Leavitt path algebras: algebraic properties

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Minicourse on Leavitt path algebras, Lecture 2

III Workshop on Dynamics, Numeration, Tilings and Graph Algebras (III FloripaDynSys)

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Overview

Recap of Lecture 1

Start with a directed graph E, build its double graph $\widehat{E}$.

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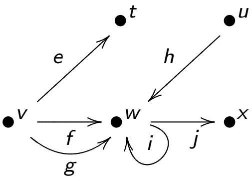

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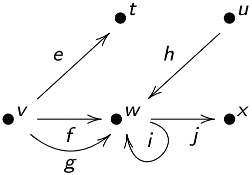

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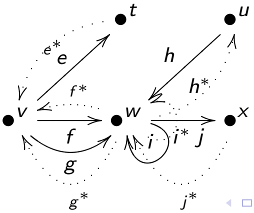

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Definition

The Leavitt path algebra of E with coefficients in K

$$L_K(E) = K\widehat{E} \ / \ < (CK1), (CK2) >$$

Standard algebras arising as Leavitt path algebras

$$E = \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

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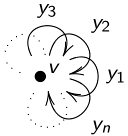
$E = R_n =$ 

Then $L_K(E) \cong L_K(1, n)$.

Recap of Lecture 1

Every element of $L_K(E)$ can be written as

$$\sum_{i=1}^{n} k_i \alpha_i \beta_i^*$$

for some $n \in \mathbb{N}$, where: $k_i \in K$, and α_i, β_j are paths in E for which $r(\alpha_i) = r(\beta_i)$ $(= s(\beta_i^*)$ $)$.

$L_K(E)$ is $\mathbb{Z}$-graded, by setting

$$\deg(\alpha\beta^*) = \ell(\alpha) - \ell(\beta).$$

Recap of Lecture 1

For E any graph, define the abelian monoid $(M_E, +)$:

M_E is generated by $\{a_v | v \in E^0\}$

Relations in M_E are given by:

$$a_v = \sum_{e \in s^{-1}(v)} a_{r(e)} \text{ (at regular vertices)}$$

Theorem

For any row-finite directed graph E,

$$\mathcal{V}(L_K(E)) \cong M_E.$$

Some ideals in $L_K(E)$

We need some graph-theoretic notation and terms

1. $v, w \in E^0$. v *connects to* w in case either $v = w$, or: there is a path p in E for which $s(p) = v, r(p) = w$.

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3. $S \subseteq E^0$ is *saturated* in case:
   For each regular vertex $v \in E^0$, if $r(s^{-1}(v)) \subseteq S$, then $v \in S$.

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$S = \{u\}$ is hereditary and saturated. (Saturated vacuously.)

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Saturated? Suppose each vertex to which the regular vertex v connects is in I; i.e., that $r(s^{-1}(v)) \subseteq I$. But

$$v = \sum_{e \in s^{-1}(v)} ee^* = \sum_{e \in s^{-1}(v)} e \cdot r(e) \cdot e^* \in I.$$

Graded ideals

Definition: If $R = \oplus_{t\in\mathbb{Z}} R_t$ is $\mathbb{Z}$-graded, and I is a two-sided ideal of R, then I is a *graded ideal* in case:

for each $a \in I$,

$$\text{if } a = \sum_{t=1}^{n} a_t \quad (\text{where } a_t \in R_t),$$

then $a_t \in I$ for all $1 \leq i \leq t$.

Graded ideals

Non-Example: In $K[x, x^{-1}]$, consider $I = \langle 1 + x \rangle$. Then I is not graded, since neither 1 nor x is in I.

In the context of $L_K(\bullet \circlearrowright x)$, this gives that $I = \langle v + x \rangle$ is nongraded. Note that $I = \langle v + x \rangle$ contains no vertices.

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Important Example: Let R be any graded ring. Suppose T is a set of idempotents in R_0. Then $\langle T \rangle$ is a graded ideal.

In particular, if T is any subset of E^0, then $\langle T \rangle$ is a graded ideal of $L_K(E)$.

Graded ideals

Let $\mathcal{H}_E$ denote the set of hereditary saturated subsets of E.

Let $\mathrm{Id}_{\mathrm{gr}}(L_K(E))$ denote the set of graded ideals of $L_K(E)$.

Proposition Let E be a row-finite graph. Then there is an order-preserving bijection

$$\mathcal{H}_E \longleftrightarrow \mathrm{Id}_{\mathrm{gr}}(L_K(E)).$$

Idea of proof. If I is any ideal of $L_K(E)$, then $I \cap E^0 \in \mathcal{H}$ by previous lemma. But if I is graded, one can show (induction) that $I = \langle I \cap E^0 \rangle$.

Conversely, if $H \in \mathcal{H}$, then one shows that the only vertices in $\langle H \rangle$ are already in H, so that $H = \langle H \rangle \cap E^0$. $\square$

Graded ideals

Note: This correspondence does not extend to non-row-finite graphs, but there is a generalization.

So: If there is a nontrivial hereditary saturated subset of E, then $L_K(E)$ cannot be simple. For instance, if E is the graph

$$\begin{tikzcd} \bullet^{u} & \bullet^{v} \arrow[l] \arrow[loop right] \end{tikzcd}$$

then $L_K(E)$ is not simple, since $\langle \{u\} \rangle$ is a proper (graded) two-sided ideal.

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Corollary: For any graph E, the Jacobson radical $J(L_K(E)) = \{0\}$.

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R is a *prime ring* in case the product $I \cdot I'$ of two nonzero two-sided ideals of R is nonzero.

Corollary: $L_K(E)$ is a prime ring if and only if each pair of vertices in E connects to a common vertex. ("downward directed")

Proof: For $\mathbb{Z}$-graded rings, it is sufficient to show that the product of any two nonzero *graded* ideals is nonzero. Now look at elements of the product.

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Note $L_K(E)$ is simple for

$$E = \bullet \longrightarrow \bullet \cdots\cdots\rightarrow \qquad \longrightarrow \bullet \quad \text{since } L_K(E) \cong M_n(K)$$

and for

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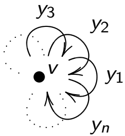

since $L_K(E) \cong L_K(1, n)$

but not simple for

$E = R_1 =$ 

since $L_K(E) \cong K[x, x^{-1}]$

Ideals in $L_K(E)$

Note: In  we obviously have $\mathcal{H}_E = \{\emptyset, E^0\}$. So there are no nontrivial **graded** ideals in $L_K($  $)$. So the absence of nontrivial hereditary saturated subsets is not sufficient to imply that the Leavitt path algebra is simple, because we have seen that, e.g., $\langle v + x \rangle$ is a nontrivial two-sided ideal.

For comparison: Why do we get $\langle v + y_i \rangle = L_K(R_n)$ for $n \geq 2$?

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For $i \neq j$,

$$y_j^*(v + y_i)y_j = y_j^* v y_j + y_j^* y_i y_j = v + 0 = v$$

Some graph definitions

1. A *cycle*

$$\begin{tikzcd} \bullet^{a} \arrow[r, bend right, "x"'] & \bullet^{b} \arrow[l, bend right, "y"'] \end{tikzcd}$$

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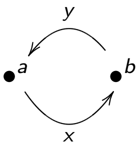
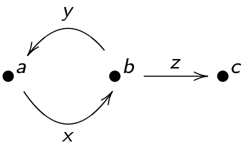


2. An *exit* for a cycle.



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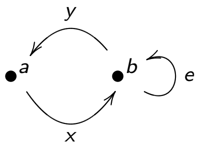

Simplicity of Leavitt path algebras

Theorem

(A -, Aranda Pino) [7] $L_K(E)$ is simple if and only if:

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Note: No role played by K.

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Idea of proof: $(\Rightarrow)$ Mimic what Leavitt did.

Step 1: Show that, in this case, if there is a nonzero element in an ideal I which is of the form $\sum_{i=1}^{n} \alpha_i$ or $\sum_{i=1}^{n} \beta_i^*$, then the ideal must be of all of $L_K(E)$.

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$(\Leftarrow)$ If $\mathcal{H}_E$ contains nontrivial elements, then there are nontrivial (graded) ideals in $L_K(E)$.

On the other hand, if there is a cycle in E which does NOT have an exit, then some piece of $L_K(E)$ contains $K[x, x^{-1}]$, which is not simple.

Simplicity of Leavitt path algebras

So the class of Leavitt path algebras yields many “new” simple algebras, over and above the Leavitt algebras $L_K(1, n)$.

1. Recap of Lecture 1
2. Ideals in $L_K(E)$, and simplicity
3. Purely infinite simplicity
4. Connections to graph C^*-algebras

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Example: $R = L_K(1, n)$ for $n \geq 2$. Then $e = 1_R$ is infinite, because $R = R1_R = Ry_1x_1 \oplus R(1_R - y_1x_1)$, and it's easy to show that $R1_R \cong Ry_1x_1$.

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Purely infinite simplicity

Leavitt's theorem, restated:

For $n \geq 2$, $L_K(1, n)$ is purely infinite simple.

Purely infinite simplicity

Side note: $\mathrm{M}_2(K)$ is simple, but not purely infinite simple.

Consider e.g., $r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} ac' & ad' \\ cc' & cd' \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Of course we do have ...

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Theorem: E finite.

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Moreover, in this situation, we can easily calculate $\mathcal{V}(L_K(E))$ using the Smith normal form of the matrix $I - A_E^t$.

Remark: It is a long but elementary task to show that $M_E \setminus \{0\}$ is a group if and only if E has the three germane properties.

Purely infinite simplicity

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Those coming from graphs with cycles, and those coming from graphs without cycles.

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Those coming from graphs with cycles, and those coming from graphs without cycles.

The only simple rings coming from graphs without cycles are $\mathrm{M}_n(K)$ for some n (by Lecture 1 result).

So the only simple Leavitt path algebras are $\mathrm{M}_n(K)$ for some $n \in \mathbb{N}$, or are purely infinite simple.

Other ring-theoretic properties of Leavitt path algebras

When is *every* ideal of $L_K(E)$ graded?

This happens, e.g., in $L_K(\begin{tikzcd}\bullet \arrow[loop above] \arrow[loop right]\end{tikzcd})$, but not in $L_K(\begin{tikzcd}\bullet \arrow[loop right]\end{tikzcd})$.

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We say a vertex v *has Condition (K)* if v is either the base of no cycles in E, or is the base of at least two simple closed paths in E. We say E has Condition (K) in case every vertex of E has Condition (K).

Other ring-theoretic properties of Leavitt path algebras

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Proposition: Every ideal of $L_K(E)$ is graded if and only if E has Condition (K).

Idea: Roughly, if a vertex v does not have Condition (K), then if c denotes the (unique) cycle based at v, the ideal $\langle v + c \rangle$ of $L_K(E)$ behaves somewhat like the ideal $\langle 1 + x \rangle$ of $K[x, x^{-1}]$.

Other ring-theoretic properties of Leavitt path algebras

We know precisely the graphs E for which $L_K(E)$ has various other properties, e.g.:

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We know precisely the graphs E for which $L_K(E)$ has various other properties, e.g.:

1. one-sided chain conditions
2. von Neumann regular
3. exchange ($\Leftrightarrow$ Condition (K))
4. two-sided chain conditions
5. primitive

Many more.

One-sided chain conditions

Proposition. Suppose c is a cycle without exit, based at v. Then $vL_K(E)v \cong K[x, x^{-1}]$.

Proof. The only paths in E which start and end at v consist of c, repeated some number of times. Also, $cc^* = v$ (by no exits). So elements of $vL_K(E)v = \sum_{i=m}^{n} k_i c^i$, where c^i is defined as $(c^*)^{-i}$ for $i < 0$.

One-sided chain conditions

Proposition. E finite. Then $R = L_K(E)$ is (one-sided) artinian if and only if E is acyclic.

Proof. If E is acyclic then $R \cong \oplus_{i=1}^{t} \mathrm{M}_i(K)$ (by Lecture 1), which is well known to be artinian.

$L_K(E)$ Artinian $\Leftrightarrow$ E acyclic

Conversely, suppose E contains a cycle c, based at v.

Case 1: Suppose c has no exit. Then $vL_K(E)v \cong K[x, x^{-1}]$, which is not artinian. So $L_K(E)$ is not artinian.

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Case 1: Suppose c has no exit. Then $vL_K(E)v \cong K[x, x^{-1}]$, which is not artinian. So $L_K(E)$ is not artinian.

Case 2: Suppose c has an exit, call it e. W.l.o.g we may assume that $s(e) = v$. Note $c^*e = 0$. Now

$$Rcc^* \supsetneqq Rc^2(c^*)^2 \supsetneqq Rc^3(c^*)^3 \supsetneqq \cdots$$

Containment? $\quad c^{i+1}(c^*)^{i+1} = c^{i+1}(c^*)^{i+1} \cdot c^i(c^*)^i$.

Proper? $\quad$ If $c^i(c^*)^i = r \cdot c^{i+1}(c^*)^{i+1}$ then multiply by $c^i e$ on the right to get $c^i e = r \cdot c^{i+1}c^*e = 0$, a contradiction.

One-sided chain conditions

Proposition. E finite. Then $R = L_K(E)$ is (one-sided) noetherian if and only if no cycle in E has an exit.

Proof. If no cycle in E has an exit, then (using ideas similar to the acyclic case),

$$R \cong (\oplus_{i=1}^{t} \mathrm{M}_i(K)) \oplus (\oplus_{j=1}^{u} \mathrm{M}_j(K[x, x^{-1}])),$$

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which is well known to be noetherian.

Conversely, suppose E contains a cycle c with an exit, again assume based at v. Then similar to above, we consider

$$R(v - cc^*) \subsetneq R(v - c^2(c^*)^2) \subsetneq R(v - c^3(c^*)^3) \subsetneq \cdots$$

Proper containments follow as above.

Connections to graph C^*-algebras.

Since around the year 2000, operator algebraists have investigated the C^*-algebra $C^*(E)$ associated with a directed graph E. [47]

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Assume for now that E is finite. With appropriate notation, and (CK1), (CK2) in mind,

$$C^*(E) = \overline{\mathrm{span}}(\{S_\mu S_\nu^*\}).$$

For us, the best way to think of the relationship between $L_{\mathbb{C}}(E)$ and $C^*(E)$ is

$$L_{\mathbb{C}}(E) = \mathrm{span}_{\mathbb{C}}(\{S_\mu S_\nu^*\}) \ \subseteq \ \overline{\mathrm{span}}_{\mathbb{C}}(\{S_\mu S_\nu^*\}) = C^*(E).$$

So $L_{\mathbb{C}}(E)$ may be viewed as a $\mathbb{C}$-subalgebra of $C^*(E)$, closed under $*$, and dense in $C^*(E)$.

Connections to graph C^*-algebras

Some relationships between $L_{\mathbb{C}}(E)$ and $C^*(E)$.

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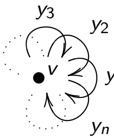
3. If $E = R_n =$ 

, then

$$L_{\mathbb{C}}(1, n) = L_{\mathbb{C}}(E) \subsetneq C^*(E) = \mathcal{O}_n,$$

the Cuntz algebra of order n.

Connections to graph C^*-algebras

Any C^*-algebra A wears two hats:

1. view A as a ring, or
2. view the ring-theoretic structure of A from a topological/analytic viewpoint.

Example: The (algebraic) simplicity of the C^*-algebra as a ring (no nontrivial two-sided ideals), or the (topological) simplicity as a topological ring (no nontrivial closed two-sided ideals).

In general, such properties need not coincide. But for graph C^*-algebras of finite graphs, they often do. AND, these properties often coincide with the corresponding (algebraic) properties of $L_{\mathbb{C}}(E)$.

Connections to graph C^*-algebras.

Simplicity:

Algebraic: No nontrivial two-sided ideals.
Analytic: No nontrivial closed two-sided ideals.

$L_{\mathbb{C}}(E)$ is simple if and only if E is cofinal and has Condition (L).

$C^*(E)$ is (topologically) simple if and only if E is cofinal and has Condition (L).

For any unital C^*-algebra A, A is topologically simple if and only if A is algebraically simple.

Result: These are equivalent for any finite graph E:

1. $L_{\mathbb{C}}(E)$ is simple
2. $C^*(E)$ is (topologically) simple
3. $C^*(E)$ is (algebraically) simple
4. E is cofinal, and satisfies Condition (L).

Connections to graph C^*-algebras.

The $\mathcal{V}$-monoid:

Algebraic: For a ring R, $\mathcal{V}(R)$ is the monoid of isomorphism classes of finitely generated left R-modules, with operation $\oplus$.

Analytic: For an operator algebra A, $\mathcal{V}_{MvN}(A)$ is the monoid of Murray - von Neumann equivalence classes of projections in $\mathrm{FM}_{\mathbb{N}}(A)$.

Whenever A is a C^*-algebra, then $\mathcal{V}(A)$ agrees with $\mathcal{V}_{MvN}(A)$.

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Result: For any finite graph E, these monoids are isomorphic:

1. The graph monoid M_E
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Note: $\mathcal{V}(L_K(E)) \cong \mathcal{V}(C^*(E))$ is very nontrivial; [36]

Connections to graph C^*-algebras.

Purely infinite simplicity:

Algebraic: R is purely infinite simple in case R is simple and every nonzero right ideal of R contains an infinite idempotent.

Analytic: The simple C^*-algebra A is called purely infinite (simple) if for every positive $x \in A$, the subalgebra $\overline{xAx}$ contains an infinite projection.

For graph C^*-algebras, $C^*(E)$ is (algebraically) purely infinite simple if and only if $C^*(E)$ is (topologically) purely infinite simple.

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Result: These are equivalent:

1. $L_{\mathbb{C}}(E)$ is purely infinite simple.
2. $C^*(E)$ is (topologically) purely infinite simple.
3. $C^*(E)$ is (algebraically) purely infinite simple.
4. E is cofinal, every cycle in E has an exit, and every vertex in E connects to a cycle.

Connections to graph C^*-algebras.

There are other properties for which this happens, e.g.:

1. exchange
2. primitivity
3. stable rank (*)

Some differences

But there are some notable differences.

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Primeness:

Some differences

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Primeness: Let $E = \begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd}$

Then $L_{\mathbb{C}}(E) = \mathbb{C}[x, x^{-1}]$ is prime (it's an integral domain), but $C^*(E) = C(\mathbb{T})$ is not prime (it's not hard to write down nonzero continuous functions on the circle which are orthogonal.)

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Tensor products:

$$\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2, \quad \text{but} \quad L_{\mathbb{C}}(1,2) \otimes L_{\mathbb{C}}(1,2) \ncong L_{\mathbb{C}}(1,2)$$

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Proposition: For finite graphs E, F:

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$L_K(E) \otimes L_K(F) \cong L_K(G)$ for some graph $G \Leftrightarrow E$ or F is acyclic.

Connections to C^*-algebras

The Isomorphism Conjecture:

If $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$, must we have $C^*(E) \cong C^*(F)$?

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This had been established (2010) in case E has $L_{\mathbb{C}}(E)$ simple.

This is now the Isomorphism **Theorem** (for E^0 finite).

"The complete classification of unital graph C^*-algebras: geometric and strong",

Eilers, Restorff, Ruiz, Sørensen posted on arXiv November 2016.

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- current / future lines of research, and some still-open questions