Leavitt path algebras: some (surprising?) connections

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Leavitt path algebras and symbolic dynamics

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Overview

1. Isomorphisms between matrix rings over the Leavitt algebras $L_K(1, n)$

2. Some classical questions, answered using Leavitt path algebras
Definition of Leavitt path algebra

Start with a directed graph $E$, build its double graph $\hat{E}$. 

(CK1) $e^*e = r(e)$; and $f^*e = 0$ for $f \neq e$ (for all edges $e$, $f$ in $E$).

(CK2) $v = \sum\{e \in E_1 | s(e) = v\}e^*e$ for each vertex $v$ in $E$.

(just at “regular” vertices)

Definition

The Leavitt path algebra of $E$ with coefficients in $K$ is $L_K(E) = K\hat{E}/<\text{(CK1)}, \text{(CK2)}>$. 

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Construct the path algebra $K\hat{E}$. Consider these relations in $K\hat{E}$:

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$$L_K(E) = K\hat{E} / \langle (CK1), (CK2) \rangle$$
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Matrices over Leavitt algebras

Let $R = L_{\mathbb{C}}(1, n)$. So $RR \cong R^n$ as left $R$-modules.
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Matrices over Leavitt algebras

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Which then (for free) gives some additional isomorphisms, e.g.

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for any $i \geq 1$. 
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Also, $RR \cong R^n \cong R^{2n-1} \cong R^{3n-2} \cong \ldots$, which also in turn yield ring isomorphisms

$$R \cong M_n(R) \cong M_{2n-1}(R) \cong M_{3n-2}(R) \cong \ldots$$
Matrices over Leavitt algebras

**Question:** Are there other matrix sizes $d$ for which $R \cong M_d(R)$?

**Answer:** In general, yes.
Matrices over Leavitt algebras

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**Answer:** In general, yes.

For instance, if $R = L(1, 4)$, then it’s not hard to show that $R \cong M_2(R)$ as rings (even though $R \ncong R R^2$ as modules).

**Idea:** These eight matrices inside $M_2(L(1, 4))$ “work”:

$X_1 = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} x_3 & 0 \\ x_4 & 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix}$, $X_4 = \begin{pmatrix} 0 & x_3 \\ 0 & x_4 \end{pmatrix}$

together with their duals

$Y_1 = \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix}$, $Y_2 = \begin{pmatrix} y_3 & y_4 \\ 0 & 0 \end{pmatrix}$, $Y_3 = \begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}$, $Y_4 = \begin{pmatrix} 0 & 0 \\ y_3 & y_4 \end{pmatrix}$

E.g., $Y_1X_1 + Y_2X_2 = e_{1,1}$, $Y_1X_3 + Y_2X_4 = e_{1,2}$, etc...
Matrices over Leavitt algebras

In general, using this same idea, we can show that:

if $d | n^t$ for some $t \in \mathbb{N}$, then $L(1, n) \cong M_d(L(1, n))$. 

Conjecture: $L(1, n) \cong M_d(L(1, n)) \iff g \cdot c \cdot d \cdot (d, n-1) = 1$.

(Note: $d | n^t \implies g \cdot c \cdot d \cdot (d, n-1) = 1$.)
Matrices over Leavitt algebras

In general, using this same idea, we can show that:

if \(d \mid n^t\) for some \(t \in \mathbb{N}\), then \(L(1, n) \cong M_d(L(1, n))\).

On the other hand ...

If \(R = L(1, n)\), then the “type” of \(R\) is \(n - 1\). (Think: “smallest difference”). Bill Leavitt showed the following in his 1962 paper:

The type of \(M_d(L(1, n))\) is \(\frac{n-1}{g.c.d.(d,n-1)}\).

In particular, if \(g.c.d.(d, n - 1) > 1\), then \(L(1, n) \not\cong M_d(L(1, n))\).
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In particular, if $\text{g.c.d.}(d, n - 1) > 1$, then $L(1, n) \ncong M_d(L(1, n))$.

**Conjecture:** $L(1, n) \cong M_d(L(1, n)) \iff g.c.d.(d, n - 1) = 1$.

(Note: $d|n^t \Rightarrow g.c.d.(d, n - 1) = 1.$)
Matrices over Leavitt algebras

Smallest interesting pair: Is $L(1, 5) \cong M_3(L(1, 5))$?

In trying to mimic the $d|n^t$ case, we are led “naturally” to consider these five matrices (and their duals) in $M_3(L(1, 5))$:

$$
\begin{pmatrix}
  x_1 & 0 & 0 \\
  x_2 & 0 & 0 \\
  x_3 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
  x_4 & 0 & 0 \\
  x_5 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
  0 & 0 & x_1^2 \\
  0 & 0 & x_2x_1 \\
  0 & 0 & x_3x_1
\end{pmatrix},
\begin{pmatrix}
  0 & 0 & x_4x_1 \\
  0 & 0 & x_5x_1 \\
  0 & 0 & x_2
\end{pmatrix},
\begin{pmatrix}
  0 & 0 & x_3 \\
  0 & 0 & x_4 \\
  0 & 0 & x_5
\end{pmatrix}
$$

$$
\begin{pmatrix}
  y_1 & y_2 & y_3 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
  y_4 & y_5 & 0 \\
  0 & 0 & 1 \\
  0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  y_1^2 & y_1y_2 & y_1y_3
\end{pmatrix},
\begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  y_1y_4 & y_1y_5 & y_2
\end{pmatrix},
\begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  y_3 & y_4 & y_5
\end{pmatrix}
$$
Matrices over Leavitt algebras

These ten matrices form a Leavitt $R_5$-family in $M_3(L(1, 5))$. 
Matrices over Leavitt algebras

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But we couldn’t see how to generate, for example, the matrix units $e_{1,3}$ and $e_{3,1}$ inside $M_3(L(1, 5))$. 
Matrices over Leavitt algebras

Breakthrough came from an analysis of isomorphisms between more general Leavitt path algebras.

There are a few “graph moves” which preserve the isomorphism classes of Leavitt path algebras.

“Shift” and ”outsplittting”. 
Isomorphisms between matrix rings over the Leavitt algebras $L_K(1, n)$

Matrices over Leavitt algebras

There exists a sequence of graphs

$$R_5 = E_1, E_2, ..., E_7 = R_5(3)$$

for which $E_{i+1}$ is gotten from $E_i$ by one of these two "graph moves".
Matrices over Leavitt algebras

\[ E = R_5 = \begin{array}{c}
\bullet \\
\end{array} \quad \text{and} \quad R_5(3) = \begin{array}{c}
\bullet \\
\rightarrow \\
\bullet \\
\end{array} \]

There exists a sequence of graphs

\[ R_5 = E_1, E_2, \ldots, E_7 = R_5(3) \]

for which \( E_{i+1} \) is gotten from \( E_i \) by one of these two “graph moves”.

So \( L_K(R_5) \cong L_K(E_2) \cong \cdots \cong L_K(R_5(3)) \cong M_3(R_5) \).

Note: For \( 2 \leq i \leq 6 \) it is not immediately obvious how to view \( L_K(E_i) \) in terms of a matrix ring over a Leavitt algebra.
Matrices over Leavitt algebras

Original set of elements in $M_3(L_K(1, 5))$ (plus duals):

\[
\begin{pmatrix}
  x_1 & 0 & 0 \\
  x_2 & 0 & 0 \\
  x_3 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
  x_4 & 0 & 0 \\
  x_5 & 0 & 0 \\
  0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
  0 & 0 & x_1^2 \\
  0 & 0 & x_2 x_1 \\
  0 & 0 & x_3 x_1 \\
\end{pmatrix}
\begin{pmatrix}
  0 & 0 & x_4 x_1 \\
  0 & 0 & x_5 x_1 \\
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\end{pmatrix}
\quad
\begin{pmatrix}
  0 & 0 & x_3 \\
  0 & 0 & x_4 \\
  0 & 0 & x_5 \\
\end{pmatrix}
\]

Instead, the image of the set $x_1, \ldots, x_5$ in $L_K(1,5)$ under the above isomorphism is this set of elements in $M_3(L_K(1,5))$:

\[
\begin{pmatrix}
  x_1 & 0 & 0 \\
  x_2 & 0 & 0 \\
  x_3 & 0 & 0 \\
\end{pmatrix}
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  x_4 & 0 & 0 \\
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  0 & 0 & x_4 \\
  0 & 0 & x_5 \\
\end{pmatrix}
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Matrices over Leavitt algebras

Now consider this set, which we will call “The List”:

\[ x_1^{d-1}, x_2x_1^{d-2}, x_3x_1^{d-2}, \ldots, x_n x_1^{d-2} \]
\[ x_2 x_1^{d-3}, x_3 x_1^{d-3}, \ldots, x_n x_1^{d-3} \]
\[ \vdots \]
\[ x_2 x_1, x_3 x_1, \ldots, x_n x_1 \]
\[ x_2, x_3, \ldots, x_n \]
Lemma / Key Observation. The elements of The List satisfy:

$$y_1^{d-1}x_1^{d-1} + \sum_{i=0}^{d-2} \sum_{j=2}^{n} y_1^i y_j x_j x_1^i = 1_K.$$
Matrices over Leavitt algebras

For integers $n, d$ for which $\gcd(d, n - 1) = 1$, there is an algorithm to partition $\{1, 2, ..., d\}$ as $S_1 \cup S_2$ in a specified way.

This induces a partition of $\{1, 2, ..., n\}$ as $\hat{S}_1 \cup \hat{S}_2$ by extending mod $d$. 
Matrices over Leavitt algebras

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This induces a partition of \( \{1, 2, ..., n\} \) as \( \hat{S}_1 \cup \hat{S}_2 \) by extending mod \( d \).

**Proposition.** It is possible to place the elements of The List in the “to be specified” entries of the “to be completed” matrices in such a way that each entry of the form \( x_u x_1^t \) for \( u \in S_k \) \((k = 1, 2)\) is placed in a row indexed by \( \hat{u} \) where \( \hat{u} \in \hat{S}_k \) \((k = 1, 2)\).
Matrices over Leavitt algebras

**Theorem**

\[(A-, Ánh, Pardo; Crelle’s J. 2008)\]

\[L(1, n) \cong M_d(L(1, n)) \iff \text{g.c.d.}(d, n - 1) = 1.\]
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\[ L(1, n) \cong M_d(L(1, n)) \iff \text{g.c.d.}(d, n-1) = 1. \]

More generally,

\[ M_d(L(1, n)) \cong M_{d'}(L(1, n)) \iff \text{g.c.d.}(d, n-1) = \text{g.c.d.}(d', n-1). \]
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More generally,

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Moreover, we can write down the isomorphisms explicitly.
Matrices over Leavitt algebras

**Computations when** $n = 5, d = 3$.

$\gcd(3, 5 - 1) = 1$. Now $5 = 1 \cdot 3 + 2$, so that $r = 2, r - 1 = 1$, and define $s = d - (r - 1) = 3 - 1 = 2$.

Consider the sequence starting at 1, and increasing by $s$ each step, and interpret mod $d$ ($1 \leq i \leq d$). This will necessarily give all integers between 1 and $d$. 
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So here we get the sequence $1, 3, 2$. 
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Now break this set into two pieces: those integers up to and including $r - 1$, and those after. Since $r - 1 = 1$, here we get

$$\{1, 2, 3\} = \{1\} \cup \{2, 3\}.$$
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Now extend these two sets mod 3 to all integers up to 5.

$$\{1, 4\} \cup \{2, 3, 5\}$$
Here are those matrices again:

Original set (plus duals):

\[
\begin{bmatrix}
x_1 & 0 & 0 \\
x_2 & 0 & 0 \\
x_3 & 0 & 0 \\
x_4 & 0 & 0 \\
x_5 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & x_1^2 \\
0 & 0 & x_2 x_1 \\
0 & 0 & x_3 x_1 \\
0 & 0 & x_4 x_1 \\
0 & 0 & x_5 x_1 \\
0 & 0 & x_2 \\
0 & 0 & x_3 \\
0 & 0 & x_4 \\
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\end{bmatrix}
\]

Instead, this set (plus duals) works:

\[
\begin{bmatrix}
x_1 & 0 & 0 \\
x_2 & 0 & 0 \\
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0 & 0 & x_2 \\
0 & 0 & x_3 \\
0 & 0 & x_4 \\
0 & 0 & x_5 \\
\end{bmatrix}
\]

The Partition for \( n = 5, d = 3 \):

\{1, 4\} \cup \{2, 3, 5\}.

"The List":

\[x_2 x_1, x_2 x_1 x_1, x_3 x_1, x_4 x_1, x_5 x_1, x_2 x_1, x_3 x_1, x_5 x_1, x_2, x_3, x_5, x_2, x_3, x_5, x_2, x_3, x_5, x_2, x_3, x_5.\]

The point is that \{x_2 x_1, x_4 x_1, x_4\} appear in row 1, while \{x_2 x_1, x_3 x_1, x_5 x_1, x_2, x_3, x_5\} appear in either rows 2 or 3.
Here are those matrices again: Original set (plus duals):

\[
\begin{pmatrix}
  x_1 & 0 & 0 \\
  x_2 & 0 & 0 \\
  x_3 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
  x_4 & 0 & 0 \\
  x_5 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix}, \begin{pmatrix}
  0 & 0 & x_1^2 \\
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  0 & 0 & x_5x_1 \\
  0 & 0 & x_2 \\
\end{pmatrix}
\begin{pmatrix}
  0 & 0 & x_4 \\
  0 & 0 & x_3 \\
  0 & 0 & x_5 \\
\end{pmatrix}
\]

The Partition for \( n = 5, d = 3 \): \( \{1, 4\} \cup \{2, 3, 5\} \).

"The List": \( x_1^2, x_2x_1, x_3x_1, x_4x_1, x_5x_1, x_2, x_3, x_4, x_5 \).

The point is that \( \{x_1^2, x_4x_1, x_4\} \) appear in row 1, while \( \{x_2x_1, x_3x_1, x_5x_1, x_2, x_3, x_5\} \) appear in either rows 2 or 3.
Matrices over Leavitt algebras

**Another Example of the Partition.** Suppose $n = 35$, $d = 13$. Then $\gcd(13, 35 - 1) = 1$, so we are in the desired situation. Now $35 = 2 \cdot 13 + 9$, so that $r = 9$, $r - 1 = 8$, and $s = d - (r - 1) = 13 - 8 = 5$. Then we consider the sequence starting at 1, and increasing by $s$ each step, and interpret mod $d$. 
Matrices over Leavitt algebras

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Matrices over Leavitt algebras

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$$\{1, 2, \ldots, 13\} = \{1, 3, 6, 8, 11\} \cup \{2, 4, 5, 7, 9, 10, 12, 13\}.$$
Matrices over Leavitt algebras

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Then \( \gcd(13, 35 - 1) = 1 \), so we are in the desired situation. Now
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Now break this set into two pieces: those integers up to and including \( r - 1 \), and those after. Since \( r - 1 = 8 \), here we get
\[ \{1, 2, \ldots, 13\} = \{1, 3, 6, 8, 11\} \cup \{2, 4, 5, 7, 9, 10, 12, 13\}. \]
Now extend these two sets mod 13 to all integers up to 35.
\[ \{1, 3, 6, 8, 11, 14, 16, 19, 21, 24, 27, 29, 32, 34\} \cup \]
\[ \{2, 4, 5, 7, 9, 10, 12, 13, 15, 17, 18, 20, 22, 23, 25, 26, 28, 30, 31, 33, 35\} \]
Matrices over Leavitt algebras

Does this elementary number theory seem familiar??
Matrices over Leavitt algebras

Corollary. (Matrices over the Cuntz C*-algebras)

\[ \mathcal{O}_n \cong M_d(\mathcal{O}_n) \iff \text{g.c.d.}(d, n-1) = 1. \]

More generally,

\[ M_d(\mathcal{O}_n) \cong M_{d'}(\mathcal{O}_n) \iff \text{g.c.d.}(d, n-1) = \text{g.c.d.}(d', n-1). \]

(And the isomorphisms are explicitly described.)
Application to the theory of simple groups

Here is an important recent application of the isomorphism theorem.
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For each pair of positive integers \( n, r \), there exists an infinite, finitely presented simple group \( G^{+}_{n,r} \). These were introduced in:


Higman knew some conditions regarding isomorphisms between these groups, but did not have a complete classification.
Matrices over Leavitt algebras

**Theorem.** (E. Pardo, 2011)

\[ G^{+}_{n,r} \cong G^{+}_{m,s} \iff m = n \text{ and } \gcd(r, n-1) = \gcd(s, n-1). \]
Matrices over Leavitt algebras

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**Proof.** Fix \( n \). Consider the set of invertible elements \( U_r(n) \) in \( M_r(L_C(1, n)) \) for which \( u^{-1} = u^* \), and for which each of the entries of \( u \) is a sum of terms of the form \( y_Ix_J \).
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For example, let

\[ u = y_1 x_2 + y_2 y_1 x_1^2 + y_2^2 x_2 x_1 \in L_C(1, 2) = M_1(L_C(1, 2)). \]

Then \( u^* = y_2 x_1 + y_1^2 x_1 x_2 + y_1 y_2 x_2^2 \), and easily \( uu^* = 1 = u^* u \) so that \( u \in U_1(2) \subseteq U(M_1(L_C(1, 2))). \)
Matrices over Leavitt algebras

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Now one shows that \( G_{n,r}^+ \cong U_r(n) \), and that the explicit isomorphisms provided in the A -, Ánh, Pardo result take \( U_r(n) \) onto \( U_s(n) \).
1. Isomorphisms between matrix rings over the Leavitt algebras $L_K(1, n)$

2. Some classical questions, answered using Leavitt path algebras
Ring theory reminders

1. *R* is von Neumann regular (or just regular) in case

\[ \forall a \in R \ \exists \ x \in R \ \text{with} \ a = axa. \]

2. *R* is prime if the product of any two nonzero two-sided ideals of *R* is nonzero.

3. *R* is primitive if *R* admits a faithful simple left *R*-module.

These are still valid for nonunital rings, in particular, for \( L_K(E) \) with *E* infinite.
Connections and Applications: Kaplansky’s question

**Lemma**: Every primitive ring is prime.
Connections and Applications: Kaplansky’s question

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**Proof.** Let $M$ denote a simple faithful left $R$-module. Suppose $I \cdot J = \{0\}$. We want to show either $I = \{0\}$ or $J = \{0\}.$
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**Lemma:** Every primitive ring is prime.

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So $(I \cdot J)M = 0$. If $JM = \{0\}$ then $J = \{0\}$ as $M$ is faithful. So suppose $JM \neq 0$. Then $JM = M$ (as $M$ is simple), so $(I \cdot J)M = 0$ gives $IM = 0$, so $I = \{0\}$ as $M$ is faithful. □

But the converse is not true: e.g. $\{0\}$ is a prime ideal of $A = K[x,x−1]$, but not primitive. ($R$ has no simple faithful modules.)
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So \((I \cdot J)M = 0\). If \( JM = \{0\} \) then \( J = \{0\} \) as \( M \) is faithful. So suppose \( JM \neq 0 \). Then \( JM = M \) (as \( M \) is simple), so \((I \cdot J)M = 0\) gives \( IM = 0 \), so \( I = \{0\} \) as \( M \) is faithful. □

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Kaplansky, 1970: *Is a regular prime ring necessarily primitive?*

Answered in the negative (Domanov, 1977), a group-algebra example.
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**Theorem.** (A-, K.M. Rangaswamy 2010) \( L_K(E) \) is von Neumann regular \( \iff \) \( E \) is acyclic.
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Condition (MT3) or “Downward Directed”
Connections and Applications: Kaplansky’s question

**Theorem.** (A-, Jason Bell, Ranga 2011) $L_K(E)$ is primitive $\iff$

1. $L_K(E)$ is prime,
2. every cycle in $E$ has an exit, and
3. there exists a countable set of vertices $S$ in $E$ for which every vertex of $E$ connects to an element of $S$.

(Countable Separation Property)
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Idea of proof:

1. A unital ring $R$ is left primitive if and only if there is a left ideal $M \neq R$ of $R$ such that for every nonzero two-sided ideal $I$ of $R$, $M + I = R$. 

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Leavitt path algebras: some (surprising?) connections
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2. Embed $L_K(E)$ in a unital algebra $L_K(E)_1$ in the usual way; primitivity is preserved.
3. Show that the lack of the CSP implies that no such left ideal can exist in $L_K(E)_1$. 

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Connections and Applications: Kaplansky’s question

It’s not hard to find acyclic graphs $E$ for which $L_K(E)$ is prime but for which C.S.P. fails.

**Example:** $X$ uncountable, $S$ the set of finite subsets of $X$. Define the graph $E$:

1. vertices indexed by $S$, and
2. edges induced by proper subset relationship.

Then $L_K(E)$ is regular, prime, not primitive.
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1. vertices indexed by $S$, and
2. edges induced by proper subset relationship.

Then $L_K(E)$ is regular, prime, not primitive.

Note: Adjoining $1_K$ in the usual way (Dorroh extension by $K$) gives unital, regular, prime, not primitive algebras.

Remark: These examples are also “Cohn algebras”.

Lie algebras arising from associative algebras

Definitions / Notation.

$R$ an associative $K$-algebra.

For $x, y \in R$ let $[x, y]$ denote $xy - yx$.

Let $[R, R]$ denote the $K$-subspace of $R$ spanned by $\{[x, y] \mid x, y \in R\}$. 
Lie algebras arising from associative algebras

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$[R, R]$ with $[-, -]$ is a Lie $K$-algebra.
Lie algebras arising from associative algebras: general ideas

Let $L$ denote a Lie $K$-algebra. A subset $I$ of $L$ is called a **Lie $K$-ideal** if $I$ is a $K$-subspace of $L$ and $[L, I] \subseteq I$.

**Important Observation:** If $K1_R \subseteq [R, R]$, then $K1_R$ is a Lie $K$-ideal of $[R, R]$. But we need not have $K1_R \subseteq [R, R]$ in general. (Cheap example: $R$ commutative.)
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**Definition:** The Lie $K$-algebra $L$ is called *simple* if $[L, L] \neq 0$ and the only Lie $K$-ideals of $L$ are 0 and $L$. 
Lie algebras arising from associative algebras: general ideas

**Question**: For which graphs $E$ and fields $K$ is the Lie algebra $[L_K(E), L_K(E)]$ simple?
Lie algebras arising from associative algebras: general ideas

**Question**: For which graphs $E$ and fields $K$ is the Lie algebra $[L_K(E), L_K(E)]$ simple?

Of great help here:

**Theorem** (Herstein, 1965). Let $S$ be a simple associative $K$-algebra. Assume either that $\text{char}(S) \neq 2$, or that $S$ is not 4-dimensional over $Z(S)$, where $Z(S)$ is a field.

Let $U$ be any proper Lie $K$-ideal of the Lie algebra $[S, S]$.

Then $U \subseteq Z(S) \cap [S, S]$. 
Lie algebras arising from associative algebras: general ideas

Intuition ...
If the center $Z(S)$ is ‘small’, then usually we have good control over all the Lie ideals of the Lie algebra $[S, S]$. 

Main Consequence of Herstein’s Theorem:
Let $R$ be a unital $K$-algebra which satisfies the hypotheses of Herstein’s Theorem. Suppose that $1[R, R] \neq 0$, and $Z(R) = K$. Then $[R, R]$ is a simple Lie $K$-algebra if and only if $1 \in [R, R]$. 

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Leavitt path algebras: some (surprising?) connections
Lie algebras arising from associative algebras: general ideas

Intuition ...

If the center $Z(S)$ is ‘small’, then usually we have good control over all the Lie ideals of the Lie algebra $[S, S]$.

**Main Consequence of Herstein’s Theorem:** Let $R$ be a unital simple $K$-algebra which satisfies the hypotheses of Herstein’s Theorem. Suppose that

1. $[[R, R], [R, R]] \neq 0$, and
2. $Z(R) = K1_R$.

Then $[R, R]$ is a simple Lie $K$-algebra if and only if $1_R \notin [R, R]$. 
Step 1 towards: When is $1_{L_K(E)} \in [L_K(E), L_K(E)]$?

**Definition.** Write $E^0 = \{v_i \mid i \in I\}$.

For each $i \in I$, let $\epsilon_i \in \mathbb{Z}^{(I)}$ denote the element with 1 as the $i$-th coordinate and zeros elsewhere.

If $v_i$ is a regular vertex, for all $j \in I$ let $a_{ij}$ denote the number of edges $e \in E^1$ such that $s(e) = v_i$ and $r(e) = v_j$.

Define

$$B_i = (a_{ij})_{j \in I} - \epsilon_i \in \mathbb{Z}^{(I)}.$$  

(If $v_i$ is not a regular vertex, define $B_i = (0)_{j \in I} \in \mathbb{Z}^{(I)}$. )
Step 1 towards: When is $1_{L_K(E)} \in [L_K(E), L_K(E)]$?

Example

Then

\[ B_1 = (1, 1, 0, 0) - \epsilon_1 = (0, 1, 0, 0), \]
\[ B_2 = (1, 0, 0, 1) - \epsilon_2 = (1, -1, 0, 1), \]
\[ B_3 = (0, 1, 1, 0) - \epsilon_3 = (0, 1, 0, 0), \]
\[ B_4 = (0, 0, 1, 0) - \epsilon_4 = (0, 0, 1, -1). \]
Step 2 towards: When is $1_{L_K(E)} \in [L_K(E), L_K(E)]$?

**Definition**

Let $K$ be a field, and let $E$ be a directed graph. The *Cohn path $K$-algebra* $C_K(E)$ of $E$ with coefficients in $K$ is the path algebra $K\hat{E}$, modulo only the (CK1) relation

$$(\text{CK1}) \quad e^* e' = \delta_{e,e'} r(e) \quad \text{for all } e, e' \in E^1.$$
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**Remark:** Cohn path algebras might be interesting to study in their own right ...

Let $N \subseteq C_K(E)$ denote the ideal of $C_K(E)$ generated by elements of the form $v = \sum\{e \in E^1 | s(e) = v\} \quad ee^*$, where $v \in E^0$ is a regular vertex.
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**Remark:** Cohn path algebras might be interesting to study in their own right ...

Let $N \subseteq C_K(E)$ denote the ideal of $C_K(E)$ generated by elements of the form $\nu = \sum\{e \in E^1 | s(e) = \nu\}ee^*$, where $\nu \in E^0$ is a regular vertex.

So

$$L_K(E) \cong C_K(E)/N.$$
Step 3 towards: When is $1_{L_K(E)} \in [L_K(E), L_K(E)]$?

Using the “standard basis” available in $C_K(E)$, we can define a $K$-linear transformation

$$T : C_K(E) \to K^{(I)}$$

having

1. $T([x, y]) = 0$ for all $x, y \in C_K(E)$,
2. $T(v_i) = \epsilon_i$ for all $i \in I$, and
3. $T(w) \in \text{span}_K \{B_i \mid i \in I\} \subseteq K^{(I)}$ for all $w \in N$. 

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Leavitt path algebras: some (surprising?) connections
Final step: When is $1_{L_K(E)} \in [L_K(E), L_K(E)]$?

**Theorem**

*If $E^0$ is finite (so that $L_K(E)$ is unital), then*

$1_{L_K(E)} \in [L_K(E), L_K(E)] \iff (1, \ldots, 1) \in \text{span}_K \{B_i \mid i \in I\} \subseteq K(I)$. 
A few known results which complete the picture.

1. $K \cong L_K(\bullet)$ is the only simple commutative Leavitt path $K$-algebra. (So we call a simple Leavitt path algebra $L_K(E)$ nontrivial in case $L_K(E) \not\cong K$.)
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3. The Simplicity Theorem for finite $E$

4. (The Centers Theorem for finite $E$) If $L_K(E)$ is simple, then $Z(L_K(E)) = K1_{L_K(E)}$. 
Simplicity of $L_K(E)$

Now from Herstein’s Theorem, the Centers Theorem, and our theorem about when $1_{L_K(E)}$ is (or is not) an element of $[L_K(E), L_K(E)]$, we get

**Theorem (A-, Mesyan 2012)**

Let $K$ be a field, and let $E$ be a finite graph for which $L_K(E)$ is a nontrivial simple Leavitt path algebra. Write $E^0 = \{v_1, \ldots, v_m\}$, and for each $1 \leq i \leq m$ let $B_i$ be as above. Then

$$[L_K(E), L_K(E)] \text{ is simple as a Lie } K\text{-algebra}$$

if and only if

$$(1, \ldots, 1) \not\in \text{span}_K\{B_1, \ldots, B_m\}.$$
\[ L_K(E), L_K(E) \] simple \( \iff \) \( (1, \ldots, 1) \notin \text{span}_K\{B_1, \ldots, B_m\} \)

Previous example.

\[
\begin{array}{c}
\bullet^v_1 \rightarrow \rightarrow \rightarrow \bullet^v_2 \\
\downarrow \downarrow \downarrow \downarrow \\
\bullet^v_3 \rightarrow \rightarrow \rightarrow \bullet^v_4
\end{array}
\]

\[
B_1 = (0, 1, 0, 0), \quad B_2 = (1, -1, 0, 1), \quad B_3 = (0, 1, 0, 0), \quad B_4 = (0, 0, 1, -1).
\]

Is \( (1, 1, 1, 1) \) in \( \text{span}_K\{B_1, B_2, B_3, B_4\} \)? That is, can we find \( k_1, k_2, k_3, k_4 \in K \) for which

\[
(1, 1, 1, 1) = k_1(0, 1, 0, 0) + k_2(1, -1, 0, 1) + k_3(0, 1, 0, 0) + k_4(0, 0, 1, -1)
\]
[\[L_K(E), L_K(E)\]] simple ⇔ (1, . . . , 1) \notin \text{span}_K\{B_1, \ldots, B_m\}

So we want to solve a system. Here’s the augmented matrix of the system:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \vdots & 1 \\
1 & -1 & 1 & 0 & 0 & \vdots & 1 \\
0 & 0 & 0 & 0 & 1 & \vdots & 1 \\
0 & 1 & 0 & -1 & 1 & \vdots & 1
\end{pmatrix}
\]
\[ [L_K(E), L_K(E)] \text{ simple } \iff (1, \ldots, 1) \not\in \text{span}_K \{B_1, \ldots, B_m\} \]

After row-reducing we get
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 1 \\
1 & -1 & 1 & 0 & \cdots & 1 \\
0 & 0 & 0 & 1 & \cdots & 1 \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]

The system has no solution (regardless of the characteristic of \(K\)). So \([L_K(E), L_K(E)]\) is simple for any field \(K\).
[\mathcal{L}_K(E), \mathcal{L}_K(E)] \text{ simple } \iff (1, \ldots, 1) \notin \text{ span}_K\{B_1, \ldots, B_m\}

More examples. Let \( \mathcal{P} = \{p_1, p_2, \ldots, p_t\} \) be a finite set of primes, let \( q = p_1 p_2 \cdots p_t \in \mathbb{N} \), and let \( E_q \) be this graph.

Here \( B_1 = (0,1,0,0) \), \( B_2 = (1,-1,0,1) \), \( B_3 = (0,1,0,0) \), and \( B_4 = (0,0,1,q) \).

When is \((1,1,1,1)\) in \( \text{ span}_K\{B_1, B_2, B_3, B_4\}\)?
$[L_K(E), L_K(E)]$ simple $\iff (1, \ldots, 1) \not\in \text{span}_K \{B_1, \ldots, B_m\}$

Elementary row-operations on the augmented matrix yield:

$$
\begin{pmatrix} 1 & -1 & 1 & 0 & \vdots & 1 \\
0 & 1 & 0 & 0 & \vdots & 1 \\
0 & 0 & 0 & 1 & \vdots & 1 \\
0 & 0 & 0 & 0 & \vdots & -q \\
\end{pmatrix}.
$$

So the system has solutions precisely when $\text{char}(K)$ divides $q$, i.e., when $\text{char}(K) \in \{p_1, p_2, \ldots, p_t\}$. So by the Main Theorem,

$[L_K(E_q), L_K(E_q)]$ is simple if and only if $\text{char}(K)$ is NOT in $\{p_1, p_2, \ldots, p_t\}$. 
Isomorphisms between matrix rings over the Leavitt algebras $L_K(1, n)$ Some classical questions, answered using Leavitt path algebra

$[L_K(E), L_K(E)]$ simple $\Leftrightarrow (1, \ldots, 1) \not\in \text{span}_K\{B_1, \ldots, B_m\}$

So the characteristic of the field $K$ plays a role here!
Isomorphisms between matrix rings over the Leavitt algebras $L_K(1, n)$: Some classical questions, answered using Leavitt path algebras

\[
[L_K(E), L_K(E)] \text{ simple } \iff (1, \ldots, 1) \not\in \text{span}_K \{B_1, \ldots, B_m\}
\]

So the characteristic of the field $K$ plays a role here!

**Remark.** On the other end of the spectrum, we can also build graphs where \([L_K(E), L_K(E)]\) is simple if and only if $\text{char}(K)$ is in \(\{p_1, p_2, \ldots, p_t\}\).

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Leavitt path algebras: some (surprising?) connections
\[ [L_K(E), L_K(E)] \text{ simple } \Leftrightarrow (1, \ldots, 1) \notin \text{span}_K \{B_1, \ldots, B_m\} \]

So the characteristic of the field \( K \) plays a role here!

**Remark.** On the other end of the spectrum, we can also build graphs where \([L_K(E), L_K(E)] \) is simple if and only if \( \text{char}(K) \) IS in \( \{p_1, p_2, \ldots, p_t\} \).

Let \( q = p_1p_2 \cdots p_t \in \mathbb{N} \). Then using the previous result for matrices over Leavitt algebras, the Lie \( K \)-algebra \([L_K(1, q + 1), L_K(1, q + 1)] \) is simple if and only if \( \text{char}(K) \in \mathcal{P} \).
Connections and Applications:
The realization question for von Neumann regular rings

**Fundamental problem:** (Goodearl, 1994) What monoids $M$ appear as $\mathcal{V}(R)$ for von Neumann regular $R$?
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**Corollary:** the realization question has affirmative answer for graph monoids $M_E$.
Thank you.