Leavitt path algebras:

applications, generalizations, and future directions of research

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Minicourse on Leavitt path algebras, Lecture 3

III Workshop on Dynamics, Numeration, Tilings and Graph Algebras (III FloripaDynSys)

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Overview

Definition of Leavitt path algebra

Start with a directed graph E, build its double graph $\widehat{E}$.

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(CK1) $e^*e = r(e)$; and $f^*e = 0$ for $f \neq e$ (for all edges e, f in E).

(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for each vertex v in E.
(just at "regular" vertices)

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Definition

The Leavitt path algebra of E with coefficients in K

$$L_K(E) = K\widehat{E} \; / \; < (CK1), (CK2) >$$

Matrices over Leavitt algebras

Easy (but key) observation:

Let S be any unital ring.

Suppose S contains $2n$ elements which behave exactly in the way as do the elements $x_1, \dots, x_n, y_1, \dots, y_n$ of $L(1, n)$.

Suppose also that these $2n$ elements generate S; that is, every $s \in S$ can be written as a sum of products of these elements.

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Then $S \cong L(1, n)$.

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Also, ${}_R R \cong {}_R R^n \cong {}_R R^{2n-1} \cong {}_R R^{3n-2} \cong \ldots$, which also in turn yield ring isomorphisms

$$R \cong \mathrm{M}_n(R) \cong \mathrm{M}_{2n-1}(R) \cong \mathrm{M}_{3n-2}(R) \cong \ldots$$

Matrices over Leavitt algebras

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For instance, if $R = L(1,4)$, then it's not hard to show that $R \cong \mathrm{M}_2(R)$ as rings (even though $R \ncong {}_R R^2$ as modules).
Idea: These eight matrices inside $\mathrm{M}_2(L(1,4))$ "work":

$$X_1 = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} x_3 & 0 \\ x_4 & 0 \end{pmatrix},\ X_3 = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix},\ X_4 = \begin{pmatrix} 0 & x_3 \\ 0 & x_4 \end{pmatrix}$$

together with their duals

$$Y_1 = \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix},\ Y_2 = \begin{pmatrix} y_3 & y_4 \\ 0 & 0 \end{pmatrix},\ Y_3 = \begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix},\ Y_4 = \begin{pmatrix} 0 & 0 \\ y_3 & y_4 \end{pmatrix}$$

E.g., $Y_1X_1 + Y_2X_2 = e_{1,1}$, $Y_1X_3 + Y_2X_4 = e_{1,2}$, etc ...

Matrices over Leavitt algebras

In general, using this same idea, we can show that:

if $d | n^t$ for some $t \in \mathbb{N}$, then $L(1, n) \cong \mathrm{M}_d(L(1, n))$.

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On the other hand ...

If $R = L(1,n)$, then the "type" of R is $n-1$. (Think: "smallest difference"). Bill Leavitt showed the following in his 1962 paper:

The type of $\mathrm{M}_d(L(1,n))$ is $\frac{n-1}{g.c.d.(d,n-1)}$.

In particular, if $g.c.d.(d,n-1) > 1$, then $L(1,n) \not\cong \mathrm{M}_d(L(1,n))$.

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Conjecture: $L(1,n) \cong \mathrm{M}_d(L(1,n)) \Leftrightarrow g.c.d.(d,n-1) = 1$.

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Conjecture: $L(1,n) \cong \mathrm{M}_d(L(1,n)) \;\Leftrightarrow\; g.c.d.(d,n-1) = 1$.

(Note: $d|n^t \;\Rightarrow\; g.c.d.(d,n-1) = 1$.)

Matrices over Leavitt algebras

Smallest interesting pair: Is $L(1,5) \cong \mathrm{M}_3(L(1,5))$?

In trying to mimic the $d|n^t$ case, we are led "naturally" to consider these five matrices (and their duals) in $\mathrm{M}_3(L(1,5))$:

$$\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_4 & 0 & 0 \\ x_5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & {x_1}^2 \\ 0 & 0 & x_2x_1 \\ 0 & 0 & x_3x_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_4x_1 \\ 0 & 0 & x_5x_1 \\ 0 & 0 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_3 \\ 0 & 0 & x_4 \\ 0 & 0 & x_5 \end{pmatrix}$$

$$\begin{pmatrix} y_1 & y_2 & y_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_4 & y_5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_1^2 & y_1y_2 & y_1y_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_1y_4 & y_1y_5 & y_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_3 & y_4 & y_5 \end{pmatrix}$$

Matrices over Leavitt algebras

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And we can generate *much of* $\mathrm{M}_3(L(1,5))$, using these ten matrices.

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And we can generate *much of* $\mathrm{M}_3(L(1,5))$, using these ten matrices.

But we couldn't see how to generate, for example, the matrix units $e_{1,3}$ and $e_{3,1}$ inside $\mathrm{M}_3(L(1,5))$.

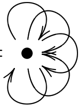
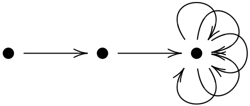
Matrices over Leavitt algebras

Breakthrough came from an analysis of isomorphisms between more general Leavitt path algebras.

There are a few “graph moves” which preserve the isomorphism classes of Leavitt path algebras.

“Shift” and ”outsplitting”.

Matrices over Leavitt algebras

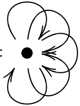
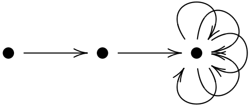
$R_5 =$  and $R_5(3) =$ 

There exists a sequence of graphs

$$R_5 = F_1, F_2, ..., F_7 = R_5(3)$$

for which F_{i+1} is gotten from F_i by one of these two "graph moves".

Matrices over Leavitt algebras

$R_5 =$  and $R_5(3) =$ 

There exists a sequence of graphs

$$R_5 = F_1, F_2, ..., F_7 = R_5(3)$$

for which F_{i+1} is gotten from F_i by one of these two "graph moves". So

$$L(1,5) \cong L(R_5) \cong L(F_2) \cong \cdots \cong L(R_5(3)) \cong \mathrm{M}_3(L(R_5)) \cong \mathrm{M}_3(L(1,5)).$$

Note: For $2 \leq i \leq 6$ it is not immediately obvious how to view $L(F_i)$ in terms of a matrix ring over a Leavitt algebra.

Matrices over Leavitt algebras

Original set of elements in $\mathrm{M}_3(L(1,5))$ (plus duals):

$$\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_4 & 0 & 0 \\ x_5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & {x_1}^2 \\ 0 & 0 & x_2x_1 \\ 0 & 0 & x_3x_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_4x_1 \\ 0 & 0 & x_5x_1 \\ 0 & 0 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_3 \\ 0 & 0 & x_4 \\ 0 & 0 & x_5 \end{pmatrix}$$

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Instead, the image of the set $x_1, ..., x_5$ in $L(1,5)$ under the above isomorphism is this set of elements in $\mathrm{M}_3(L(1,5))$:

$$\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_4 & 0 & 0 \\ x_5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & {x_1}^2 \\ 0 & 0 & x_2x_1 \\ 0 & 0 & x_3x_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_4x_1 \\ 0 & 0 & x_5x_1 \\ 0 & 0 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_4 \\ 0 & 0 & x_3 \\ 0 & 0 & x_5 \end{pmatrix}$$

Matrices over Leavitt algebras

Now consider this set, which we will call "The List":

$$x_1^{d-1}$$

$$x_2x_1^{d-2}, x_3x_1^{d-2}, \ldots, x_nx_1^{d-2}$$

$$x_2x_1^{d-3}, x_3x_1^{d-3}, \ldots, x_nx_1^{d-3}$$

$$\vdots$$

$$x_2x_1, x_3x_1, \ldots, x_nx_1$$

$$x_2, x_3, \ldots, x_n$$

Matrices over Leavitt algebras

Lemma / Key Observation. The elements of The List satisfy:

$$y_1^{d-1}x_1^{d-1} + \sum_{i=0}^{d-2}\sum_{j=2}^{n} y_1^i y_j x_j x_1^i = 1_K.$$

Matrices over Leavitt algebras

For integers n, d for which $\mathrm{g.c.d.}(d, n-1) = 1$, there is an algorithm to partition $\{1, 2, ..., d\}$ as $S_1 \cup S_2$ in a specified way.

This induces a partition of $\{1, 2, ..., n\}$ as $\hat{S}_1 \cup \hat{S}_2$ by extending $\mathrm{mod}\ d$.

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Proposition. It is possible to place the elements of The List in the "to be specified" entries of the "to be completed" matrices in such a way that each entry of the form $x_u x_1^t$ for $u \in S_k$ $(k = 1, 2)$ is placed in a row indexed by $\hat{u}$ where $\hat{u} \in \hat{S}_k$ $(k = 1, 2)$.

Matrices over Leavitt algebras

Theorem

(A-, Ánh, Pardo; Crelle's J. 2008)

$$L(1,n) \cong \mathrm{M}_d(L(1,n)) \;\Leftrightarrow\; g.c.d.(d, n-1) = 1.$$

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More generally,

$$\mathrm{M}_d(L(1,n)) \cong \mathrm{M}_{d'}(L(1,n)) \;\Leftrightarrow\; g.c.d.(d, n{-}1) = g.c.d.(d', n{-}1).$$

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Moreover, we can write down the isomorphisms explicitly.

Matrices over Leavitt algebras

Computations when $n = 5, d = 3$.

$\gcd(3, 5-1) = 1$. Now $5 = 1 \cdot 3 + 2$, so that $r = 2, r-1 = 1$, and define $s = d - (r-1) = 3 - 1 = 2$.

Consider the sequence starting at 1, and increasing by s each step, and interpret mod d $(1 \leq i \leq d)$. This will necessarily give all integers between 1 and d.

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Now break this set into two pieces: those integers up to and including $r - 1$, and those after. Since $r - 1 = 1$, here we get

$$\{1, 2, 3\} = \{1\} \cup \{2, 3\}.$$

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Now extend these two sets mod 3 to all integers up to 5.

$$\{1, 4\} \cup \{2, 3, 5\}$$

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The Partition for $n = 5, d = 3$: $\{1, 4\} \cup \{2, 3, 5\}$.

"The List": $x_1^2, \quad x_2x_1, x_3x_1, x_4x_1, x_5x_1, \quad x_2, x_3, x_4, x_5$.

The point is that $\{x_1^2, x_4x_1, x_4\}$ appear in row 1, while $\{x_2x_1, x_3x_1, x_5x_1, x_2, x_3, x_5\}$ appear in either rows 2 or 3.

Matrices over Leavitt algebras

Another Example of the Partition. Suppose $n = 35, d = 13$. Then $\gcd(13, 35-1) = 1$, so we are in the desired situation. Now $35 = 2 \cdot 13 + 9$, so that $r = 9, r-1 = 8$, and $s = d - (r-1) = 13 - 8 = 5$. Then we consider the sequence starting at 1, and increasing by s each step, and interpret mod d.

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So here we get the sequence $1, 6, 11, 3, 8, 13, 5, 10, 2, 7, 12, 4, 9.$

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Now break this set into two pieces: those integers up to and including $r - 1$, and those after. Since $r - 1 = 8$, here we get

$$\{1, 2, ..., 13\} = \{1, 3, 6, 8, 11\} \cup \{2, 4, 5, 7, 9, 10, 12, 13\}.$$

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$$\{1, 2, ..., 13\} = \{1, 3, 6, 8, 11\} \cup \{2, 4, 5, 7, 9, 10, 12, 13\}.$$

Now extend these two sets mod 13 to all integers up to 35.

$$\{1, 3, 6, 8, 11, 14, 16, 19, 21, 24, 27, 29, 32, 34\} \cup$$

$$\{2, 4, 5, 7, 9, 10, 12, 13, 15, 17, 18, 20, 22, 23, 25, 26, 28, 30, 31, 33, 35\}$$

Matrices over Leavitt algebras

Does this elementary number theory seem familiar to anyone??

Details are in [Appendix 2].

Matrices over Leavitt algebras

Corollary. (Matrices over the Cuntz C*-algebras)

$$\mathcal{O}_n \cong \mathrm{M}_d(\mathcal{O}_n) \iff g.c.d.(d, n-1) = 1.$$

More generally,

$$\mathrm{M}_d(\mathcal{O}_n) \cong \mathrm{M}_{d'}(\mathcal{O}_n) \iff g.c.d.(d, n-1) = g.c.d.(d', n-1).$$

(And the isomorphisms are explicitly described.)

Application to the theory of simple groups

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For each pair of positive integers n, r, there exists an infinite, finitely presented simple group $G_{n,r}^{+}$.

"Higman - Thompson" groups.

Application to the theory of simple groups

For each pair of positive integers n, r, there exists an infinite, finitely presented simple group $G^+_{n,r}$.

"Higman - Thompson" groups.
Higman knew *some* conditions regarding isomorphisms between these groups, but did not have a complete classification.

Matrices over Leavitt algebras

Theorem. (E. Pardo, 2011)

$$G_{n,r}^{+} \cong G_{m,s}^{+} \quad \Leftrightarrow \quad m = n \text{ and } \mathrm{g.c.d.}(r, n-1) = \mathrm{g.c.d.}(s, n-1).$$

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Proof. Fix n. Consider the set of invertible elements $U_r(n)$ in $M_r(L_{\mathbb{C}}(1,n))$ for which $u^{-1} = u^*$, and for which each of the entries of u is a sum of terms of the form $y_I x_J$.

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For example, let

$$u = y_1x_2 + y_2y_1x_1^2 + y_2^2x_2x_1 \in L_{\mathbb{C}}(1,2) = \mathrm{M}_1(L_{\mathbb{C}}(1,2)).$$

Then $u^* = y_2x_1 + y_1^2x_1x_2 + y_1y_2x_2^2$, and easily $uu^* = 1 = u^*u$. so that $u \in U_1(2) \subseteq U(\mathrm{M}_1(L_{\mathbb{C}}(1,2)))$.

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Now one shows that $G^+_{n,r} \cong U_r(n)$, and that the explicit isomorphisms provided in the A -, Ánh, Pardo result take $U_r(n)$ onto $U_s(n)$.

Ring theory reminders

1. R is *von Neumann regular* (or just *regular*) in case
$$\forall a \in R\ \exists\ \ x \in R \ \text{ with } a = axa.$$
2. R is *prime* if the product of any two nonzero two-sided ideals of R is nonzero.
3. R is *primitive* if R admits a faithful simple left R-module.

These are still valid for nonunital rings, in particular, for $L_K(E)$ with E infinite.

Connections and Applications: Kaplansky's question

Lemma: Every primitive ring is prime.

Connections and Applications: Kaplansky's question

Lemma: Every primitive ring is prime.

But the converse is not true: e.g. $\{0\}$ is a prime ideal of $A = K[x, x^{-1}]$, but not primitive. (R has no simple faithful modules.)

Connections and Applications: Kaplansky's question

Kaplansky, 1970: *Is a regular prime ring necessarily primitive?*

Answered in the negative (Domanov, 1977), a group-algebra example.

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Theorem. [16] $L_K(E)$ is von Neumann regular $\Leftrightarrow$ E is acyclic.

Theorem. [44] (mentioned previously) $L_K(E)$ is prime $\Leftrightarrow$ for each pair of vertices u, v in E there exists a vertex w in E for which $u \geq w$ and $v \geq w$.

Connections and Applications: Kaplansky's question

Theorem. [10] $L_K(E)$ is primitive $\Leftrightarrow$

1. $L_K(E)$ is prime,
2. every cycle in E has an exit, and
3. there exists a countable set of vertices S in E for which every vertex of E connects to an element of S.
   (Countable Separation Property)

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Proof: comment later.

Connections and Applications: Kaplansky's question

It's not hard to find acyclic graphs E for which $L_K(E)$ is prime but for which C.S.P. fails.

Example: X uncountable, S the set of finite subsets of X. Define the graph E:

1. vertices indexed by S, and
2. edges induced by proper subset relationship.

Then $L_K(E)$ is regular, prime, not primitive.

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1. vertices indexed by S, and
2. edges induced by proper subset relationship.

Then $L_K(E)$ is regular, prime, not primitive.

Note: Adjoining 1_K in the usual way (Dorroh extension by K) gives unital, regular, prime, not primitive algebras.

Remark: These examples are also "Cohn algebras".

Connections and Applications: The realization question for von Neumann regular rings

Fundamental problem: (Goodearl, 1994) What monoids M appear as $\mathcal{V}(R)$ for von Neumann regular R?

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Theorem: (Ara / Brustenga, 2007) For any row-finite graph E and field K there exists a von Neumann regular K-algebra $Q_K(E)$ for which $L_K(E)$ embeds in $Q_K(E)$, and

$$\mathcal{V}(L_K(E)) \cong \mathcal{V}(Q_K(E)).$$

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$$\mathcal{V}(L_K(E)) \cong \mathcal{V}(Q_K(E)).$$

Corollary: the realization question has affirmative answer for graph monoids M_E.

(and MANY monoids can be viewed as graph monoids; 2017 work by Ara / Pardo.)

Generalizations: relative Cohn path algebras

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OR, we only impose the (CK2) condition at a specified subset X of the regular vertices of E?

$C_K^X(E)$, the "relative Cohn path algebras"

Leavitt path algebras of separated graphs

In the (CK2) condition, the edges coming out from a given regular vertex v are treated as a single entity, and the single relation $v = \sum_{e \in s^{-1}(v)} ee^*$ is imposed.

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More generally, one may partition the set $s^{-1}(v)$ into disjoint nonempty subsets, and then impose a (CK2)-type relation corresponding exactly to those finite subsets.

(SCK1) for each $X \in C$, $e^* f = \delta_{e,f} r(e)$ for all $e, f \in X$, and

(SCK2) for each non-sink $v \in E^0$,

$$v = \sum_{e \in X} ee^* \text{ for every finite } X \in C_v.$$

Leavitt path algebras of separated graphs

So the usual Leavitt path algebra $L_K(E)$ is exactly $L_K(E, C)$, where each $C_v = \{s^{-1}(v)\}$ if v is not a sink, and $\emptyset$ otherwise.

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$L_K(m, n)$ for $m \geq 2$ originally studied by Leavitt do not arise as $L_K(E)$ for any graph E.

But $L_K(m, n)$ ($m \geq 2$) appears as a full corner of the Leavitt path algebra of an explicitly described separated graph (having two vertices and $m + n$ edges). In particular, $L_K(m, n)$ is Morita equivalent to the Leavitt path algebra of a separated graph.

Leavitt path algebras of separated graphs

AND,

Theorem

(Ara/Goodearl [31]) Let M be any conical abelian monoid. Then there exists a graph E, and partition $C = \sqcup_{v \in E^0} C_v$, for which $\mathcal{V}(L_K(E, C)) \cong M$.

Some additional generalizations

- Kumjian-Pask algebras.

  (a.k.a. "Leavitt path algebras of higher rank graphs")

The usual notion of a graph is a 1-graph in this more general context.

- Leavitt path algebras of weighted graphs.

Non-field coefficients; dependence on scalars

For a commutative unital ring R and graph E one may form the *path ring RE of E with coefficients in R* in the expected way.

It is then easy to see how to subsequently define the *Leavitt path ring $L_R(E)$ of E with coefficients in R*.

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I'll comment later about a very important property of $L_{\mathbb{Z}}(E)$ for which there are no (currently) known corresponding results about Leavitt path algebras $L_K(E)$ for K a field.

The groupoid approach: Steinberg algebras

A *groupoid* $\mathcal{G}$ is a small category in which every morphism has an inverse.

A *topological groupoid* is a groupoid in which the underlying set is equipped with a topology, in which both the product (i.e., composition) and inversion functions are continuous.

(Here the set of pairs of composable morphisms is given the topology induced from the product topology.)

The groupoid approach: Steinberg algebras

In 2010 Ben Steinberg introduced, for any topological groupoid $\mathcal{G}$ satisfying various additional topological conditions (Hausdorff and ample), and any commutative unital ring K, the *K-algebra of the groupoid $\mathcal{G}$*, denoted $K\mathcal{G}$.

("A groupoid approach to inverse semigroup algebras", Adv. Math. **223** (2010), 689–727.)

Formally, $K\mathcal{G}$ is the K-module spanned by the functions from $\mathcal{G}$ to K which have compact open support and which are continuous on their support (where K has the discrete topology).

The algebra $K\mathcal{G}$ is now known as the *Steinberg K-algebra* of the groupoid; in addition, the more common notation for $K\mathcal{G}$ has become $A_K(\mathcal{G})$.

The groupoid approach: Steinberg algebras

Key Proposition: Given a directed graph E, there exists a groupoid (the "graph groupoid" $\mathcal{G}_E$) for which:

1) $\mathcal{G}_E$ has the appropriate topological properties, and

2) $L_K(E) \cong A_K(\mathcal{G}_E)$.

The groupoid approach: Steinberg algebras

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The groupoid approach: Steinberg algebras

The importance of being able to interpret Leavitt path algebras as Steinberg algebras is (at least) twofold.

• the groupoid approach provides a context in which both Leavitt path algebras over $\mathbb{C}$ and graph C^*-algebras live.

• a number of results have been established for various types of Steinberg algebras (i.e., those associated to various types of groupoids). Many of these results have thereby been used to re-establish known results about Leavitt path algebras (e.g. Simplicity, Uniqueness Theorems) and provide some new results as well.

The groupoid approach: Steinberg algebras

In particular, Steinberg has established (2016) necessary and sufficient conditions which give the primitivity of $A_K(\mathcal{G})$ in terms of the structure of $\mathcal{G}$.

In the case where $\mathcal{G} = \mathcal{G}_E$ is the graph groupoid of the graph E, then the *effectiveness* of $\mathcal{G}_E$ corresponds to Condition (L) in E, while the *existence of a dense orbit* in E corresponds to the downward directedness of E. In fact, the dense orbit condition turns out to automatically yield the (CSP) condition on E in case E is not row-finite. So this gives a (much more natural) proof of the previously mentioned primitivity result.

The groupoid approach: Steinberg algebras

From Ben Steinberg's presentation "Partial actions and representations symposium", Gramado, May 2014 (modified slightly)

• *In 2009 I introduced a discrete analogue of groupoid C^*-algebras for ample groupoids. My hope was that it would explain many of the similarities between the discrete and continuous setting, especially for Leavitt path algebras.*

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• *Groupoid algebras over $\mathbb{C}$ were rediscovered later by L.O. Clark, C. Farthing, A. Sims and M. Tomforde, who kindly dubbed them "Steinberg algebras".*

("A groupoid generalization of Leavitt path algebras", Semigroup Forum **89** (2014), 501–517.)

The groupoid approach: Steinberg algebras

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“The graph algebra problem page”

Built / maintained by Mark Tomforde

www.math.uh.edu/∼tomforde/GraphAlgebraProblems/ListOfProblems.html

(See page 38 of survey.)

"The Algebraic Kirchberg Phillips Question"

Using some very deep results from symbolic dynamics (Theorems of Franks [64] and Huang), together with some nontrivial constructions of isomorphisms and Morita equivalences between Leavitt path algebras based on certain "graph moves", the following was established in [13].

(Recall that $K_0(L_K(E)) = \mathcal{V}(L_K(E)) \setminus \{[0]\}$ if $L_K(E)$ is purely infinite simple.)

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(Recall that $K_0(L_K(E)) = \mathcal{V}(L_K(E)) \setminus \{[0]\}$ if $L_K(E)$ is purely infinite simple.)

Notation: A_E denotes the "incidence matrix" or "edge matrix" of the graph E.

"The Algebraic Kirchberg Phillips Question"

Theorem

("The Restricted Algebraic Kirchberg Phillips Theorem") If E and F are finite graphs for which $L_K(E)$ and $L_K(F)$ are purely infinite simple, for which there is an isomorphism

$$\varphi : K_0(L_K(E)) \to K_0(L_K(F)) \quad \textit{having} \quad \varphi([1_{L_K(E)}]) = [1_{L_K(F)}],$$

and for which

$$\det(I - A_E^t) = \det(I - A_F^t),$$

then $L_K(E) \cong L_K(F)$ as K-algebras.

"The Algebraic Kirchberg Phillips Question"

What is generally agreed to be the most compelling unresolved question in the subject of Leavitt path algebras (as of March 2017) is:

(The Algebraic Kirchberg Phillips (KP) Question) Can the hypothesis on the determinants in the Restricted Algebraic Kirchberg Phillips Theorem be dropped?

"The Algebraic Kirchberg Phillips Question"

For comparison:

Theorem

If E and F are finite graphs for which $C^*(E)$ and $C^*(F)$ are purely infinite simple, for which there is a homeomorphism

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This follows from a very deep result about C^*-algebras established independently by Kirchberg and Phillips in 2000.

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then $C^*(E) \cong C^*(F)$ as C^*-algebras.

This follows from a very deep result about C^*-algebras established independently by Kirchberg and Phillips in 2000. This also follows somewhat more 'directly' from some results of Cuntz and Rørdam, which use KK-theory.

Note: The isomorphism in the Kirchberg Phillips Theorem is NOT explicitly constructed.

"The Algebraic Kirchberg Phillips Question"

With the Restricted Algebraic KP Theorem in mind, there are three possible answers to the Algebraic KP Question:

No. That is, if the two graphs E and F have $\det(I - A_E^t) \neq \det(I - A_F^t)$, then $L_K(E) \ncong L_K(F)$ for any field K.

Yes. That is, the existence of an isomorphism of the indicated type between the K_0 groups is sufficient to yield an isomorphism of the associated Leavitt path algebras, for any field K.

Sometimes. That is, for some pairs of graphs E and F, and/or for some fields K, the answer is *No*, and for other pairs the answer is *Yes*.

"The Algebraic Kirchberg Phillips Question"

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If the answer is *No*, then isomorphism classes of purely infinite simple unital Leavitt path algebras will match exactly the flow equivalence classes of the germane set of graphs.

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If the answer is *Yes*, this would yield further evidence for an as-yet-not-discovered direct connection between Leavitt path algebra results and graph C^*-algebra results.

“The Algebraic Kirchberg Phillips Question”

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If the answer is *Yes*, this would yield further evidence for an as-yet-not-discovered direct connection between Leavitt path algebra results and graph C^*-algebra results.

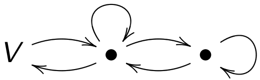
If the answer is *Sometimes*, then this would likely require the development and utilization of a completely new set of tools in the subject. (The *Sometimes* answer might be the most interesting of the three!)

"The Algebraic Kirchberg Phillips Question"

Suppose E is a finite graph for which $L_K(E)$ is purely infinite simple. There is a way to associate with E a new (finite) graph E_-, for which $L_K(E_-)$ is purely infinite simple, for which

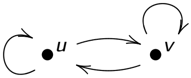
$$K_0(L_K(E)) \cong K_0(L_K(E_-)) \quad \text{and} \quad \det(I - A_E) = -\det(I - A_{E_-}).$$

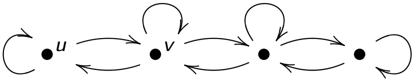
This is called the "Cuntz splice" process, which appends to a vertex $V \in E^0$ two additional vertices and six additional edges:



"The Algebraic Kirchberg Phillips Question"

The most basic pair of such algebras arises from the following two graphs:

$E_2 =$ 

and $E_4 =$ 

.

“The Algebraic Kirchberg Phillips Question”

The most basic pair of such algebras arises from the following two graphs:

$$E_2 = \begin{tikzcd} \bullet^u \arrow[loop left] \arrow[r, bend left] & \bullet^v \arrow[l, bend left] \arrow[loop above] \end{tikzcd} \quad \text{and} \quad E_4 = \begin{tikzcd} \bullet^u \arrow[loop left] \arrow[r, bend left] & \bullet^v \arrow[l, bend left] \arrow[loop above] \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[loop above] \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[loop right] \end{tikzcd}.$$

Is $L_K(E_2) \cong L_K(E_4)$?

"The Algebraic Kirchberg Phillips Question"

An extremely interesting recent development:

Theorem

(R. Johansen and A. Sørensen, J. Pure Appl. Algebra **220***(12), 2016.)*

The Leavitt path algebras $L_{\mathbb{Z}}(E_2)$ and $L_{\mathbb{Z}}(E_4)$ are not isomorphic as $*$-algebras.

“The Algebraic Kirchberg Phillips Question”

Similarly, the Algebraic KP Question can be modified to ask whether the Leavitt path algebras are isomorphic if the isomorphism is required to preserve additional properties.

For example, isomorphisms which preserve the diagonal subalgebra.

Tensor products

We saw previously that $L_K(1,2) \otimes_K L_K(1,2) \not\cong L_K(1,2)$.

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But we can ask:

Question: For a field K, does there exist a unital homomorphism $\phi\colon L_K(1,2) \otimes L_K(1,2) \to L_K(1,2)$?

This question is open. But there are recent germane results:

Theorem

(N. Brownlowe and A. Sørensen, J. Algebra **456***, 2016.)*

There is no unital $*$-embedding of $L_{\mathbb{Z}}(1,2) \otimes L_{\mathbb{Z}}(1,2)$ into $L_{\mathbb{Z}}(1,2)$.

Tensor products

There are a number of additional unresolved questions regarding tensor products of Leavitt path algebras, for example:

Question:

Is $L_K(1,2) \otimes_K L_K(1,3)$ isomorphic to $L_K(1,2) \otimes_K L_K(1,2)$

as K-algebras?

Focus on the $\mathbb{Z}$-graded structure

The Algebraic Kirchberg Phillips Question, motivated by the corresponding C^*-algebra result, is not the only natural classification-type question to ask in the context of Leavitt path algebras.

Focus on the $\mathbb{Z}$-graded structure

There is a very well-developed theory of graded modules over group-graded rings, which is especially robust in case the group is $\mathbb{Z}$.

See R. Hazrat, "Graded rings and graded Grothendieck groups", London Mathematical Society Lecture Note Series #435, 2016.

Focus on the $\mathbb{Z}$-graded structure

In the expected way, one defines *graded finitely generated projective module*, and the monoid $\mathcal{V}^{\mathrm{gr}}$, and the *graded Grothendieck group* $K_i^{\mathrm{gr}}(A)$ for each $i \geq 0$.

If $[M] \in \mathcal{V}^{\mathrm{gr}}$, then $[M(j)] \in \mathcal{V}^{\mathrm{gr}}$ for each $j \in \mathbb{Z}$, which yields a $\mathbb{Z}$-action on $\mathcal{V}^{\mathrm{gr}}$. Each of the Grothendieck groups becomes a $\mathbb{Z}[x, x^{-1}]$-module, via the suspension operation.

Focus on the $\mathbb{Z}$-graded structure

Question: Does the *graded* version of the Kirchberg Phillips Theorem hold?

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Hazrat's Conjecture: [68] Let E and F be any pair of finite graphs and K any field. Then $L_K(E) \cong L_K(F)$ as $\mathbb{Z}$-graded K-algebras if and only if $K_0^{\mathrm{gr}}(L_K(E)) \cong K_0^{\mathrm{gr}}(L_K(F))$ as $\mathbb{Z}[x, x^{-1}]$-modules, via an order-preserving isomorphism which takes $[1_{L_K(E)}]$ to $[1_{L_K(F)}]$.

Focus on the $\mathbb{Z}$-graded structure

Hazrat has verified the Conjecture in case the graphs E and F are *polycephalic* (essentially, mixtures of acyclic graphs, or graphs which can be described as “multiheaded comets” or “multiheaded roses” in which the cycles and/or roses have no exits.)

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But a complete resolution of Hazrat's Conjecture remains open.

Connections to noncommutative algebraic geometry

One of the basic ideas of (standard) algebraic geometry is the correspondence between geometric spaces and commutative algebras. Over the past few decades, significant research energy has been focused on appropriately extending this correspondence to the noncommutative case; the resulting theory is called *noncommutative algebraic geometry*.

Connections to noncommutative algebraic geometry

Suppose A is a $\mathbb{Z}^+$-graded algebra (i.e., a $\mathbb{Z}$-graded algebra for which $A_n = \{0\}$ for all $n < 0$).

Let $\mathrm{Gr}(A)$ denote the category of $\mathbb{Z}$-graded left A-modules (with graded homomorphisms), and let $\mathrm{Fdim}(A)$ denote the full subcategory of $\mathrm{Gr}(A)$ consisting of the graded A-modules which equal the sum of their finite dimensional submodules.

Denote by $\mathrm{QGr}(A)$ the quotient category $\mathrm{Gr}(A)/\mathrm{Fdim}(A)$. This category $\mathrm{QGr}(A)$ turns out to be one of the fundamental constructions in noncommutative algebraic geometry.

Connections to noncommutative algebraic geometry

In particular, if E is a directed graph, then the path algebra KE is $\mathbb{Z}^+$-graded in the usual way, and so one may construct the category $\mathrm{QGr}(KE)$.

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Let E^{nss} denote the graph gotten by repeatedly removing all sinks and sources (and their incident edges) from E.

Theorem

(S.P. Smith [94]) Let E be a finite graph. Then there is an equivalence of categories $\mathrm{QGr}(KE) \ \sim \ \mathrm{Gr}(L_K(E^{\mathrm{nss}}))$.

So the Leavitt path algebra construction arises naturally in the context of noncommutative algebraic geometry. In specific situations there are some geometric perspectives available, but the general case is not well understood.

Thank you.