Primitive graph algebras

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Overview

1. Primitive Leavitt path algebras
2. Primitive graph C*-algebras

1 Primitive Leavitt path algebras

2 Primitive graph C*-algebras

Throughout R is associative, but not necessarily with identity.

Assume R at least has "local units":

Prime rings

Definition: I, J two-sided ideals of R. The product IJ is the two-sided ideal

$$IJ = \{\sum_{\ell=1}^{n} i_\ell j_\ell \mid i_\ell \in I, j_\ell \in J, n \in \mathbb{N}\}.$$

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R is *prime* if the product of any two nonzero two-sided ideals of R is nonzero.

Examples:

1. Commutative domains, e.g. fields, $\mathbb{Z}$, $K[x]$, $K[x, x^{-1}]$, ...
2. Simple rings
3. $\mathrm{End}_K(V)$ where $\dim_K(V)$ is infinite. $(\cong \mathrm{RFM}(K))$

Prime rings

Note: Definition makes sense for nonunital rings.

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Lemma: R prime. Then R embeds as an ideal in a unital prime ring R_1. (Dorroh extension of R.)

If R is a K-algebra then we can construct R_1 a K-algebra for which $\dim_K(R_1/R) = 1$.

Primitive rings

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- Simple rings (note: need local units to build irreducibles)

NON-Examples:

- $\mathbb{Z}$, $K[x]$, $K[x, x^{-1}]$

Primitive rings

Primitive rings are "natural" generalizations of matrix rings.

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Definition of "primitive" makes sense for non-unital rings.

Prime and primitive rings

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So $(I \cdot J)M = 0$. If $JM = \{0\}$ then $J = \{0\}$ as M is faithful. So suppose $JM \neq 0$. Then $JM = M$ (as M is simple), so $(I \cdot J)M = 0$ gives $IM = 0$, so $I = \{0\}$ as M is faithful. □

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If R is prime, then in previous embedding,

$$R \text{ is primitive} \Leftrightarrow R_1 \text{ is primitive.}$$

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Remark for later:

From a ring-theoretic point of view, the question of finding prime, non-primitive rings is uninteresting (since there are so many of them!)

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Note $n \cdot (r + N) = nr + N$ need not be $\overline{0}$ in R/N since nr is not necessarily in N.

Example: K any field, V an infinite dimensional K-vector space. $R = \mathrm{End}_K(V) \cong \mathrm{RFM}(K)$ is primitive, not simple.

Here $M = Re_{11}$ is simple. Easy to show $\mathrm{Ann}_R(M) = \{0\}$, but R contains a nontrivial ideal (the finite rank transformations).

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But we always have $\mathrm{Ann}_R(R/N) \subseteq N$, since if $r(1 + N) = 0 + N$ then $r \in N$.

Leavitt path algebras

Let K be a field, and let $E = (E^0, E^1, s, r)$ be **any** directed graph.

The Leavitt path K-algebra $L_K(E)$ of E with coefficients in K

is the K-algebra generated by a set $\{v \mid v \in E^0\}$, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

(V) $vw = \delta_{v,w}v$ for all $v, w \in E^0$,

(E1) $s(e)e = er(e) = e$ for all $e \in E^1$,

(E2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$, and

(CK1) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.

(CK2) $v = \sum_{\{e\in E^1 | s(e)=v\}} ee^*$ for every regular vertex $v \in E^0$.

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Theorem. (Aranda Pino, Pardo, Siles Molina 2009) E arbitrary. Then $L_K(E)$ is prime $\Leftrightarrow$ for each pair $v, w \in E^0$ there exists $u \in E^0$ with $v \geq u$ and $w \geq u$.

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Idea of Proof. ($\Rightarrow$) Let R denote $L_K(E)$. Let $v, w \in E^0$. But $RvR \neq \{0\}$ and $RwR \neq \{0\} \Rightarrow RvRwR \neq \{0\} \Rightarrow vRw \neq \{0\} \Rightarrow v\alpha\beta^*w \neq 0$ for some paths α, β in E. Then $u = r(\alpha)$ works.

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($\Leftarrow$) $L_K(E)$ is graded by $\mathbb{Z}$, so need only check primeness on nonzero graded ideals I, J. But each nonzero graded ideal contains a vertex. Let $v \in I \cap E^0$ and $w \in J \cap E^0$. By downward directedness there is $u \in E^0$ with $v \geq u$ and $w \geq u$. But then $u = p^* vp \in I$ and $u = q^* wq \in J$, so that $0 \neq u = u^2 \in IJ$.

The Countable Separation Property

‘ **Definition.** Let E be any directed graph. E has the *Countable Separation Property* (CSP) if there exists a countable set of vertices S in E for which every vertex of E connects to an element of S.

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Same idea for any subset X of E^0: X has CSP (with respect to S_X) in case S_X is countable, and every element of X connects to an element of S_X.

Note for later: If $X = \emptyset$, then X vacuously has CSP (with respect to any countable subset of vertices).

So if X does not have CSP, then $X \neq \emptyset$.

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Observe: If E^0 is countable, then E has CSP.

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2) **Example**: X uncountable, S the set of finite subsets of X. Define the graph E:

1. vertices indexed by S, and
2. edges induced by proper subset relationship.

Then E does not have CSP.

Primitive Leavitt path algebras

Can we describe the (left) primitive Leavitt path algebras?

Note: Since $L_K(E) \cong L_K(E)^{op}$, left primitivity and right primitivity coincide. So we can just say "primitive" Leavitt path algebra.

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2. every cycle in E has an exit (Condition (L)), and
3. E has the Countable Separation Property.

$L_K(E)$ primitive $\Leftrightarrow$ E has (MT3), (L), and CSP

Strategy of Proof:

1. A unital ring R is left primitive if and only if there is a left ideal $N \neq R$ of R such that for every nonzero two-sided ideal I of R, $N + I = R$.

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Strategy of Proof:

1. A unital ring R is left primitive if and only if there is a left ideal $N \neq R$ of R such that for every nonzero two-sided ideal I of R, $N + I = R$.

Idea: ($\Leftarrow$) Embed N in a maximal left ideal T (this is OK since R is unital). So ${}_R R/T$ is simple.
Then $\mathrm{Ann}_R(R/T) \subseteq T$ (noted previously). Thus $N + \mathrm{Ann}_R(R/T) \subseteq T$. If to the contrary $\mathrm{Ann}_R(R/T) \neq \{0\}$, the hypotheses would yield $N + \mathrm{Ann}_R(R/T) = R$, impossible.

($\Rightarrow$) If M is the supposed simple having $\mathrm{Ann}_R(M) = \{0\}$, write $M \cong R/T$ for some maximal left ideal T. (In particular $T \neq R$.) So if $I \neq \{0\}$ then $I \cdot R/T = R/T$, so that $I + T = R$.

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2. Embed a prime $L_K(E)$ in a unital algebra $L_K(E)_1$ in the usual way; primitivity is preserved.

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4. Then show that the lack of the CSP implies that no such left ideal can exist in $L_K(E)_1$.

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We will use:

Proposition: If E has Condition (L) then every nonzero two-sided ideal of E contains a vertex.

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($\Leftarrow$). Suppose E downward directed, E has Condition (L), and E has CSP.

Suffices to establish primitivity of $L_K(E)_1$. Let T denote a set of vertices w/resp. to which E has CSP.

T is countable: label the elements $T = \{v_1, v_2, ...\}$.

$L_K(E)$ primitive $\Leftarrow$ E has (MT3), (L), and CSP

Inductively define a sequence $\lambda_1, \lambda_2, \ldots$ of paths in E for which, for each $i \in \mathbb{N}$,

1. λ_i is an initial subpath of λ_j whenever $i \leq j$, and
2. $v_i \geq r(\lambda_i)$.

Define $\lambda_1 = v_1$.

Suppose $\lambda_1, \ldots, \lambda_n$ have the indicated properties. By downward directedness, there is u_{n+1} in E^0 for which $r(\lambda_n) \geq u_{n+1}$ and $v_{n+1} \geq u_{n+1}$. Let $p_{n+1} : r(\lambda_n) \rightsquigarrow u_{n+1}$.

Define $\lambda_{n+1} = \lambda_n p_{n+1}$. $\square$

$L_K(E)$ primitive $\Leftarrow$ E has (MT3), (L), and CSP

Since λ_i is an initial subpath of λ_t for all $i \leq t$, we get that

$$\lambda_i\lambda_i^*\lambda_t\lambda_t^* = \lambda_t\lambda_t^* \quad \text{for each pair of positive integers} \quad i \leq t.$$

In particular $(1 - \lambda_i\lambda_i^*)\lambda_t\lambda_t^* = 0$ for $i \leq t$.

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In particular $(1 - \lambda_i\lambda_i^*)\lambda_t\lambda_t^* = 0$ for $i \leq t$.

Define $N = \sum_{i=1}^{\infty} L_K(E)_1(1 - \lambda_i\lambda_i^*)$.

$N \neq L_K(E)_1$: otherwise, $1 = \sum_{i=1}^{t} r_i(1 - \lambda_i\lambda_i^*)$ for some $r_i \in L_K(E)_1$, but then

$$0 \neq 1 \cdot \lambda_t\lambda_t^* = (\sum_{i=1}^{t} r_i(1 - \lambda_i\lambda_i^*)) \cdot \lambda_t\lambda_t^* = 0.$$

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Claim: Every nonzero two-sided ideal I of $L_K(E)_1$ contains some $\lambda_n \lambda_n^*$.

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Idea: E is downward directed, so $L_K(E)$, and therefore $L_K(E)_1$, is prime. Since $L_K(E)$ embeds in $L_K(E)_1$ as a two-sided ideal, we get $I \cap L_K(E)$ is a nonzero two-sided ideal of $L_K(E)$. So Condition (L) gives that I contains some vertex w.

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Claim: Every nonzero two-sided ideal I of $L_K(E)_1$ contains some $\lambda_n\lambda_n^*$.

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Then $w \geq v_n$ for some n by CSP. But $v_n \geq r(\lambda_n)$ by construction, so $w \geq r(\lambda_n)$. So $w \in I$ gives $r(\lambda_n) \in I$, so $\lambda_n\lambda_n^* \in I$.

Now we're done. Show $N + I = L_K(E)_1$ for every nonzero two-sided ideal I of $L_K(E)_1$. But $1 - \lambda_n\lambda_n^* \in N$ (all $n \in \mathbb{N}$) and $\lambda_n\lambda_n^* \in I$ (some $n \in \mathbb{N}$) gives $1 \in N + I$.

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

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1) E not downward directed $\Rightarrow$ $L_K(E)$ not prime $\Rightarrow$ $L_K(E)$ not primitive.

2) General ring theory result: If R is primitive and $f = f^2$ is nonzero then fRf is primitive.

If E contains a cycle c (based at v) without exit then $vL_K(E)v \cong K[x, x^{-1}]$, which is not primitive, so $L_K(E)$ is not primitive.

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

3) (The hard part.) Show if E does not have CSP then $L_K(E)$ is not primitive.

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Lemma. Let N be a left ideal of a unital ring A. If there exist $x, y \in A$ such that $1 + x \in N$, $1 + y \in N$, and $xy = 0$, then $N = A$.

Proof: Since $1 + y \in N$ then $x(1 + y) = x + xy = x \in N$, so that

$$1 = (1 + x) - x \in N.$$

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

We show that if E does not have CSP, then there does NOT exist a left ideal $N \neq L_K(E)_1$ for which $N + I = L_K(E)_1$ for all two-sided ideals I of $L_K(E)_1$.

To do this: assume N is such an ideal, show $N = L_K(E)_1$.

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Strategy: If N has this property, then for each $v \in E^0$ we have $N + \langle v \rangle = L_K(E)_1$. So for each $v \in E^0$ there exists $y_v \in \langle v \rangle$, $n_v \in N$ for which $n_v + y_v = 1$. Let $x_v = -y_v$. This gives a set $\{x_v \mid v \in E^0\} \subseteq L_K(E)_1$ for which $1 + x_v \in N$ for all $v \in E^0$.

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

Now show that the lack of CSP in E^0 forces the existence of a pair of vertices v, w for which $x_v \cdot x_w = 0$. (This is the technical part.)

Then use the Lemma.

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

Key pieces of the technical part:

1. Every element ℓ of $L_K(E)$ can be written as $\sum_{i=1}^{n} k_i\alpha_i\beta_i^*$ for some $n = n(\ell)$, and paths α_i, β_i. In particular, we can "cover" all elements of $L_K(E)$ by specifying n and lengths of paths. This is a countable covering of $L_K(E)$. (Not a partition.)

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2. Collect up the x_v according to this covering. Since E does not have CSP, then some specific subset in the cover does not have CSP.
3. Show that, in this specific subset Z, there exists $v \in Z$ for which the set
$$\{w \in Z \mid x_v x_w = 0\}$$
does not have CSP. In particular, this set is nonempty. Pick such v and w. Then we are done by the Lemma. $\square$

von Neumann regular rings

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Examples:

1. Division rings
2. Direct sums of matrix rings over division rings
3. Direct limits of von Neumann regular rings

R is regular $\Leftrightarrow$ R_1 is regular.

Kaplansky's Question

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I. KAPLANSKY, *Algebraic and analytic aspects of operator algebras*, AMS, 1970.

Is every regular prime algebra primitive?

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Is every regular prime algebra primitive?

Answered in the negative (Domanov, 1977), a group-algebra example. (Clever, but very ad hoc.)

Kaplansky's Question

Theorem. (A-, K.M. Rangaswamy 2010)

$L_K(E)$ is von Neumann regular $\Leftrightarrow$ E is acyclic.

Kaplansky's Question

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$$L_K(E) \text{ is von Neumann regular} \Leftrightarrow E \text{ is acyclic.}$$

Idea of Proof: ($\Leftarrow$) If E contains a cycle c based at v, can show that $a = v + c$ has no "regular inverse".

($\Rightarrow$) Show that if E is acyclic then every element of $L_K(E)$ can be trapped in a subring of $L_K(E)$ which is isomorphic to a finite direct sum of finite matrix rings.

Application to Kaplansky's question

It's not hard to find acyclic graphs E for which $L_K(E)$ is prime but for which C.S.P. fails.

Example (mentioned previously): X uncountable, S the set of finite subsets of X. Define the graph E:

- vertices indexed by S, and

- edges induced by proper subset relationship.

Then for this graph E,

1. $L_K(E)$ is regular (E is acyclic)
2. $L_K(E)$ is prime (E is downward directed)
3. $L_K(E)$ is not primitive (E does not have CSP).

Application to Kaplansky's question

Note: Embedding $L_K(E)$ in $L_K(E)_1$ in the usual way gives unital, regular, prime, not primitive algebras.

Remark: These examples are also "Cohn path algebras".

Application to Kaplansky's question

A second construction of such graphs:

Let $\kappa > 0$ be any ordinal. Define E_κ as follows:

$$E_\kappa^0 = \{\alpha \mid \alpha < \kappa\}, \quad E_\kappa^1 = \{e_{\alpha,\beta} \mid \alpha, \beta < \kappa, \text{ and } \alpha < \beta\},$$

$s(e_{\alpha,\beta}) = \alpha$, and $r(e_{\alpha,\beta}) = \beta$ for each $e_{\alpha,\beta} \in E_\kappa^1$.

Application to Kaplansky's question

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$s(e_{\alpha,\beta}) = \alpha$, and $r(e_{\alpha,\beta}) = \beta$ for each $e_{\alpha,\beta} \in E_\kappa^1$.

Suppose κ has *uncountable cofinality*. Then $L_K(E_\kappa)$ is regular, prime, not primitive.

Prime graph C^*-algebras

For a ring R with a topology in which multiplication is continuous, then R is prime as a ring iff R is prime with respect to closed ideals.

So for a C^*-algebra, primeness as a ring and primeness in the usual C^* sense mean the same thing.

Prime graph C*-algebras

For a ring R with a topology in which multiplication is continuous, then R is prime as a ring iff R is prime with respect to closed ideals.

So for a C*-algebra, primeness as a ring and primeness in the usual C* sense mean the same thing.

Proposition. Let E be any graph. Then $C^*(E)$ is prime if and only if

1. E is downward directed, and
2. E satisfies Condition (L).

Proof. This was established by Takeshi Katsura (2006), in the more general context of topological graphs.

$C^*(E)$ prime $\Leftarrow$ E has (MT3) and (L)

Idea of Proof:

Suppose E is downward directed and has (L).

If I and J are nonzero ideals in $C^*(E)$, then (L) with the Cuntz Krieger Uniqueness Theorem gives $u, v \in E^0$ such that $p_u \in I$ and $p_v \in J$.

Then downward directed gives $w \in E^0$ such that $u \geq w$ and $v \geq w$. So $p_w \in I$ and $p_w \in J$, so $0 \neq p_w = p_w^2 \in IJ$.

$C^*(E)$ prime $\Rightarrow$ E has (MT3) and (L)

Conversely: Suppose E does not satisfy (L). Then there exists a cycle $\alpha = e_1 \dots e_n$ in E without exits. If $H = \alpha^0$, then $I_H = I_{\overline{H}}$ is Morita equivalent to $C^*(\mathbb{T})$.

But this is impossible, since

1. any ideal of a prime C*-algebra is itself prime as a C*-algebra,
2. primeness is preserved under Morita equivalence, and
3. $C^*(\mathbb{T})$ is easily shown to not be prime.

So E satisfies Condition (L).

$C^*(E)$ prime $\Rightarrow$ E has (MT3) and (L)

Now show E is downward directed. Let $u, v \in E^0$. For $w \in E^0$

$$H(w) := \{x \in E^0 : w \geq x\}.$$

Let $\overline{H(w)}$ denote the saturated closure of $H(w)$.

For $u, v \in E^0$, the ideals $I_{H(u)} = I_{\overline{H(u)}}$ and $I_{H(v)} = I_{\overline{H(v)}}$ are each nonzero.

Since $C^*(E)$ is prime, $I_{\overline{H(u)}} \cap I_{\overline{H(v)}} \neq \{0\}$.

But $I_{\overline{H(u)} \cap \overline{H(v)}} = I_{\overline{H(u)}} \cap I_{\overline{H(v)}}$, so $\overline{H(u)} \cap \overline{H(v)} \neq \emptyset$, which gives that $H(u) \cap H(v) \neq \emptyset$.

Then $w \in H(u) \cap H(v)$ works.

Prime graph C*-algebras

So the "answer" to the primeness question in the graph C*-algebra setting differs from that of the Leavitt path algebra setting.

For example:

$$K[x, x^{-1}] = L(\bullet \circlearrowright) \text{ is prime,}$$

but

$$C^*(\mathbb{T}) = C^*(\bullet \circlearrowright) \text{ is not prime.}$$

Primitive C^*-algebras

Definition. The C^*-algebra A is *primitive* if there exists an irreducible faithful $*$-representation of A.

Rephrased: A is primitive if there is an irreducible faithful representation $\pi : A \to B(\mathcal{H})$ as bounded operators on a Hilbert space $\mathcal{H}$. $\square$

Primitive C*-algebras

This will be useful:

Proposition: Suppose A is a C*-algebra. Suppose there exists a modular left ideal $N \neq A$ of A such that $N + I = A$ for every nonzero closed two-sided ideal I of A. Then A is left primitive.

Primitive C*-algebras

Idea of Proof. Suppose u is a modular element for N; so $a - au \in N$ for all $a \in A$.

Standard: $u \notin N$ (otherwise $N = A$).

Standard: N embeds in a maximal (necessarily modular) left ideal T of A.

Standard: T is closed.

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Standard: T is closed.

Since T is maximal, A/T is irreducible. Using closure of T and approximate identities for elements of A, standard to show that $\mathrm{Ann}_A(A/T) \subseteq T$.

Now argue as in the unital ring case.

Primitive C^*-algebras

Lemma (well-known): Any primitive C^*-algebra is prime.

Primitive C*-algebras

Lemma (well-known): Any primitive C*-algebra is prime.

Proof. Let $\pi : A \to B(\mathcal{H})$ be the supposed irreducible faithful representation of the C*-algebra A, and let I, J be (closed) two-sided ideals of A. Suppose $IJ = \{0\}$; we show that either $I = \{0\}$ or $J = \{0\}$. If $J \neq \{0\}$ then the faithfulness of π gives $\pi(J)\mathcal{H} \neq \{0\}$. But $\pi(J)\mathcal{H}$ is then a nonzero closed subrepresentation of the irreducible representation π, so $\pi(J)\mathcal{H} = \mathcal{H}$. Then $\{0\} = IJ$ gives $\{0\} = \pi(IJ)\mathcal{H} = \pi(I)\pi(J)\mathcal{H} = \pi(I)\mathcal{H}$, so that, again invoking the faithfulness of π, we get $I = \{0\}$. $\square$

Primitive C*-algebras

Theorem (Dixmier, 1960) Every prime separable C*-algebra is primitive.

Remark: It's an existence proof; the faithful irreducible representation is not explicitly constructed.

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Then $C^*(E)$ is primitive if and only if

E is downward directed, and satisfies Condition (L).

... and, in this case, if and only if $L_K(E)$ is primitive.

Primitive graph C*-algebras

Can we identify the primitive graph C*-algebras for arbitrary graphs?

Note: "Primeness + Separability" of $C^*(E)$ is not the appropriate pairing of properties to achieve "Primitivity" in general.

For example $C^*(E)$ is primitive for E the graph with one vertex and uncountably many loops, but $C^*(E)$ is not separable.

Primitive graph C*-algebras

Theorem. (A-, Mark Tomforde, *Münster J. Math.*, to appear)

Let E be any graph. Then $C^*(E)$ is primitive if and only if ...

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1. E is downward directed,
2. E satisfies Condition (L), and
3. E satisfies the Countable Separation Property.

... if and only if $C^*(E)$ is prime and E satisfies the Countable Separation Property.

$C^*(E)$ primitive $\Leftarrow$ E has (MT3), (L), and CSP

Proof of sufficiency. *A lot of this will look familiar.*

Let X be a set of vertices with respect to which E satisfies the Countable Separation Property. Label the elements of X as $\{v_1, v_2, ...\}$. We know (previous proof) there is a sequence $\lambda_1, \lambda_2, ...$ of paths in E having the following properties for each $i \in \mathbb{N}$:

(i) $v_i \geq r(\lambda_i)$, and

(ii) $\lambda_{i+1} = \lambda_i \mu_{i+1}$ for some path μ_{i+1} in E.

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(i) $v_i \geq r(\lambda_i)$, and

(ii) $\lambda_{i+1} = \lambda_i \mu_{i+1}$ for some path μ_{i+1} in E.

Note: since by construction $\lambda_1 = v$, $S_{\lambda_1} S_{\lambda_1}^* = P_v$.

$C^*(E)$ primitive $\Leftarrow$ E has (MT3), (L), and CSP

By construction, for $i < t$ we have

$$S_{\lambda_i}S^*_{\lambda_i}S_{\lambda_t}S^*_{\lambda_t} = S_{\lambda_t}S^*_{\lambda_t} \quad \text{for each pair of positive integers} \quad i \leq t.$$

Claim: Every nonzero (closed) two-sided ideal J of $C^*(E)$ contains $S_{\lambda_n}S^*_{\lambda_n}$ for some $n \in \mathbb{N}$.

Reason: By Condition (L), the Cuntz-Krieger Uniqueness Theorem applies to yield that J contains some vertex projection P_w.

By the CSP there exists $v_n \in X$ for which $w \geq v_n$. But $v_n \geq r(\lambda_n)$.

So there is a path μ in E for which $s(\mu) = w$ and $r(\mu) = r(\lambda_n)$. Since $P_w \in J$ we get $P_{r(\lambda_n)} \in J$, so $S_{\lambda_n}S^*_{\lambda_n} = S_{\lambda_n}P_{r(\lambda_n)}S^*_{\lambda_n} \in J$.

$C^*(E)$ primitive $\Leftarrow$ E has (MT3), (L), and CSP

Let A denote $C^*(E)$, and let v denote v_1. Consider the left ideal L of A defined by:

$$L = \{\sum_{i=1}^{n}(x_i - x_i S_{\lambda_i} S_{\lambda_i}^*) \mid x_i \in A, n \in \mathbb{N}\}.$$

L is modular (with $a - aP_v \in L$ for all $a \in A$).

$P_v \notin L$. (Same proof as for Leavitt path algebras:)

$C^*(E)$ primitive $\Leftarrow$ E has (MT3), (L), and CSP

We use previous Proposition; need only show that $I + L = A$ for any nonzero closed two-sided ideal I of A. But any such two-sided ideal contains $S_{\lambda_n} S_{\lambda_n}^*$ for some $n \in \mathbb{N}$, hence contains $a S_{\lambda_n} S_{\lambda_n}^*$ for all $a \in A$, but then

$$a = aS_{\lambda_n} S_{\lambda_n}^* + (a - aS_{\lambda_n} S_{\lambda_n}^*) \in I + L.$$

$\square$

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

Proof of Converse.

Show that if $A = C^*(E)$ is primitive, then E has Condition (L), is downward directed, and has CSP.

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

Proof of Converse.

Show that if $A = C^*(E)$ is primitive, then E has Condition (L), is downward directed, and has CSP.

Since primitive implies prime we get that E satisfies Condition (L) and is downward directed.

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

So suppose to the contrary that E does not satisfy the Countable Separation Property. We show that $C^*(E)$ admits no faithful irreducible representations.

We actually show more, that $C^*(E)$ admits no faithful *cyclic* representations. Suppose $\psi : A \to B(\mathcal{H})$ is a cyclic representation of A; so there exists $\xi \in \mathcal{H}$ for which $\psi(A)\mathcal{H} = \overline{\psi(A)\xi}$.

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

We will use this general result:

Lemma. Let ψ be a representation of a C*-algebra B as bounded operators on a Hilbert space $\mathcal{H}$, and let $\xi \in \mathcal{H}$. Suppose $\{Q_i \mid i \in I\}$ is a set of nonzero mutually orthogonal projections in B for which, for each $i \in I$, $\psi(Q_i)\xi \neq 0$. Then I is at most countable.

Proof. Use the Pythagorean Theorem in B.

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

This graph-theoretic definition will also be useful.

Let E be any graph. For $w \in E^0$, let

$$U(w) = \{v \in E^0 \mid v \geq w\}.$$

Observation: E does *not* satisfy the Countable Separation Property in case for every countable subset X of E^0, there exists some vertex v in $E^0 \setminus \cup_{x \in X} U(x)$.

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

For every integer $n \geq 0$ define

$$\Gamma_n = \{\mu \in \mathrm{Path}(E) \mid \psi(S_\mu S_\mu^*)\xi \neq 0, \text{ and } |\mu| = n\}.$$

(We view paths of length 0 as vertices, and in this case interpret $S_\mu S_\mu^*$ as $P_{s(\mu)}$.)

Because the paths in Γ_n are of fixed length, the set $\{S_\mu S_\mu^* \mid \mu \in \Gamma_n\}$ consists of nonzero orthogonal projections.

So by the Lemma, each Γ_n is at most countable.

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

For every integer $n \geq 0$ define

$$\Omega_n = \{w \in E^0 \mid w \in \mu^0 \text{ for some } \mu \in \Gamma_n\}.$$

Since each Γ_n is countable, and any path in E contains finitely many vertices, we get that each Ω_n is countable.

For every integer $n \geq 0$ define

$$\Theta_n = \cup_{w \in \Omega_n} U(w), \quad \text{and} \quad \Theta = \cup_{n=0}^{\infty} \Theta_n.$$

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

Since $\Theta = \cup_{n=0}^{\infty}(\cup_{w\in\Omega_n} U(w))$, and each Ω_n is countable, we have that Θ is the union of a countable number of sets of the form $U(w)$.

So by the hypothesis that E does not satisfy the Countable Separation Property, we conclude that there exists some $v \in E^0 \setminus \Theta$.

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

Since $\Theta = \cup_{n=0}^{\infty}(\cup_{w \in \Omega_n} U(w))$, and each Ω_n is countable, we have that Θ is the union of a countable number of sets of the form $U(w)$.

So by the hypothesis that E does not satisfy the Countable Separation Property, we conclude that there exists some $v \in E^0 \setminus \Theta$.

But $v \in E^0 \setminus \Theta$ means that for every path γ having $s(\gamma) = v$, then every path ν for which $r(\gamma) \in \nu^0$ has $\psi(S_\nu S_\nu^*)\xi = 0$.

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

Let J denote the (nonzero) closed two-sided ideal of $C^*(E)$ generated by P_v. Let $H(v)$ denote the set $\{w \in E^0 \mid v \geq w\}$.

Consider the set

$$T = \mathrm{span}_{\mathbb{C}}\{S_\mu S_\nu^* \mid \mu, \nu \in \mathrm{Path}(E) \text{ with } r(\mu) = r(\nu) \in H(v)\}.$$

Then T is dense in J.

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

Claim: $\psi(t)\xi = 0$ for all $t \in T$.

Reason: Suffices to show that $\psi(S_\mu S_\nu^*)\xi = 0$ for any $\mu, \nu \in \mathrm{Path}(E)$ for which $r(\mu) = r(\nu) \in H(v)$. But by the above description of $E^0 \setminus \Theta$ we have $\psi(S_\nu S_\nu^*)\xi = 0$, so that

$$\psi(S_\mu S_\nu^*)\xi = \psi(S_\mu S_\nu^* S_\nu S_\nu^*)\xi = \psi(S_\mu S_\nu^*)\psi(S_\nu S_\nu^*)\xi = \psi(S_\mu S_\nu^*)0 = 0.$$

$C^*(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

So $\psi(T)\xi = 0$, so that $\psi(\overline{T})\xi = 0$, and thus $\psi(J)\xi = 0$, which gives $\overline{\psi(J)\xi} = 0$. But then

$$\psi(J)\mathcal{H} = \psi(J \cdot A)\mathcal{H} = \psi(J)\psi(A)\mathcal{H} = \psi(J)\overline{\psi(A)\xi}$$

$$\subseteq \overline{\psi(J \cdot A)\xi} = \overline{\psi(J)\xi} = 0,$$

so that $J \subseteq \mathrm{Ker}(\psi)$. Since J is nonzero, ψ is not faithful. $\square$

Primitive C*-algebras

We actually have shown more.

Definition. Let π be a representation of a C*-algebra A on a Hilbert space $\mathcal{H}$. We say π is *countably generated* in case there exists a countable subset $\{h_i \mid i \in \mathbb{N}\}$ of $\mathcal{H}$ for which

$$\mathcal{H} = \overline{\mathrm{span}}\{\pi(A)h_i \mid i \in \mathbb{N}\}.$$

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Proposition. If E does not have CSP, then $C^*(E)$ admits no countably generated faithful representations.

Primitive C*-algebras

Proof. Same idea as above. Suppose $\{h_i \mid i \in \mathbb{N}\} \subseteq \mathcal{H}$ has $\mathcal{H} = \overline{\mathrm{span}}\{\pi(A)h_i \mid i \in \mathbb{N}\}$. For $n \geq 0, i \in \mathbb{N}$ define

$$\Gamma_n = \{\mu \in \mathrm{Path}(E) \mid \psi(S_\mu S_\mu^*)\xi_i \neq 0 \text{ for some } i, \text{ and } |\mu| = n\}.$$

Now argue as before.

Prime, non-primitive C*-algebras

The theorem gives us a machine to build prime, non-primitive C*-algebras.

Example: The graph E as considered previously. X an uncountable set, S the set of finite subsets of X. E is the graph with:

1. vertices indexed by S, and
2. edges induced by proper subset relationship.

Then E is downward directed, has Condition (L), and does not have CSP.

So $C^*(E)$ is a prime, non-primitive C*-algebra.

Note that $C^*(E)$ is an AF algebra.

Prime, non-primitive C*-algebras

Modify E by adding a loop at each vertex. Call the new graph E'.

Then E' is still downward directed, has Condition (L), and does not have CSP.

So $C^*(E')$ is a prime, non-primitive C*-algebra.

Note $C^*(E)$ is not AF. Also, since E' does not have Condition (K), $C^*(E)$ does not have real rank 0.

Prime, non-primitive C*-algebras

Modify E' by adding a second loop at each vertex. Call the new graph E''.

Then E'' is downward directed, has Condition (L), and does not have CSP.

So $C^*(E'')$ is a prime, non-primitive C*-algebra.

Note that $C^*(E'')$ also has Condition (K), so has real rank 0.

Summary

Theorem. For an arbitrary graph E, these are equivalent.

1. E is downward directed, has Condition (L), and satisfies the Countable Separation Property.
2. $L_K(E)$ is primitive for every field K.
3. $L_{\mathbb{C}}(E)$ is primitive.
4. $C^*(E)$ is primitive.

Summary

Theorem. For an arbitrary graph E, these are equivalent.

1. E is downward directed, has Condition (L), and satisfies the Countable Separation Property.
2. $L_K(E)$ is primitive for every field K.
3. $L_{\mathbb{C}}(E)$ is primitive.
4. $C^*(E)$ is primitive.

Moreover, using this result, we can easily construct infinite classes of:

1. prime, non-primitive, von Neumann regular algebras, and
2. prime, non-primitive C^*-algebras.

Summary

Questions?

(Thanks to the Simons Foundation)