Leavitt path algebras: introduction, motivation, and basic properties

Gene Abrams
University of Colorado
Colorado Springs

Minicourse on Leavitt path algebras, Lecture 1

III Workshop on Dynamics, Numeration, Tilings and Graph Algebras (III FloripaDynSys)

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Overview

1. Leavitt path algebras: Introduction and Motivation
2. Multiplicative properties
3. Projective modules
1. Leavitt path algebras: Introduction and Motivation

2. Multiplicative properties

3. Projective modules
Brief history, and motivating examples

One of the first theorems you saw as an undergraduate student:

**Dimension Theorem for Vector Spaces.** Every nonzero vector space $V$ has a basis. Moreover, if $\mathcal{B}$ and $\mathcal{B}'$ are two bases for $V$, then $|\mathcal{B}| = |\mathcal{B}'|$. 
Brief history, and motivating examples

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Note: $V$ has a basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\} \iff V \cong \bigoplus_{i=1}^{n} \mathbb{R}$ as vector spaces. So:

**One result of Dimension Theorem, Rephrased:**

$$\bigoplus_{i=1}^{n} \mathbb{R} \cong \bigoplus_{i=1}^{m} \mathbb{R} \iff m = n.$$
The same Dimension Theorem holds, with the identical proof, if $K$ is any division ring (i.e., any ring for which every nonzero element has a multiplicative inverse).
Brief history, and motivating examples

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**Idea**: Show any maximal linearly independent subset of $V$ actually spans $V$. Why are there *any* linearly independent subsets at all?

If $0 \neq v \in V$, then $\{v\}$ is linearly independent. If $kv = 0$, need to show $k = 0$. But $k \neq 0 \Rightarrow 1/k \cdot kv = 0 \Rightarrow v = 0$, contradiction. Similar idea (multiply by the inverse of a nonzero element of $K$) shows that a maximal linearly independent subset of $V$ actually spans $V$. 
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Similar idea (multiply by the inverse of a nonzero element of $K$) shows that a maximal linearly independent subset of $V$ actually spans $V$. 
Question: Is the Dimension Theorem true for rings in general? That is, if $R$ is a ring, and $\bigoplus_{i=1}^{n} R \cong \bigoplus_{i=1}^{m} R$ as $R$-modules, must $m = n$? ("module" = "left module")

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Answer: NO

But the answer is YES for many rings, e.g. commutative, or having chain conditions, e.g. \( \mathbb{Z}, M_2(\mathbb{R}), C(\mathbb{R}), \ldots \)
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**Example**: Consider the ring $S$ of linear transformations from an infinite dimensional $\mathbb{R}$-vector space $V$ to itself.

Think of $V$ as $\bigoplus_{i=1}^{\infty} \mathbb{R}$. Then think of $S$ as $RFM(\mathbb{R})$. 
Brief history, and motivating examples

Intuitively, $S$ and $S \oplus S$ have a chance to be “the same”.

$M \mapsto (\text{Odd numbered columns of } M, \text{Even numbered columns of } M)$

More formally: Let

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\[ X_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
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Brief history, and motivating examples

Then $MY_1$ gives the Odd Columns of $M$, while $MY_2$ gives the Even Columns of $M$. 
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Similarly, we should be able to 'go back' from pairs of matrices to a single matrix, by interweaving the columns.

That is, more formally, $(M_1, M_2) \mapsto M_1X_1 + M_2X_2$ is a reasonable way to associate a pair of matrices with a single one.
Brief history, and motivating examples

Here’s what’s really going on. These equations are easy to verify:

\[ Y_1X_1 + Y_2X_2 = I, \]
\[ X_1 Y_1 = I = X_2 Y_2, \quad \text{and} \quad X_1 Y_2 = 0 = X_2 Y_1. \]
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Using these, we get inverse maps \( S \to S \oplus S \) and \( S \oplus S \to S \):

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\[ (M_1, M_2) \mapsto M_1 X_1 + M_2 X_2 \]
\[ \mapsto ((M_1 X_1 + M_2 X_2)Y_1, (M_1 X_1 + M_2 X_2)Y_2) = (M_1, M_2) \]
Brief history, and motivating examples

Using exactly the same idea, let $R$ be ANY ring which contains four elements $y_1, y_2, x_1, x_2$ satisfying

$$y_1x_1 + y_2x_2 = 1_R,$$

$$x_1y_1 = 1_R = x_2y_2, \text{ and } x_1y_2 = 0 = x_2y_1.$$ 

Then $R \cong R \oplus R$. 
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and

$$x_1y_2 = 0 = x_2y_1.$$

Then $R \cong R \oplus R$.

Note for later: i.e., $\sum_{i=1}^{2} y_ix_i = 1_R$ and $x_iy_j = \delta_{i,j}1_R$. 

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Remark: Here the sets $\{1_R\}$ and $\{x_1, x_2\}$ are each bases for $R$. 
Brief history, and motivating examples

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Actually, when \(R \cong R \oplus R\) as \(R\)-modules, then \(\bigoplus_{i=1}^{m} R \cong \bigoplus_{i=1}^{n} R\) for all \(m, n \in \mathbb{N}\).
Leavitt path algebras

Natural question:
Does there exist $R$ with, e.g., $R \cong R \oplus R \oplus R$, but $R \not\cong R \oplus R$?
Leavitt algebras

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Theorem


For every $m < n \in \mathbb{N}$ and field $K$ there exists a $K$-algebra $R = L_K(m, n)$ with $\bigoplus_{i=1}^m R \cong \bigoplus_{i=1}^n R$, and all isomorphisms between free left $R$-modules result precisely from this one. Moreover, $L_K(m, n)$ is universal with this property.
Leavitt algebras

The $m = 1$ situation of Leavitt’s Theorem is now somewhat familiar. Similar to the $n = 2$ case that we saw above, $R \cong R^n$ if and only if there exist

$$x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in R$$

for which

$$\sum_{i=1}^{n} y_i x_i = 1_R$$

and

$$x_i y_j = \delta_{i,j} 1_R.$$
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for which

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\sum_{i=1}^{n} y_i x_i = 1_R \quad \text{and} \quad x_i y_j = \delta_{i,j} 1_R.
\]

\( L_K(1, n) \) is the quotient

\[
K < X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n > / \ < (\sum_{i=1}^{n} Y_i X_i) - 1_K; X_i Y_j - \delta_{i,j} 1_K >
\]

Note: \( RFM(K) \) is much bigger than \( L_K(1, 2) \).
Leavitt path algebras

As a result, we have this: Let $S$ denote $L_K(1, n)$. Then

$$S^a \cong S^b \iff a \equiv b \text{ mod}(n - 1).$$

In particular, $S \cong S^n$, and $n > 1$ is minimal with this property.
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It turns out:

**Theorem.** (Leavitt, Duke J. Math, 1964)

For every field $K$ and $n \geq 2$, $L_K(1, n)$ is simple.

(On the other hand, for $m \geq 2$, $L_K(m, n)$ is not simple.)

Remember, a ring $R$ being *simple* means:

\[ \forall 0 \neq r \in R, \exists \alpha_i, \beta_i \in R \text{ with } \sum_{i=1}^{n} \alpha_i r \beta_i = 1_R. \]
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Actually, $L_K(1, n)$ is VERY simple:

$\forall \ 0 \neq r \in L_K(1, n), \ \exists \ \alpha, \beta \in L_K(1, n) \text{ with } \alpha r \beta = 1_{L_K(1,n)}.$
Building rings from combinatorial objects

Here’s a familiar idea. Consider the set $T = \{x^0, x^1, x^2, \ldots\}$. Define multiplication on $T$ in the usual way: $x^i \cdot x^j = x^{i+j}$.

Consider formal symbols of the form

$$k_1 t_1 + k_2 t_2 + \cdots + k_n t_n$$

where $t_i \in T$, and $k_i \in \mathbb{R}$. Denote this set of symbols by $KT$. We can add and multiply two symbols of this form, as usual, e.g. $(kt)(k't') = kk'(t \cdot t')$. 
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Also, e.g. if we impose the relation $x^n = x^0$ on $T$, call the new semigroup $\overline{T}$, then $\overline{T} = \{x^0, x^1, x^2, ..., x^{n-1}\}$, and

$$\mathbb{R}\overline{T} \cong \mathbb{R}[x]/\langle x^n - 1 \rangle$$
Building rings from combinatorial objects

This is a standard construction to produce rings:

Start with some binary operation on a set $S$, and some field $K$, and form the formal symbols as above. Add and multiply based on addition and 'multiplication' in $K$ and $S$. 
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For instance:

matrix rings, group rings, multivariable polynomial rings, etc ... can all be thought of in this way.
General path algebras

Let $E$ be a directed graph. $E = (E^0, E^1, r, s)$

$$s(e) \cdot e \rightarrow r(e)$$

The *path algebra of $E$ with coefficients in $K$* is the $K$-algebra $KS$ as above, where the underlying set $S$ is the set of all directed paths in $E$ (including vertices), and multiplication of paths is just concatenation. Denote by $KE$. In particular, in $KE$,

For each edge $e$, $s(e) \cdot e = e = e \cdot r(e)$

For each vertex $v$, $v \cdot v = v$
Building Leavitt path algebras

Start with $E$, build its *double graph* $\hat{E}$. 
Building Leavitt path algebras

Start with $E$, build its *double graph* $\hat{E}$. Example:

\[ E = \]

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet \\
& \Downarrow & \Downarrow & \swarrow & \swarrow \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
& \swarrow & & \swarrow & \\
\bullet & & & \bullet & \bullet \\
& & \Downarrow & \swarrow & \\
\bullet & & & \bullet & \bullet \\
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Building Leavitt path algebras

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\[ \begin{array}{cc}
  & t \\
 v & w & \cdot \\
  & j \\
 f & g & i \\
  & w \\
 & x \\
  & u \\
 e & h \\
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Building Leavitt path algebras

Construct the path algebra $K\hat{E}$. 

Consider these relations in $K\hat{E}$:

(CK1) $e^* e = r(e)$; and $f^* e = 0$ for $f \neq e$ (for all edges $e$, $f$ in $E$).

(CK2) $v = \sum_{\{e \in E_1 | s(e) = v\}} ee^*$ for each vertex $v$ in $E$.

(just at those vertices $v$ which are not sinks, and which emit only finitely many edges: "regular" vertices)

Definition

The Leavitt path algebra of $E$ with coefficients in $K$ is $L(K)(E) = K\hat{E}/\langle (CK1), (CK2) \rangle$. 

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University of Colorado @ Colorado Springs

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**Definition**

The Leavitt path algebra of $E$ with coefficients in $\mathbb{K}$

$$L_\mathbb{K}(E) = \mathbb{K}\hat{E} / < (CK1), (CK2) >$$
Leavitt path algebras: Examples

Some sample computations in $L_{\mathbb{C}}(E)$ from the Example:

\[ \hat{E} = \]

\[ ee^* + ff^* + gg^* = v \quad g^*g = w \quad g^*f = 0 \]

\[ h^*h = w \ (CK1) \quad hh^* = u \ (CK2) \]
Leavitt path algebras: Examples

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$$ee^* + ff^* + gg^* = v \quad g^*g = w \quad g^*f = 0$$
$$h^*h = w \ (CK1) \quad hh^* = u \ (CK2)$$

$ff^* = \ldots \ (\text{no simplification})$  Note: $(ff^*)^2 = f(f^*f)f^* = ff^*$

Gene Abrams
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Leavitt path algebras: Examples

Standard algebras arising as Leavitt path algebras:
Leavitt path algebras: Examples

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\[ E = v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \ldots \ldots \ldots \ldots v_{n-1} \xrightarrow{e_{n-1}} v_n \]

Then \( L_K(E) \cong M_n(K) \).
Leavitt path algebras: Examples

Standard algebras arising as Leavitt path algebras:

\[
E = \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots \cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}
\]

Then \(L_K(E) \cong M_n(K)\).

\[
E = \bullet^v \xrightarrow{\x}
\]

Then \(L_K(E) \cong K[x, x^{-1}]\).
Leavitt path algebras: Examples

\[ E = R_n = \]

Then \( L_K(E) \cong L_K(1, n) \).
Leavitt path algebras: Examples

Then $L_K(E) \cong L_K(1, n)$.

Remember: $L_K(1, n)$ has generators and relations:
$x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in R;$
Leavitt path algebras: Examples

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Remember: $L_K(1, n)$ has generators and relations:
$x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in R; \sum_{i=1}^n y_i x_i = 1_R$, and $x_i y_j = \delta_{i,j} 1_R$.
while $L_K(R_n)$ has these SAME generators and relations, where we identify $y_i^*$ with $x_i$.  

Gene Abrams
University of Colorado @ Colorado Springs
Historical note, part 1

1962: Leavitt gives construction of $L_K(1, n)$.
Historical note, part 1

1962: Leavitt gives construction of $L_K(1, n)$.
1979: Cuntz gives construction of the $C^*$-algebras $O_n$. 
Historical note, part 1

1962: Leavitt gives construction of $L_K(1, n)$.

1979: Cuntz gives construction of the $C^*$-algebras $\mathcal{O}_n$.

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June 2004: Various algebraists attend the CBMS lecture series “Graph $C^*$-algebras: algebras we can see”, held at University of Iowa, given by Iain Raeburn.

Algebraic analogs of graph $C^*$-algebras are defined and investigated starting Fall 2004.
| 1 | Leavitt path algebras: Introduction and Motivation |
| 2 | Multiplicative properties |
| 3 | Projective modules |
Elements in $L_K(E)$

Notation: If $p = e_1 e_2 \cdots e_n$ is a directed path in $E$ then $s(p)$ denotes $s(e_1)$, and $r(p)$ denotes $r(e_n)$.

Denote $n$ by $\ell(p)$.

**Lemma**: Every element of $L_K(E)$ can be written as

$$\sum_{i=1}^{n} k_i \alpha_i \beta_i^*$$

for some $n \in \mathbb{N}$, where: $k_i \in K$, and $\alpha_i, \beta_j$ are paths in $E$ for which $r(\alpha_i) = r(\beta_i) = s(\beta_i^*)$.

Idea: any expression with a $\ast$-term on the left reduces either to 0, or to the appropriate vertex.
Remark: Elements of the form $\alpha_i \beta_i^*$ are each nonzero in $L_K(E)$ (as long as $r(\alpha) = r(\beta)$), and they span, but they are not in general $K$-linearly independent.
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**Lemma:** $L_K(E)$ is unital if and only if $E^0$ is finite, in which case

$$1 = \sum_{v \in E^0} v.$$
**Elements in $L_K(E)$**

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**Lemma**: $L_K(E)$ is unital if and only if $E^0$ is finite, in which case

$$1 = \sum_{v \in E^0} v.$$ 

If $E^0$ is infinite we get a set of enough idempotents in $L_K(E)$. (Module theory is still well-understood in this situation.)
$L_K(E)$ as a $\mathbb{Z}$-graded algebra

For each vertex $v$, and each edge $e$, define
\[\deg(v) = 0, \quad \deg(e) = 1, \quad \deg(e^*) = -1.\]

Extend this to terms of the form $\alpha \beta^*$ by setting
\[\deg(\alpha \beta^*) = \ell(\alpha) - \ell(\beta).\]

For $d \in \mathbb{Z}$, let $L_K(E)_d$ denote expressions of the form
\[\sum_{i=1}^{n} \alpha_i \beta_i^* \quad \text{where} \quad \deg(\alpha_i \beta_i^*) = d.\]

Then $L_K(E)_d$ is clearly a $K$-subspace of $L_K(E)$, and for all $d, d' \in \mathbb{Z}$ we can show: $L_K(E)_d \cdot L_K(E)_{d'} \subseteq L_K(E)_{d+d'}$.

$L_K(E)$ is “$\mathbb{Z}$-graded”.
More Examples of Leavitt path algebras.

Mentioned above: If

\[
E = \bullet v_1 \xrightarrow{f_1} \bullet v_2 \xrightarrow{f_2} \bullet v_3 \ldots \ldots \bullet v_{n-1} \xrightarrow{f_{n-1}} \bullet v_n
\]

then \( L_K(E) \cong M_n(K) \).

Any expression \( p_u p_t^* \) has a unique start / end vertex, say \( v_i \) and \( v_j \).
Then the isomorphism \( L_K(E) \to M_n(K) \) is given by extending:

\[
p_u p_t^* \mapsto e_{i,j}.
\]

Note that we may wlog assume that each of \( p_u \) and \( p_t \) ends at \( v_n \).
More Examples of Leavitt path algebras.

Note also: the graph $E$ contains no (directed) closed paths, contains exactly one sink (namely, $v_n$), and that there are exactly $n$ paths which end in $v_n$ (including the path of length 0).

Using this idea, we can generalize to the following.

**Proposition:** Suppose $E$ is a finite graph which contains no (directed) closed paths. Let $v_1, v_2, \ldots, v_t$ denote the sinks of $E$. (At least one must exist.) For each $1 \leq i \leq t$, let $n_i$ denote the number of paths in $E$ which end in $v_i$. Then

$$L_K(E) \cong \bigoplus_{i=1}^{t} M_{n_i}(K).$$
More Examples of Leavitt path algebras.

So the “finite, no (directed) closed paths” case gives algebras which are well-understood.

Note: If

\[ E = \bullet \rightarrow \bullet \rightarrow \bullet \quad \text{and} \quad F = \bullet \rightarrow \bullet \leftarrow \bullet \]

then \( E \) and \( F \) are not isomorphic as graphs, but \( L_K(E) \cong L_K(F) \cong M_3(K) \).

So, nonisomorphic graphs might give rise to isomorphic Leavitt path algebras.
More Examples of Leavitt path algebras.

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So, nonisomorphic graphs might give rise to isomorphic Leavitt path algebras.

**A fundamental question in Leavitt path algebras:** Can we identify graphical connections between graphs \(E\) and \(F\) which will guarantee that \(L_K(E) \cong L_K(F)\)?
More Examples of Leavitt path algebras.

We use this same idea to produce more descriptions of Leavitt path algebras. Let $R_n(d)$ denote this graph:

$$
\begin{array}{c}
\bullet \quad w_1 \quad \rightarrow \quad \bullet \quad w_2 \quad \rightarrow \quad \cdots \quad \bullet \quad w_{d-1} \quad \rightarrow \quad \bullet \quad v \\
\end{array}
$$

Then $L_K(R_n(d)) \cong M_d(L_K(1,n))$. The idea is the same as before, but now at the end of each trip into the "end", you pick up an element of $L_K(1,n)$. For this result $n=1$ is included as well.
More Examples of Leavitt path algebras.

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![Graph Image]

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More Examples of Leavitt path algebras.

Even more generally:

**Proposition**: Let $E$ be a finite graph, and $d \in \mathbb{N}$. Let $S_dE$ be the graph constructed from $E$ by taking the “straight line” graph of length $d$ and appending it at each vertex of $E$. Then

$$L_K(S_dE) \cong M_n(L_K(E)).$$
More Examples of Leavitt path algebras.

Using similar ideas:

**Proposition:** Let $E$ be a graph consisting of a single cycle, with $t$ vertices. Then $L_K(E) \cong M_t(K[x, x^{-1}])$. 
More Examples of Leavitt path algebras.

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**Proposition:** Let $E$ be a graph consisting of a single cycle, with $t$ vertices. Then $L_K(E) \cong M_t(K[x, x^{-1}])$.

More generally, if $E$ is a graph which contains a single cycle $c$, and $c$ has no “exits”, then

$$L_K(E) \cong M_{n(v)}(K[x, x^{-1}]),$$

where, if $v$ denotes any (fixed) vertex of $c$, $n(v)$ is the number of distinct paths in $E$ which end at $v$ and do not contain $c$. 
More Examples of Leavitt path algebras.

There are some non-standard (surprising?) isomorphisms between Leavitt path algebras. Let $E = R_3$, so that $S = L_K(E) \cong L_K(1, 3)$. Then as left $S$-modules we have $S^1 \cong S^3$. So $\text{End}_S(S) \cong \text{End}_S(S^3)$, which gives that, as rings,

$$S \cong M_3(S).$$

So using the previous Proposition, these two graphs have isomorphic Leavitt path algebras:

$$R_3 = \bullet \quad \text{and} \quad R_3(3) = \bullet \rightarrow \bullet \rightarrow \bullet$$

That is, $L_K(R_3) \cong L_K(R_3(3))$. 
More Examples of Leavitt path algebras.

On the other hand, $R_3$ and

$$R_3(2) = \bullet \xrightarrow{} \bullet$$

do NOT have isomorphic Leavitt path algebras.

(Leavitt showed this in the 1962 paper.)
Morita equivalence

General definition: Let $R$ and $S$ be rings. $R$ and $S$ are *Morita equivalent* in case the module categories $R - \text{Mod}$ and $S - \text{Mod}$ are equivalent.
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$\iff$ there exist bimodules (with additional properties) $R_P S$ and $S_Q R$ with $P \otimes Q \cong R$ (as $R-R$-bimodules) and $Q \otimes P \cong S$ (as $S-S$-bimodules).
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$\iff$ there exists $n \in \mathbb{N}$, $e = e^2 \in M_n(S)$ for which $M_n(S)eM_n(S) = M_n(S)$ and $R \cong eM_n(S)e$. 
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$\iff$ $FM_N(R) \cong FM_N(S)$ as rings.
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Note: In particular, $R$ and $M_n(R)$ are always Morita equivalent.
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So another reasonable question to here is to ask: For graphs $E$ and $F$, when is $L_K(E) \sim_M L_K(F)$?
Note: In particular, $R$ and $M_n(R)$ are always Morita equivalent.

So another reasonable question to here is to ask: For graphs $E$ and $F$, when is $L_K(E) \sim_M L_K(F)$?

Note that this is a courser equivalence relation on rings than isomorphism. So e.g. even though $R_3$ and $M_2(R_3)$ are not isomorphic, they are Morita equivalent (and therefore share many of the same properties).

Appropriate generalizations hold in case $R$ and $S$ have enough idempotents.
1. Leavitt path algebras: Introduction and Motivation

2. Multiplicative properties

3. Projective modules
The monoid $\mathcal{V}(R)$

Recall: $P$ is a \textit{finitely generated projective} $R$-module in case $P \oplus Q \cong R^n$ for some $Q$, some $n \in \mathbb{N}$. 

Example: In $R = M_2(\mathbb{R})$, $P = M_2(\mathbb{R})e_1$, $1 = (\ast \ 0 \ 0 \ast)$ is a finitely projective $R$-module. Note $P \not\cong R^n$ for any $n$. 

Example: $L_K(E)$ contains projective modules of the form $L_K(E)p$ for each path $p$ in $E$. 

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Additional examples: $Rf$ where $f$ is idempotent (i.e., $f^2 = f$), since $Rf \oplus R(1 - f) = R^1$.

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$\mathcal{V}(R)$ denotes the isomorphism classes of finitely generated projective (left) $R$-modules. With operation $\oplus$, this becomes an abelian monoid. Note $R$ itself plays a special role in $\mathcal{V}(R)$. 

Example. $R = \mathbb{K}$, a field. Then $\mathcal{V}(R) \cong \mathbb{Z}^+$. 

Lemma: If $R \cong M \cong S$, then $\mathcal{V}(R) \cong \mathcal{V}(S)$.

So, in particular:

Example. $S = \text{Mod}(\mathbb{K})$, $\mathbb{K}$ a field. Then $\mathcal{V}(S) \cong \mathbb{Z}^+$. (But note that the 'position' of $S$ in $\mathcal{V}(S)$ is different than the position of $R$ in $\mathcal{V}(R)$.)

Remark: $\mathcal{V}(R) \{[0]\}$ is a semigroup (i.e., is closed under $\oplus$). 

Remark: Given a ring $R$, it is in general not easy to compute $\mathcal{V}(R)$. 

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The monoid $M_E$

Here’s a ‘natural’ monoid arising from any directed graph $E$. 
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Here’s a ‘natural’ monoid arising from any directed graph $E$. Associate to $E$ the abelian monoid $(M_E, +)$:

$M_E$ is generated by $\{a_v | v \in E^0\}$

So $M_E = \{n_1 a_{v_1} + n_2 a_{v_2} + \cdots + n_t a_{v_t}\}$ with $n_i \in \mathbb{Z}^+$. Relations in $M_E$ are given by: $a_v = \sum_{e \in s^{-1}(v)} a_{r(e)}$. 
**The monoid \( M_E \)**

**Example.** Let \( F \) be the graph

![Graph Diagram]

So \( M_F \) consists of elements \( \{n_1a_1 + n_2a_2 + n_3a_3\} \) \((n_i \in \mathbb{Z}^+)\), subject to: \( a_1 = a_2 + a_3 \); \( a_2 = a_1 + a_3 \); \( a_3 = a_1 + a_2 \).
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It's not hard to get: $M_F = \{0, \ a_1, \ a_2, \ a_3, \ a_1 + a_2 + a_3\}$. 
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It’s not hard to get: $M_F = \{0, a_1, a_2, a_3, a_1 + a_2 + a_3\}$.

We see that the semigroup $M_F \setminus \{0\}$ is actually a group, $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. 
The monoid $\mathcal{V}(L_K(E))$

Example:

$$E = R_n = \begin{array}{c}
y_1 \\
y_2 \\
y_3 \\
y_n
\end{array}$$

Then $M_E$ is the set of symbols of the form

$$n_1 a_v \ (n_1 \in \mathbb{Z}^+)$$

subject to the relation: $a_v = na_v$
The monoid $\mathcal{V}(L_K(E))$

**Example:**

$$E = R_n = \begin{array}{c}
\text{\begin{array}{c} y_3 \\
y_2 \\
y_1 \\
y_n 
\end{array}}
\end{array}$$

Then $M_E$ is the set of symbols of the form

$$n_1 a_v \quad (n_1 \in \mathbb{Z}^+)$$

subject to the relation: \( a_v = na_v \)

So here, $M_E = \{0, a_v, 2a_v, \ldots, (n - 1)a_v\}$.

Again we have a situation where the semigroup $M_E \setminus \{0\}$ is a group, \( \cong \mathbb{Z}_{n-1} \).
A graph $E$ is \textit{row-finite} if $|s^{-1}(v)| < \infty$ for all $v \in E^0$. (In other words, if $E$ contains no “infinite emitters”.)
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Theorem

(P. Ara, M.A. Moreno, E. Pardo, 2007) [36]

For any row-finite directed graph $E$,

$$\mathcal{V}(L_K(E)) \cong M_E.$$
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For any row-finite directed graph $E$,  

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Moreover, there is an appropriate universal property that $L_K(E)$ satisfies.
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A graph $E$ is *row-finite* if $|s^{-1}(v)| < \infty$ for all $v \in E^0$. (In other words, if $E$ contains no “infinite emitters”.)

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(P. Ara, M.A. Moreno, E. Pardo, 2007) \cite{ara2007}

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Aside: The proof uses a deep result by G. Bergman \cite{bergman1978}.
The monoid $\mathcal{V}(L_K(E))$

One (very nontrivial) consequence: Let $S$ denote $L_K(1, n)$. Then

$$\mathcal{V}(S) = \{0, S, S^2, \ldots, S^{n-1}\}.$$
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Another nice consequence: The class of Leavitt path algebras consists of algebras other than those arising in the context of Leavitt algebras.
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Yet another: It’s clear that $L_K(1, n) \cong L_K(1, n') \iff n = n'$. But we also get:

$$L_K(1, n) \sim_M L_K(1, n') \iff n = n'.$$
Historical Note, Part 2

So we can think of Leavitt path algebras in two ways:

1) the “quotient of a path algebra” approach, and
2) the “universal algebra which supports $M_E$ as its $\mathcal{V}$-monoid” approach.

These were developed in parallel.

The two approaches together have complemented each other in the development of the subject.
What’s ahead?

**Lecture 2:** (Wednesday) Some theorems of the form

\[ L_K(E) \text{ has ring-theoretic property } \mathcal{P} \iff E \text{ has graph-theoretic property } \mathcal{Q}. \]

In particular, we’ll consider the ideal structure of \( L_K(E) \).
Also: connections / similarities with graph C*-algebras.

**Lecture 3:** (Friday) Some applications of, generalizations of, and open questions in Leavitt path algebras.