and only if the graph E satisfies the following conditions:*

(1) *The graph E is cofinal, and*
(2) *E satisfies Condition (L).*

Proof. We will use the characterizarion of cofinality given in Lemma 2.9.5. Suppose first that $L_K(E)$ is simple. By Theorem 2.4.8, $\mathscr{H}_E = \{\emptyset, E^0\}$. On the other hand, if E does not satisfy Condition (L), then there exists a cycle c in E which has no exits. This implies that $I(P_c(E))$ is a nonzero ideal of $L_K(E)$, and so by the simplicity of $L_K(E)$, we must have $I(P_c(E)) = L_K(E)$. But, by Theorem 2.7.3, the algebra $I(P_c(E))$ is not simple. This is a contradiction and, therefore, E must satisfy Condition (L).

Now, suppose that the graph E satisfies Conditions (1) and (2) in the statement, and let I be a nonzero ideal of $L_K(E)$. By Corollary 2.2.14, $I \cap E^0 \neq \emptyset$. Since $I \cap E^0 \in \mathscr{H}_E$ (by Lemma 2.4.3), the hypothesis in the statement imply $I \cap E^0 = E^0$ or, in other words, $E^0 \subseteq I$. This immediately gives $I = L_K(E)$.

We now record the two-sided chain condition results for Leavitt path algebras. Since the verifications of these results follow from the Structure Theorem for Ideals, using arguments similar to those presented in Theorem 2.9.1 and Lemma 2.5.12, we omit the proofs. We note, however, that with the Structure Theorem for Ideals in hand, such proofs are significantly shorter than those offered originally in [11, Theorems 3.6 and 3.9].

Proposition 2.9.8. *Let E be an arbitrary graph and K any field.*

(i) *$L_K(E)$ is two-sided artinian if and only if E satisfies Condition (K), $\mathscr{H}_E$ satisfies the descending chain condition with respect to inclusion, and, for each $H \in \mathscr{H}_E$, the set B_H of breaking vertices is finite.*
(ii) *$L_K(E)$ is two-sided noetherian if and only if $\mathscr{H}_E$ satisfies the ascending chain condition with respect to inclusion, and, for each $H \in \mathscr{H}_E$, the set B_H of breaking vertices is finite.*

We note that, by Proposition 2.5.13(ii), $L_K(E)$ is noetherian if and only if $L_K(E)$ is graded noetherian (as the two graph-theoretic conditions on E are identical). The same cannot be said for the artinian condition: for instance, $K[x,x^{-1}] \cong L_K(R_1)$ is graded artinian, but is well known to not be artinian. In addition, we note that if E does not satisfy Condition (K), then there is some hereditary saturated subset H of E^0 for which the quotient graph E/H contains a cycle without an exit; this is how Condition (K) becomes incorporated into the Structure Theorem for Ideals.

{gradedCondK}

For the next consequence of the Structure Theorem for Ideals, we record the previously promised result regarding a characterization of Condition (K) in terms of the graded ideals of $L_K(E)$.

Proposition 2.9.9. *Let E be an arbitrary graph and K any field. Then every ideal of $L_K(E)$ is graded if and only if E satisfies Condition* (K)*.*

Proof. If E satisfies Condition (K), then $C_u(E) = \emptyset$ and so, by the Structure Theorem for Ideals 2.8.10, every ideal of $L_K(E)$ is of the form $I(H \cup S^H)$, and hence is graded.

Conversely, suppose that E does not satisfy Condition (K). Then there exists a cycle c in $C_u(E)$. Let H denote the saturated closure of the tree of the ranges of the exits of c. Then $H \in \mathscr{H}_E$, $c^0 \cap H = \emptyset$, and the range of every exit of c belongs to H. Therefore $c \in C_H$ and so, choosing for example $p(x) = 1 + x \in K[x]$, we have that $\varphi(((H,\emptyset),\{c\},\{p(x)\})) = I(H \cup \{1+c\})$ is a nongraded ideal of $L_K(E)$. □

{idealsInfiniteClock}

Example 2.9.10. As one specific consequence of Proposition 2.9.9, we conclude that the list of graded ideals of the Leavitt path algebra of the infinite clock graph $C_{\mathbb{N}}$, presented in Example 2.5.10, indeed represents the list of *all* ideals of $L_K(C_{\mathbb{N}})$.

Yet another immediate application of the Structure Theorem for Ideals is the following result, in which we present (among other things) the converse of Corollary 2.5.23 regarding the structure of graded ideals in $L_K(E)$.

{idLPA}

Corollary 2.9.11. *Let E be an arbitrary graph and K any field. For an ideal I of the Leavitt path algebra $L_K(E)$, the following are equivalent.*

(1) *I is a graded ideal.*
(2) *I is generated by idempotents.*
(3) $I = I^2$.
(4) *I is K-algebra isomorphic to a Leavitt path K-algebra.*

In particular, by Proposition 2.9.9, E satisfies Condition (K) *if and only if every ideal of $L_K(E)$ is generated by idempotents.*

Proof. (1) $\Longrightarrow$ (2) follows by Theorem 2.4.8.

(2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1) follows from the observation made in Remark 2.8.15.

(1) $\Longrightarrow$ (4) is Corollary 2.5.23.

(4) $\Longrightarrow$ (3) follows because any Leavitt path algebra has local units (Lemma 1.2.12). □

{idOfid}

Corollary 2.9.12. *Let E be an arbitrary graph and K any field. If J is an ideal of a graded ideal I of $L_K(E)$, then J is an ideal of $L_K(E)$.*

Proof. Let $a \in L_K(E)$ and $y \in J \subseteq I$. By Corollary 2.9.11(4) and Lemma 1.2.12(v) there exists $x \in I$ such that $y = xy$. Then $ay = (ax)y \in IJ \subseteq J$. □

We finish Chapter 2 by presenting a result which serves as an appropriate bridge to Chapter 3, in that this result relates an ideal structure property to a property of idempotents. Rings for which every nonzero one-sided ideal contains a nonzero idempotent were studied in [109].

{charCondL}

Proposition 2.9.13. *Let E be an arbitrary graph and K any field. The following conditions are equivalent:*

(1) *E satisfies Condition* (L).
(2) *Every nonzero two-sided ideal of $L_K(E)$ contains a vertex.*
(3) *Every nonzero one-sided ideal of $L_K(E)$ contains a nonzero idempotent.*

Proof. (1) $\Rightarrow$ (3). Let a be a nonzero element in a left ideal I of $L_K(E)$. Apply the Reduction Theorem 2.2.11 to find $\mu,\nu \in \text{Path}(E)$, $v \in E^0$ and $k \in K^\times$ such that $0 \neq \mu^* a \nu = kv$. Then $k^{-1}\nu\mu^* a$ is nonzero, because $0 \neq v = v^2 = k^{-2}\mu^* a(\nu\mu^* a)\nu$, and it is an idempotent inside I, as $(k^{-1}\nu\mu^* a)(k^{-1}\nu\mu^* a) = k^{-1}\nu v\mu^* a = k^{-1}\nu\mu^* a$. An analogous proof, or an appeal to Corollary 2.0.9, establishes the result for right ideals as well.

(3) $\Rightarrow$ (1). If E does not satisfy Condition (L), then there exists a cycle without exits c in E. Denote by I the (graded) ideal of $L_K(E)$ generated by the vertices of c. Lemma 2.7.1 implies that I is isomorphic to $\mathrm{M}_\Lambda(K[x,x^{-1}])$ for some set Λ. Since the ideals of I are ideals of $L_K(E)$ by Corollary 2.9.12, the hypothesis implies that every nonzero ideal of $\mathrm{M}_\Lambda(K[x,x^{-1}])$ contains a nonzero idempotent, which is not true. This shows our claim.

An argument similar to the one given in the previous paragraph also establishes (2) $\Rightarrow$ (1). That (1) $\Rightarrow$ (2) is Corollary 2.2.14. □

Chapter 3
Idempotents, and finitely generated projective modules

{simplandidempotents}

ABSTRACT: The richness of the idempotent structure of Leavitt path algebras lies at the heart of the subject; in this chapter we present a number of topics which fall under this umbrella. These include: the purely infinite property (for both simple and non-simple algebras); the structure of the monoid of finitely generated projective modules; the exchange property; von Neumann regularity; and primitive idempotents.

In this chapter we consider various topics related to the structure of the idempotents in $L_K(E)$. We start with a discussion of the purely infinite simplicity of a Leavitt path algebra, a topic which has fueled much of the investigative effort in the subject. In the subsequent section we analyze the structure of the monoid $\mathscr{V}(L_K(E))$ of isomorphism classes of finitely generated projective modules over a Leavitt path algebra $L_K(E)$. This will allow us to more fully describe Bergman's construction (presented earlier in Section 1.4), which was essential to the genesis of the subject. In Section 3.3 we remind the reader of the definition of an exchange ring, and subsequently show that the exchange Leavitt path algebras are exactly those arising from graphs which satisfy Condition (K). Von Neumann regularity is taken up in Section 3.4; in addition to showing that the von Neumann regular Leavitt path algebras are precisely those arising from acyclic graphs, we identify the set of vertices which generate the largest von Neumann regular ideal of $L_K(E)$. We continue our discussion of the idempotents in $L_K(E)$ in Section 3.5 by identifying the collection of primitive idempotents which are not minimal.

We consider in Section 3.6 the monoid-theoretic structure of $\mathscr{V}(L_K(E))$. While the monoid $\mathscr{V}(R)$ for a general ring R necessarily satisfies certain properties (e.g., $\mathscr{V}(R)$ is conical), we will show that when E is a row-finite graph and $R = L_K(E)$ then $\mathscr{V}(R)$ enjoys many additional properties over and above the conical property, including refinement and separativity. In the subsequent Section 3.7 we consider the *extreme* cycles in a graph, and show that the ideal of $L_K(E)$ generated by the vertices in such cycles may be appropriately viewed as the "purely infinite socle" of $L_K(E)$. We conclude the chapter with Section 3.8, in which we remind the reader of the general notion of a purely infinite (but not necessarily simple) ring, and then identify those graphs E for which $L_K(E)$ is purely infinite.

We start by presenting an easily established but fundamental result regarding isomorphisms between various left $L_K(E)$-modules. This result expands on the idea presented in Lemma 2.6.10.

{fundisoprop}

Proposition 3.0.1. *Let E be an arbitrary graph and K any field. Let $\mu \in \text{Path}(E)$ for which $s(\mu) = v$ and $r(\mu) = w$.*

(i) *There is a direct sum decomposition*

$$L_K(E)v = L_K(E)\mu\mu^* \oplus L_K(E)(v - \mu\mu^*)$$

as left ideals of $L_K(E)$.

(ii) *There is an isomorphism of left $L_K(E)$-modules*

$$L_K(E)w \cong L_K(E)\mu\mu^*.$$

Consequently, there is an isomorphism $L_K(E)v \cong L_K(E)w \oplus T$ for some left ideal T of $L_K(E)$.

Proof. (i) Since $\mu\mu^*$ is an idempotent which commutes with v, we have that $v-\mu\mu^*$ is also an idempotent. But $\mu\mu^*(v-\mu\mu^*) = \mu\mu^* - \mu\mu^* = 0 = (v-\mu\mu^*)\mu\mu^*$, which gives easily that $L_K(E)v = L_K(E)\mu\mu^* \oplus L_K(E)(v-\mu\mu^*)$ as left $L_K(E)$-modules. (We note that in general the second summand might be $\{0\}$.)

(ii) We define $\varphi = \rho_{\mu^*} : L_K(E)w \to L_K(E)\mu\mu^*$ to be the right multiplication by μ^* map, so $(rw)\phi = rw\mu^* = r\mu^*$. The observation that $\mu^*\mu\mu^* = \mu^*$ shows that φ indeed maps into $L_K(E)\mu\mu^*$. Now define $\psi = \rho_\mu : L_K(E)\mu\mu^* \to L_K(E)w$ to be the right multiplication by μ map, so $(r\mu\mu^*)\psi = r\mu\mu^*\mu = r\mu$. Using that $\mu^*\mu = w$ and that $\mu\mu^*\mu = \mu$ shows that φ and ψ are inverses. The second part of the statement now follows from (i). ☐

{dichotomy}

{infiniteidempotentdef}

3.1 Purely infinite simplicity, and the Dichotomy Principle.

In Section 2.9 we identified the simple Leavitt path algebras. Intuitively speaking, such algebras can be partitioned into two types: those which behave much like full matrix rings over K, and those which behave much like the Leavitt algebras $L_K(1,n)$. The goal of this section is to make this dichotomy precise.

{infiniteidempotents}

{infiniteisnotacc}

Definitions 3.1.1. (See e.g. [33, Definitions 1.2]) Let R be a ring. An idempotent e in R is said to be *infinite* if there exist orthogonal idempotents $f,g \in R$ such that $e = f+g$, $g \neq 0$, and $Re \cong Rf$ as left R-modules. Rephrased, the idempotent e is infinite in case Re is isomorphic to a proper direct summand of itself. In such a situation we say Re is a *directly infinite module*.

{Leavitthasinfiniteidempotents}

Remark 3.1.2. We note that if e is an infinite idempotent in a ring R, then the left R-module Re can satisfy neither the ascending nor the descending chain condition on submodules. In particular, a Noetherian ring contains no infinite idempotents.

{infinitesummandgivesinfinite}

Example 3.1.3. In our context, the quintessential example of an infinite idempotent is provided in the Leavitt algebra $R = L_K(R_2) \cong L_K(1,2)$. We show that 1_R is an infinite idempotent. If e,f are the loops based at v in R_2, then by (CK2) we have $v = 1_R = ee^* + ff^*$. By Proposition 3.0.1(i) we get $L_K(R_2) = L_K(R_2)1_R = L_K(R_2)ee^* \oplus L_K(R_2)(v-ee^*) = L_K(R_2)ee^* \oplus L_K(R_2)ff^*$ (where each of the two summands is clearly nonzero), and by Proposition 3.0.1(ii) we have that $L_K(R_2)1_R \cong L_K(R_2)ee^*$. A similar conclusion can be drawn in any of the Leavitt algebras $L_K(1,n)$. (Indeed, we will show in Example 3.2.7 that *every* nonzero idempotent of $L_K(1,n)$ is infinite.)

{lemmacycleswithexits}

Remark 3.1.4. Suppose e is an infinite idempotent in a ring R, and suppose that g is an idempotent of R such that $Rg \cong Re \oplus Q$ for some left R-module Q. Then g is infinite as well. This is easy to see, as by hypothesis, $Re \cong Re \oplus P$ for some nonzero left R-module P, so that $Rg \cong Re \oplus Q \cong (Re \oplus P) \oplus Q \cong (Re \oplus Q) \oplus P \cong Rg \oplus P$.

There is a strong connection between infinite idempotents in $L_K(E)$ and cycles having exits in E.

Lemma 3.1.5. *Let E be an arbitrary graph and K any field. Suppose c is a cycle based at w, and suppose e is an exit for c with $s(e) = w$. Then $L_K(E)w = P \oplus Q$, where P and Q are nonzero left ideals of $L_K(E)$, and $L_K(E)w \cong P$ as left $L_K(E)$-modules. In particular, w is an infinite idempotent of $L_K(E)$.*

{infiniteverticesprop}

Proof. By Proposition 3.0.1(i), we get a decomposition $L_K(E)w = L_K(E)cc^* \oplus L_K(E)(w-cc^*)$. But since $r(c) = w$, we get by Proposition 3.0.1(ii) that $L_K(E)w \cong L_K(E)cc^*$. Since e is an exit for c we have $c^*e = 0$ (by (CK1)). This yields that $w - cc^* \neq 0$, since, if otherwise $w - cc^* = 0$, then multiplying on the right by e would give $e = 0$ in $L_K(E)$, violating Corollary 1.5.13. Thus $P = L_K(E)cc^*$ and $Q = L_K(E)(w-cc^*)$ give the desired result. ☐

We now identify those vertices of E which are infinite idempotents of $L_K(E)$.

Proposition 3.1.6. *Let E be an arbitrary graph and K any field. Let $v \in E^0$. Then v is an infinite idempotent in $L_K(E)$ if and only if v connects to a cycle with exits in E.*

Proof. Suppose first that v connects to a cycle with exits. Specifically, suppose there exists a cycle c in E with an exit e to which v connects. Let w denote $s(e)$. Since v connects to c, there exists $\mu \in \mathrm{Path}(E)$ with $s(\mu) = v$ and $r(\mu) = w$. By Proposition 3.0.1(i) we have $L_K(E)v \cong L_K(E)w \oplus T$ for some left ideal T of $L_K(E)$. But $L_K(E)w$ is infinite by Lemma 3.1.5, so that Remark 3.1.4 yields the result.

Conversely, assume that $T(v)$ does not contain any cycle with exits. By Theorem 1.6.10, it suffices to consider the case of a finite graph E. (Observe that if F is a finite complete subgraph of E containing a cycle c which has no exits in E, then c is also a cycle without exits in the graph $F(\mathrm{Reg}(E) \cap \mathrm{Reg}(F))$ built in Definition 1.5.16, because the vertices in c are regular both in E and in F.)

Now, by Corollary 2.7.6, we have

$$I(v) \cong \mathrm{M}_{r_1}(K) \oplus \cdots \oplus \mathrm{M}_{r_k}(K) \oplus \mathrm{M}_{s_1}(K[x,x^{-1}]) \oplus \cdots \oplus \mathrm{M}_{s_\ell}(K[x,x^{-1}]),$$

and by Remark 3.1.2 this ring contains no infinite idempotents. $\square$

{purelyinfinitesimpleequivconds}

We now utilize a result which we will discuss in further detail in Section 3.8 below.

Proposition 3.1.7. *Let R be a (not necessarily unital) ring. Then the following conditions are equivalent:*

(1) *For each nonzero $x \in R$ there exist elements $s,t \in R$ such that sxt is an infinite idempotent.*
(2) *Every nonzero one-sided ideal of R contains an infinite idempotent.*

{purelyinfinitesimpledef}

Proof. (1) $\Rightarrow$ (2). Let a be a nonzero element of R. By (1) there are $s,t \in R$ such that $e := sat$ is an infinite idempotent. Observe that we can assume that $s = es$ and $t = te$. It then follows that $a(ts)$ is an infinite idempotent in aR, because $(ats)R \cong (sat)R$.

(2) $\Rightarrow$ (1). Let x be a nonzero element in R. Then, for some $t \in R$ we have that $e := xt$ is an infinite idempotent. Hence $e = ext$ is an infinite idempotent. $\square$

Definition 3.1.8. A simple ring R which satisfies the equivalent conditions of Proposition 3.1.7 is called a *purely infinite simple* ring.

{Leavittispis}

Remark 3.1.9. It is of historical importance to note that the proof given by Leavitt of the simplicity of $L_K(1,n)$ for each $n \geq 2$ [101, Theorem 2] in fact demonstrates that $L_K(1,n)$ is purely infinite simple. The fact that $1_{L_K(1,n)}$ is an infinite idempotent was observed in Example 3.1.3.

{purelyinfinitesimple}

We now have all the tools necessary to characterize the purely infinite simple Leavitt path algebras in terms of properties of the associated graph.

Theorem 3.1.10. *(***The Purely Infinite Simplicity Theorem***) Let E be an arbitrary graph and K any field. Then the Leavitt path algebra $L_K(E)$ is purely infinite simple if and only if E satisfies the following conditions:*

(i) $\mathscr{H}_E = \{\emptyset, E^0\}$,
(ii) *E satisfies Condition* (L)*, and*
(iii) *every vertex in E^0 connects to a cycle.*

Equivalently, (iii) *may be replaced by:*
(iii') *E contains at least one cycle.*

Proof. Suppose first that conditions (i), (ii) and (iii) are satisfied. By the Simplicity Theorem 2.9.1 we have that $L_K(E)$ is a simple ring. Note that (ii) and (iii) together give that every vertex connects to a cycle with exits. So by Proposition 3.1.6 we get that all the vertices of E are infinite idempotents in $L_K(E)$. Now let $0 \neq \alpha \in L_K(E)$. Since E satisfies Condition (L), by the Reduction Theorem 2.2.11 there exist $\mu, \kappa \in \mathrm{Path}(E)$ and $k \in K^\times$ with $k^{-1}\mu\alpha\kappa = v$ for some vertex v. Since v is an infinite idempotent by the previous paragraph, we see from Proposition 3.1.7(i) that $L_K(E)$ is purely infinite.

Conversely, suppose that $L_K(E)$ is purely infinite simple. Again invoking the Simplicity Theorem 2.9.1, the graph E satisfies Conditions (i) and (ii) in the statement. Now we see that Condition (iii) holds as well. By Proposition 3.1.6, it suffices to show that every vertex v of E is an infinite idempotent in $L_K(E)$. By hypothesis (using Proposition 3.1.7(ii)), the nonzero left ideal $L_K(E)v$ contains an infinite idempotent y;

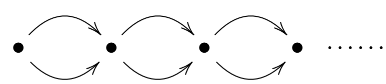
write $y = rv$ for some $r \in L_K(E)$. As y is infinite, necessarily $y \neq 0$. Then, since $rv \cdot rv = rv$, it is easy to show that $x = vrv$ is an idempotent as well; moreover, $x \neq 0$, as otherwise $x = 0$ would give $rx = 0$, which would give $rvrv = rv = 0$, contrary to the choice of $y = rv$. Thus x is a nonzero idempotent in $L_K(E)v$ which commutes with v, and so $L_K(E)v = L_K(E)x \oplus L_K(E)(v-x)$. But $L_K(E)vrv = L_K(E)rv$; the inclusion $\subseteq$ is clear, while $\supseteq$ follows from $rv = rvrv$. Rephrased, $L_K(E)x = L_K(E)y$. Thus $L_K(E)v = L_K(E)y \oplus L_K(E)(v-x)$. As y is infinite, we get that v must be infinite as well, using Remark 3.1.4.

We finish by showing that Conditions (iii) and (iii') are equivalent in the presence of Conditions (i) and (ii). By Theorem 2.9.7, condition (i) may be replaced by the condition that E is cofinal. In particular, every vertex of E must connect to every cycle of E (as each cycle gives rise to an infinite path in E). So the existence of at least one cycle suffices to give (iii), and conversely. □

With both the Simplicity Theorem 2.9.7 and Purely Infinite Simplicity Theorem 3.1.10 now established, Proposition 2.6.20 immediately yields the following.

Theorem 3.1.11. *(***The Dichotomy Principle for simple Leavitt path algebras***) Let E be an arbitrary graph and K any field. If $L_K(E)$ is simple, then either $L_K(E)$ is locally matricial or $L_K(E)$ is purely infinite simple.*

Example 3.1.12. Any algebra of the form $M_\Lambda(K)$ (for any set Λ) is an example of a locally matricial simple Leavitt path algebra (see Corollary 2.6.6). Additional such examples exist as well, for instance, let E denote the "doubly infinite line graph"



The corresponding Leavitt path algebra $L_K(E)$ is simple, but is not isomorphic to $M_\Lambda(K)$ for any set Λ, as $\mathrm{Soc}(L_K(E)) = \{0\}$ by Theorem 2.6.14.

Remark 3.1.13. We note that as a result of Condition (3) in Theorem 3.1.10, if E is a graph for which $L_K(E)$ is purely infinite simple, then necessarily E contains no sinks.

Indeed, the cofinality condition yields a version of the Dichotomy Principle with respect to graded simplicity.

Proposition 3.1.14. *(***The Trichotomy Principle for graded simple Leavitt path algebras***) Let E be an arbitrary graph and K any field. If $L_K(E)$ is graded simple, then exactly one of the following occurs:*

(i) *$L_K(E)$ is locally matricial, or*
(ii) *$L_K(E) \cong M_\Lambda(K[x,x^{-1}])$ for some set Λ, or*
(iii) *$L_K(E)$ is purely infinite simple.*

Proof. By Corollary 2.5.15 and Lemma 2.9.6, the graded simplicity of $L_K(E)$ is equivalent to the cofinality of E. The three possibilities given in the statement correspond precisely to whether E contains no cycles; resp., contains exactly one cycle; resp., contains two or more cycles. If E contains no cycles then (i) follows by Proposition 2.6.20. If E contains at least two cycles then by cofinality each cycle in E must connect to each of the other cycles in E. Consequently, each cycle in E has an exit, and (iii) follows by the Purely Infinite Simplicity Theorem 3.1.10.

Now suppose that E contains exactly one cycle c. Then c has no exits (otherwise, if e were an exit for c then by cofinality $r(e)$ would connect to c, and would thus produce a second cycle in E). So $P_c(E)$ is nonempty, which yields that $I(P_c(E))$ is a nonzero (necessarily graded) ideal of $L_K(E)$. But then graded simplicity gives that $L_K(E) = I(P_c(E))$, from which Theorem 2.7.3 yields the desired result. □

3.2 Finitely generated projective modules: the $\mathcal{V}$-monoid

The goal of this section is to establish Theorem 1.4.3, the fundamental result which was presented (without proof) in the first chapter. This result provided one of the main springboards from which the entire subject

{Vmonoiddef}

of Leavitt path algebras was launched. We restate the result below as Theorem 3.2.6. We recall now the definitions of its two main ingredients.

Definition 3.2.1. Let R be a unital ring. We denote by $\mathscr{V}(R)$ the set of isomorphism classes (denoted using $[\ \ ]$) of finitely generated projective left R-modules. We endow $\mathscr{V}(R)$ with the structure of a commutative monoid by defining

$$[P]+[Q] := [P \oplus Q]$$

for $[P],[Q] \in \mathscr{V}(R)$.

Suppose more generally that R is a not-necessarily-unital ring. We consider any unital ring S containing R as a two-sided ideal, and denote by $FP(R,S)$ the class of finitely generated projective left S-modules P for which $P = RP$. In this situation, $\mathscr{V}(R)$ is defined as the monoid of isomorphism classes of objects in $FP(R,S)$. This definition of $\mathscr{V}(R)$ does not depend on the particular unital ring S in which R sits as a two-sided ideal, as can be seen from the following alternative description: $\mathscr{V}(R)$ is the set of equivalence classes of idempotents in $\mathrm{M}_{\mathbb{N}}(R)$, where $e \sim f$ in $\mathrm{M}_{\mathbb{N}}(R)$ if and only if there are $x,y \in \mathrm{M}_{\mathbb{N}}(R)$ such that $e = xy$ and $f = yx$. (See [105, page 296].)

For an idempotent $e \in R$ we will sometimes denote the element $[Re]$ of $\mathscr{V}(R)$ simply by $[e]$.

We note that if R is a ring with local units, then the well-studied *Grothendieck group* $K_0(R)$ of R is the universal group corresponding to the monoid $\mathscr{V}(R)$, see [105, Proposition 0.1]. We will study the Grothendieck group of Leavitt path algebras in great depth throughout Chapter 6.

{MsubEdefinition}

For any graph E one can associate a monoid M_E; this monoid will play a central role in the theory of Leavitt path algebras. We now present the definition of the monoid M_E associated to a row-finite graph; we will extend this definition to arbitrary graphs later in this section.

Definition 3.2.2. Let E be a row-finite graph. We define M_E to be the free abelian monoid (written additively), having generating set $\{a_v \mid v \in E^0\}$, and with relations given by setting

$$a_v = \sum_{\{e \in E^1 \mid s(e)=v\}} a_{r(e)} \qquad \text{for every } v \in \mathrm{Reg}(E).$$

For notational clarity, we often denote the zero element of M_E by z.

{MsubEexamples}

Examples 3.2.3 Some examples of the construction of the monoid M_E will be helpful.

(i) As noted in Section 1.4, if R_n is the rose with n petals graph ($n \geq 2$), then

$$M_{R_n} = \{z, a_v, 2a_v, \ldots, (n-1)a_v\}, \text{ with relation } na_v = a_v.$$

Although this observation is somewhat counterintuitive at first glance, we see that the subset $M_{R_n} \setminus \{z\}$ of M_{R_n} is not only closed under $+$ (and thereby forms a subsemigroup of M_{R_n}), $M_{R_n} \setminus \{z\}$ is in fact a *group*, isomorphic to $\mathbb{Z}/(n-1)\mathbb{Z}$, with identity element $(n-1)a_v$.

(ii) For the graph R_1 having one vertex v and one loop, we see that M_{R_1} is the monoid $\{z, a_v, 2a_v, \ldots\} \cong \mathbb{Z}^+$.

(iii) For the straight line graph A_n ($n \geq 1$), M_{A_n} is generated by the n elements $a_{v_1}, a_{v_2}, \ldots, a_{v_n}$, with relations $a_{v_i} = a_{v_{i+1}}$ for $1 \leq i \leq n-1$. Thus $M_{A_n} = \{z, a_{v_n}, 2a_{v_n}, \ldots\} \cong \mathbb{Z}^+$.

(iv) For the Toeplitz graph E_T of Example 1.3.6, M_{E_T} is the free abelian monoid generated by $\{a_u, a_v\}$, modulo the single relation $a_u = a_u + a_v$.

{def:row-finite-category}

Definition 3.2.4. The category $\mathscr{RG}$ of row-finite graphs is the full subcategory of the category $\mathscr{G}$ (given in Definition 1.6.2) whose objects are the pairs $(E, \mathrm{Reg}(E))$, where E is a row-finite graph. We identify the objects of $\mathscr{RG}$ with the row-finite graphs. Note that the morphisms between two objects E and F of $\mathscr{RG}$ are precisely the complete homomorphisms $\psi\colon E \to F$, that is, the graph homomorphisms $\psi\colon E \to F$ such that ψ^0 and ψ^1 are injective and such that, for each $v \in \mathrm{Reg}(E)$, the map ψ^1 induces a bijection from $s_E^{-1}(v)$ onto $s_F^{-1}(\psi^0(v))$. The subcategory $\mathscr{RG}$ of $\mathscr{G}$ is closed under direct limits, and the assignment $E \mapsto L_K(E)$ $(= C_K^{\mathrm{Reg}(E)}(E))$ extends to a continuous functor from $\mathscr{RG}$ to the category of K-algebras (cf. Proposition 1.6.4).

{mondirectlimit}

Lemma 3.2.5. *The assignment $E \mapsto M_E$ can be extended to a continuous functor from the category $\mathscr{RG}$ of row-finite graphs and complete graph homomorphisms to the category of abelian monoids. Moreover, this assignment commutes with direct limits. It follows that every graph monoid M_E arising from a row-finite graph E is the direct limit of graph monoids corresponding to finite graphs.*

Proof. Every complete graph homomorphism $f\colon E \to F$ induces a natural monoid homomorphism

$$M(f)\colon M_E \to M_F,$$

and so we get a functor M from the category $\mathscr{RG}$ to the category of abelian monoids. The fact that M commutes with direct limits is established in the same way as in Proposition 1.6.4. □

{VLE}

We recall that a unital ring R is called *left hereditary* in case every left ideal of R is projective. We are ready to prove Theorem 1.4.3, slightly restated and expanded here.

Theorem 3.2.6. *Let E be a row-finite graph and K any field. Then there is a natural monoid isomorphism $\mathscr{V}(L_K(E)) \cong M_E$. Moreover, if E is finite, then $L_K(E)$ is hereditary.*

Proof. Because of the defining relations used to build M_E, for each row-finite graph E there is a unique monoid homomorphism $\gamma_E\colon M_E \to \mathscr{V}(L_K(E))$ such that $\gamma_E(a_v) = [L_K(E)v]$. Clearly these homomorphisms induce a natural transformation from the functor M to the functor $\mathscr{V} \circ L$; that is, if $f\colon E \to F$ is a complete graph homomorphism, then the following diagram commutes:

$$\begin{tikzcd} M_E \arrow[r, "\gamma_E"] \arrow[d, "M(f)"'] & \mathscr{V}(L_K(E)) \arrow[d, "\mathscr{V}(L_K(f))"] \\ M_F \arrow[r, "\gamma_F"] & \mathscr{V}(L_K(F)) \end{tikzcd}$$

We need to show that γ_E is a monoid isomorphism for every row-finite graph E. By Lemma 5.3.3 and Corollary 1.6.16, we see that it is enough to show that γ_E is an isomorphism for any finite graph E.

So let E be a finite graph, and let $\{v_1, \dots, v_m\} = \mathrm{Reg}(E)$ (i.e., the non-sinks of E). We start by defining the algebra

$$A_0 = \prod_{v \in E^0} K.$$

In A_0 we clearly have a family $\{p_v : v \in E^0\}$ of orthogonal idempotents such that $\sum_{v\in E^0} p_v = 1$. Now we consider the two finitely generated projective left A_0-modules $P = A_0 p_{v_1}$ and $Q = \oplus_{\{e\in E^1 | s(e)=v_1\}} A_0 p_{r(e)}$. By a beautiful (and delicate) construction of Bergman (see [51, page 38]), there exists an algebra $A_1 := A_0\langle i, i^{-1} : \overline{P} \cong \overline{Q}\rangle$ which admits a universal isomorphism $i\colon \overline{P} := A_1 \otimes_{A_0} P \to \overline{Q} := A_1 \otimes_{A_0} Q$. By examining the construction, we see that this algebra is precisely the algebra $L_K(X_1)$, where X_1 is the graph having $X_1^0 = E^0$, and where v_1 emits the same edges as it does in E, but all other vertices do not emit any edges. More explicitly, the row $(x_e : s(e) = v_1)$ implements an isomorphism $\overline{P} = A_1 p_{v_1} \to \overline{Q} = \oplus_{\{e\in E^1 | s(e)=v_1\}} A_1 p_{r(e)}$, with inverse given by the column $(y_e : s(e) = v_1)^T$, which is clearly universal. By [51, Theorem 5.2], the monoid $\mathscr{V}(A_1)$ is obtained from $\mathscr{V}(A_0)$ by adjoining the relation $[P] = [Q]$. Because in our situation we have that $\mathscr{V}(A_0)$ is the free abelian monoid on generators $\{a_v \mid v \in E^0\}$, where $a_v = [p_v]$, we get that $\mathscr{V}(A_1)$ is given by generators $\{a_v \mid v \in E^0\}$ and a single relation

$$a_{v_1} = \sum_{\{e\in E^1 | s(e)=v_1\}} a_{r(e)}.$$

Now we proceed inductively. For $k \geq 1$, let A_k be the Leavitt path algebra $A_k = L_K(X_k)$, where X_k is the graph with the same vertices as E, but where only the first k vertices $v_1, \dots, v_k$ emit edges, and these vertices emit the same edges as they do in E. We assume by induction that $\mathscr{V}(A_k)$ is the abelian monoid given by generators $\{a_v \mid v \in E^0\}$ and relations

$$a_{v_i} = \sum_{\{e\in E^1 | s(e)=v_i\}} a_{r(e)},$$

for $i = 1, \ldots, k$. Let X_{k+1} be the analogous graph, corresponding to vertices $v_1, \ldots, v_k, v_{k+1}$. Then we have $A_{k+1} = A_k\langle i, i^{-1} : \overline{P} \cong \overline{Q}\rangle$ for $P = A_k p_{v_{k+1}}$ and $Q = \oplus_{\{e \in E^1 | s(e) = v_{k+1}\}} A_k p_{r(e)}$, and so we can again apply [51, Theorem 5.2] to deduce that $\mathscr{V}(A_{k+1})$ is the monoid with the same generators as before, and with relations corresponding to those given in the displayed equations. This establishes the desired isomorphism of monoids.

It follows from a related result of Bergman ([51, Theorem 6.2]) that the global dimension of $L_K(E)$ is at most 1, i.e., that $L_K(E)$ is hereditary. $\square$

{VmonoidforRn}

Example 3.2.7. By Theorem 3.2.6 and Examples 3.2.3(i), we see that, for $n \geq 2$,

$$\mathscr{V}(L_K(R_n)) \cong \{z, a_v, 2a_v, \ldots, (n-1)a_v\}, \text{ with relation } na_v = a_v.$$

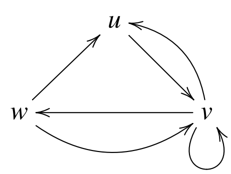
We note that this conclusion regarding the explicit description of the $\mathscr{V}$-monoid of the Leavitt algebras $L_K(1,n) \cong L_K(R_n)$ is quite non-trivial; we do not know of a "direct" or "first principles" proof of this statement.

Further, this property implies that every nonzero finitely generated projective module over $L_K(1,n)$ is necessarily infinite, as the regular module $L_K(1,n)$ itself is infinite.

Of course we may also apply Theorem 3.2.6 to the graphs R_1 and A_n to get the well-known facts that the $\mathscr{V}$-monoid of each of the algebras $L_K(R_1) \cong K[x, x^{-1}]$ and $L_K(A_n) \cong \mathrm{M}_n(K)$ is isomorphic to $\mathbb{Z}^+$.

{Z3graph}

Example 3.2.8. Let E be the graph



Then M_E is the monoid generated by $\{a_u, a_v, a_w\}$, modulo the relations $a_u = a_v$; $a_v = a_u + a_v + a_w$; and $a_w = a_u + a_v$. By some tedious computations, it is not hard to show that $M_E = \{z, a_u, 2a_u, 3a_u\}$. (We will give a streamlined approach to the computation of M_E in Section 6.1.) We note that, as was the case with the M_{R_n} examples ($n \geq 2$), this monoid M_E has the property that $M_E \setminus \{z\}$ is a group (isomorphic to $\mathbb{Z}/3\mathbb{Z}$).

{remarkMEminuszeroisagroup}

Remark 3.2.9. Of all the specific examples of graphs presented in this section, the R_n graphs of Examples 3.2.3(i), and the graph E of Example 3.2.8, are precisely the graphs which have the property that the corresponding Leavitt path algebra is purely infinite simple (by Theorem 3.1.10). That these are also precisely the graphs for which $M_E \setminus \{z\}$ is a group is not coincidental, as we will show in Proposition 6.1.12 below.

We now describe the monoid M_E corresponding to an arbitrary graph E. Indeed, we do more than this: we describe the monoid corresponding to any object (E, X) in the category $\mathscr{G}$ investigated in Chapter 1. As the reader can guess, this assignment will be extended to a continuous functor from $\mathscr{G}$ to the category of abelian monoids. (A complete treatment in the more general framework of separated graphs appears in [31].)

Recall the category $\mathscr{G}$ presented in Definition 1.6.2, whose objects are the pairs (E, X), where E is a directed graph and X is a subset of $\mathrm{Reg}(E)$.

{MEXdefinition}

Definition 3.2.10. Let (E, X) be an object of the category $\mathscr{G}$. We define the *graph monoid* $M(E, X)$ as the abelian monoid given by the set of generators

$$E^0 \sqcup \{q'_Z \mid Z \subseteq s^{-1}(v),\ v \in E^0,\ 0 < |Z| < \infty\},$$

together with the following relations:

(1) $v = \mathbf{r}(Z) + q'_Z$ for $v \in E^0$, $Z \subseteq s^{-1}(v)$, and $0 < |Z| < \infty$, where for a finite subset Y of E^1 we set $\mathbf{r}(Y) := \sum_{e \in Y} r(e)$,
(2) $q'_{Z_1} = \mathbf{r}(Z_2 \setminus Z_1) + q'_{Z_2}$ for finite nonempty subsets Z_1 and Z_2 of $s^{-1}(v)$, $v \in E^0$, with $Z_1 \subsetneq Z_2$, and

(3) $q'_Z = 0$ for $Z = s^{-1}(v)$ when $v \in X$.

Of course the elements q'_Z are intended to represent the equivalence classes of the idempotents $v - \sum_{e\in Z} ee^*$ in $C_K^X(E)$, for Z a finite nonempty subset of $s^{-1}(v)$, $v \in E^0$.

Clearly we see that $M(E,\mathrm{Reg}(E)) = M_E$ when E is a row-finite graph, so these monoids $M(E,X)$ generalize the monoids M_E defined above for row-finite graphs.

In order to simplify notation, we will denote elements in the monoid $M(E,X)$ corresponding to vertices v in E simply by the same symbol v. Of course these correspond to the elements denoted by a_v in the monoid $M_E = M(E,\mathrm{Reg}(E))$. Due to the various decorations of the generators of $M(E,X)$, we think this simplification will be helpful for the reader.

There is some redundancy among these generators and relations. In particular, we could omit the generators q'_Z for nonempty proper subsets Z of $s^{-1}(v)$ for $v \in \mathrm{Reg}(E)$, since relation (2) gives q'_Z in terms of $q'_{s^{-1}(v)}$, and relation (1) for Z follows from the corresponding relation for $s^{-1}(v)$ in light of (2). In general, (1) may be viewed as a form of (2) with $Z_1 = \emptyset$, except that the notation $q'_\emptyset$ would not be well-defined.

Taking into account these comments, an alternative definition of the monoid $M(E,X)$ is as follows: the monoid $M(E,X)$ is the abelian monoid given by the set of generators

$$E^0 \sqcup \{q_v \mid v \in \mathrm{Reg}(E)\setminus X\} \sqcup \{q'_Z \mid Z \subseteq s^{-1}(v),\ v \in \mathrm{Inf}(E),\ 0 < |Z| < \infty\}$$

and the following relations:

(1′) $v = \mathbf{r}(Z) + q'_Z$ for $v \in \mathrm{Inf}(E)$, $Z \subseteq s^{-1}(v)$, and $0 < |Z| < \infty$,
(2′) $q'_{Z_1} = \mathbf{r}(Z_2 \setminus Z_1) + q'_{Z_2}$ for finite nonempty subsets Z_1 and Z_2 of $s^{-1}(v)$, $v \in \mathrm{Inf}(E)$, with $Z_1 \subsetneqq Z_2$,
(3′) $v = \mathbf{r}(s^{-1}(v))$ for each $v \in X$, and
(4′) $v = \mathbf{r}(s^{-1}(v)) + q_v$ for each $v \in \mathrm{Reg}(E)\setminus X$.

Of course the elements q_v for $v \in \mathrm{Reg}(E)\setminus X$ are intended to represent the equivalence classes of the idempotents $v - \sum_{e\in s^{-1}(v)} ee^*$ in $C_K^X(E)$, and correspond to the elements $q_{s^{-1}(v)}$ in the above notation.

Although this second definition might seem more intuitive, the reason to work instead with the first definition becomes apparent when we look for the natural definition of the morphism associated to a map in $\mathscr{G}$. Consider a morphism $\phi : (F,Y) \to (E,X)$ in $\mathscr{G}$. There is a unique monoid homomorphism $M(\phi) : M(F,Y) \to M(E,X)$ sending $v \mapsto \phi^0(v)$ for $v \in F^0$, and sending $q'_Z \mapsto q'_{\phi^1(Z)}$ for nonempty finite sets $Z \subseteq s^{-1}(v)$, $v \in E^0$. The latter assignments are well-defined because if Z is a nonempty finite subset of $s^{-1}(v)$ for some $v \in E^0$, then $\phi^1(Z)$ is a nonempty finite subset of $s^{-1}(\phi^0(v))$. Moreover, the conditions (2) and (3) in Definition 1.6.2 make clear that relation (3) above is preserved by $M(\phi)$. The assignments $(E,X) \mapsto M(E,X)$ and $\phi \mapsto M(\phi)$ define a functor M from $\mathscr{G}$ to the category of abelian monoids. It is easily checked (just as for the functor C_K^X in Proposition 1.6.4) that M is continuous.

{comput VMEX}

We denote by **Mon** the category of abelian monoids.

Theorem 3.2.11. *Let E be an arbitrary graph and K any field. Let $\mathscr{G}$ be the category presented in Definition 1.6.2. For each object (E,X) of $\mathscr{G}$, define*

$$\Gamma(E,X)\colon M(E,X) \to \mathscr{V}(C_K^X(E))$$

to be the monoid homomorphism sending $v \mapsto [v]$ for $v \in E^0$, and $q'_Z \mapsto [v - \sum_{e\in Z} ee^*]$ for each finite nonempty subset $Z \subseteq s^{-1}(v)$. Then $\Gamma : M \to \mathscr{V} \circ C_K$ is an isomorphism of functors $\mathscr{G} \to$ **Mon**.

Proof. It is easily seen that the maps $\Gamma(E,X)$ are well-defined monoid homomorphisms, and that Γ defines a natural transformation from M to $\mathscr{V} \circ C_K$.

We have observed that M is continuous, as is $\mathscr{V} \circ C_K$ (by taking into account that $\mathscr{V}$ is continuous, and invoking Proposition 1.6.4). Thus, by Theorem 1.6.10, we see that it is sufficient to show that $\Gamma(E,X)$ is an isomorphism in the case where E is a finite graph.

We use induction on $|\mathrm{Reg}(E)|$ (i.e., the number of non-sinks in E) to establish the result for finite objects (E,X) in $\mathscr{G}$. The result is trivial if $|\mathrm{Reg}(E)| = 0$ (i.e., if there are no edges in E). Assume that $\Gamma(F,Y)$ is an isomorphism for all finite objects (F,Y) of $\mathscr{G}$ for which $|\mathrm{Reg}(F)| \le n-1$ for some $n \ge 1$, and let (E,X) be a finite object in $\mathscr{G}$ such that $|\mathrm{Reg}(E)| = n$. Select $v \in E^0$ such that $s^{-1}(v) \neq \emptyset$. We can apply induction to the object (F,Y) obtained from (E,X) by deleting all the edges in $s^{-1}(v)$, and leaving intact the structure corresponding to the remaining vertices (keeping $F^0 = E^0$).

Assume first that $v \in X$. Then $M(E,X)$ is obtained from $M(F,Y)$ by factoring out the relation $v = \mathbf{r}(s^{-1}(v))$. On the other hand, the algebra $C_K^X(E)$ is the Bergman algebra obtained from $C_K^Y(F)$ by adjoining a universal isomorphism between the pair of finitely generated projective modules $C_K^Y(F)v$ and $\bigoplus_{e\in s^{-1}(v)} C_K^Y(F)r(e)$.

Accordingly, it follows from [51, Theorem 5.2] that $\mathscr{V}(C_K^X(E))$ is the quotient of $\mathscr{V}(C_K^Y(F))$ modulo the relation $[v] = [\mathbf{r}(s^{-1}(v))]$. Since $\Gamma(F,Y) : M(F,Y) \to \mathscr{V}(C_K^Y(F))$ is an isomorphism by the induction hypothesis, we obtain that $\Gamma(E,X)$ is an isomorphism in this case. (The proof in this case is indeed similar to the proof of Theorem 3.2.6.)

Assume now that $v \notin X$. In this case, $M(E,X)$ is obtained from $M(F,Y)$ by adjoining a new generator q_v and factoring out the relation $v = \mathbf{r}(s^{-1}(v)) + q_v$. On the K-algebra side, we shall make use of another of Bergman's constructions, namely "the creation of idempotents". Write $s^{-1}(v) = \{e_1, \dots, e_m\}$. Let R be the algebra obtained from $C_K^Y(F)$ by adjoining $m+1$ pairwise orthogonal idempotents $g_1, \dots, g_m, q'_v$ with

$$v = g_1 + \cdots + g_m + q'_v.$$

It follows from [51, Theorem 5.1] that $\mathscr{V}(R)$ is the monoid obtained from $\mathscr{V}(C_K^Y(F))$ by adjoining $m+1$ new generators $z_1, \dots, z_m, q''_v$, and factoring out the relation $[v] = \sum_{j=1}^m z_j + q''_v$.

It is then clear that $C_K^X(E)$ is isomorphic to the Bergman algebra obtained from R by consecutively adjoining universal isomorphisms between the left modules generated by the idempotents $r(e_i)$ and g_i, for $i = 1, \dots, m$. It follows that $\mathscr{V}(C_K^X(E))$ is the monoid obtained from $\mathscr{V}(C_K^Y(F))$ by adjoining a new generator q''_v and factoring out the relation $[v] = [\mathbf{r}(s^{-1}(v))] + q''_v$. Therefore, applying the induction hypothesis to (F,Y), we again conclude that $\Gamma(E,X)$ is an isomorphism. $\square$

We can now obtain the description of $\mathscr{V}(L_K(E))$ for an arbitrary graph E. To match the notation utilized in the row-finite case, we set $M_E := M(E, \mathrm{Reg}(E))$. From Definition 3.2.10 we see that M_E is the abelian monoid given by the set of generators

$$E^0 \sqcup \{q'_Z \mid Z \subseteq s^{-1}(v),\ v \in \mathrm{Inf}(E),\ 0 < |Z| < \infty\},$$

and the following relations:

(1) $v = \mathbf{r}(Z) + q'_Z$ for $v \in \mathrm{Inf}(E)$, $Z \subseteq s^{-1}(v)$, and $0 < |Z| < \infty$,
(2) $q'_{Z_1} = \mathbf{r}(Z_2 \setminus Z_1) + q'_{Z_2}$ for finite nonempty subsets Z_1 and Z_2 of $s^{-1}(v)$, $v \in \mathrm{Inf}(E)$, with $Z_1 \subsetneqq Z_2$, and
(3) $v = \mathbf{r}(s^{-1}(v))$ for each $v \in \mathrm{Reg}(E)$.

Corollary 3.2.12. *Let E be an arbitrary graph and K any field. Then $\mathscr{V}(L_K(E)) \cong M_E$.* {mon-for-arbitrary-graphs}

3.3 The exchange property

{exchangeLPAs}

Our next excursion into the idempotent structure of Leavitt path algebras brings us to the notion of an *exchange ring*. The exchange property for modules was introduced by Crawley and Jónsson in [62]. Roughly speaking, it is the suitable condition which yields a version of the Krull Schmidt Theorem even in situations where the modules do not decompose as direct sums of indecomposables. Following [140], the (unital) ring R is an *exchange ring* if ${}_RR$ has the property that for every left R-module M and any two decompositions of M as $M = M' \oplus N$ and $M = \bigoplus_{i=1}^n M_i$, for which $M' \cong {}_RR$, then there exist submodules $M'_i \subseteq M_i$ such that $M = M' \oplus (\bigoplus_{i=1}^n M'_i)$.

A multiplicative characterization of unital exchange rings was obtained independently by Goodearl [80] and by Nicholson [110]. Concretely, R is an exchange ring if and only if for every element $a \in R$ there exists an idempotent $e \in R$ such that $e \in Ra$ and $1 - e \in R(1-a)$. The appropriate generalization of the notion of exchange ring to not-necessarily unital rings was provided in [19]: R is exchange in case there

{exchangeexamples}

is a unital ring S containing R as an ideal, for which, for every $x \in R$, there exists $e = e^2 \in R$ for which $e - x \in S(x - x^2)$.

Many classes of rings are exchange rings. In that regard, for our purposes the next three results are key. Because the exchange property in a ring can be formulated as the existence of a solution to a specific type of equation in the ring, and because it is easy to show that any finite dimensional matrix algebra $\mathrm{M}_n(K)$ is an exchange ring, we get the following.

{pisisexchange}

Proposition 3.3.1. *The direct limit of exchange rings is an exchange ring. In particular, let K be a field. Then any locally matricial K-algebra is an exchange ring. Specifically, $\mathrm{M}_\Lambda(K)$ is an exchange ring for any set Λ.*

{Laurentnotexchange}

In the current context, the most important class of exchange rings is the following.

Theorem 3.3.2. *[21, Corollary 1.2] Let R be a purely infinite simple ring. Then R is an exchange ring.*

On the other hand, the K-algebra $R = K[x, x^{-1}]$ is not an exchange ring, as follows. Since the only idempotents in R are 0 and 1, and $a = 1 + x + x^2$ is not invertible in R, and $1 - a = -x - x^2$ is also not invertible in R, the exchange condition fails for the element a. More generally,

{IdealExchange}

Lemma 3.3.3. *For any field K, and for any set Λ, the matrix algebra $\mathrm{M}_\Lambda(K[x, x^{-1}])$ is not an exchange ring.*

We will need the following additional property of exchange rings (which we state here in less than its full generality).

{cycleswithexits}

Theorem 3.3.4. *([19, Lemma 3.1(a) and Theorem 2.2]) Let R be a ring and let I be an ideal of R. Then R is an exchange ring if and only if I and R/I are exchange rings, and the natural map $\mathscr{V}(R) \mapsto \mathscr{V}(R/I)$ is surjective.*

Having given this background information, we now focus on our goal of identifying those Leavitt path algebras $L_K(E)$ which are exchange rings. Recall that for $X \subseteq E^0$, we denote by $\overline{X}$ the hereditary saturated closure of X.

{CondKeveryCondL}

Proposition 3.3.5. *Let E be a graph and suppose that c is a cycle with exits such that for every $v \in c^0$ there is only one cycle based at v. Let $v \in c^0$, and consider the set*

$$X = \{w \in E^0 \mid v \geq w \text{ and } w \not\geq v\}.$$

Then X is a hereditary subset of E^0 and $H := \overline{X}$ is a hereditary saturated subset of E^0 for which $c^0 \cap H = \emptyset$. In particular, c is a cycle without exits in the quotient graph E/H.

Proof. Clearly X is a hereditary subset of E^0 with $X \cap c^0 = \emptyset$. Since the hypotheses yield that no vertex in $\overline{X} \setminus X$ can be contained in a cycle, we see that $\overline{X} \cap c^0 = \emptyset$ as well. □

{exchangeImpliesL}

Lemma 3.3.6. *Let E be an arbitrary graph. If E does not satisfy Condition (K), then there exists a hereditary saturated subset H in E^0 such that E/H does not satisfy Condition (L).*

Proof. Since E does not satisfy Condition (K), then there exists $u \in E^0$ which is the base of a unique closed simple path, hence of a unique cycle; denote it by c. As in Proposition 3.3.5, the hereditary set $X = \{w \in E^0 \mid v \geq w, w \not\geq v\}$ has the property that $\overline{X} \cap c^0 = \emptyset$. Set $H := \overline{X}$. Then c is a cycle without exits in E/H, so that E/H does not satisfy Condition (L). □

Lemma 3.3.7. *Let E be an arbitrary graph and K any field. If $L_K(E)$ is an exchange ring, then E satisfies Condition (L).*

Proof. Suppose on the contrary that E does not satisfy Condition (L). Then there exists a cycle c in E which has no exits. Denote by I the ideal of $L_K(E)$ generated by c^0. Then Lemma 2.7.1 gives that I is isomorphic to $\mathrm{M}_\Lambda(K[x, x^{-1}])$ for some set Λ, which is not an exchange ring by Lemma 3.3.3. But every ideal of an exchange ring is exchange (Theorem 3.3.4), so I must be exchange, a contradiction. □

{limitswithK}

Lemma 3.3.7, together with relationships between Condition (K) and Condition (L), will help us reach the main goal in this section, namely, to show that the exchange Leavitt path algebras are precisely those arising from graphs having Condition (K). One of the fundamental steps in the proof of that result is the following graph theoretic property.

Lemma 3.3.8. *Let E be a graph satisfying Condition (K), and let X be a finite subgraph of E. Then there is a finite complete subgraph F of E, containing X, such that F satisfies Condition (K).*

Proof. By Theorem 1.6.10 there is a finite complete subgraph G of E such that $X \subseteq G$. The goal is to embed G in a finite complete subgraph F of E such that F satisfies Condition (K). Let $\sim_E$ be the symmetric closure of the relation $\geq$ on E^0: that is, for $v,w \in E^0$, $v \sim_E w$ in case either $v = w$, or there is a closed path in E containing both v and w.

We claim that if $v \sim_E w$ then $|CSP_E(v)| > 1$ if and only if $|CSP_E(w)| > 1$. Indeed, it suffices to show one of the implications. Assume that $|CSP_E(v)| > 1$ and that $v \neq w$ and $v \sim_E w$. Since $v \sim_E w$, one can easily show that there is a closed simple path $e_1e_2\cdots e_n \in CSP_E(v)$ such that $s(e_i) = w$ for exactly one i with $1 < i \leq n$. By hypothesis, there is a distinct path $\gamma = f_1f_2\cdots f_m$ in $CSP_E(v)$. If γ^0 does not contain w, then $e_ie_{i+1}\cdots e_ne_1e_2\cdots e_{i-1}$ and $e_ie_{i+1}\cdots e_n\gamma e_1e_2\cdots e_{i-1}$ are distinct elements of $CSP_E(w)$. If γ^0 contains w, and $e_1e_2\cdots e_{i-1} \neq f_1f_2\cdots f_{i-1}$, then taking j such that $s(f_j) = w$, we obtain that $e_ie_{i+1}\cdots e_nf_1f_2\cdots f_{j-1}$ and $e_ie_{i+1}\cdots e_ne_1e_2\cdots e_{i-1}$ are distinct elements of $CSP_E(w)$. Similarly, if γ^0 contains w, $f_{m-(n-i)}\cdots f_m \neq e_i\cdots e_n$, and j is as above, then $f_j\cdots f_me_1\cdots e_{i-1}$ and $e_i\cdots e_ne_1\cdots e_{i-1}$ are distinct elements of $CSP_E(w)$. Finally if both $e_1e_2\cdots e_{i-1} = f_1f_2\cdots f_{i-1}$ and $f_{m-(n-i)}\cdots f_m = e_i\cdots e_n$, then $e_ie_{i+1}\cdots e_ne_1e_2\cdots e_{i-1}$ and $f_if_{i+1}\cdots f_{j-1}$ are two different elements of $CSP_E(w)$, where j is the first index for which $j > i$ and $s(f_j) = w$. This establishes the claim.

There is a finite number of cycles $c_1,\dots,c_r$ in G, based at $v_1,\dots,v_r$ respectively, for which $|CSP_G(v_i)| = 1$ for all i. We form a new graph G' by adding to G the vertices and edges in a closed simple path $\gamma_i \neq c_i$ based at v_i, for $i = 1,\dots,r$. Let F be the completion of G' in E, so that F is formed by adding the edges departing from vertices $v \in (G')^0$ such that $v \in \mathrm{Reg}(E)$ and $s_{G'}^{-1}(v) \neq \emptyset$, together with the corresponding range vertices (in case these edges were not already in G').

We show that F satisfies Condition (K). Note that, for $v \in (G')^0$, either $|CSP_{G'}(v)| \geq 2$ or $|CSP_{G'}(v)| = 0$, as follows. If $v \in G^0$ and $|CSP_G(v)| = 1$ then $v \in \cup_{i=1}^r c_i^0$ and thus $|CSP_{G'}(v)| \geq 2$. If $v \in \gamma_i^0$ for some i then $v \sim_{G'} v_i$ and so $|CSP_{G'}(v)| \geq 2$, because $|CSP_{G'}(v_i)| \geq 2$, using the observation above. Finally if $v \in G^0$, $|CSP_G(v)| = 0$ and $|CSP_{G'}(v)| \neq 0$, then $v \sim_{G'} v_i$ for some i, and so $|CSP_{G'}(v)| \geq 2$.

Since all vertices in $F^0 \setminus (G')^0$ are sinks in F, it therefore suffices to show that $|CSP_F(w)| \neq 1$ for all $w \in (G')^0$ having $|CSP_{G'}(w)| = 0$. Suppose that there is a cycle $c = e_1e_2\cdots e_m$ based at w in F and that $|CSP_{G'}(w)| = 0$. If $w \notin G^0$, then $w \in \gamma_i^0$ for some i, and so $|CSP_{G'}(w)| \geq 2$, because $w \sim_{G'} v_i$. Therefore, $w \in G^0$. Let p be the smallest index with $e_p \notin G^1$. Then we have $s(e_p) \in G^0$. Since G is complete, the vertex $s(e_p)$ is a sink in G, and is not a sink in G'. It follows that $s(e_p) \in \gamma_i^0$ for some i, and so $|CSP_{G'}(s(e_p))| \geq 2$ as before. Hence

$$|CSP_F(s(e_p))| \geq |CSP_{G'}(s(e_p))| \geq 2.$$

Since $w \sim_F s(e_p)$, we get that $|CSP_F(w)| \geq 2$, as desired. $\square$

{FinIdeImpliesK}

Lemma 3.3.9. *Let E be a graph and K a field for which the ideal lattice $\mathscr{L}(L_K(E))$ of $L_K(E)$ is finite. Then E satisfies Condition* (K).

Proof. By Lemma 3.3.6, it suffices to show that the quotient graph E/H satisfies Condition (L) for every $H \in \mathscr{H}_E$. Suppose on the contrary that there exists a hereditary saturated subset H of E^0 such that E/H does not satisfy Condition (L). This means that E/H contains a cycle without exits, say c. Since $L_K(E/H) \cong L_K(E)/I(H \cup B_H^H)$ (see Theorem 2.4.15) has a finite number of ideals, we may assume that $H = \emptyset$.

Denote by I the ideal generated by c^0. By Lemma 2.7.1 the ideal $\bar{I}$ is isomorphic to $\mathrm{M}_\Lambda(K[x,x^{-1}])$ for some set Λ, so that I has infinitely many ideals. Since I is a graded ideal, the ideals of I are also ideals of $L_K(E)$ (by Lemma 2.9.12), so $L_K(E)$ has infinitely many ideals, a contradiction. $\square$

We note that the converse of Lemma 3.3.9 is clearly not true, with any graph having infinitely many vertices and no edges providing a counterexample.

Although at first glance the following result might seem to be quite limited in its scope, it will indeed provide the basis of the key theorem of this section.

{CondKfinite}

Proposition 3.3.10. *Let E be a row-finite graph for which the ideal lattice $\mathscr{L}(L_K(E))$ is finite. Then $L_K(E)$ is an exchange ring.*

Proof. Observe first that Lemma 3.3.9 implies that the graph E satisfies Condition (K). Since $\mathscr{L}(L_K(E))$ is finite, we can build an ascending chain of ideals

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = L_K(E)$$

such that, for every $i \in \{1, \dots, n-1\}$ the ideal I_i is maximal among the ideals of $L_K(E)$ contained in I_{i+1}. Now we prove the result by induction on n.

If $n = 1$, then $L_K(E)$ is a simple ring. By the Dichotomy Principle 3.1.11, $L_K(E)$ is either locally matricial or purely infinite simple. But then Proposition 3.3.1 together with Theorem 3.3.2 imply that $L_K(E)$ is an exchange ring.

Now suppose the result holds for any Leavitt path algebra in which there are a finite number of ideals, and a maximal chain of two-sided ideals has length $k < n$. Since the graph satisfies Condition (K) (by Lemma 3.3.9), Proposition 2.9.9 can be applied to get that every ideal of $L_K(E)$ is graded. Since E is row-finite, by Theorem 2.5.9 there exist $H_i \in \mathscr{H}_E$, for $i \in \{1, \dots, n\}$ such that:

(1) $I_i = I(H_i)$ for every $1 \le i \le n$,
(2) $H_i \subsetneqq H_{i+1}$ for every $i \in \{1, \dots, n-1\}$, and
(3) for every $i \in \{1, \dots, n-1\}$, there is no hereditary and saturated set T such that $H_i \subsetneqq T \subsetneqq H_{i+1}$.

At this point we may apply the induction hypothesis to I_{n-1}, which is the Leavitt path algebra of a row-finite graph by Proposition 2.5.19, and has finitely many ideals by Corollary 2.9.12. Thus we have that I_{n-1} is an exchange ring. But $L_K(E)/I_{n-1} \cong L_K(E/H_{n-1})$ is a simple Leavitt path algebra (it is simple by the maximality of I_{n-1} inside $L_K(E)$, and it is a Leavitt path algebra by Corollary 2.4.13(i)). By the first step of the induction hypothesis, $L_K(E)/I_{n-1} \cong L_K(E/H_{n-1})$ is an exchange ring. Since $\mathscr{V}(L_K(E/H_{n-1}))$ is generated by the isomorphism classes arising from its vertices (by Theorem 3.2.6), we obviously have that the natural map $\mathscr{V}(L_K(E)) \to \mathscr{V}(L_K(E)/I_{n-1})$ is surjective. So Theorem 3.3.4 can be applied, and finishes the proof. $\square$

{charCondK}

We are now in position to present the main result of the section.

Theorem 3.3.11. *Let E be an arbitrary graph and K any field. Then the following are equivalent.*

(1) *$L_K(E)$ is an exchange ring.*
(2) *E/H satisfies Condition* (L) *for every hereditary saturated subset H of E^0.*
(3) *E satisfies Condition* (K).
(4) *$\mathscr{L}_{gr}(L_K(E)) = \mathscr{L}(L_K(E))$; that is, every two-sided ideal of $L_K(E)$ is graded.*
(5) *The graphs E_H and E/H both satisfy Condition* (K) *for every hereditary saturated subset H of E^0.*
(6) *The graphs E_H and E/H both satisfy Condition* (K) *for some hereditary saturated subset H of E^0.*

Proof. (1) $\Rightarrow$ (2). Consider a hereditary saturated subset $H \in \mathscr{H}_E$. By Corollary 2.4.13(ii) we have that $L_K(E)/I(H \cup B_H^H)$ is isomorphic to the Leavitt path algebra $L_K(E/H)$. Since the quotient of an exchange ring by an ideal is an exchange ring (Theorem 3.3.4), Lemma 3.3.7 applies to get (2).

(2) $\Rightarrow$ (3) is Lemma 3.3.6.

(3) $\Rightarrow$ (1). By Lemma 3.3.8 and Theorem 1.6.10, we can write

$$L_K(E) \cong \varinjlim_{F \in \mathscr{F}} C_K^{X_F}(F) \cong \varinjlim_{F \in \mathscr{F}} L_K(F(X_F)),$$

where $\mathscr{F}$ is the family of finite complete subgraphs of E satisfying Condition (K), and $F(X_F)$ is the finite graph obtained from F by applying Theorem 1.5.18. Recalling Definition 1.5.16, we see that the graph $F(X_F)$ satisfies Condition (K) if F satisfies Condition (K), because both graphs contain the same closed paths, and the new vertices added to F in order to form $F(X_F)$ are sinks. Since the class of exchange rings is closed under direct limits (Proposition 3.3.1), it suffices to prove the result for finite graphs.

Let E be a finite graph with Condition (K). Then all the ideals of $L_K(E)$ are graded by Proposition 2.9.9, and so, by Theorem 2.5.9, the lattice of ideals of $L_K(E)$ is finite. The result follows therefore from Proposition 3.3.10.

(3) $\Leftrightarrow$ (4) is Proposition 2.9.9.

(3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). It is easy to see that for every $H \in \mathscr{H}_E$ we have $\mathrm{CSP}_E(v) = \mathrm{CSP}_{E_H}(v)$ for all $v \in H$, and $\mathrm{CSP}_E(w) = \mathrm{CSP}_{E/H}(w)$ for all $w \in E^0 \setminus H$. This gives the result. □

{CharactCondKself}

We close the section by giving another characterization of the exchange Leavitt path algebras. Recall that an ideal I of $L_K(E)$ is self-adjoint in case $\alpha^* \in I$ for every $\alpha \in I$.

Proposition 3.3.12. *Let E be an arbitrary graph and K any field. Then E satisfies Condition* (K) *if and only if every two-sided ideal of $L_K(E)$ ideal is self adjoint.*

Proof. Suppose E satisfies Condition (K). Then by Theorem 3.3.11 every ideal of $L_K(E)$ is graded, and by Corollary 2.4.10 every such ideal is self adjoint.

Conversely, suppose every ideal of $L_K(E)$ is self-adjoint. Let H be a hereditary saturated subset of E^0. We will show that E/H satisfies Condition (L), and thus Theorem 3.3.11 will yield the desired result. On the contrary, if E/H does not satisfy Condition (L), then there exists a cycle without exits c in E/H. By Lemma 2.7.1 the ideal I of $L_K(E/H)$ generated by c^0 is isomorphic to $\mathrm{M}_\Lambda(K[x,x^{-1}])$ for some set Λ. By Corollary 2.4.10 $I(H \cup B_H^H)$ is a self adjoint ideal, hence the hypothesis implies that every ideal of $L_K(E)/I(H \cup B_H^H)$ is self adjoint. Since $L_K(E/H) \cong L_K(E)/I(H \cup B_H^H)$ (by Corollary 2.4.13(ii)), the Leavitt path algebra $L_K(E/H)$ satisfies that each of its ideals is self adjoint. But every ideal of I is an ideal of $L_K(E/H)$ (by Lemma 2.9.12), hence every ideal of I, and consequently of $\mathrm{M}_\Lambda(K[x,x^{-1}])$, is self adjoint. But this is a contradiction, as can easily be seen by using the same ideas as presented in the $|\Lambda| = 1$ case given prior to Corollary 2.4.10. □

{vNregularsection}

3.4 Von Neumann regularity

In this section we will show that the Leavitt path algebras arising from acyclic graphs are precisely the von Neumann regular Leavitt path algebras. Subsequently, we will give an explicit description of the largest von Neumann regular ideal of a Leavitt path algebra.

Recall that an element a in a ring R is said to be *von Neumann regular* if there exists $b \in R$ such that $aba = a$. The ring R is called a *von Neumann regular ring* if every element in R is von Neumann regular. Note that in this situation the element $x = ba$ is idempotent. Indeed, von Neumann regular rings are characterized as those rings for which every finitely generated left ideal is generated by an idempotent, so that the topic of von Neumann regularity fits well with the theme of this chapter.

{vNr}

Theorem 3.4.1. *Let E be an arbitrary graph and K any field. Then the following are equivalent.*

(1) *$L_K(E)$ is von Neumann regular.*
(2) *E is acyclic.*
(3) *$L_K(E)$ is locally K-matricial.*

Proof. (1) $\Rightarrow$ (2). Suppose that there exists a cycle c in E; denote $s(c)$ by v. We will prove that the element $v - c$ cannot be von Neumann regular. Suppose otherwise that there exists an element $\beta \in L_K(E)$ such that $(v-c)\beta(v-c) = (v-c)$. Replacing β by $v\beta v$ if necessary, there is no loss of generality in assuming that $\beta = v\beta v$. We write β as a sum of homogeneous elements $\beta = \sum_{i=m}^{n} \beta_i$, where $m,n \in \mathbb{Z}$, $\beta_m \neq 0$, $\beta_n \neq 0$, and $\deg(\beta_i) = i$ for all nonzero β_i with $m \leq i \leq n$. Since $\deg(v) = 0$, we have $v\beta_i v = \beta_i$ for all i. Then

$$v - c = (v-c)\left(\sum_{i=m}^{n} \beta_i\right)(v-c).$$

Equating the lowest degree terms on both sides, we get $\beta_m = v$. Since $\deg(v) = 0$, we conclude that $m = 0$, and that $\beta_0 = v$. Thus $\beta = \sum_{i=0}^{n} \beta_i$. Suppose $\deg(c) = s > 0$. By again equating terms of like degree in the displayed equation, we see that $\beta_i = 0$ whenever i is nonzero and not a multiple of s, so that

$$\sum_{i=m}^{n} \beta_i = v + \sum_{t=1}^{n/s} \beta_{ts}.$$

So upon rewriting the equation above, we have

$$v - c = (v-c)v(v-c) + (v-c)\left(\sum_{t=1}^{k} \beta_{ts}\right)(v-c), \text{ which gives } 0 = -c + c^2 + (v-c)\left(\sum_{t=1}^{k} \beta_{ts}\right)(v-c).$$

By equating the degree s components on both sides we obtain $\beta_s = c$. Similarly, by equating the degree $2s$ components, we get $0 = c^2 - c\beta_s - \beta_s c + \beta_{2s}$. But substituting $\beta_s = c$ yields $\beta_{2s} = c^2$, and continuing in this manner we get $\beta_{ts} = c^t$, for every $t \in \mathbb{N}$. But this is not possible, as $\beta_{ts} = 0$ for $t > n/s$.

(2) $\Rightarrow$ (3) is Proposition 2.6.20.

(3) $\Rightarrow$ (1). It is well known that every matricial K-algebra is a von Neumann regular ring, and hence easily so too is any direct union of such algebras. □

{BMradicalofLaurentiszero}

Remark 3.4.2. It was established in [14, Theorem 1] that in the context of Leavitt path algebras, the three properties von Neumann regularity, π-regularity, and strong π-regularity are equivalent.

Every ring R contains a largest von Neumann regular ideal (see e.g., [78, Proposition 1.5]), which we denote here by $U(R)$. Specifically, $U(R)$ is an ideal of R, which is von Neumann regular as a ring, with the property that if J is any ideal of R which is von Neumann regular as a ring, then $J \subseteq U(R)$. This ideal is often called the *Brown-McCoy radical* of R. It is not hard to show that $R/U(R)$ contains no nonzero von Neumann regular ideals.

Remark 3.4.3. It is clear that if R is matricial, then $U(R) = R$. On the other hand, using an idea which amounts to a special case of the idea used in the proof of Theorem 3.4.1, it is easy to show that $U(K[x,x^{-1}]) = \{0\}$. This in turn can be used to show that $U(R) = \{0\}$ for any K-algebra R of the form which arises in Theorem 2.7.3. In particular, $U(I(P_c(E)) = \{0\}$, where $P_c(E)$ is the set of vertices in E^0 which lie in a cycle without exits (cf. Notation 2.2.4).

{vNrIsGraded}

We begin by showing that every von Neumann regular ideal of a Leavitt path algebra is graded.

Lemma 3.4.4. *Let E be an arbitrary graph and K any field. Then every von Neumann regular ideal of $L_K(E)$ is a graded ideal.*

Proof. Clearly the result holds for the zero ideal, so let I be a nonzero von Neumann regular ideal of $L_K(E)$. By the Structure Theorem for Ideals 2.8.10 we have that $I = I(H \cup S^H \cup P_C)$, where H, S^H and P_C are as described therein. If I were not graded, then necessarily $P_C \neq \emptyset$, and the ideal $I/I(H \cup S^H)$ of the Leavitt path algebra $L_K(E)/I(H \cup S^H)$ would be contained in $I(P_c(E/(H,S)))$. By Theorem 2.7.3, $L_K(E)/I(H \cup S^H)$ would then be isomorphic to a subalgebra of $\bigoplus_{\overline{c}\in\overline{C}} M_{\Lambda_{\overline{c}}}(p_c(x)K[x,x^{-1}])$. But algebras of the latter form contain no nonzero von Neumann regular elements, which contradicts the von Neumann regularity of $I/I(H \cup S^H)$ (which is a consequence of it being the quotient of the von Neumann regular ring I). Therefore I must be graded, as required. □

In the context of Leavitt path algebras, we are able to describe the Brown-McCoy radical of $L_K(E)$ in terms of a specific subset of E^0.

{PncDefinition}

Definition 3.4.5. For a graph E, we denote by $P_{nc}(E)$ the set of all vertices in E^0 which do not connect to any cycle in E.

It is clear from the definition that $P_{nc}(E)$ is both hereditary and saturated. As well, we see immediately that $P_l(E) \subseteq P_{nc}(E)$.

{biggestVNR}

Proposition 3.4.6. *Let E be an arbitrary graph and K any field. Let H denote $P_{nc}(E)$. Then $U(L_K(E)) = I(H \cup B_H^H)$.*

Proof. We first establish that $I(H \cup B_H^H)$ is a von Neumann regular ideal of $L_K(E)$. Indeed, by Theorem 2.5.22, this ideal (viewed as a ring) is isomorphic to the Leavitt path algebra of the graph ${}_{(H,B_H)}E$. Since none of the vertices in H connects to a cycle in E then it is straightforward from the definition of ${}_{(H,B_H)}E$ that this graph is necessarily acyclic. So by Theorem 3.4.1, $L_K({}_{(H,B_H)}E)$, and hence $I(H \cup B_H^H)$, is a von Neumann regular ring. Thus $I(H \cup B_H^H) \subseteq U(L_K(E))$.

To establish the reverse inclusion, we first invoke Lemma 3.4.4 to get that $U(L_K(E))$ is a graded ideal. So by the Structure Theorem for Graded Ideals 2.5.8 we have $U(L_K(E)) = I(H' \cup S^{H'})$ for some $S \subseteq B_{H'}$, where $H' = U(L_K(E)) \cap E^0$. We claim that $H' \subseteq H$; to establish the claim, we consider the ideal $I(H')$. By Theorem 2.5.19, $I(H')$ is isomorphic to $L_K({}_{H'}E)$. On the other hand, $I(H') \subseteq I(H' \cup S^{H'}) = U(L_K(E))$, so that $I(H')$ is von Neumann regular (as it is an ideal of the von Neumann regular ring $U(L_K(E))$). Thus Theorem 3.4.1 applies to yield that ${}_{H'}E$ is acyclic and, consequently, that H' has no cycles. By definition, this gives that $H' \subseteq P_{nc}(E) = H$, which establishes the claim.

Now, use that $H' \subseteq H$ implies $B_{H'} \subseteq H \cup B_H$ and, consequently, that $S^{H'} \subseteq H \cup B_H^H$, to get $U(L_K(E)) = I(H' \cup S^{H'}) \subseteq I(H \cup B_H^H)$. □

{vNrReestablished}

Remark 3.4.7. Since $P_{nc}(E) \in \mathscr{H}_E$, we have $I(P_{nc}(E)) = L_K(E)$ if and only if $P_{nc}(E) = E^0$. But by definition, the latter statement is equivalent to E being acyclic. So Proposition 3.4.6 can be viewed as a generalization of Theorem 3.4.1.

We recall the following subset of E^0 given in Definitions 2.6.1: the set of line points of E, denote $P_l(E)$, is the set of those vertices of E which connect neither to bifurcations nor to cycles. In particular, $P_l(E)$ contains all the sinks of E. Additionally, by definition we have $P_l(E) \subseteq P_{nc}(E)$, so that $I(P_l(E)) \subseteq I(P_{nc}(E))$ for any graph E.

{socleequalsBrownMcCoyforEfinite}

Corollary 3.4.8. *Let E be a finite graph and K any field. Then* $\mathrm{Soc}(L_K(E)) = U(L_K(E))$*; that is, the socle coincides with the Brown-McCoy radical for the Leavitt path algebra of a finite graph.*

Proof. Using Theorem 2.6.14 and Proposition 3.4.6, we need only show that $I(P_l(E)) = I(P_{nc}(E))$. As noted immediately above, the containment $I(P_l(E)) \subseteq I(P_{nc}(E))$ holds for any graph E. Conversely, recall that for a finite graph E, each vertex connects either to a cycle or to a sink. So $v \in P_{nc}(E)$ and the finiteness of E implies that there is an integer N for which every path starting at v ends in a sink in at most N steps. But then using the (CK2) relation as many times as necessary at each of these N steps (together with the finiteness of the graph), we see that v is in the saturated closure of the sinks of E, and hence $v \in I(P_l(E))$. So $I(P_{nc}(E)) \subseteq I(P_l(E))$, completing the proof. □

{PncExample}

Example 3.4.9. In the particular case of the Toeplitz algebra $\mathscr{T}_K = L_K(E_T)$ (see Example 1.3.6), the largest von Neumann regular ideal $U(\mathscr{T}_K)$ is the ideal generated by the sink, which by Corollary 3.4.8 is precisely $\mathrm{Soc}(\mathscr{T}_K)$.

{RemarkVNR}

Remark 3.4.10. Corollary 3.4.8 does not extend to infinite graphs, not even to infinite acyclic graphs. For example, let E denote the graph

$$\begin{tikzcd}
\bullet^{v_1} \arrow[r] \arrow[d] & \bullet^{v_2} \arrow[r] \arrow[d] & \bullet^{v_3} \arrow[r] \arrow[d] & \bullet^{v_4} \arrow[r, dotted] \arrow[d, dotted] & {} \\
\bullet^{w_1} & \bullet^{w_2} & \bullet^{w_3} & {} &
\end{tikzcd}$$

Then $P_l(E) = \{w_n\}_{n \in \mathbb{N}}$. It is easy to see that $I(P_l(E))$ is not all of $L_K(E)$ (since $v_i \notin I(P_l(E))$ for all $i \in \mathbb{N}$), so that by Theorem 2.6.14 we have $\mathrm{Soc}(L_K(E)) \neq L_K(E)$. But by Theorem 3.4.1 we have that $L_K(E)$ is von Neumann regular, so that $U(L_K(E)) = L_K(E)$.

3.5 Primitive non-minimal idempotents

{primnonminimalidemptssectio

We continue our description of idempotent-related topics by considering the primitive, non-minimal idempotents of $L_K(E)$. We focus first on the ideal generated by these elements; this ideal will play a role similar to that played by $\mathrm{Soc}(L_K(E))$, but with respect to the vertices which lie on cycles without exits. We will utilize the following general ring-theoretic result.

{primitivecharacterization}

Proposition 3.5.1. *([96, Proposition 21.8]) Let e be an idempotent in a (not-necessarily unital) ring R. The following are equivalent.*

(1) *Re is an indecomposable right R-module (equivalently, Re is an indecomposable left R-module).*
(2) *eRe is a ring without nontrivial idempotents.*
(3) *e cannot be decomposed as $a+b$, where a,b are nonzero orthogonal idempotents in R.*

A nonzero idempotent of R which satisfies these conditions is called a primitive *idempotent.*

Clearly (by (i)) any minimal idempotent of R (Definitions 2.6.7) is necessarily primitive.

primitivegraphcharacterization}

Proposition 3.5.2. *Let E be an arbitrary graph and K any field. Let $v \in E^0$. Then v is a primitive idempotent of $L_K(E)$ if and only if $T(v)$ has no bifurcations.*

Proof. Suppose that $T(v)$ has bifurcations; say $T(v)$ has its first bifurcation at w, with μ being the shortest path which connects v to w. Since there are no bifurcations in μ, the (CK2) relation at each non-final vertex of μ yields $\mu\mu^* = v$. Hence we get $L_K(E)v = L_K(E)\mu\mu^*$. Let e and f be two different edges emitted by w; then $ee^* \neq w$ (as otherwise $w - ee^* = 0$, which on right multiplication by f would give $f = 0$), and so by Proposition 3.0.1(i) we get $L_K(E)w = L_K(E)ee^* \oplus L_K(E)(w - ee^*)$ is a decomposition of the desired type.

Conversely, suppose that $T(v)$ has no bifurcations. Two cases can occur. First, suppose $T(v)$ does not contain vertices in cycles. In this case, $v \in P_l(E)$, which means that v is minimal by Proposition 2.6.11, and so necessarily primitive. On the other hand, suppose $T(v) \cap P_c(E) \neq \emptyset$. Since $T(v)$ has no bifurcations, there can be only one cycle $c \in L_K(E)$ such that $T(v) \cap c^0 \neq \emptyset$, which in addition has no exits. Furthermore, every vertex of $T(v)$ is either in c^0 or connects to a vertex w in c^0 via a path μ, where there are no bifurcations at any of the vertices of μ. Since then $\mu\mu^* = v$, we get $L_K(E)v \cong L_K(E)w$ as left $L_K(E)$-modules by Proposition 3.0.1(ii). Since w is in a cycle without exits, by Proposition 2.2.7 we have $wL_K(E)w \cong K[x,x^{-1}]$, which is a ring without nontrivial idempotents. Now Proposition 3.5.1 gives that w and v are both primitive, and completes the proof. $\square$

Remark 3.5.3. If $vL_K(E)v$ is a ring with no nontrivial idempotents, then v is a primitive idempotent and, as a consequence of the proof of Proposition 3.5.2, we have either $vL_K(E)v \cong K$ (if v is minimal) or $vL_K(E)v \cong K[x,x^{-1}]$ (if it is not).

We have found a close relationship between the primitive and the minimal vertices of the Leavitt path algebra of any graph: the minimal vertices are those whose trees do not contain bifurcations nor connect to cycles, while the primitive vertices see this second condition suppressed. In particular,

{trulyprimitivecorners}

Remark 3.5.4. A vertex $v \in E^0$ is a primitive non-minimal idempotent of $L_K(E)$ if and only if $vL_K(E)v \cong K[x,x^{-1}]$. In particular, the vertices in $P_c(E)$ are primitive non-minimal.

Proposition 3.5.2 provides us with a tool to distinguish between those cycles with exits and those cycles without exits in a graph, giving us a characterization of Condition (L) in terms of primitive vertices.

{(L)characterization}

Corollary 3.5.5. *Let E be an arbitrary graph and K any field. Then E satisfies Condition* (L) *if and only if every primitive vertex in $L_K(E)$ is minimal.*

In particular, if every vertex in $L_K(E)$ is infinite, then E satisfies Condition (L).

Proof. By Proposition 3.5.2 and Remark 3.5.4, $L_K(E)$ contains a primitive non-minimal vertex if and only if E contains a cycle without exits. The additional statement follows vacuously. $\square$

In Theorem 3.5.7 we extend Corollary 3.5.5 from the primitive non-minimal vertices to the primitive non-minimal idempotents of a Leavitt path algebra. As one consequence, this will show (Corollary 3.5.8) that Condition (L) is a ring isomorphism invariant of Leavitt path algebras, in the sense that if E,F are two graphs such that $L_K(E) \cong L_K(F)$ as rings, then E satisfies Condition (L) if and only if F does as well.

{primitivegoestovertex}

Proposition 3.5.6. *Let E be an arbitrary graph and K any field. If $z \in L_K(E)$ is a primitive idempotent and we can write $\alpha z \beta = kv$ for $\alpha, \beta \in L_K(E)$, $k \in K^\times$, and $v \in E^0$, then $L_K(E)z \cong L_K(E)v$. If, moreover, z is primitive non-minimal, then $zL_K(E)z \cong K[x,x^{-1}]$.*

Proof. We may assume $\alpha = v\alpha$ and $\beta = \beta v$. Define $a = k^{-1}\alpha z$ and $b = z\beta$. Then $ab = v$, and $e := ba = k^{-1}z\beta\alpha z$ is in $zL_K(E)z$. Moreover, $e^2 = baba = bva = ba = e$ and thus $L_K(E)e \cong L_K(E)v$ as left ideals of $L_K(E)$ by a standard ring theory result. (The maps $\rho_b : L_K(E)e \to L_K(E)v$ and $\rho_a : L_K(E)v \to L_K(E)e$ give the isomorphisms.) Note in particular that this implies $L_K(E)e \neq \{0\}$. Since z is a primitive idempotent, $zL_K(E)z$ is a ring without nontrivial idempotents, so that $e \in \{0,z\}$; since $e \neq 0$, we have $z = e$, so that $L_K(E)z \cong L_K(E)v$ as desired. If in addition z is primitive non-minimal, then so necessarily is v, and hence $zL_K(E)z \cong vL_K(E)v \cong K[x,x^{-1}]$ by Remark 3.5.4. □

We are now in position to establish a result similar to Corollary 3.5.5, but with respect to *all* idempotents in $L_K(E)$.

{Condition(L)characterization}

Theorem 3.5.7. *Let E be an arbitrary graph and K any field. Then E satisfies Condition* (L) *if and only if every primitive idempotent in $L_K(E)$ is minimal.*

Proof. If $L_K(E)$ has no primitive non-minimal idempotents, in particular it has no primitive non-minimal vertices, so that by Corollary 3.5.5, E satisfies Condition (L).

Now suppose E satisfies Condition (L), and let x be a primitive non-minimal idempotent of $L_K(E)$. By the Reduction Theorem 2.2.11 there exist a vertex v, a nonzero scalar k and elements $\mu, \kappa \in \text{Path}(E)$ such that $\mu^* x \kappa = kv$. Note that, by Corollary 3.5.5, v cannot be primitive non-minimal. But this is a contradiction since by Proposition 3.5.6, $L_K(E)v \cong L_K(E)x$. □

Because Theorem 3.5.7 yields a characterization of Condition (L) in E as a ring-theoretic condition on $L_K(E)$, we immediately get the following.

{primmin}

Corollary 3.5.8. *Let E,F be arbitrary graphs and K any field, and suppose $L_K(E) \cong L_K(F)$ as rings. Then E satisfies Condition (L) if and only if F satisfies Condition (L).*

Observe that Corollary 3.5.8 also follows from Proposition 2.9.13.

The tools developed above will allow us to reformulate, in terms of idempotents, the Simplicity and Purely Infinite Simplicity Theorems. By the Simplicity Theorem 2.9.1, $L_K(E)$ is simple if and only if $\mathscr{H}_E = \{\emptyset, E^0\}$, and E satisfies Condition (L). The condition $\mathscr{H}_E = \{\emptyset, E^0\}$ is equivalent to the nonexistence of nontrivial two-sided ideals of $L_K(E)$ generated by idempotents (see Theorem 2.5.8 and Corollary 2.9.11). So Theorem 3.5.7 yields the following.

{simpleiffnonminprimidempotents}

Corollary 3.5.9. *Let E be an arbitrary graph and K any field. Then $L_K(E)$ is simple if and only if every primitive idempotent in $L_K(E)$ is minimal, and $L_K(E)$ contains no nontrivial two-sided ideals generated by idempotents.*

By the Purely Infinite Simplicity Theorem 3.1.10, $L_K(E)$ is purely infinite simple if and only if $L_K(E)$ is simple, and every vertex of E connects to a cycle. If E is finite, then the latter condition may be replaced by the condition that there are no minimal idempotents in $L_K(E)$, as follows. On the one hand, if every vertex connects to a cycle (necessarily with an exit), then there are no minimal vertices in E (indeed, by Proposition 3.1.6, every vertex is infinite in this case). On the other hand, if there are no minimal vertices then there are no sinks, and since E is finite, this yields that every vertex must connect to a cycle. But $\text{Soc}(L_K(E)) = I(P_l(E))$ (Theorem 2.6.14), and $P_l(E) = \emptyset$ (because E is finite and there are no sinks), so that $\text{Soc}(L_K(E)) = \{0\}$. Specifically, there are no minimal idempotents in $L_K(E)$. So we have established

{pischarEfinite}

Corollary 3.5.10. *Let E be a finite graph and K any field. Then $L_K(E)$ is purely infinite simple if and only if $L_K(E)$ contains no primitive idempotents and no nontrivial two-sided ideals generated by idempotents.*

{Vmonoidstructure}

3.6 Structural properties of the $\mathscr{V}$-monoid

{monoiddefinitions}

For a ring R with enough idempotents, the monoid $\mathscr{V}(R)$ of isomorphism classes of finitely generated projective left R-modules was discussed in Section 3.2. The monoid $\mathscr{V}(R)$ is clearly *conical*; that is, if $p,q \in \mathscr{V}(R)$ have $p+q=0$, then $p=q=0$. In the specific case of a Leavitt path algebra $L_K(E)$, we show in this section that the monoid $\mathscr{V}(L_K(E))$ satisfies some additional monoid-theoretic properties (properties which, unlike the conical property, fail for some monoids of the form $\mathscr{V}(S)$ for some rings S). These properties arise in various contexts associated with decomposition and cancellation properties among finitely generated projective left $L_K(E)$-modules.

Definitions 3.6.1. Let $(M,+)$ denote an abelian monoid.

(i) M is called a *refinement monoid* if whenever $a+b=c+d$ in M, there exist $x,y,z,t \in M$ such that $a=x+y$ and $b=z+t$, while $c=x+z$ and $d=y+t$.

(ii) There is a canonical preorder on any abelian monoid M (the *algebraic preorder*), defined by setting $x \le y$ if and only if there exists $m \in M$ such that $y = x+m$. Following [32], M is called a *separative monoid* in case M satisfies the following condition: if $a,b,c \in M$ satisfy $a+c=b+c$, and $c \le na$ and $c \le nb$ for some $n \in \mathbb{N}$, then $a=b$.

{ringdecomdefinitions}

There are analogous definitions from a ring-theoretic point of view.

{V(R)differentfromFP(R)}

Definitions 3.6.2. Let R be a ring with enough idempotents. The class of finitely generated projective left R-modules is denoted by $FP(R)$.

{refinementandseparative}

(i) We say that $FP(R)$ satisfies the *refinement property* if whenever $A_1,A_2,B_1,B_2 \in FP(R)$ satisfy $A_1 \oplus A_2 \cong B_1 \oplus B_2$, then there exist decompositions $A_i = A_{i1} \oplus A_{i2}$ for $i=1,2$ such that $A_{1j} \oplus A_{2j} \cong B_j$ for $j=1,2$.

(ii) We say that R is *separative* if whenever $A,B,C \in FP(R)$ satisfy $A \oplus C \cong B \oplus C$ and C is isomorphic to direct summands of both nA and nB for some $n \in \mathbb{N}$, then $A \cong B$.

Remark 3.6.3. We note that, while the monoid $\mathscr{V}(R)$ of *isomorphism classes* of finitely generated projective left R-modules has been, and will continue to be, a key player in the subject of Leavitt path algebras, it is more common in the literature to focus on the class of *all* finitely generated projective left R-modules in a discussion of the properties of R presented in Definitions 3.6.2.

The following is then clear.

Proposition 3.6.4. *Let R be a ring with enough idempotents.*

(i) *$\mathscr{V}(R)$ is a refinement monoid if and only if $FP(R)$ satisfies the refinement property.*

(ii) *$\mathscr{V}(R)$ is separative if and only if R is separative.*

We will show in this section that $\mathscr{V}(L_K(E))$ is both separative and a refinement monoid for every graph E and field K. The approach will be to first establish these results for row-finite graphs, and subsequently invoke appropriate direct limit theorems from Chapter 1. For context, we note that it has been shown [32, Proposition 1.2] that every exchange ring satisfies the refinement property. On the other hand, it is an outstanding open question to determine whether every exchange ring is separative.

We recall here the definition of the monoid M_E (Definition 3.2.2). For any row-finite graph E, M_E denotes the abelian monoid given by the generators $\{a_v \mid v \in E^0\}$, with the relations:

{(M)}

$$a_v = \sum_{\{e \in E^1 \mid s(e)=v\}} a_{r(e)} \qquad \text{for every } v \in E^0 \text{ that emits edges.} \tag{M}$$

We introduce some helpful notation. Let E be a row-finite graph, and let $\mathbb{F}$ be the free abelian monoid on the set E^0. Each of the nonzero elements of $\mathbb{F}$ can be written in a unique form (up to permutation) as $\sum_{i=1}^n x_i$, where $x_i \in E^0$ (and repeats are allowed). Now we will give a description of the congruence on $\mathbb{F}$ generated by the relations (M) on $\mathbb{F}$. For $x \in \mathrm{Reg}(E)$, write

$$\mathbf{r}(x) := \sum_{\{e\in E^1 \mid s(e)=x\}} r(e) \in \mathbb{F}.$$

With this notation, the relations (M) are expressed more efficiently as $x = \mathbf{r}(x)$ for every $x \in \operatorname{Reg}(E)$.

{binary}

Definition 3.6.5. Let $\mathbb{F}$ be the free abelian monoid on the set of vertices E^0 of a row-finite graph E. Define a binary relation $\to_1$ on $\mathbb{F}\setminus\{0\}$ as follows. Let $\sum_{i=1}^n x_i$ be an element in $\mathbb{F}\setminus\{0\}$ as above and let $j\in\{1,\dots,n\}$ be an index such that x_j emits edges. Then $\sum_{i=1}^n x_i \to_1 \sum_{i\neq j} x_i + \mathbf{r}(x_j)$. Let $\to$ be the transitive and reflexive closure of $\to_1$ on $\mathbb{F}\setminus\{0\}$, that is, $\alpha\to\beta$ if and only if there is a finite string $\alpha=\alpha_0\to_1\alpha_1\to_1\cdots\to_1\alpha_t=\beta$. Let $\sim$ be the congruence on $\mathbb{F}\setminus\{0\}$ generated by the relation $\to_1$ (or, equivalently, by the relation $\to$). Namely $\alpha\sim\alpha$ for all $\alpha\in\mathbb{F}\setminus\{0\}$ and, for $\alpha,\beta\neq 0$, we have $\alpha\sim\beta$ if and only if there is a finite string $\alpha=\alpha_0,\alpha_1,\dots,\alpha_n=\beta$, such that, for each $i=0,\dots,n-1$, either $\alpha_i\to_1\alpha_{i+1}$ or $\alpha_{i+1}\to_1\alpha_i$. The number n above will be called the *length* of the string. The congruence $\sim$ on $\mathbb{F}\setminus\{0\}$ is extended to $\mathbb{F}$ by adding the single pair $0\sim 0$. It is clear that $\sim$ is the congruence on $\mathbb{F}$ generated by relations (M), and so $M_E=\mathbb{F}/{\sim}$.

{division}

The *support* of an element γ in $\mathbb{F}$, denoted $\operatorname{supp}(\gamma)\subseteq E^0$, is the set of basis elements appearing in the canonical expression of γ.

Lemma 3.6.6. *Let $\to$ be the binary relation on $\mathbb{F}$ given in Definition 3.6.5. Suppose $\alpha,\beta,\alpha_1,\beta_1\in\mathbb{F}\setminus\{0\}$ with $\alpha=\alpha_1+\alpha_2$ and $\alpha\to\beta$. Then β can be written as $\beta=\beta_1+\beta_2$ with $\alpha_1\to\beta_1$ and $\alpha_2\to\beta_2$.*

Proof. By induction, it is enough to show the result in the case where $\alpha\to_1\beta$. If $\alpha\to_1\beta$, then there is an element x in the support of α such that $\beta=(\alpha-x)+\mathbf{r}(x)$. The element x belongs either to the support of α_1 or to the support of α_2. Assume, for instance, that the element x belongs to the support of α_1. Then we set $\beta_1=(\alpha_1-x)+\mathbf{r}(x)$ and $\beta_2=\alpha_2$. The case where x is in the support of α_2 is similar. □

Note that the elements β_1 and β_2 in Lemma 3.6.6 are not uniquely determined by α_1 and α_2 in general, because the element $x\in E^0$ considered in the proof could belong to *both* the support of α_1 and the support of α_2.

{confluence}

The following lemma gives the important "confluence" property of the congruence $\sim$ on the free abelian monoid $\mathbb{F}$.

Lemma 3.6.7. *(**The Confluence Lemma**) Let α and β be nonzero elements in $\mathbb{F}$. Then $\alpha\sim\beta$ if and only if there is $\gamma\in\mathbb{F}\setminus\{0\}$ such that $\alpha\to\gamma$ and $\beta\to\gamma$.*

Proof. Assume that $\alpha\sim\beta$. Then there exists a finite string $\alpha=\alpha_0,\alpha_1,\dots,\alpha_n=\beta$, such that, for each $i=0,\dots,n-1$, either $\alpha_i\to_1\alpha_{i+1}$ or $\alpha_{i+1}\to_1\alpha_i$. We proceed by induction on n. If $n=0$, then $\alpha=\beta$ and there is nothing to prove. Assume the result is true for strings of length $n-1$, and let $\alpha=\alpha_0,\alpha_1,\dots,\alpha_n=\beta$ be a string of length n. By the induction hypothesis, there is $\lambda\in\mathbb{F}$ such that $\alpha\to\lambda$ and $\alpha_{n-1}\to\lambda$. Now there are two cases to consider. If $\beta\to_1\alpha_{n-1}$, then $\beta\to\lambda$ and we are done. Assume that $\alpha_{n-1}\to_1\beta$. By definition of $\to_1$, there is a basis element $x\in E^0$ in the support of α_{n-1} such that $\alpha_{n-1}=x+\alpha'_{n-1}$ and $\beta=\mathbf{r}(x)+\alpha'_{n-1}$. By Lemma 3.6.6, we have $\lambda=\lambda(x)+\lambda'$, where $x\to\lambda(x)$ and $\alpha'_{n-1}\to\lambda'$. If the length of the string from x to $\lambda(x)$ is positive, then we have $\mathbf{r}(x)\to\lambda(x)$ and so $\beta=\mathbf{r}(x)+\alpha'_{n-1}\to\lambda(x)+\lambda'=\lambda$. In case that $x=\lambda(x)$, we define $\gamma=\mathbf{r}(x)+\lambda'$. Then $\lambda\to_1\gamma$ and so $\alpha\to\gamma$, and also $\beta=\mathbf{r}(x)+\alpha'_{n-1}\to\mathbf{r}(x)+\lambda'=\gamma$. This concludes the proof. □

{refinement}

We are now ready to show the refinement property of M_E.

Proposition 3.6.8. *The monoid M_E associated with any row-finite graph E is a refinement monoid.*

Proof. We use the identification $M_E=\mathbb{F}/\sim$. Let $\alpha=\alpha_1+\alpha_2\sim\beta=\beta_1+\beta_2$, with $\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbb{F}$. By Lemma 3.6.7, there is $\gamma\in\mathbb{F}$ such that $\alpha\to\gamma$ and $\beta\to\gamma$. By Lemma 3.6.6, we can write $\gamma=\alpha'_1+\alpha'_2=\beta'_1+\beta'_2$, with $\alpha_i\to\alpha'_i$ and $\beta_i\to\beta'_i$ for $i=1,2$. Since $\mathbb{F}$ is a free abelian monoid, $\mathbb{F}$ has the refinement property and so there are decompositions $\alpha'_i=\gamma_{i1}+\gamma_{i2}$ for $i=1,2$ such that $\beta'_j=\gamma_{1j}+\gamma_{2j}$ for $j=1,2$. The result follows. □

Let E be row-finite. Our next goal is to establish a lattice isomorphism between the lattice $\mathscr{H}_E$ of hereditary saturated subsets of E^0 and the lattice of order-ideals of the associated monoid M_E. This in turn can then be interpreted as a lattice isomorphism with the graded ideals of $L_K(E)$ (Theorem 2.5.9), and thereby also an isomorphism with the lattice of the ideals of $L_K(E)$ generated by idempotents (Corollary 2.9.11).

An *order-ideal* of a monoid M is a submonoid I of M such that $x+y=z$ in M and $z \in I$ imply that both x,y belong to I. An order-ideal can also be described as a submonoid I of M which is hereditary with respect to the canonical preorder $\leq$ on M: $x \leq y$ and $y \in I$ imply $x \in I$. Recall that the preorder $\leq$ on M is defined by setting $x \leq y$ if and only if there exists $m \in M$ such that $y = x+m$.

The set $\mathscr{L}(M)$ of order-ideals of M forms a (complete) lattice $\big(\mathscr{L}(M), \subseteq, \overline{\sum}, \cap\big)$. Here, for a family of order-ideals $\{I_i\}$, we denote by $\overline{\sum} I_i$ the set of elements $x \in M$ such that $x \leq y$ for some y belonging to the algebraic sum $\sum I_i$ of the order-ideals I_i. Note that $\sum I_i = \overline{\sum} I_i$ whenever M is a refinement monoid.

{lattmonoid}

Let $\mathbb{F}_E$ be the free abelian monoid on E^0, and recall that $M_E = \mathbb{F}_E/\sim$. For $\gamma \in \mathbb{F}_E$ we will denote by $[\gamma]$ its class in M_E. Note that any order-ideal I of M_E is generated *as a monoid* by the set $\{[v] \mid v \in E^0\} \cap I$.

The set $\mathscr{H}_E$ of hereditary saturated subsets of E^0 is also a complete lattice $(\mathscr{H}_E, \subseteq, \overline{\cup}, \cap)$ (Remark 2.5.2).

Proposition 3.6.9. *Let E be a row-finite graph. Then there are order-preserving mutually inverse maps*

$$\varphi\colon \mathscr{H}_E \longrightarrow \mathscr{L}(M_E) \quad \text{and} \quad \psi\colon \mathscr{L}(M_E) \longrightarrow \mathscr{H}_E,$$

where $\varphi(H)$ is the order-ideal of M_E generated by $\{[v] \mid v \in H\}$, for $H \in \mathscr{H}_E$, and $\psi(I)$ is the set of elements v in E^0 such that $[v] \in I$, for $I \in \mathscr{L}(M_E)$.

Proof. The maps φ and ψ are obviously order-preserving. We claim that to establish the result it suffices to show

(1) for $I \in \mathscr{L}(M_E)$, the set $\psi(I)$ is a hereditary saturated subset of E^0, and
(2) if $H \in \mathscr{H}_E$ then $[v] \in \varphi(H)$ if and only if $v \in H$.

To see this, if (1) and (2) hold, then ψ is well-defined by (1), and $\psi(\varphi(H)) = H$ for $H \in \mathscr{H}_E$, by (2). On the other hand, if I is an order-ideal of M_E, then obviously $\varphi(\psi(I)) \subseteq I$, and since I is generated as a monoid by $\{[v] \mid v \in E^0\} \cap I = [\psi(I)]$, it follows that $I \subseteq \varphi(\psi(I))$.

Proof of (1). Let I be an order-ideal of M_E, and set $H := \psi(I) = \{v \in E^0 \mid [v] \in I\}$. To see that H is hereditary, we have to prove that, whenever we have $\gamma = e_1e_2, \cdots e_n$ in $\mathrm{Path}(E)$ with $s(e_1) = v$ and $r(e_n) = w$ and $v \in H$, then $w \in H$. If we consider the corresponding sequence $v \to_1 \gamma_1 \to_1 \gamma_2 \to_1 \cdots \to_1 \gamma_n$ in $\mathbb{F}_E$, we see that w belongs to the support of γ_n, so that $w \leq \gamma_n$ in $\mathbb{F}_E$. This implies that $[w] \leq [\gamma_n] = [v]$, and so $[w] \in I$ because I is hereditary.

To show saturation, take a non-sink $v \in E^0$ such that $r(e) \in H$ for every $e \in E^1$ such that $s(e) = v$. We then have $\mathrm{supp}(\mathbf{r}(v)) \subseteq H$, so that $[\mathbf{r}(v)] \in I$ because I is a submonoid of M_E. But $[v] = [\mathbf{r}(v)]$, so that $[v] \in I$ and $v \in H$.

Proof of (2). Let H be a hereditary saturated subset of E^0, and let $I := \varphi(H)$ be the order-ideal of M_E generated by $\{[v] \mid v \in H\}$. Clearly $[v] \in I$ if $v \in H$. Conversely, suppose that $[v] \in I$. Then $[v] \leq [\gamma]$, where $\gamma \in \mathbb{F}_E$ satisfies $\mathrm{supp}(\gamma) \subseteq H$. Thus we can write $[\gamma] = [v] + [\delta]$ for some $\delta \in \mathbb{F}_E$. By Lemma 3.6.7, there is $\beta \in F_E$ such that $\gamma \to \beta$ and $v + \delta \to \beta$. Since H is hereditary and $\mathrm{supp}(\gamma) \subseteq H$, we get $\mathrm{supp}(\beta) \subseteq H$. By Lemma 3.6.6, we have $\beta = \beta_1 + \beta_2$, where $v \to \beta_1$ and $\delta \to \beta_2$. Observe that $\mathrm{supp}(\beta_1) \subseteq \mathrm{supp}(\beta) \subseteq H$. Using that H is saturated, it is a simple matter to check that, if $\alpha \to_1 \alpha'$ and $\mathrm{supp}(\alpha') \subseteq H$, then $\mathrm{supp}(\alpha) \subseteq H$. Using this and induction, we obtain that $v \in H$, as desired. $\square$

We now show that the monoid M_E associated with a row-finite graph E is always a separative monoid. Recall (Definitions 3.6.1) this means that for elements $x,y,z \in M_E$, if $x+z = y+z$ and $z \leq nx$ and $z \leq ny$ for some positive integer n, then $x = y$.

The separativity of M_E follows from results of Brookfield [53] on primely generated monoids; see also [143, Chapter 6]. Indeed the class of primely generated refinement monoids satisfies many other nice cancellation properties. We will highlight unperforation later, and refer the reader to [53] for further information.

Definition 3.6.10. Let M be a monoid. An element $p \in M$ is *prime* if for all $a_1, a_2 \in M$, $p \le a_1 + a_2$ implies $p \le a_1$ or $p \le a_2$. A monoid is *primely generated* if each of its elements is a sum of primes.

Proposition 3.6.11. *[53, Corollary 6.8] Any finitely generated refinement monoid is primely generated.*

It follows from Propositions 3.6.8 and 3.6.11 that, for a finite graph E, the monoid M_E is primely generated. Note that the primely generated property does not extend in general to row-finite graphs, as is demonstrated by the following graph G:

$$
\begin{tikzcd}
p_0 \arrow[r] \arrow[d] & p_1 \arrow[r] \arrow[dl] & p_2 \arrow[r] \arrow[dll] & p_3 \arrow[r] \arrow[dlll] & \cdots \\
a & & & &
\end{tikzcd}
$$

The corresponding monoid M_G has generators $a, p_0, p_1, \dots$, and relations given by $p_i = p_{i+1} + a$ for all $i \ge 0$. One can easily see that the only prime element in M is a, so that M is not primely generated.

Theorem 3.6.12. *Let E be a row-finite graph. Then the monoid M_E is separative.*

Proof. By Lemma 5.3.3, we get that M_E is the direct limit of monoids M_{X_i} corresponding to finite graphs X_i. Therefore, in order to check separativity, we can assume that the graph E is finite. In this situation, we have that M_E is generated by the finite set E^0 of vertices of E, and thus M_E is finitely generated. By Proposition 3.6.8, M_E is a refinement monoid, so it follows from Proposition 3.6.11 that M_E is a primely generated refinement monoid. By [53, Theorem 4.5], the monoid M_E is separative. $\square$

As remarked previously, primely generated refinement monoids satisfy many nice cancellation properties, as shown in [53]. Some of these properties are preserved in direct limits, so they are automatically true for the graph monoids corresponding to any row-finite graph. Especially important in several applications is the property of unperforation.

Definition 3.6.13. The monoid $(M, +)$ is said to be *unperforated* in case, for all elements $a, b \in M$ and all positive integers n, we have $na \le nb \implies a \le b$.

Proposition 3.6.14. *Let E be a row-finite graph. Then the monoid M_E is unperforated.*

Proof. As in the proof of Theorem 3.6.12, we can reduce to the case of a finite graph E. In this case, the result follows from [53, Corollary 5.11(5)]. $\square$

Corollary 3.6.15. *Let E be a row-finite graph. Then $FP(L_K(E))$ satisfies the refinement property, and $L_K(E)$ is a separative ring. Moreover, the monoid $\mathscr{V}(L_K(E))$ is unperforated.*

Proof. By Theorem 3.2.6, we have $\mathscr{V}(L_K(E)) \cong M_E$. So the result follows from Proposition 3.6.8, Theorem 3.6.12 and Proposition 3.6.14. $\square$

Another useful technique to deal with graph monoids of finite graphs consists of considering composition series of order-ideals in the monoid. These composition series correspond via Proposition 3.6.9 and Theorem 2.5.9 to composition series of graded ideals in $L_K(E)$. (Using [49, Theorem 4.1(b)], they also correspond to composition series of closed gauge-invariant ideals of the graph C^*-algebra $C^*(E)$; this approach will be used in the proof of Theorem 5.3.4 below.) The composition series approach can be used to achieve a different proof of the separativity of M_E (Theorem 3.6.12), an approach we sketch in Remark 3.6.19.

Definition 3.6.16. Given an order-ideal S of a monoid M we define a congruence $\sim_S$ on M by setting $a \sim_S b$ if and only if there exist $e, f \in S$ such that $a + e = b + f$. Let M/S be the factor monoid obtained from the congruence $\sim_S$. (See [32].) We denote by $[x]_S$ the class of an element $x \in M$ in M/S.

In particular, If I is any ideal of a ring R, the monoid $\mathscr{V}(I)$ is an order-ideal of $\mathscr{V}(R)$. Using the construction of the factor monoid given in Definition 3.6.16, it can be shown that for a large class of rings R, one has $\mathscr{V}(R/I) \cong \mathscr{V}(R)/\mathscr{V}(I)$ for any ideal I of R; see [32, Proposition 1.4]. We present here some general useful facts about $\mathscr{V}$-monoids.

{LVAisoIdA}

Proposition 3.6.17. *Let R be any ring with local units.*

(i) *Assume that $\mathscr{V}(R)$ is a refinement monoid. Then the map*

$$I \mapsto \mathscr{V}(I)$$

gives a lattice isomorphism between the lattice $\mathscr{L}_{\mathrm{idem}}(R)$ consisting of those ideals of R which are generated by idempotents, and the lattice $\mathscr{L}(\mathscr{V}(R))$ of order-ideals of $\mathscr{V}(R)$.

(ii) *If I is an ideal of R generated by idempotents, then there is a canonical injective map*

$$\omega\colon \mathscr{V}(R)/\mathscr{V}(I) \to \mathscr{V}(R/I),$$

such that $\omega([e]_{\mathscr{V}(I)}) = [e+I]$ for every idempotent e in R.

Proof. (i) Since R has local units and $\mathscr{V}(R)$ is a refinement monoid, every idempotent E in $M_{\mathbb{N}}(R)$ is equivalent to an idempotent of the form $e_1 \oplus \cdots \oplus e_n$ for some idempotents $e_1, \ldots, e_n$ of R. It follows that the set of trace ideals considered in [31, Definition 10.9] is exactly the set of ideals of R generated by idempotents. Therefore the bijective correspondence follows from [31, Proposition 10.10] (see [70, Theorem 2.1(c)] for the unital case).

(ii) Since R has local units, the proof of [28, Proposition 5.3(c)] can be easily adapted to get that the map ω is injective. Note that ω is just the map induced by the canonical projection $\pi\colon R \to R/I$. $\square$

{factoring}

Observe that Proposition 3.6.9 can be obtained by combining Theorem 2.5.9, Corollary 2.9.11, Theorem 3.2.6 and Theorem 3.6.8, by using Proposition 3.6.17(i). A similar route can be used to show the following result.

Lemma 3.6.18. *Let E be a row-finite graph. For a hereditary saturated subset H of E^0, consider the order-ideal $S = \varphi(H)$ of M_E associated with H, as in Proposition 3.6.9. Let E/H be the quotient graph (recall Definition 2.4.11). Then there are natural monoid isomorphisms*

$$M_E/S \cong \mathscr{V}(L_K(E))/\mathscr{V}(I(H)) \cong \mathscr{V}(L_K(E)/I(H)) \cong \mathscr{V}(L_K(E/H)) \cong M_{E/H}.$$

{remarkseparativity}

Proof. By Theorem 3.2.6 we have $M_E \cong \mathscr{V}(L_K(E))$. By Proposition 3.6.17, the map

$$\omega\colon \mathscr{V}(L_K(E))/\mathscr{V}(I(H)) \to \mathscr{V}(L_K(E)/I(H)) \quad \text{defined by} \quad \omega([e]_{\mathscr{V}(I(H))}) = [e+I(H)]$$

is injective. Moreover, there is an isomorphism $L_K(E)/I(H) \cong L_K(E/H)$, given in Corollary 2.4.13(i). Since $\mathscr{V}(L_K(E/H)) \cong M_{E/H}$, the monoid $\mathscr{V}(L_K(E/H))$ is generated by the classes of vertices v in $E^0 \setminus H$, so we get that the map ω is surjective. The result follows. $\square$

Remark 3.6.19. We sketch a proof of the separativity of M_E, different from the one presented in Theorem 3.6.12, using the theory of order-ideals. For a row-finite graph E, we call M_E *simple* in case the only order-ideals of M_E are trivial. This corresponds by Proposition 3.6.9 to the situation where the hereditary saturated subset generated by any vertex of E is E^0. By Lemma 2.9.6, this happens if and only if E is cofinal (Definition 2.9.4).

As in the proof of Theorem 3.6.12, we can assume that E is a finite graph. In this case it is obvious that E^0 has a finite number of hereditary saturated subsets, so M_E has a finite number of order-ideals. Take a finite chain $0 = S_0 \le S_1 \le \cdots \le S_n = M_E$ such that each S_i is an order-ideal of M_E, and all the quotients S_i/S_{i-1} are simple. By Proposition 3.6.9, we have $S_i \cong M_{H_i}$, for some finite graph H_i, and by Lemma 3.6.18, we have $S_i/S_{i-1} \cong M_{G_i}$ for some cofinal finite graph G_i. By Proposition 3.6.8, S_i is a refinement monoid for all i, so the Extension Theorem for refinement monoids ([32, Theorem 4.5]) tells us that S_i is separative if and only if so are S_{i-1} and S_i/S_{i-1}. It follows by induction that it is enough to show the case where E is a cofinal finite graph.

Let E be a cofinal finite graph. We distinguish three cases. First, suppose that E is acyclic. Then there is a sink v, and by cofinality for every vertex w of E there is a path from w to v. It follows that M_E is a free abelian monoid of rank one (i.e., isomorphic to $\mathbb{Z}^+$), generated by a_v. In particular M_E is a separative monoid. Secondly, assume that E has a cycle without exits, and let v be any vertex in this cycle. By using the cofinality condition, it is easy to see that there are no other cycles in E, and that every vertex in E connects to v. It follows again that M_E is a free abelian monoid of rank one, generated by a_v. Finally, we consider the case where every cycle in E has an exit. By cofinality, every vertex connects to every cycle. Using this and the property that every cycle has an exit, it is easy to show that for every nonzero element x in M_E there is a nonzero element y in M_E such that $x = x + y$. It follows that $M_E \setminus \{0\}$ is a group; see for example [32, Proposition 2.4]. In particular M_E is a separative monoid. □

{examplebis}

Example 3.6.20. This example will be useful later on. Consider the following graph E:

$$\begin{tikzcd} a \arrow[loop above] \arrow[loop below] & b \arrow[l] \arrow[r] & c \arrow[loop above] \arrow[loop below] \arrow[r] & d \end{tikzcd} .$$

Then M_E is the monoid generated by a, b, c, d with defining relations $a = 2a$, $b = a + c$, $c = 2c + d$.

A composition series of order-ideals for M_E is obtained from the graph monoids corresponding to the following chain of hereditary saturated subsets of E:

$$\emptyset \, , \qquad d \, , \quad \begin{tikzcd} c \arrow[loop above] \arrow[loop below] \arrow[r] & d \end{tikzcd} , \quad \begin{tikzcd} a \arrow[loop above] \arrow[loop below] & b \arrow[l] \arrow[r] & c \arrow[loop above] \arrow[loop below] \arrow[r] & d \end{tikzcd} .$$

By Lemma 3.6.18, the corresponding simple quotient monoids are the graph monoids corresponding to the following graphs:

$$d \, , \quad \begin{tikzcd} c \arrow[loop above] \arrow[loop below] \end{tikzcd} , \quad \begin{tikzcd} a \arrow[loop above] \arrow[loop below] & b \arrow[l] \end{tikzcd} .$$

It is a relatively straightforward matter to generalize the previously established structural results about graph monoids of row-finite graphs to arbitrary graphs, using the direct limit machinery from Section 1.6. We complete this section by providing the details.

{refetal-forgeneralgraphs}

Theorem 3.6.21. *Let E be an arbitrary graph, let K be a field, and let X be a subset of* $\mathrm{Reg}(E)$*. Then the monoid $\mathscr{V}(C_K^X(E))$ is an unperforated, separative, refinement monoid. In particular, the monoid $\mathscr{V}(L_K(E))$ is an unperforated, separative, refinement monoid.*

Proof. Since the properties in the statement are preserved under direct limits, and since the functor $\mathscr{V}$ is continuous, we see from Theorem 1.6.10 that it suffices to show the result for a finite graph E. So suppose that E is a finite graph and that X is a finite subset of $\mathrm{Reg}(E)$. By Theorem 1.5.18, we have that $C_K^X(E) \cong L_K(E(X))$ for a certain finite graph $E(X)$. By Proposition 3.6.8, Theorem 3.6.12, Proposition 3.6.14 and Theorem 3.2.6, $\mathscr{V}(L_K(E(X))$ is an unperforated, separative, refinement monoid, and thus so is $\mathscr{V}(C_K^X(E))$. □

{rem:unperf-implies-separative}

Remark 3.6.22. For a refinement monoid, unperforation implies separativity. This follows immediately from [59, Theorem 1], and it was noted independently in [142, Corollary 2.4].

{order-ideals-arbitrary}

Theorem 3.6.23. *Let E be an arbitrary graph and K any field.*

(i) *The map*

$$I \mapsto \mathscr{V}(I)$$

gives a lattice isomorphism between the lattice $\mathscr{L}_{\mathrm{gr}}(L_K(E))$ of graded ideals of $L_K(E)$ and the lattice $\mathscr{L}(\mathscr{V}(L_K(E)))$ of order-ideals of $\mathscr{V}(L_K(E))$.

(ii) *Let I be a graded ideal of $L_K(E)$. Then there is a natural monoid isomorphism*

$$\omega\colon \frac{\mathscr{V}(L_K(E))}{\mathscr{V}(I)} \longrightarrow \mathscr{V}(L_K(E)/I).$$

Proof. (i) Since $\mathscr{V}(L_K(E))$ is a refinement monoid (Theorem 3.6.21) and the graded ideals of $L_K(E)$ are precisely the idempotent-generated ideals (Corollary 2.9.11), the result follows directly from Proposition 3.6.17(i).

(ii) Again by Corollary 2.9.11, we have that I is an idempotent-generated ideal, so the map ω is injective by Proposition 3.6.17(ii). Now by Theorem 2.5.8 there exist $H \in \mathscr{H}_E$ and $S \subseteq B_H$ such that $I = I(H \cup S^H)$. Therefore, by using Theorem 2.4.15 and Corollary 3.2.12, we get

$$\mathscr{V}(L_K(E)/I) = \mathscr{V}(L_K(E)/I(H \cup S^H)) \cong \mathscr{V}(L_K(E/(H,S))) \cong M_{E/(H,S)}.$$

It follows that $\mathscr{V}(L_K(E)/I)$ is generated by elements of the form $[v - \sum_{f\in Z} ff^*]$, where $v \in E^0 \setminus H$ and Z is a finite (possibly empty) subset of $s_E^{-1}(v)$ such that $r(f) \notin H$ for every $f \in Z$. Thus the map ω is surjective, and consequently a monoid isomorphism. $\square$

{extreme}

3.7 Extreme cycles

In Chapter 1 we described the three "primary colors" of Leavitt path algebras: $n \times n$ matrix rings $\mathrm{M}_n(K) \cong L_K(A_n)$, Laurent polynomials $K[x,x^{-1}] \cong L_K(R_1)$, and Leavitt algebras $L_K(1,n) \cong L_K(R_n)$ (for $n \geq 2$). In Theorem 2.6.14 we showed that the ideal of $L_K(E)$ generated by the set of line points $P_l(E)$ yields a piece of $L_K(E)$ similar in appearance to the first color, while in Theorem 2.7.3 we showed that the ideal of $L_K(E)$ generated by the vertices which lie on cycles without exits $P_c(E)$ is similar in appearance to the second color. Intuitively, in this section we complete the picture by describing the piece of $L_K(E)$ which most resembles the third color. Specifically, we identify sets of vertices which generate ideals in $L_K(E)$ which are purely infinite simple as a K-algebra.

{ece}

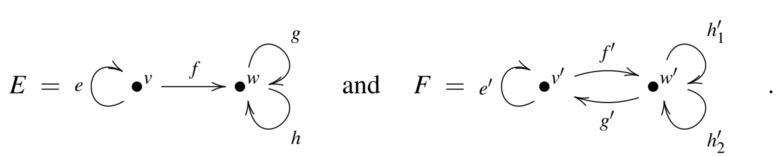
Definitions 3.7.1. Let E be a graph and c a cycle in E. We say that c is an *extreme cycle* if c has exits and, for every path λ starting at a vertex in c^0, there exists $\mu \in \mathrm{Path(E)}$ such that $r(\lambda) = s(\mu)$, and $r(\lambda\mu) \in c^0$. We will denote by $P_{ec}(E)$ the set of vertices which belong to extreme cycles. Intuitively, c is an extreme cycle in case every path which leaves c can be lengthened in such a way that the longer path returns to c.

Let X'_{ec} be the set of all extreme cycles in a graph E. We define in X'_{ec} the following relation: given $c, d \in X'_{ec}$, we write $c \sim d$ whenever c and d are connected, that is, $T(c^0) \cap d^0 \neq \emptyset$, equivalently, $T(d^0) \cap c^0 \neq \emptyset$. It is not difficult to see that $\sim$ is an equivalence relation. The set of all $\sim$-equivalence classes is denoted by $X_{ec} = X'_{ec}/\sim$. When we want to emphasize a specific graph E under consideration we will write $X'_{ec}(E)$ and $X_{ec}(E)$ for X'_{ec} and X_{ec}, respectively.

For $c \in X'_{ec}$, we let $\tilde{c}$ denote the class of c. We write $\tilde{c}^0$ to represent the set of all vertices which are in the cycles belonging to $\tilde{c}$.

{Pec(E)Examples}

Examples 3.7.2. Consider the following graphs.



Then straightforward computations yield that $P_{ec}(E) = \{w\}$, $X'_{ec}(E) = \{g,h\}$, and $X_{ec}(E) = \{\tilde{g}\}$. Similarly, $P_{ec}(F) = \{v', w'\}$, $X'_{ec}(F) = \{e', f'g', g'f', h'_1, h'_2\}$, and $X_{ec}(F) = \{\tilde{e'}\}$.

{Pec(E)Toeplitz}

Example 3.7.3. Let E_T be the Toeplitz graph $e \circlearrowright \bullet^u \xrightarrow{f} \bullet^v$. Then clearly $P_{ec}(E_T) = \emptyset$.

{propECE}

Remark 3.7.4. Let E be an arbitrary graph. These two observations are straightforward to verify.

(i) For any $c \in X'_{ec}$, $\tilde{c}^0 = T(c^0)$. Consequently, $\tilde{c}^0$ is a hereditary subset of E^0, which in turn yields that $P_{ec}(E)$ is a hereditary subset of E^0.
(ii) Given $c, d \in X'_{ec}$, $\tilde{c} \neq \tilde{d}$ if and only if $\tilde{c}^0 \cap \tilde{d}^0 = \emptyset$.

We analyze the structure of the ideal generated by $P_{ec}(E)$.

{eceIsPIS}

Lemma 3.7.5. *Let E be an arbitrary graph and K any field. For every cycle c such that $c \in X'_{ec}$, the ideal $I(\tilde{c}^0)$ is isomorphic to a purely infinite simple Leavitt path algebra. Concretely, $I(\tilde{c}^0) \cong L_K({}_HE)$, where $H = \tilde{c}^0$.*

Proof. Observing that H is a hereditary subset of E^0, we may use Theorem 2.5.19 and Remark 2.5.21(iii) to get that $I(\tilde{c}^0)$ is isomorphic to the Leavitt path algebra $L_K({}_HE)$. We will show that this Leavitt path algebra is purely infinite and simple by invoking the Purely Infinite Simplicity Theorem 3.1.10.

To show that every vertex of ${}_HE$ connects to a cycle, take $v \in {}_HE^0$. If $v \in H$ then it connects to c by definition of $H = \tilde{c}^0$. If $v \notin H$ then there is $f \in ({}_HE)^1$ such that $s(f) = v$ and $r(f) \in H$. Hence v connects to c too.

Next, we show that every cycle in ${}_HE$ has an exit. Pick such a cycle d; then necessarily by the definition of ${}_HE$, d is a cycle in H. Since by construction we have $\tilde{d} = \tilde{c}$, this means that d connects to c and hence it has an exit in E, which is also an exit in ${}_HE$.

Finally, to show that the only hereditary saturated subsets of $({}_HE)^0$ are $\emptyset$ and $({}_HE)^0$, let $\emptyset \neq H' \in \mathscr{H}_{{}_HE}$, and consider $v \in H'$. Note that every pair of vertices in H is connected by a path, and that $({}_HE)^0$ is the saturation of H in ${}_HE$. Hence, if $v \in H$ then $H' = ({}_HE)^0$. If $v \notin H$ then there exists $f \in ({}_HE)^1$ such that $v = s(f)$ and $r(f) \in H$. This implies $({}_HE)^0 \subseteq H'$, so we get equality too. ☐

{DescriptionPece}

Proposition 3.7.6. *Let E be an arbitrary graph and K any field. Then*

$$I(P_{ec}(E)) = \oplus_{\tilde{c} \in X_{ec}} I(\tilde{c}^0).$$

Furthermore, $I(\tilde{c}^0)$ is isomorphic to a purely infinite simple Leavitt path algebra for each $\tilde{c} \in X_{ec}$.

Proof. The hereditary set $P_{ec}(E)$ can be partitioned as $P_{ec}(E) = \sqcup_{\tilde{c} \in X_{ec}} \tilde{c}^0$. By the Remark 3.7.4(ii) and Proposition 2.4.7, $I(P_{ec}(E)) = I(\sqcup_{\tilde{c} \in X_{ec}} \tilde{c}^0) = \oplus_{\tilde{c} \in X_{ec}} I(\tilde{c}^0)$. Finally, each $I(\tilde{c}^0)$ is isomorphic to a purely infinite simple Leavitt path algebra by Lemma 3.7.5. ☐

{intersection}

Lemma 3.7.7. *Let E be an arbitrary graph and K any field. Then the hereditary sets $P_l(E)$, $P_c(E)$ and $P_{ec}(E)$ are pairwise disjoint. Consequently, the ideal of $L_K(E)$ generated by their union is $I(P_l(E)) \oplus I(P_c(E)) \oplus I(P_{ec}(E))$.*

Proof. By the definition of $P_l(E)$, $P_c(E)$ and $P_{ec}(E)$, they are pairwise disjoint. To get the result, apply Proposition 2.4.7. ☐

The following ideal will be of use later on, so we name it here.

{lce}

Definition 3.7.8. For a graph E we define

$$I_{lce} := I(P_l(E)) \oplus I(P_c(E)) \oplus I(P_{ec}(E)).$$

As mentioned at the start of this section, the ideal I_{lce} captures the essential structural properties of the three primary colors of Leavitt path algebras, a statement we now make more precise.

{idealDenso}

Theorem 3.7.9. *Let E be an arbitrary graph and K any field. Consider I_{lce}, the ideal of $L_K(E)$ presented in Definition 3.7.8. Then*

$$I_{lce} \cong \left(\oplus_{i \in \Gamma_1} \mathrm{M}_{\Lambda_i}(K)\right) \oplus \left(\oplus_{j \in \Gamma_2} \mathrm{M}_{\Lambda_j}(K[x, x^{-1}])\right) \oplus \left(\oplus_{l \in \Gamma_3} I(\tilde{c}_l^0)\right),$$

where Γ_1 is the index set of the disjoint decomposition of $P_l(E)$ into hereditary sets, i.e., $P_l(E) = \sqcup_{i\in\Gamma_1} H_i$ as in Lemma 2.6.13, and for every $i \in \Gamma_1$ we have that Λ_i denotes the cardinal of the set $\{\mu\mu^* \mid \mu \in \mathrm{Path}(E),\ r(\mu) \in H_i\}$; where Γ_2 is the index set of the cycles without exits in E and for every $j \in \Gamma_2$ it happens that Λ_j is the cardinal of the set of different paths ending at the basis of cycle without exits c_j and not containing all the edges of the cycle; and where Γ_3 indexes $X_{ec}(E)$ (see Definition 3.7.1).

{threetypesofverticesE0finite}

Proof. We know that $I(P_l(E))$ is the socle of the Leavitt path algebra $L_K(E)$ and also its structure, which is precisely the given in the statement, by Theorem 2.6.14. The structure of $I(P_c(E))$ was established in Theorem 2.7.3. Finally, the structure of the third summand in I_{lce} follows by Proposition 3.7.6. That each direct summand $I(\tilde{c}^0_{n_l})$ is purely infinite simple (loosely speaking, is of the same primary color as the Leavitt algebras) follows from Proposition 3.7.6. □

In general the ideal I_{lce} of $L_K(E)$ need not be essential in $L_K(E)$. For example, let F denote the "doubly infinite line graph" of Example 3.1.12. Since there are no cycles in F, we get vacuously that $P_{ec}(F) = \emptyset = P_c(E)$. Since there are no line points in F, we have $P_l(F) = \emptyset$, so that, by definition, $I_{lce}(F) = \{0\}$. So we have produced an example of the desired type. However, when E^0 is finite, we show below that the ideal I_{lce} is "large" in $L_K(E)$. The key is the following.

Lemma 3.7.10. *Let E be a graph for which E^0 is finite. Let $v \in E^0$. Then v connects to at least one of: a sink, a cycle without exits, or an extreme cycle.*

{IsublcedensewhenEfinite}

Proof. Recall the preorder $\geq$ on E^0 presented in Definition 2.0.5. Consider partial order resulting from the antisymmetric closure of $\geq$; denote it by $\geq'$. The statement will be proved once we show that the minimal elements in $(E^0, \geq')$ are sinks, vertices in cycles without exits, and vertices in extreme cycles.

Indeed, let $v \in E^0$ be a minimal element. If v is a sink, we are done. Otherwise, there exists $w \in E^0$ such that $v \geq w$. The minimality of v implies $w \geq' v$, hence there is a cycle c in E such that $v, w \in c^0$. If c has no exits, we are done. Otherwise, let μ be a path in E of length ≥ 1 such that the first edge appearing in μ is an exit for c. Then $v \geq s(\mu)$. Again by the minimality of v we have $s(\mu) \geq' v$. This implies that every path starting at a vertex of c^0 returns to c^0 and so c is an extreme cycle as required. □

Proposition 3.7.11. *Let E be a graph for which E^0 is finite. Then I_{lce} is an essential ideal of $L_K(E)$.*

Proof. Let $v \in E^0$. Since E^0 is finite then Lemma 3.7.10 ensures that v connects to a line point, or to a cycle without exists, or to an extreme cycle. This means that every vertex of E connects to the hereditary set $P_l(E) \cup P_c(E) \cup P_{ec}(E)$ and, consequently, to its hereditary saturated closure, which we denote by H. By Proposition 2.7.10 this means that $I(H)$ is an essential ideal of $L_K(E)$, and by Lemma 3.7.7 it coincides with I_{lce}. □

We note that although I_{lce} is an essential ideal of $L_K(E)$ when E^0 is finite, I_{lce} need not equal all of $L_K(E)$. We see this behavior in $L_K(E_T)$, where E_T is the Toeplitz graph as discussed in Example 3.7.3. Here we have $P_{ec}(E_T) = \emptyset = P_c(E_T)$, and $P_l(E_T)$ is the sink v. So $I_{lce}(E_T) = I(\{v\}) \neq L_K(E_T)$ (because $\{v\} \in \mathscr{H}_{E_T}$).

{Nonsimplepi}

3.8 Purely infinite without simplicity

We conclude Chapter 3 by presenting a description of the purely infinite (but not necessarily simple) Leavitt path algebras arising from row-finite graphs. As happened in the purely infinite simple case (Section 3.1), an in-depth analysis of the idempotent structure of $L_K(E)$ will be required. Many of the fundamental ideas in this section can be found in the seminal paper [42].

{def::s-unital}

The general theory of purely infinite rings works smoothly for s-unital rings, defined here.

Definition 3.8.1. A ring R is said to be *s-unital* provided for each a in R there exist b in R such that $a = ab = ba$. By [22, Lemma 2.2], if R is s-unital then for each finite subset F of R there is an element u in R such that $ux = x = xu$ for all $x \in F$.

{blocksumsdefinition}

Of course all rings with local units are s-unital, so that all Leavitt path algebras fall under this umbrella. For an example of an s-unital ring without nonzero idempotents, consider the algebra $C_c(\mathbb{R})$ of those continuous functions on the real line having compact support.

We start by recalling the definitions of the properties *properly purely infinite* and *purely infinite* in a general non-unital, non-simple ring, introduced in [42]. We will then specialize to the simple case.

Definition 3.8.2. Let R be a ring, and suppose x and y are square matrices over R, say $x \in \mathrm{M}_k(R)$ and $y \in \mathrm{M}_n(R)$ for $k,n \in \mathbb{N}$. We use $\oplus$ to denote block sums of matrices; thus,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in \mathrm{M}_{k+n}(R),$$

and similarly for block sums of more than two matrices. We define a relation $\precsim$ on matrices over R by declaring that $x \precsim y$ if and only if there exist $\alpha \in \mathrm{M}_{kn}(R)$ and $\beta \in \mathrm{M}_{nk}(R)$ such that $x = \alpha y \beta$.

It is not hard to show that if x and y are idempotent matrices, then $x \precsim y$ if and only if $x \sim f$, where f is an idempotent such that $f \leq y$.

{infdef}

For any ring R and element $a \in R$, the expression RaR denotes the set of all finite sums $\sum_{i=1}^n z_i a t_i$, where $z_i, t_i \in R$. In case R is s-unital, then RaR is precisely the ideal of R generated by a.

{EveryPropInfImpliesPi}

Definitions 3.8.3. Let R be any ring.

(i) We call an element $a \in R$ *properly infinite* if $a \neq 0$ and $a \oplus a \precsim a$.
(ii) We call R *purely infinite* if the following two conditions are satisfied:
  (1) no quotient of R is a division ring, and
  (2) whenever $a \in R$ and $b \in RaR$, then $b \precsim a$ (i.e., $b = xay$ for some $x,y \in R$).
(iii) We call R *properly purely infinite* if every nonzero element of R is properly infinite.

Lemma 3.8.4. *Let R be an s-unital ring.*

(i) *If R is properly purely infinite, then R is purely infinite.*
(ii) *If $\mathrm{M}_2(R)$ is purely infinite, then R is properly purely infinite.*

Proof. (i) Suppose first that R/I is a division ring for some ideal I of R. Take a nonzero element $\overline{a}$ of R/I. Then a is a nonzero element in R, and thus by hypothesis is properly infinite. So there exist elements $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & 0 \end{pmatrix}.$$

But then in R/I we have that

$$\begin{pmatrix} \overline{a} & 0 \\ 0 & \overline{a} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}_1 \overline{a} \overline{\beta}_1 & \overline{\alpha}_1 \overline{a} \overline{\beta}_2 \\ \overline{\alpha}_2 \overline{a} \overline{\beta}_1 & \overline{\alpha}_2 \overline{a} \overline{\beta}_2 \end{pmatrix}.$$

Since $\overline{a} \neq 0$, it follows that $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\beta}_1, \overline{\beta}_2$ are all nonzero. Now, since R/I is a division ring, $\overline{\alpha}_1 \overline{a} \overline{\beta}_2 = 0$ implies $\overline{a} = 0$, a contradiction. This shows that no quotient of R is a division ring, so that Condition (1) of Definitions 3.8.3(ii) holds.

Now let $a \in R$ be properly infinite and $b \in RaR$. By using that R is s-unital, one can easily see that $x_1 + x_2 + \cdots + x_r \precsim x_1 \oplus x_2 \oplus \cdots \oplus x_r$ for all $x_1, \ldots, x_r \in R$, cf. [42, Lemma 2.2]. Write $b = \sum_{i=1}^n x_i a y_i$ for some $x_i, y_i \in R$. We have $x_i a y_i \precsim a$ for all $1 \leq i \leq n$, whence by the above, we have

$$b \precsim x_1 a y_1 \oplus x_2 a y_2 \oplus \cdots \oplus x_n a y_n \precsim a \oplus a \oplus \cdots \oplus a \precsim a,$$

with the final $\precsim$ being a consequence of $a \oplus a \precsim a$. This establishes Condition (2) of Definitions 3.8.3(ii), and yields the result.

(ii) As R is s-unital, given $a \in R$ there exists $u \in R$ such that $ua = au = a$. Hence,

$$a \oplus a = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(R)(a \oplus 0)\mathrm{M}_2(R).$$

Since $\mathrm{M}_2(R)$ is assumed to be purely infinite, it follows that $a \oplus a \precsim a \oplus 0$, and so $a \oplus a \precsim a$. Therefore a is either zero or properly infinite. $\square$

{lem:first-stepi-e}

The concepts of properly purely infinite and purely infinite agree for *simple* s-unital rings. Moreover, in this case we can relate these conditions to the existence of infinite idempotents in all nonzero right (or left) ideals, see Proposition 3.8.7 below. However there are simple, non s-unital rings, which are purely infinite but not properly purely infinite ([42, Example 3.5]).

We first show, in the next few lemmas, that every simple s-unital purely infinite ring contains nonzero idempotents.

{rels:existsid}

Lemma 3.8.5. *Let R be a ring (without a unit), and suppose that R contains nonzero elements x,y,u,v satisfying the relations*

$$vu = uv = u, \quad yu = y, \quad vx = x, \quad v = yx. \tag{3.1}$$

Then R contains a nonzero idempotent.

Proof. Let $\widetilde{R}$ denote a ring obtained by adjoining a unit to R. Then in $\widetilde{R}$ we have

$$(y+(1-v))(x+(1-u)) = yx + y(1-u) + (1-v)x + (1-v)(1-u) = v + 0 + 0 + (1-v) = 1.$$

{idem-exists}

It follows that $e = (x+(1-u))(y+(1-v))$ is an idempotent in $\widetilde{R}$, whence $1-e$ is an idempotent which is easily seen to belong to R. If $e \neq 1$, then $1-e$ is the desired nonzero idempotent in R.

Suppose that $e = 1$. Then $y = yeu = y(x+(1-u))(y+(1-v))u = yxy \in R$, which shows that $v = yx$ is a (nonzero) idempotent in R. $\square$

Lemma 3.8.6. *If R is s-unital, simple, and purely infinite then R contains a nonzero idempotent.*

Proof. Let $0 \neq x \in R$, so (as R is s-unital) there exists $a \in R$ with $ax = xa = x$. Then $0 \neq x = xa = xa^2$, so that $a^2 \neq 0$. Using two times the s-unitality, we see that there are $b,c \in R$ such that $ab = ba = a$ and $bc = cb = b$. Since R is purely infinite, there are $s,t \in R$ such that $c = sa^2t$. Thus we have the following relations between a,b,c and s,t:

$$ab = ba = a, \quad bc = cb = b, \quad \text{and } \ c = sa^2t.$$

{simples-unital}

Define $x = at$, $y = sa$, $v = c$, and $u = b$. Then $vu = uv = u$, $yx = sa^2t = v$, $vx = cat = cbat = bat = at = x$, and $yu = sab = sa = y$. So x,y,u,v are nonzero elements of R satisfying the relations (3.1), and so it follows from Lemma 3.8.5 that R contains a nonzero idempotent. $\square$

We now obtain the promised characterization of purely infinite simple s-unital rings. In particular all the conditions below are equivalent for a simple Leavitt path algebra.

Proposition 3.8.7. *Let R be a simple s-unital ring. Then the following are equivalent:*

(1) *R is properly purely infinite.*
(2) *R is purely infinite.*
(3) *For every nonzero $a \in R$ there are elements $s,t \in R$ such that sat is a nonzero, infinite idempotent.*
(4) *Every nonzero one-sided ideal of R contains a nonzero infinite idempotent.*

Proof. (1) $\Rightarrow$ (2) follows from Lemma 3.8.4(i).

(2) $\Rightarrow$ (3). By Lemma 3.8.6 R contains a nonzero idempotent e. So given a nonzero element a in R there exist s,t in R such that $e = sat$. It remains to check that every nonzero idempotent in R is infinite. Let e be a nonzero idempotent. Assume first that e is a unit for R. Then, since R is not a division ring, there is a nonzero a in R such that a is not left invertible in R. Let $s,t \in R$ be such that $sat = e$. Then $f := tsa$ is an idempotent in R with $e \sim f$ and $f \neq e$, which implies that e is infinite. Finally assume that e is not a unit for R. We may assume that $(1-e)x \neq 0$ for some $x \in R$, where here $1 \in \widetilde{R}$ if R is not unital. As before we can find an idempotent $f \in (1-e)xR$ such that $f \sim e$. But now $g := f(1-e)$ is an idempotent in R orthogonal to e, and equivalent to e. Since $e+g = uev$ for some $u,v \in R$, there is an idempotent $h \leq e$ such that $h \sim e+g \sim e \oplus e$, showing indeed that e is properly infinite. This completes the argument.

(3) $\Rightarrow$ (4) is contained in Proposition 3.1.7.

(4) $\Rightarrow$ (1). First observe that, as R is a simple ring, every infinite idempotent in R is indeed properly infinite. Now let a be a nonzero element in R. By assumption, there is a properly infinite idempotent e in R such that $e \precsim a$. Since R is simple there exists $n \geq 1$ such that $a \precsim n \cdot e = e \oplus e \oplus \cdots \oplus e$. Thus we get

$$a \oplus a \precsim n \cdot e \oplus n \cdot e \precsim e \precsim a,$$

showing that a is properly infinite. $\square$

{pinftoidealquo}

Lemma 3.8.8. *Let I be an ideal of an arbitrary ring R.*

(i) *If R is (properly) purely infinite, then so is R/I.*
(ii) *Suppose that I is s-unital when viewed as a ring. If R is (properly) purely infinite, then so is I.*

Proof. (i) It is clear that proper pure infiniteness passes from R to R/I. Now assume only that R is purely infinite. Since any quotient of R/I is also a quotient of R, no quotient of R/I is a division ring. Consider $a, b \in R$ such that $\overline{b} \in (R/I)\overline{a}(R/I)$. Then there is some $c \in RaR$ such that $\overline{c} = \overline{b}$. By hypothesis, $c = xay$ for some $x, y \in R$, and therefore $\overline{b} = \overline{c} = \overline{xay}$.

(ii) Assume first the specific case in which R is properly purely infinite, and let $0 \neq a \in I$. Then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} a \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$. Since I is s-unital, we also have $a = ua = au$ for some $u \in I$. Then

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha_1 u \\ \alpha_2 u \end{pmatrix} a \begin{pmatrix} u\beta_1 & u\beta_2 \end{pmatrix}$$

with $\alpha_1 u, \alpha_2 u, u\beta_1, u\beta_2 \in I$. This proves that I is properly purely infinite.

Now assume the general case, so we assume only that R is purely infinite. Suppose first that I has an ideal J such that I/J is a division ring. Since I is s-unital, J is an ideal of R. Since R/J is purely infinite by (i), it suffices to find a contradiction working in R/J. Thus, there is no loss of generality in assuming that $J = 0$. If e is the unit of I, then $I = eI = Ie$, and so $I = eR = Re$. It follows that $er = ere = re$ for all $r \in R$, whence e is a central idempotent of R. But then the annihilator of e in R is an ideal T such that $R = I \oplus T$, and $R/T \cong I$ is a division ring, contradicting the assumption that R is purely infinite. Therefore no quotient of I is a division ring.

Secondly, if $a \in I$ and $b \in IaI$, then we at least have $b = xay$ for some $x, y \in R$. Since also $a = ua = au$ for some $u \in I$, we have $b = (xu)a(uy)$ with $xu, uy \in I$. Thus I satisfies the two required conditions, and is therefore purely infinite. $\square$

{cornerpurinf}

Lemma 3.8.9. *Let e be an idempotent in a ring R. If R is (properly) purely infinite, then so is eRe.*

Proof. Assume first that R is properly purely infinite. Any nonzero element $a \in R$ is properly infinite in R, and so $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} a \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$. Then

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} e\alpha_1 e \\ e\alpha_2 e \end{pmatrix} a \begin{pmatrix} e\beta_1 e & e\beta_2 e \end{pmatrix},$$

which shows that a is properly infinite in eRe. Therefore eRe is properly purely infinite in this case.

Now assume only that R is purely infinite. We first show that a prime purely infinite ring does not contain idempotents e such that eRe is a division ring. To do so, suppose that R is a prime purely infinite ring, and we have an idempotent $e \in R$ such that eRe is a division ring. Since R is prime, eR is a simple right R-module.

If $eR = R$, then $(R(1-e))^2 = 0$ and so $R(1-e) = 0$ because R is prime. (Here we are writing $R(1-e)$ for the left ideal $\{r - re \mid r \in R\}$.) But then $R = eRe$ and R is a division ring, contradicting the hypothesis that R is purely infinite. Thus, $eR \neq R$ and so $(1-e)R \neq 0$. Now $(1-e)ReR \neq 0$ because R is prime, and hence there exists a nonzero element $a \in (1-e)Re$. Note that aR is a nonzero homomorphic image of eR, whence aR is a simple right R-module. Since R is prime, $aR = gR$ for some idempotent g, and $eg = 0$

because $ea = 0$. Observe that $g - ge$ is an idempotent which generates gR, so we can replace g by $g - ge$. Hence, there is no loss of generality in assuming that $e \perp g$.

Now $f = e + g$ is an idempotent such that $fR = eR \oplus aR$, and $f \in ReR$ because $gR = aR \subseteq ReR$. Since R is purely infinite, $f = xey$ for some $x, y \in R$. But then fR is a homomorphic image of eR, implying that fR is simple or zero, which is impossible in light of $fR = eR \oplus aR$. This contradiction establishes our claim.

Suppose now that I is an ideal of eRe such that eRe/I is a division ring. In this case I is a maximal ideal of eRe. Moreover, $e \notin (eRe)I(eRe) = eRIRe$, and so $e \notin RIR$. Consequently, $\overline{e}$ is a nonzero idempotent in R/RIR, and in particular, $\overline{e}$ cannot be in the Jacobson radical of R/RIR. Hence, there exists a (left) primitive ideal P of R such that $e \notin P$ and $RIR \subseteq P$. Now $I \subseteq P \cap eRe \subsetneqq eRe$, and by maximality of I in eRe we have $I = P \cap eRe$. This yields $eRe/I = eRe/(P \cap eRe) \cong \overline{e}(R/P)\overline{e}$. But this means that the purely infinite prime ring R/P has a corner which is a division ring, contradicting the claim above. Therefore no quotient of eRe is a division ring.

Establishing the second condition is easier. Suppose that $a \in eRe$ and $b \in (eRe)a(eRe) \subseteq RaR$. Since R is purely infinite, there exist $x, y \in R$ such that $b = xay$, and hence $b = (exe)a(eye)$ with $exe, eye \in eRe$. This shows that eRe is purely infinite. $\square$

Our next goal is to characterize the properly infinite vertices of a Leavitt path algebra. Recall that a characterization of the infinite vertices has been given in Proposition 3.1.6.

{vInfiniteIdempotent}

Lemma 3.8.10. *Let E be an arbitrary graph. If $v \in E^0$ and $|CSP(v)| \geq 2$, then v is a properly infinite idempotent in $L_K(E)$.*

Proof. Let $e_1 \cdots e_m$ and $f_1 \cdots f_n$ be two different closed simple paths in E based at v. Then there is some positive integer t such that $e_i = f_i$ for $i = 1, \ldots, t-1$ while $e_t \neq f_t$. Thus, we have at least two different edges leaving the vertex $r(e_{t-1}) = r(f_{t-1})$. We compute that

$$v = s(e_1) \gtrsim r(e_1) \gtrsim \cdots \gtrsim r(e_{t-1}) \gtrsim r(e_t) \oplus r(f_t) \gtrsim r(e_{t+1}) \oplus r(f_{t+1}) \gtrsim \cdots \gtrsim r(e_m) \oplus r(f_n) = v \oplus v.$$

Therefore v is properly infinite. $\square$

We now obtain a characterization of the properly infinite vertices in a Leavitt path algebra. Recall that for $X \subseteq E^0$, we denote by $\overline{X}$ the hereditary saturated closure of X.

{proinf-charac}

Proposition 3.8.11. *Let E be an arbitrary graph and K any field. Let $v \in E^0$. Then v is a properly infinite idempotent in $L_K(E)$ if and only if there are vertices $w_1, \ldots, w_n$ in $T(v)$ such that $|CSP(w_i)| \geq 2$ for all i and $v \in \overline{\{w_1, \ldots, w_n\}}$.*

Proof. Assume that v is properly infinite. Let W be the set of vertices w in $T(v)$ such that $|CSP(w)| \geq 2$. If $I(v) = I(W)$ then there is a finite number $w_1, \ldots, w_n$ of elements of W such that $I(v) = I(\{w_1, \ldots, w_n\})$. It then follows that $v \in \overline{\{w_1, \ldots, w_n\}}$. It suffices therefore to show that $I(v) = I(W)$. On the contrary, suppose $I(W)$ is strictly contained in $I(v)$. Then by Zorn's Lemma there exists a hereditary saturated subset H properly contained in $\overline{T(v)}$ and containing $\overline{W}$. Then $L_K(E)/I(H \cup B_H^H) \cong L_K(E/H)$, and $X := \overline{T(v)} \setminus H$ is a hereditary saturated subset of E/H not containing any non-trivial hereditary saturated subsets. By Theorem 2.5.19 we have $I(v)/I(H \cup B_H^H) \cong L_K({}_X(E/H))$, and $L_K({}_X(E/H))$ is graded simple. Moreover, v is a properly infinite idempotent in $L_K({}_X(E/H))$, and it follows from the Trichotomy Principle (Proposition 3.1.14) that $L_K({}_X(E/H))$ is purely infinite simple. Therefore there exists $w \in T_{E/H}(v)$ such that $|CSP_{E/H}(w)| \geq 2$. Thus we obtain $w \in T(v) \setminus H$ and $|CSP_E(w)| \geq 2$, so that $w \in W \setminus H$, which is a contradiction, and thereby establishes one direction.

Conversely, assume that there are distinct vertices $w_1, \ldots, w_n$ in $T(v)$ such that $|CSP(w_i)| \geq 2$ for all i and $v \in \overline{\{w_1, \ldots, w_n\}}$. By Lemma 3.8.10, $e := w_1 + w_2 + \cdots + w_n$ is a properly infinite idempotent of $L_K(E)$. We claim that $e \lesssim v$. If $w_j \in T(w_i)$ for $i \neq j$, then $w_i \oplus w_j \lesssim w_j \oplus w_j \lesssim w_j$ and so we can eliminate such w_i. Thus we may assume without loss of generality that $w_i \notin T(w_j)$ for all $i \neq j$. For each i, let $\gamma_i \in \mathrm{Path}(E)$ with $s(\gamma_i) = v$ and $r(\gamma_i) = w_i$. Since $w_i \notin T(w_j)$ for all i, we see that the paths $\gamma_1, \gamma_2, \cdots, \gamma_n$ are pairwise incomparable, so that $\gamma_i^* \gamma_j = 0$ if $i \neq j$, and thus

$$g := \gamma_1 \gamma_1^* + \gamma_2 \gamma_2^* + \cdots + \gamma_n \gamma_n^*$$

is an idempotent such that $g \leq v$, and such that

$$e = w_1 + w_2 + \cdots + w_n \sim g.$$

It follows that $w_1 + w_2 + \cdots + w_n \lesssim v$. Since $I(v) = I(w_1, \ldots, w_n) = I(w_1 + \cdots + w_n)$, we have $v \lesssim \ell \cdot (w_1 + \cdots w_n) = \ell \cdot e$ for some $\ell \in \mathbb{N}$. Finally we have

$$v \oplus v \lesssim 2\ell \cdot (w_1 + \cdots + w_n) \lesssim w_1 + \cdots + w_n \lesssim v,$$

which shows that v is properly infinite. $\square$

{remonCSPs}

Remark 3.8.12. It follows easily from Proposition 3.8.11 that, for a vertex v of an arbitrary graph E, if v is a properly infinite idempotent in $L_K(E)$, then $|CSP(v)|$ is either 0 or ≥ 2.

{def:infiniteelement}

Definition 3.8.13. An element a of a ring R is said to be an *infinite element* in case $a \oplus b \precsim a$ for some nonzero element b in R. Obviously, a properly infinite element of R is an infinite element of R.

{lem:infinite-andH}

Lemma 3.8.14. *Let E be an arbitrary graph and K be any field. Suppose that every nonzero ideal of every quotient of $L_K(E)$ contains an infinite element. Then E satisfies Condition (K), and $B_H = \emptyset$ for every $H \in \mathscr{H}_E$.*

Proof. To show that E satisfies Condition (K), we have to check that $C_H = \emptyset$ for every $H \in \mathscr{H}_E$ (see the proof of Corollary 2.9.9). If $C_H \neq \emptyset$ for some $H \in \mathscr{H}_E$, then by the Structure Theorem for Ideals 2.8.10 there is a subquotient of $L_K(E)$ isomorphic to $\mathrm{M}_\Lambda(p(x)K[x,x^{-1}])$, for some set Λ, where $p(x)$ is a polynomial of the form $1 + a_1x + \cdots + a_nx^n$, with $n > 0$ and $a_n \neq 0$. Since $K[x,x^{-1}]$ embeds into a field, rank considerations show immediately that there are no infinite elements in the ring $\mathrm{M}_\Lambda(p(x)K[x,x^{-1}])$. Therefore our hypothesis implies that $C_H = \emptyset$ for all $H \in \mathscr{H}_E$.

Now suppose that, for some $H \in \mathscr{H}_E$, we have $B_H \neq \emptyset$. Then the algebra $L_K(E)/I(H) \cong L_K(E/(H,\emptyset))$ has a nonzero socle, indeed the ideal $I(H \cup B_H^H)/I(H)$ is a nonzero ideal of $L_K(E)/I(H)$ contained in the socle of $L_K(E)/I(H)$ (see Theorem 2.4.15). Since clearly the socle (of any semiprime ring) cannot contain infinite elements, we obtain a nonzero subquotient of $L_K(E)$ with no infinite elements, contradicting our hypothesis. $\square$

Recall that a nonzero element u of a conical monoid V is said to be *irreducible* in case u cannot be written as a sum of two nonzero elements ([42, Definitions 6.1]). Observe that, for an idempotent e of a ring R, we have that $[e]$ is irreducible in $\mathscr{V}(R)$ if and only if e is a primitive idempotent of R.

We are now in position to present the main result of this section, in which we characterize the purely infinite Leavitt path algebras.

{LeavittPi}

Theorem 3.8.15. *Let E be an arbitrary graph and K any field. The following are equivalent.*

(1) *Every nonzero ideal of every quotient of $L_K(E)$ contains an infinite vertex, i.e., if $I \subsetneq J$ are ideals of $L_K(E)$, then there exists $v \in E^0$ such that $v \in J \setminus I$ and such that $v + I$ is an infinite idempotent of $L_K(E)/I$.*
(2) *Every nonzero right ideal of every quotient of $L_K(E)$ contains an infinite idempotent.*
(3) *Every nonzero left ideal of every quotient of $L_K(E)$ contains an infinite idempotent.*
(4) *$L_K(E)$ is properly purely infinite.*
(5) *$L_K(E)$ is purely infinite.*
(6) *Every vertex $v \in E^0$ is properly infinite as an idempotent in $L_K(E)$, and $B_H = \emptyset$ for all $H \in \mathscr{H}_E$.*

Proof. We recall that $L_K(E)$ has local units (cf. Lemma 1.2.12(v)), so that all previously established results about s-unital rings apply here.

(1) $\Rightarrow$ (2) and (3). Observe that Lemma 3.8.14 gives that E satisfies Condition (K) and that $B_H = \emptyset$ for every $H \in \mathscr{H}_E$. Therefore all the ideals of $L_K(E)$ are of the form $I(H)$ for some $H \in \mathscr{H}_E$. So a nonzero quotient of $L_K(E)$ will be of the form $L_K(E/H)$. Moreover, by Theorem 3.3.11, each such E/H necessarily satisfies Condition (L).

Let v be a vertex of E/H. If v does not connect to any cycle in E/H, then $\overline{T_{E/H}(v)}$ is an acyclic graph, and thus the ideal generated by v in $L_K(E/H)$ does not contain any infinite vertex, by Proposition 3.1.6, contradicting (1). Therefore every vertex of E/H connects to a cycle with exits, and again by Proposition 3.1.6, we get that every vertex is infinite.

By Proposition 2.9.13, every nonzero one-sided ideal of $L_K(E/H)$ contains a nonzero idempotent. By Corollary 3.2.12, it only remains to show that every idempotent of the form $v - \sum_{e\in Z} ee^*$, where $v \in \mathrm{Inf}(E/H)$ and Z is a non-empty finite subset of $s^{-1}_{E/H}(v)$, is infinite. But in this situation we can choose $f \in s^{-1}_{E/H}(v) \setminus Z$, and $ff^* \leq v - \sum_{e\in Z} ee^*$, with $ff^* \sim f^*f = r(f)$, which is an infinite idempotent in $L_K(E/H)$ by the above. It follows that every nonzero idempotent of $L_K(E/H)$ is infinite, and so every nonzero one-sided ideal of $L_K(E/H)$ contains an infinite idempotent.

(2) or (3) $\Rightarrow$ (4). This holds in any s-unital ring, see e.g. [42, Proposition 3.13].

(4) $\Rightarrow$ (5). This implication also holds in any s-unital ring, by Lemma 3.8.4(i).

(5) $\Rightarrow$ (6). Let v be a vertex in E. By Proposition 3.6.21, $\mathscr{V}(L_K(E))$ is a refinement monoid. Hence, by [42, Theorem 6.10], in order to show that v is properly infinite as an idempotent of $L_K(E)$, it suffices to show that $[v]$ is not irreducible in any quotient of $\mathscr{V}(L_K(E))$.

By Theorem 3.6.23(i), any order-ideal I of $\mathscr{V}(L_K(E))$ is of the form $\mathscr{V}(I(H\cup S^H))$, where H is a hereditary saturated subset of E^0 and $S \subseteq B_H$. Moreover, it follows from Theorem 3.6.23(ii) that we have monoid isomorphisms

$$\mathscr{V}(L_K(E))/I \cong \mathscr{V}(L_K(E)/I(H\cup S^H)) \cong \mathscr{V}(L_K(E/(H,S))).$$

Since there is nothing to do if $[v] \in I$, we may assume that $v \notin H$. By Lemma 3.8.8(i), $L_K(E/(H,S)) \cong L_K(E)/I(H\cup S^H)$ is purely infinite, and so for this part of the proof we may replace $L_K(E)$ by $L_K(E/(H,S))$. Thus, we need only show that $[v]$ is not irreducible in $\mathscr{V}(L_K(E))$, or equivalently, that v is not a primitive idempotent.

By Proposition 3.5.2, if v is a primitive idempotent then there cannot be any bifurcations in $T(v)$. So either v is a line point, or there is a unique shortest path connecting v to a cycle without exits. So we get that either $vL_K(E)v \cong K$, or $vL_K(E)v \cong K[x,x^{-1}]$. In any case $vL_K(E)v$ is not properly infinite, contradicting Lemma 3.8.9.

We now show that $B_H = \emptyset$ for every $H \in \mathscr{H}_E$. Let $H \in \mathscr{H}_E$. Then $L_K(E)/I(H) \cong L_K(E/(H,\emptyset))$ is properly infinite by Lemma 3.8.8(i), so by the preceeding argument every vertex of $L_K(E/(H,\emptyset))$ is properly infinite. But if $v \in B_H$ then the idempotent v' in the graph $E/(H,\emptyset)$ (which corresponds to the class of v^H) belongs to the socle of $L_K(E/(H,\emptyset))$ and so cannot be properly infinite. This shows that $B_H = \emptyset$.

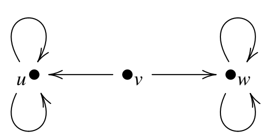
{graphconditionrequired}

(6) $\Rightarrow$ (1). By Proposition 3.8.11, for every $v \in E^0$ there are $w_1,\dots,w_n \in T(v)$ such that $|CSP(w_i)| \geq 2$ for all i such that $I(v) = I(\{w_1,\dots,w_n\})$. It follows in particular that E satisfies Condition (L). Since the same is true for every graph E/H, where H is a hereditary saturated subset of E^0, we conclude that E satisfies Condition (K) by Theorem 3.3.11. It follows from Proposition 2.9.9 that every ideal of $L_K(E)$ is a graded ideal. Since $B_H = \emptyset$ for every $H \in \mathscr{H}_E$, it follows from the Structure Theorem for Graded Ideals 2.5.8 that every ideal of $L_K(E)$ is of the form $I(H)$ for some $H \in \mathscr{H}_E$. Thus every nonzero ideal of every quotient $L_K(E)/I(H) \cong L_K(E/H)$ of $L_K(E)$ contains a vertex (by Proposition 2.9.13), which is necessarily (properly) infinite. $\square$

{examplebis}

Remark 3.8.16. As a result of Proposition 3.8.11, Condition (6) of Theorem 3.8.15 provides a characterization of purely infinite Leavitt path algebras $L_K(E)$ which depends solely on properties of the graph E.

Example 3.8.17. We present an example of a purely infinite non-simple Leavitt path algebra. Consider the following graph E:



.

We note that $L_K(E)$ is purely infinite because all vertices in E are properly infinite, and E is row-finite. (Observe that v is properly infinite by using Proposition 3.8.11.) On the other hand, it is non-simple because $\{u\}$ and $\{w\}$ are hereditary saturated subsets.

{piimpliesexchange}

We close the chapter by recording the following consequence of Theorem 3.8.15.

Corollary 3.8.18. *Let E be an arbitrary graph and K any field. If $L_K(E)$ is purely infinite then $L_K(E)$ is an exchange ring.*

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