Leavitt path algebras: introduction, motivation, and basic properties

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Minicourse on Leavitt path algebras, Lecture 1

III Workshop on Dynamics, Numeration, Tilings and Graph Algebras (III FloripaDynSys)

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Overview

Brief history, and motivating examples

One of the first theorems you saw as an undergraduate student:

Dimension Theorem for Vector Spaces. Every nonzero vector space V has a basis. Moreover, if $\mathcal{B}$ and $\mathcal{B}'$ are two bases for V, then $|\mathcal{B}| = |\mathcal{B}'|$.

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Note: V has a basis $\mathcal{B} = \{b_1, b_2, ..., b_n\}$ $\Leftrightarrow$ $V \cong \oplus_{i=1}^{n}\mathbb{R}$ as vector spaces. So:

One result of Dimension Theorem, Rephrased:

$\oplus_{i=1}^{n}\mathbb{R} \cong \oplus_{i=1}^{m}\mathbb{R} \ \Leftrightarrow \ m = n.$

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If $0 \neq v \in V$, then $\{v\}$ is linearly independent.

If $kv = 0$, need to show $k = 0$. But $k \neq 0 \Rightarrow \frac{1}{k}kv = 0 \Rightarrow v = 0$, contradiction.

Similar idea (multiply by the inverse of a nonzero element of K) shows that a maximal linearly independent subset of V actually spans V.

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Question: Is the Dimension Theorem true for rings in general? That is, if R is a ring, and $\oplus_{i=1}^{n} R \cong \oplus_{i=1}^{m} R$ as R-modules, must $m = n$? ("module" = "left module")

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Example: Consider the ring S of linear transformations from an infinite dimensional $\mathbb{R}$-vector space V to itself.

Think of V as $\oplus_{i=1}^{\infty} \mathbb{R}$. Then think of S as $\mathrm{RFM}(\mathbb{R})$.

Brief history, and motivating examples

Intuitively, S and $S \oplus S$ have a chance to be "the same".

$M \mapsto$ (Odd numbered columns of M , Even numbered columns of M)

More formally: Let

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

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Similarly, we should be able to 'go back' from pairs of matrices to a single matrix, by interweaving the columns.

That is, more formally, $(M_1, M_2) \mapsto M_1X_1 + M_2X_2$ is a reasonable way to associate a pair of matrices with a single one.

Brief history, and motivating examples

Here's what's really going on. These equations are easy to verify:

$$Y_1X_1 + Y_2X_2 = I,$$

$$X_1Y_1 = I = X_2Y_2, \quad \text{and} \quad X_1Y_2 = 0 = X_2Y_1.$$

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$$(M_1, M_2) \mapsto M_1X_1 + M_2X_2$$

$$\mapsto (\ (M_1X_1 + M_2X_2)Y_1, (M_1X_1 + M_2X_2)Y_2\ ) = (M_1, M_2)$$

Brief history, and motivating examples

Using exactly the same idea, let R be ANY ring which contains four elements y_1, y_2, x_1, x_2 satisfying

$$y_1 x_1 + y_2 x_2 = 1_R,$$

$$x_1 y_1 = 1_R = x_2 y_2, \quad \text{and} \quad x_1 y_2 = 0 = x_2 y_1.$$

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Then $R \cong R \oplus R$.

Note for later: i.e., $\sum_{i=1}^{2} y_ix_i = 1_R$ and $x_iy_j = \delta_{i,j}1_R$.

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Remark: Here the sets $\{1_R\}$ and $\{x_1, x_2\}$ are each bases for R.

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Actually, when $R \cong R \oplus R$ as R-modules, then $\oplus_{i=1}^{m} R \cong \oplus_{i=1}^{n} R$ for *all* $m, n \in \mathbb{N}$.

Leavitt algebras

Natural question:

Does there exist R with, e.g., $R \cong R \oplus R \oplus R$, but $R \ncong R \oplus R$?

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Theorem

(William G. Leavitt, Trans. Amer. Math. Soc., 1962)

For every $m < n \in \mathbb{N}$ and field K there exists a K-algebra $R = L_K(m,n)$ with $\oplus_{i=1}^m R \cong \oplus_{i=1}^n R$, and all isomorphisms between free left R-modules result precisely from this one. Moreover, $L_K(m,n)$ is universal with this property.

Leavitt algebras

The $m = 1$ situation of Leavitt's Theorem is now somewhat familiar. Similar to the $n = 2$ case that we saw above, $R \cong R^n$ if and only if there exist

$$x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in R$$

for which

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$$\sum_{i=1}^{n} y_i x_i = 1_R \quad \text{and} \quad x_i y_j = \delta_{i,j} 1_R.$$

$L_K(1, n)$ is the quotient

$$K < X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n > \ / \ < (\sum_{i=1}^{n} Y_i X_i) - 1_K;\ X_i Y_j - \delta_{i,j} 1_K >$$

Note: $\mathrm{RFM}(K)$ is much bigger than $L_K(1, 2)$.

Leavitt algebras

As a result, we have this: Let S denote $L_K(1, n)$. Then

$$S^a \cong S^b \;\Leftrightarrow\; a \equiv b \bmod (n-1).$$

In particular, $S \cong S^n$, and $n > 1$ is minimal with this property.

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It turns out:

Theorem. (Leavitt, Duke J. Math, 1964)

For every field K and $n \geq 2$, $L_K(1,n)$ is simple.

(On the other hand, for $m \geq 2$, $L_K(m,n)$ is not simple.)

Remember, a ring R being *simple* means:
$\forall\ 0 \neq r \in R, \exists\ \alpha_i, \beta_i \in R$ with $\sum_{i=1}^{n} \alpha_i r \beta_i = 1_R$.

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Actually, $L_K(1,n)$ is VERY simple:

$\forall\ 0 \neq r \in L_K(1,n), \exists\ \alpha, \beta \in L_K(1,n)$ with $\alpha r \beta = 1_{L_K(1,n)}$.

Building rings from combinatorial objects

Here's a familiar idea. Consider the set $T = \{x^0, x^1, x^2, ...\}$. Define multiplication on T in the usual way: $x^i \cdot x^j = x^{i+j}$. Consider formal symbols of the form

$$k_1 t_1 + k_2 t_2 + \cdots + k_n t_n$$

where $t_i \in T$, and $k_i \in \mathbb{R}$. Denote this set of symbols by KT. We can add and multiply two symbols of this form, as usual, e.g. $(kt)(k't') = kk'(t \cdot t')$.

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Also, e.g. if we impose the relation $x^n = x^0$ on T, call the new semigroup $\overline{T}$, then $\overline{T} = \{x^0, x^1, x^2, ..., x^{n-1}\}$, and

$$\mathbb{R}\overline{T} \cong \mathbb{R}[x]/\langle x^n - 1\rangle$$

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This is a standard construction to produce rings:

Start with some binary operation on a set S, and some field K, and form the formal symbols as above. Add and multiply based on addition and 'multiplication' in K and S.

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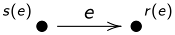
For instance:

matrix rings, group rings, multivariable polynomial rings, etc ...

can all be thought of in this way.

General path algebras

Let E be a directed graph. $E = (E^0, E^1, r, s)$



The *path algebra of E with coefficients in K* is the K-algebra KS as above, where the underlying set S is the set of all directed paths in E (including vertices), and multiplication of paths is just concatenation. Denote by KE. In particular, in KE,

For each edge e, $\quad s(e) \cdot e = e = e \cdot r(e)$

For each vertex v, $\quad v \cdot v = v$

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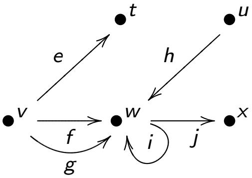

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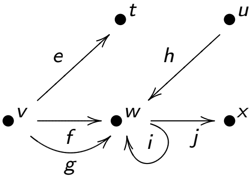

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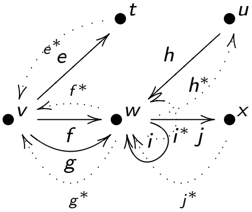

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(CK1) $e^*e = r(e)$; and $f^*e = 0$ for $f \neq e$ (for all edges e, f in E).

(CK2) $v = \sum_{\{e \in E^1 | s(e)=v\}} ee^*$ for each vertex v in E.

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Definition

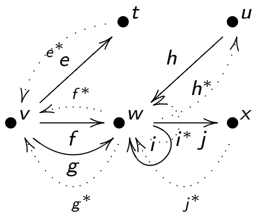
The Leavitt path algebra of E with coefficients in K

$$L_K(E) = K\widehat{E} \ / \ < (CK1), (CK2) >$$

Leavitt path algebras: Examples

Some sample computations in $L_{\mathbb{C}}(E)$ from the Example:

$$\widehat{E} =$$



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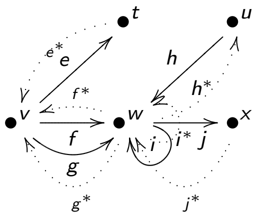

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$ff^* = ...$ (no simplification) Note: $(ff^*)^2 = f(f^*f)f^* = ff^*$

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Then $L_K(E) \cong M_n(K)$.

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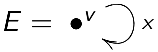

Then $L_K(E) \cong K[x, x^{-1}]$.

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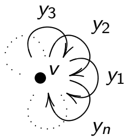
$E = R_n =$ 

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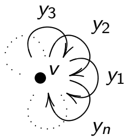
$E = R_n =$ 

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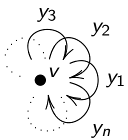
$E = R_n =$ 

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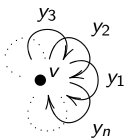
$E = R_n =$ 

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while $L_K(R_n)$ has these SAME generators and relations, where we identify y_i^* with x_i.

Historical note, part 1

1962: Leavitt gives construction of $L_K(1, n)$.

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1979: Cuntz gives construction of the C^*-algebras $\mathcal{O}_n$.

1980's: Cuntz, Krieger, and others generalize the $\mathcal{O}_n$ construction to directed graphs, and produce the *graph* C^**-algebras* $C^*(E)$.

Note: The notation (CK1) and (CK2) refer to Cuntz and Krieger.

Historical note, part 1

1962: Leavitt gives construction of $L_K(1, n)$.

1979: Cuntz gives construction of the C^*-algebras $\mathcal{O}_n$.

1980's: Cuntz, Krieger, and others generalize the $\mathcal{O}_n$ construction to directed graphs, and produce the *graph* C^**-algebras* $C^*(E)$.

Note: The notation (CK1) and (CK2) refer to Cuntz and Krieger.

June 2004: Various algebraists attend the CBMS lecture series

"Graph C^*-algebras: algebras we can see",

held at University of Iowa, given by Iain Raeburn.

Algebraic analogs of graph C^*-algebras are defined and investigated starting Fall 2004.

Elements in $L_K(E)$

Notation: If $p = e_1 e_2 \cdots e_n$ is a directed path in E then
$s(p)$ denotes $s(e_1)$, and $r(p)$ denotes $r(e_n)$.

Denote n by $\ell(p)$.

Lemma: Every element of $L_K(E)$ can be written as

$$\sum_{i=1}^{n} k_i \alpha_i \beta_i^*$$

for some $n \in \mathbb{N}$, where: $k_i \in K$, and α_i, β_j are paths in E for which $r(\alpha_i) = r(\beta_i)$ $(= s(\beta_i^*)$).

Idea: any expression with a $*$-term on the left reduces either to 0, or to the appropriate vertex.

Elements in $L_K(E)$

Remark: Elements of the form $\alpha_i\beta_i^*$ are each nonzero in $L_K(E)$ (as long as $r(\alpha) = r(\beta)$), and they span, but they are not in general K-linearly independent.

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If E^0 is infinite we get a *set of enough idempotents* in $L_K(E)$. (Module theory is still well-understood in this situation.)

$L_K(E)$ as a $\mathbb{Z}$-graded algebra

For each vertex v, and each edge e, define

$$\deg(v) = 0, \ \deg(e) = 1, \ \deg(e^*) = -1.$$

Extend this to terms of the form $\alpha\beta^*$ by setting

$$\deg(\alpha\beta^*) = \ell(\alpha) - \ell(\beta).$$

For $d \in \mathbb{Z}$, let $L_K(E)_d$ denote expressions of the form

$$\sum_{i=1}^{n} \alpha_i \beta_i^* \ \text{ where } \ \deg(\alpha_i \beta_i^*) = d.$$

Then $L_K(E)_d$ is clearly a K-subspace of $L_K(E)$, and for all $d, d' \in \mathbb{Z}$ we can show: $\quad L_K(E)_d \cdot L_K(E)_{d'} \subseteq L_K(E)_{d+d'}$.

$L_K(E)$ is "$\mathbb{Z}$-graded".

More Examples of Leavitt path algebras.

Mentioned above: If

$$E = \bullet^{v_1} \xrightarrow{f_1} \bullet^{v_2} \xrightarrow{f_2} \bullet^{v_3} \cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{f_{n-1}} \bullet^{v_n}$$

then $L_K(E) \cong \mathrm{M}_n(K)$.

Any expression $p_u p_t^*$ has a unique start / end vertex, say v_i and v_j. Then the isomorphism $L_K(E) \to \mathrm{M}_n(K)$ is given by extending:

$$p_u p_t^* \mapsto e_{i,j}.$$

Note that we may wlog assume that each of p_u and p_t ends at v_n.

More Examples of Leavitt path algebras.

Note also: the graph E contains no (directed) closed paths, contains exactly one sink (namely, v_n), and that there are exactly n paths which end in v_n (including the path of length 0).

Using this idea, we can generalize to the following.

Proposition: Suppose E is a finite graph which contains no (directed) closed paths. Let $v_1, v_2, ..., v_t$ denote the sinks of E. (At least one must exist.) For each $1 \leq i \leq t$, let n_i denote the number of paths in E which end in v_i. Then

$$L_K(E) \cong \oplus_{i=1}^{t} \mathrm{M}_{n_i}(K).$$

More Examples of Leavitt path algebras.

So the "finite, no (directed) closed paths" case gives algebras which are well-understood.

Note: If

$$E = \bullet \longrightarrow \bullet \longrightarrow \bullet \quad \text{and} \quad F = \bullet \longrightarrow \bullet \longleftarrow \bullet$$

then E and F are not isomorphic as graphs, but $L_K(E) \cong L_K(F) \cong \mathrm{M}_3(K)$.

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So, nonisomorphic graphs might give rise to isomorphic Leavitt path algebras.

A fundamental question in Leavitt path algebras: Can we identify graphical connections between graphs E and F which will guarantee that $L_K(E) \cong L_K(F)$?

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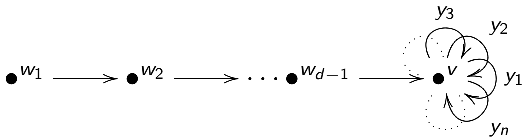
We use this same idea to produce more descriptions of Leavitt path algebras. Let $R_n(d)$ denote this graph:



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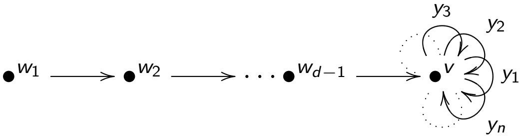

Then

$$L_K(R_n(d)) \cong \mathrm{M}_d(L_K(1,n)).$$

The idea is the same as before, but now at the end of each trip into the "end", you pick up an element of $L_K(1,n)$. For this result $n = 1$ is included as well.

More Examples of Leavitt path algebras.

Even more generally:

Proposition: Let E be a finite graph, and $d \in \mathbb{N}$. Let $S_d E$ be the graph constructed from E by taking the "straight line" graph of length d and appending it at each vertex of E. Then

$$L_K(S_d E) \cong \mathrm{M}_n(L_K(E)).$$

More Examples of Leavitt path algebras.

Using similar ideas:

Proposition: Let E be a graph consisting of a single cycle, with t vertices. Then $L_K(E) \cong \mathrm{M}_t(K[x, x^{-1}])$.

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Proposition: Let E be a graph consisting of a single cycle, with t vertices. Then $L_K(E) \cong \mathrm{M}_t(K[x, x^{-1}])$.

More generally, if E is a graph which contains a single cycle c, and c has no "exits", then

$$L_K(E) \cong \mathrm{M}_{n(v)}(K[x, x^{-1}]),$$

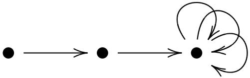
where, if v denotes any (fixed) vertex of c, $n(v)$ is the number of distinct paths in E which end at v and do not contain c.

More Examples of Leavitt path algebras.

There are some non-standard (surprising?) isomorphisms between Leavitt path algebras. Let $E = R_3$, so that $S = L_K(E) \cong L_K(1,3)$. Then as left S-modules we have $S^1 \cong S^3$. So $\mathrm{End}_S(S) \cong \mathrm{End}_S(S^3)$, which gives that, as rings,

$$S \cong \mathrm{M}_3(S).$$

So using the previous Proposition, these two graphs have isomorphic Leavitt path algebras:

$R_3 =$  and $R_3(3) =$ 

That is, $L_K(R_3) \cong L_K(R_3(3))$.

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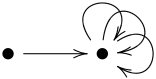
On the other hand, R_3 and

$R_3(2) =$ 

do NOT have isomorphic Leavitt path algebras.

(Leavitt showed this in the 1962 paper.)

Morita equivalence

General definition: Let R and S be rings. R and S are *Morita equivalent* in case the module categories $R - Mod$ and $S - Mod$ are equivalent.

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$\Leftrightarrow$ $\mathrm{FM}_{\mathbb{N}}(R) \cong \mathrm{FM}_{\mathbb{N}}(S)$ as rings.

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So another reasonable question to here is to ask: For graphs E and F, when is $L_K(E) \sim_M L_K(F)$?

Note that this is a courser equivalence relation on rings than isomorphism. So e.g. even though R_3 and $\mathrm{M}_2(R_3)$ are not isomorphic, they are Morita equivalent (and therefore share many of the same properties).

Appropriate generalizations hold in case R and S have enough idempotents.

The monoid $\mathcal{V}(R)$

Recall: P is a *finitely generated projective* R-module in case
$P \oplus Q \cong R^n$ for some Q, some $n \in \mathbb{N}$.

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Additional examples: Rf where f is idempotent (i.e., $f^2 = f$), since $Rf \oplus R(1-f) = R^1$.

Example: In $R = \mathrm{M}_2(\mathbb{R})$, $P = \mathrm{M}_2(\mathbb{R})e_{1,1} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$ is a finitely projective R-module. Note $P \not\cong R^n$ for any n.

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Example: $L_K(E)$ contains projective modules of the form $L_K(E)pp^*$ for each path p in E.

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$\mathcal{V}(R)$ denotes the isomorphism classes of finitely generated projective (left) R-modules. With operation $\oplus$, this becomes an abelian monoid. Note R itself plays a special role in $\mathcal{V}(R)$.

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Remark: Given a ring R, it is in general not easy to compute $\mathcal{V}(R)$.

The monoid M_E

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Associate to E the abelian monoid $(M_E, +)$:

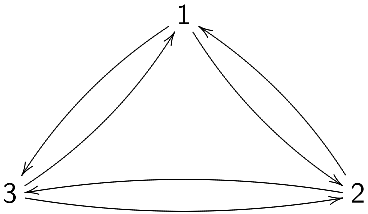
M_E is generated by $\{a_v | v \in E^0\}$

So $M_E = \{n_1 a_{v_1} + n_2 a_{v_2} + \cdots + n_t a_{v_t}\}$ with $n_i \in \mathbb{Z}^+$.

Relations in M_E are given by: $a_v = \sum_{e \in s^{-1}(v)} a_{r(e)}$.

The monoid M_E

Example. Let F be the graph



So M_F consists of elements $\{n_1 a_1 + n_2 a_2 + n_3 a_3\}$ $(n_i \in \mathbb{Z}^+)$,

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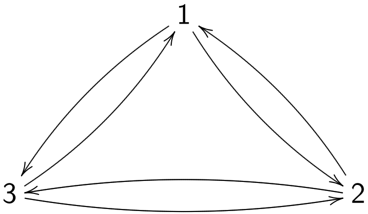

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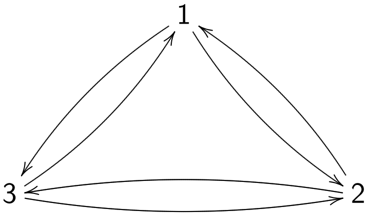

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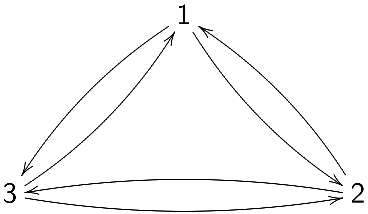

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We see that the semigroup $M_F \setminus \{0\}$ is actually a *group*, $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

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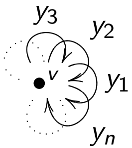
$E = R_n =$ 

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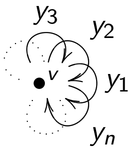
$E = R_n =$ 

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So here, $M_E = \{0, a_v, 2a_v, ..., (n-1)a_v\}$.

Again we have a situation where the semigroup $M_E \setminus \{0\}$ is a group, $\cong \mathbb{Z}_{n-1}$.

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Aside: The proof uses a deep result by G. Bergman [49].

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One (very nontrivial) consequence: Let S denote $L_K(1,n)$. Then

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Yet another: It's clear that $L_K(1,n) \cong L_K(1,n') \Leftrightarrow n = n'$. But we also get:

$$L_K(1,n) \sim_M L_K(1,n') \ \Leftrightarrow\ n = n'.$$

Historical Note, Part 2

So we can think of Leavitt path algebras in two ways:

1) the "quotient of a path algebra" approach, and

2) the "universal algebra which supports M_E as its $\mathcal{V}$-monoid" approach.

These were developed in parallel.

The two approaches together have complemented each other in the development of the subject.

What's ahead?

Lecture 2: (Wednesday) Some theorems of the form

$$L_K(E) \text{ has ring-theoretic property } \mathcal{P} \Leftrightarrow$$
$$E \text{ has graph-theoretic property } \mathcal{Q}.$$

In particular, we'll consider the ideal structure of $L_K(E)$.

Also: connections / similarities with graph C^*-algebras.

Lecture 3: (Friday) Some applications of, generalizations of, and open questions in Leavitt path algebras.