Leavitt path algebras: some (surprising?) connections

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Overview

1. Isomorphisms between matrix rings over the Leavitt algebras $L_K(1, n)$

2. Some classical questions, answered using Leavitt path algebras

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(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for each vertex v in E.
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Definition

The Leavitt path algebra of E with coefficients in K

$$L_K(E) = K\widehat{E} \ / \ < (CK1), (CK2) >$$

Matrices over Leavitt algebras

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Also, ${}_RR \cong {}_RR^n \cong {}_RR^{2n-1} \cong {}_RR^{3n-2} \cong \ldots$, which also in turn yield ring isomorphisms

$$R \cong \mathrm{M}_n(R) \cong \mathrm{M}_{2n-1}(R) \cong \mathrm{M}_{3n-2}(R) \cong \ldots$$

Matrices over Leavitt algebras

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For instance, if $R = L(1,4)$, then it's not hard to show that $R \cong \mathrm{M}_2(R)$ as rings (even though $R \not\cong {}_R R^2$ as modules).
Idea: These eight matrices inside $\mathrm{M}_2(L(1,4))$ "work":

$$X_1 = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}, \ X_2 = \begin{pmatrix} x_3 & 0 \\ x_4 & 0 \end{pmatrix}, \ X_3 = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix}, \ X_4 = \begin{pmatrix} 0 & x_3 \\ 0 & x_4 \end{pmatrix}$$

together with their duals

$$Y_1 = \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix}, \ Y_2 = \begin{pmatrix} y_3 & y_4 \\ 0 & 0 \end{pmatrix}, \ Y_3 = \begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}, \ Y_4 = \begin{pmatrix} 0 & 0 \\ y_3 & y_4 \end{pmatrix}$$

E.g., $Y_1X_1 + Y_2X_2 = e_{1,1}$, $Y_1X_3 + Y_2X_4 = e_{1,2}$, etc ...

Matrices over Leavitt algebras

In general, using this same idea, we can show that:

if $d|n^t$ for some $t \in \mathbb{N}$, then $L(1, n) \cong \mathrm{M}_d(L(1, n))$.

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On the other hand ...

If $R = L(1,n)$, then the "type" of R is $n-1$. (Think: "smallest difference"). Bill Leavitt showed the following in his 1962 paper:

The type of $\mathrm{M}_d(L(1,n))$ is $\frac{n-1}{g.c.d.(d,n-1)}$.

In particular, if $g.c.d.(d,n-1) > 1$, then $L(1,n) \not\cong \mathrm{M}_d(L(1,n))$.

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Conjecture: $L(1,n) \cong \mathrm{M}_d(L(1,n)) \iff g.c.d.(d,n-1) = 1$.

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Conjecture: $L(1,n) \cong \mathrm{M}_d(L(1,n)) \Leftrightarrow g.c.d.(d,n-1) = 1$.

(Note: $d|n^t \Rightarrow g.c.d.(d,n-1) = 1$.)

Matrices over Leavitt algebras

Smallest interesting pair: Is $L(1,5) \cong \mathrm{M}_3(L(1,5))$?

In trying to mimic the $d|n^t$ case, we are led "naturally" to consider these five matrices (and their duals) in $\mathrm{M}_3(L(1,5))$:

$$\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_4 & 0 & 0 \\ x_5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & {x_1}^2 \\ 0 & 0 & x_2 x_1 \\ 0 & 0 & x_3 x_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_4 x_1 \\ 0 & 0 & x_5 x_1 \\ 0 & 0 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_3 \\ 0 & 0 & x_4 \\ 0 & 0 & x_5 \end{pmatrix}$$

$$\begin{pmatrix} y_1 & y_2 & y_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_4 & y_5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_1^2 & y_1 y_2 & y_1 y_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_1 y_4 & y_1 y_5 & y_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_3 & y_4 & y_5 \end{pmatrix}$$

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And we can generate *much of* $\mathrm{M}_3(L(1,5))$, using these ten matrices.

But we couldn't see how to generate, for example, the matrix units $e_{1,3}$ and $e_{3,1}$ inside $\mathrm{M}_3(L(1,5))$.

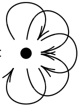
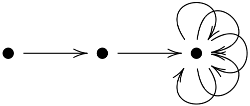
Matrices over Leavitt algebras

Breakthrough came from an analysis of isomorphisms between more general Leavitt path algebras.

There are a few “graph moves” which preserve the isomorphism classes of Leavitt path algebras.

“Shift” and ”outsplitting”.

Matrices over Leavitt algebras

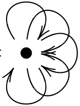
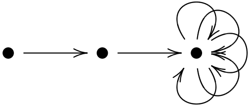
$E = R_5 =$  and $R_5(3) =$ 

There exists a sequence of graphs

$$R_5 = E_1, E_2, ..., E_7 = R_5(3)$$

for which E_{i+1} is gotten from E_i by one of these two "graph moves".

Matrices over Leavitt algebras

$E = R_5 =$  and $R_5(3) =$ 

There exists a sequence of graphs

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for which E_{i+1} is gotten from E_i by one of these two "graph moves".

$$\text{So } \; L_K(R_5) \cong L_K(E_2) \cong \cdots \cong L_K(R_5(3)) \cong \mathrm{M}_3(R_5).$$

Note: For $2 \le i \le 6$ it is not immediately obvious how to view $L_K(E_i)$ in terms of a matrix ring over a Leavitt algebra.

Matrices over Leavitt algebras

Original set of elements in $\mathrm{M}_3(L_K(1,5))$ (plus duals):

$$\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_4 & 0 & 0 \\ x_5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & {x_1}^2 \\ 0 & 0 & x_2x_1 \\ 0 & 0 & x_3x_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_4x_1 \\ 0 & 0 & x_5x_1 \\ 0 & 0 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_3 \\ 0 & 0 & x_4 \\ 0 & 0 & x_5 \end{pmatrix}$$

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Instead, the image of the set $x_1, ..., x_5$ in $L_K(1,5)$ under the above isomorphism is this set of elements in $\mathrm{M}_3(L_K(1,5))$:

$$\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_4 & 0 & 0 \\ x_5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & {x_1}^2 \\ 0 & 0 & x_2x_1 \\ 0 & 0 & x_3x_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_4x_1 \\ 0 & 0 & x_5x_1 \\ 0 & 0 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_4 \\ 0 & 0 & x_3 \\ 0 & 0 & x_5 \end{pmatrix}$$

Matrices over Leavitt algebras

Now consider this set, which we will call "The List":

$$x_1^{d-1}$$

$$x_2x_1^{d-2}, x_3x_1^{d-2}, \ldots, x_nx_1^{d-2}$$

$$x_2x_1^{d-3}, x_3x_1^{d-3}, \ldots, x_nx_1^{d-3}$$

$$\vdots$$

$$x_2x_1, x_3x_1, \ldots, x_nx_1$$

$$x_2, x_3, \ldots, x_n$$

Matrices over Leavitt algebras

Lemma / Key Observation. The elements of The List satisfy:

$$y_1^{d-1}x_1^{d-1} + \sum_{i=0}^{d-2}\sum_{j=2}^{n} y_1^i y_j x_j x_1^i = 1_K.$$

Matrices over Leavitt algebras

For integers n, d for which $\mathrm{g.c.d.}(d, n-1) = 1$, there is an algorithm to partition $\{1, 2, ..., d\}$ as $S_1 \cup S_2$ in a specified way.

This induces a partition of $\{1, 2, ..., n\}$ as $\hat{S}_1 \cup \hat{S}_2$ by extending $\bmod\ d$.

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Proposition. It is possible to place the elements of The List in the “to be specified” entries of the “to be completed” matrices in such a way that each entry of the form $x_u x_1^t$ for $u \in S_k$ $(k = 1, 2)$ is placed in a row indexed by $\hat{u}$ where $\hat{u} \in \hat{S}_k$ $(k = 1, 2)$.

Matrices over Leavitt algebras

Theorem

(A-, Ánh, Pardo; Crelle's J. 2008)

$$L(1, n) \cong \mathrm{M}_d(L(1, n)) \Leftrightarrow g.c.d.(d, n-1) = 1.$$

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Moreover, we can write down the isomorphisms explicitly.

Matrices over Leavitt algebras

Computations when $n = 5, d = 3$.

$\gcd(3, 5-1) = 1$. Now $5 = 1 \cdot 3 + 2$, so that $r = 2, r - 1 = 1$, and define $s = d - (r-1) = 3 - 1 = 2$.

Consider the sequence starting at 1, and increasing by s each step, and interpret mod d $(1 \leq i \leq d)$. This will necessarily give all integers between 1 and d.

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Now break this set into two pieces: those integers up to and including $r-1$, and those after. Since $r-1 = 1$, here we get

$$\{1, 2, 3\} = \{1\} \cup \{2, 3\}.$$

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Now extend these two sets mod 3 to all integers up to 5.

$$\{1, 4\} \ \cup \{2, 3, 5\}$$

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The Partition for $n = 5, d = 3$: $\{1,4\} \cup \{2,3,5\}$.

"The List": $x_1^2, \quad x_2x_1, x_3x_1, x_4x_1, x_5x_1, \quad x_2, x_3, x_4, x_5$.

The point is that $\{x_1^2, x_4x_1, x_4\}$ appear in row 1, while $\{x_2x_1, x_3x_1, x_5x_1, x_2, x_3, x_5\}$ appear in either rows 2 or 3.

Matrices over Leavitt algebras

Another Example of the Partition. Suppose $n = 35, d = 13$. Then $\gcd(13, 35 - 1) = 1$, so we are in the desired situation. Now $35 = 2 \cdot 13 + 9$, so that $r = 9, r - 1 = 8$, and $s = d - (r - 1) = 13 - 8 = 5$. Then we consider the sequence starting at 1, and increasing by s each step, and interpret mod d.

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So here we get the sequence $1, 6, 11, 3, 8, 13, 5, 10, 2, 7, 12, 4, 9.$

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Now break this set into two pieces: those integers up to and including $r - 1$, and those after. Since $r - 1 = 8$, here we get

$$\{1, 2, ..., 13\} = \{1, 3, 6, 8, 11\} \cup \{2, 4, 5, 7, 9, 10, 12, 13\}.$$

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Now extend these two sets mod 13 to all integers up to 35.

$$\{1, 3, 6, 8, 11, 14, 16, 19, 21, 24, 27, 29, 32, 34\} \cup$$

$$\{2, 4, 5, 7, 9, 10, 12, 13, 15, 17, 18, 20, 22, 23, 25, 26, 28, 30, 31, 33, 35\}$$

Matrices over Leavitt algebras

Does this elementary number theory seem familiar ??

Matrices over Leavitt algebras

Corollary. (Matrices over the Cuntz C*-algebras)

$$\mathcal{O}_n \cong \mathrm{M}_d(\mathcal{O}_n) \iff g.c.d.(d, n-1) = 1.$$

More generally,

$$\mathrm{M}_d(\mathcal{O}_n) \cong \mathrm{M}_{d'}(\mathcal{O}_n) \iff g.c.d.(d, n-1) = g.c.d.(d', n-1).$$

(And the isomorphisms are explicitly described.)

Application to the theory of simple groups

Here is an important recent application of the isomorphism theorem.

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For each pair of positive integers n, r, there exists an infinite, finitely presented simple group $G_{n,r}^{+}$. These were introduced in:

G. Higman, "Finitely presented infinite simple groups", Notes on Pure Mathematics, 8, Department of Pure Mathematics, I.A.S. Australian National University, Canberra, 1974.

Application to the theory of simple groups

Here is an important recent application of the isomorphism theorem.

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Higman knew *some* conditions regarding isomorphisms between these groups, but did not have a complete classification.

Matrices over Leavitt algebras

Theorem. (E. Pardo, 2011)

$$G^+_{n,r} \cong G^+_{m,s} \quad \Leftrightarrow \quad m = n \text{ and } \mathrm{g.c.d.}(r, n-1) = \mathrm{g.c.d.}(s, n-1).$$

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Proof. Fix n. Consider the set of invertible elements $U_r(n)$ in $M_r(L_{\mathbb{C}}(1,n))$ for which $u^{-1} = u^*$, and for which each of the entries of u is a sum of terms of the form $y_I x_J$.

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For example, let

$$u = y_1x_2 + y_2y_1x_1^2 + y_2^2x_2x_1 \in L_{\mathbb{C}}(1,2) = \mathrm{M}_1(L_{\mathbb{C}}(1,2)).$$

Then $u^* = y_2x_1 + y_1^2x_1x_2 + y_1y_2x_2^2$, and easily $uu^* = 1 = u^*u$. so that $u \in U_1(2) \subseteq U(\mathrm{M}_1(L_{\mathbb{C}}(1,2)))$.

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Now one shows that $G^+_{n,r} \cong U_r(n)$, and that the explicit isomorphisms provided in the A -, Ánh, Pardo result take $U_r(n)$ onto $U_s(n)$.

Ring theory reminders

1. R is *von Neumann regular* (or just *regular*) in case

$$\forall a \in R \; \exists \; x \in R \text{ with } a = axa.$$

2. R is *prime* if the product of any two nonzero two-sided ideals of R is nonzero.
3. R is *primitive* if R admits a faithful simple left R-module.

These are still valid for nonunital rings, in particular, for $L_K(E)$ with E infinite.

Connections and Applications: Kaplansky's question

Lemma: Every primitive ring is prime.

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So $(I \cdot J)M = 0$. If $JM = \{0\}$ then $J = \{0\}$ as M is faithful. So suppose $JM \neq 0$. Then $JM = M$ (as M is simple), so $(I \cdot J)M = 0$ gives $IM = 0$, so $I = \{0\}$ as M is faithful. $\square$

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But the converse is not true: e.g. $\{0\}$ is a prime ideal of $A = K[x, x^{-1}]$, but not primitive. (R has no simple faithful modules.)

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Answered in the negative (Domanov, 1977), a group-algebra example.

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Condition (MT3) or "Downward Directed"

Connections and Applications: Kaplansky's question

Theorem. (A-, Jason Bell, Ranga 2011) $L_K(E)$ is primitive $\Leftrightarrow$

1. $L_K(E)$ is prime,
2. every cycle in E has an exit, and
3. there exists a countable set of vertices S in E for which every vertex of E connects to an element of S.

   (Countable Separation Property)

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Idea of proof:

1. A unital ring R is left primitive if and only if there is a left ideal $M \neq R$ of R such that for every nonzero two-sided ideal I of R, $M + I = R$.

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2. Embed $L_K(E)$ in a unital algebra $L_K(E)_1$ in the usual way; primitivity is preserved.

3. Show that the lack of the CSP implies that no such left ideal can exist in $L_K(E)_1$.

Connections and Applications: Kaplansky's question

It's not hard to find acyclic graphs E for which $L_K(E)$ is prime but for which C.S.P. fails.

Example: X uncountable, S the set of finite subsets of X. Define the graph E:

1. vertices indexed by S, and
2. edges induced by proper subset relationship.

Then $L_K(E)$ is regular, prime, not primitive.

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Note: Adjoining 1_K in the usual way (Dorroh extension by K) gives unital, regular, prime, not primitive algebras.

Remark: These examples are also "Cohn algebras".

Lie algebras arising from associative algebras

Definitions / Notation.

R an associative K-algebra.

For $x, y \in R$ let $[x, y]$ denote $xy - yx$.

Let $[R, R]$ denote the K-subspace of R spanned by $\{[x, y] \mid x, y \in R\}$.

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$[R, R]$ with $[\ -\ ,\ -\ ]$ is a Lie K-algebra.

Lie algebras arising from associative algebras: general ideas

Let L denote a Lie K-algebra. A subset I of L is called a *Lie K-ideal* if I is a K-subspace of L and $[L, I] \subseteq I$.

Important Observation: *If* $K1_R \subseteq [R, R]$, then $K1_R$ is a Lie K-ideal of $[R, R]$. But we need not have $K1_R \subseteq [R, R]$ in general. (Cheap example: R commutative.)

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Definition: The Lie K-algebra L is called *simple* if $[L, L] \neq 0$ and the only Lie K-ideals of L are 0 and L.

Lie algebras arising from associative algebras: general ideas

Question: For which graphs E and fields K is the Lie algebra $[L_K(E), L_K(E)]$ simple?

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Of great help here:

Theorem (Herstein, 1965). Let S be a simple associative K-algebra. Assume either that $\mathrm{char}(S) \neq 2$, or that S is not 4-dimensional over $Z(S)$, where $Z(S)$ is a field.

Let U be any proper Lie K-ideal of the Lie algebra $[S, S]$.

Then $U \subseteq Z(S) \cap [S, S]$.

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If the center $Z(S)$ is 'small', then usually we have good control over all the Lie ideals of the Lie algebra $[S, S]$.

Main Consequence of Herstein's Theorem: Let R be a unital simple K-algebra which satisfies the hypotheses of Herstein's Theorem. Suppose that

1. $[[R, R], [R, R]] \neq 0$, and
2. $Z(R) = K1_R$.

Then $[R, R]$ is a simple Lie K-algebra if and only if $1_R \notin [R, R]$.

Step 1 towards: When is $1_{L_K(E)} \in [L_K(E), L_K(E)]$?

Definition. Write $E^0 = \{v_i \mid i \in I\}$.

For each $i \in I$, let $\epsilon_i \in \mathbb{Z}^{(I)}$ denote the element with 1 as the i-th coordinate and zeros elsewhere.

If v_i is a regular vertex, for all $j \in I$ let a_{ij} denote the number of edges $e \in E^1$ such that $s(e) = v_i$ and $r(e) = v_j$.

Define

$$B_i = (a_{ij})_{j \in I} - \epsilon_i \in \mathbb{Z}^{(I)}.$$

(If v_i is not a regular vertex, define $B_i = (0)_{j \in I} \in \mathbb{Z}^{(I)}$.)

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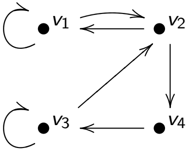
Example



Then

$$
\begin{aligned}
B_1 &= (1,1,0,0) - \epsilon_1 = (0,1,0,0),\\
B_2 &= (1,0,0,1) - \epsilon_2 = (1,-1,0,1),\\
B_3 &= (0,1,1,0) - \epsilon_3 = (0,1,0,0),\\
B_4 &= (0,0,1,0) - \epsilon_4 = (0,0,1,-1).
\end{aligned}
$$

Step 2 towards: When is $1_{L_K(E)} \in [L_K(E), L_K(E)]$?

Definition

Let K be a field, and let E be a directed graph. The *Cohn path K-algebra $C_K(E)$ of E with coefficients in K* is the path algebra $K\hat{E}$, modulo *only* the (CK1) relation

(CK1) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.

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Let $N \subseteq C_K(E)$ denote the ideal of $C_K(E)$ generated by elements of the form $v - \sum_{\{e \in E^1 | s(e) = v\}} ee^*$, where $v \in E^0$ is a regular vertex.

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So

$$L_K(E) \cong C_K(E)/N.$$

Step 3 towards: When is $1_{L_K(E)} \in [L_K(E), L_K(E)]$?

Using the "standard basis" available in $C_K(E)$, we can define a K-linear transformation

$$T : C_K(E) \to K^{(I)}$$

having

1. $T([x, y]) = 0$ for all $x, y \in C_K(E)$,
2. $T(v_i) = \epsilon_i$ for all $i \in I$, and
3. $T(w) \in \mathrm{span}_K\{B_i \mid i \in I\} \subseteq K^{(I)}$ for all $w \in N$.

Final step: When is $1_{L_K(E)} \in [L_K(E), L_K(E)]$?

Theorem

If E^0 is finite (so that $L_K(E)$ is unital), then

$$1_{L_K(E)} \in [L_K(E), L_K(E)] \Leftrightarrow (1, \ldots, 1) \in \mathrm{span}_K\{B_i \mid i \in I\} \subseteq K^{(I)}.$$

A few known results which complete the picture.

1. $K \cong L_K(\bullet)$ is the only simple commutative Leavitt path K-algebra. (So we call a simple Leavitt path algebra $L_K(E)$ *nontrivial* in case $L_K(E) \ncong K$.)

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2. For any noncommutative $R = L_K(E)$, $[[R, R], [R, R]] \neq 0$
3. The Simplicity Theorem for finite E
4. (The Centers Theorem for finite E) If $L_K(E)$ is simple, then $Z(L_K(E)) = K1_{L_K(E)}$.

Simplicity of $L_K(E)$

Now from Herstein's Theorem, the Centers Theorem, and our theorem about when $1_{L_K(E)}$ is (or is not) an element of $[L_K(E), L_K(E)]$, we get

Theorem (A-, Mesyan 2012)

Let K be a field, and let E be a finite graph for which $L_K(E)$ is a nontrivial simple Leavitt path algebra. Write $E^0 = \{v_1, \ldots, v_m\}$, and for each $1 \leq i \leq m$ let B_i be as above. Then

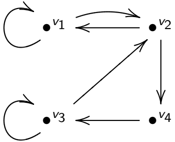
$$[L_K(E), L_K(E)] \quad \textit{is simple as a Lie } K\textit{-algebra}$$

if and only if

$$(1, \ldots, 1) \notin \mathrm{span}_K\{B_1, \ldots, B_m\}.$$

$[L_K(E), L_K(E)]$ simple $\Leftrightarrow$ $(1, \dots, 1) \notin \mathrm{span}_K\{B_1, \dots, B_m\}$

Previous example.



$B_1 = (0, 1, 0, 0), B_2 = (1, -1, 0, 1),\ B_3 = (0, 1, 0, 0),\ B_4 = (0, 0, 1, -1).$

Is $(1, 1, 1, 1)$ in $\mathrm{span}_K\{B_1, B_2, B_3, B_4\}$? That is, can we find $k_1, k_2, k_3, k_4 \in K$ for which

$$(1, 1, 1, 1) = k_1(0, 1, 0, 0)+k_2(1, -1, 0, 1)+k_3(0, 1, 0, 0)+k_4(0, 0, 1, -1)\ ?$$

$[L_K(E), L_K(E)]$ simple $\Leftrightarrow$ $(1, \ldots, 1) \notin \mathrm{span}_K\{B_1, \ldots, B_m\}$

So we want to solve a system. Here's the augmented matrix of the system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \vdots & 1 \\ 1 & -1 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 1 & 0 & -1 & \vdots & 1 \end{pmatrix}.$$

$[L_K(E), L_K(E)]$ simple $\Leftrightarrow$ $(1,\ldots,1) \notin \mathrm{span}_K\{B_1,\ldots,B_m\}$

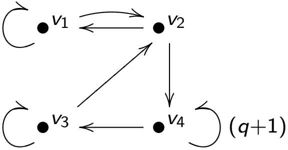
After row-reducing we get

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \vdots & 1 \\ 1 & -1 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & \vdots & 1 \end{pmatrix}.$$

The system has no solution (regardless of the characteristic of K).
So $[L_K(E), L_K(E)]$ is simple for any field K.

$[L_K(E), L_K(E)]$ simple $\Leftrightarrow$ $(1,\dots,1) \notin \mathrm{span}_K\{B_1,\dots,B_m\}$

More examples. Let $\mathcal{P} = \{p_1, p_2, \dots, p_t\}$ be a finite set of primes, let $q = p_1 p_2 \cdots p_t \in \mathbb{N}$, and let E_q be this graph.



Here $B_1 = (0,1,0,0)$, $B_2 = (1,-1,0,1)$, $B_3 = (0,1,0,0)$, and $B_4 = (0,0,1,q)$.

When is $(1,1,1,1)$ in $\mathrm{span}_K\{B_1, B_2, B_3, B_4\}$?

$[L_K(E), L_K(E)]$ simple $\Leftrightarrow$ $(1,\dots,1) \notin \mathrm{span}_K\{B_1,\dots,B_m\}$

Elementary row-operations on the augmented matrix yield:

$$\begin{pmatrix} 1 & -1 & 1 & 0 & \vdots & 1 \\ 0 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & \vdots & -q \end{pmatrix}.$$

So the system has solutions precisely when $\mathrm{char}(K)$ divides q, i.e., when $\mathrm{char}(K) \in \{p_1, p_2, \dots, p_t\}$. So by the Main Theorem,

$[L_K(E_q), L_K(E_q)]$ is simple if and only if $\mathrm{char}(K)$ is NOT in $\{p_1, p_2, \dots, p_t\}$.

$[L_K(E), L_K(E)]$ simple $\Leftrightarrow$ $(1,\ldots,1) \notin \mathrm{span}_K\{B_1,\ldots,B_m\}$

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Remark. On the other end of the spectrum, we can also build graphs where $[L_K(E), L_K(E)]$ is simple if and only if $\mathrm{char}(K)$ IS in $\{p_1, p_2, \ldots, p_t\}$.

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Let $q = p_1 p_2 \cdots p_t \in \mathbb{N}$. Then using the previous result for matrices over Leavitt algebras, the Lie K-algebra $[L_K(1,q+1), L_K(1,q+1)]$ is simple if and only if $\mathrm{char}(K) \in \mathcal{P}$.

Connections and Applications:
The realization question for von Neumann regular rings

Fundamental problem: (Goodearl, 1994) What monoids M appear as $\mathcal{V}(R)$ for von Neumann regular R?

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Theorem: (Ara / Brustenga, 2007) For any row-finite graph E and field K there exists a von Neumann regular K-algebra $Q_K(E)$ for which $L_K(E)$ embeds in $Q_K(E)$, and

$$\mathcal{V}(L_K(E)) \cong \mathcal{V}(Q_K(E)).$$

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Corollary: the realization question has affirmative answer for graph monoids M_E.

Thank you.