Mad Veterinarian Puzzles

MathPath 2011

Breakout Sessions July 12 – 15

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There is a collection of problems which have come to be known as "Mad Veterinarian Puzzles", for reasons which will become obvious.

As a warm-up, we will look at the "usual" questions which are asked in this context Then we will ask some new types of questions. The goal is to see just how much structure is contained in these ideas, and really how beautiful these ideas can be!

This will definitely be a HANDS ON, COLLABORATIVE activity!

Scenario 1

Here's your first example of a Mad Veterinarian scenario. (The wording is taken from the most famous Mad Vet Puzzle site, http://www.bumblebeagle.org/madvet/index.html The details of this particular scenario are slightly different than the details of the "Main Mad Vet Puzzle" given at that site.)

A Mad Veterinarian has created three animal transmogrifying machines.

Place a cat in the input bin of the first machine, press the button, and .... whirrr .... bing! Open the output bins to find four dogs and one mouse. The second machine can convert a dog into three cats and three mice, and the third machine can convert a mouse into a cat and a dog. Each machine can also operate in reverse. (So, for example, if you've got four dogs and one mouse, you can convert them into one cat.)

This Mad Vet has one cat.

Question 1.1:
Can he convert it into seven mice? (If so, show how. If not, give a precise explanation as to why not. "Seven mice" means *exactly* seven mice, with no dogs or cats left over.)

Question 1.2:
Can he convert it into a kennel of dogs, with no cats or mice left over? (If so, show how. If not, why not?)

(over)

Scenario 2

A second Mad Vet has these three machines:

Machine #1 turns one cat into one dog.
Machine #2 turns one dog into one cat, one dog, and one mouse
Machine #3 turns one mouse into one cat and one dog

This Mad Vet has one cat.

Question 2.1

Can she convert it into two cats? (If so, show how. If not, explain why not. Similar to the wording from Scenario 1, and from now on, a phrase such as "two cats" means exactly two cats, with no dogs or mice left over.)

Question 2.2

Can she convert it into three cats? (If so, show how. If not, explain why not.)

Question 2.3

Can she convert it into four cats? (If so, show how. If not, explain why not.)

Question 2.4

Can she convert it into one mouse? (If so, show how. If not, explain why not.)

Question 2.5

Using these three Mad Vet machines, how many different "classes" of animals are there? Describe all of the classes.

Question 2.6

A friend of this Mad Vet brings her, as a birthday present, a cat and a mouse. But our Mad Vet prefers to have just one cat. Can she transmogrify her new collection of animals back to the way things were when she only had one cat?

Question 2.7

Does you answer to the previous question depend on the fact that our Mad Vet started with one cat? That is, if she started with ANY collection of animals, and her friend brought her a cat and a mouse, would she be able to transmogrify her collection back to the original collection? Explain.

Question 2.8

The Mad Vet is looking to retire, but she'd like to close down her Mad Vet business and wind up owning only one mouse (and no other animals). From a previous question, you know she can't simply use the cat she has to produce one mouse. So she'll go to the pet shop down the street and buy some animals, and then use those animals together with her one cat to produce a mouse. What animals should she buy at the pet shop?

Question 2.9

By adding animals from the pet shop to the cat she currently owns, is it possible for the Mad Vet to close down her Mad Vet business owning whatever collection of animals she'd like?

Question 2.10

Does your answer to the previous question depend on the fact that she happens to currently own one cat?

Scenario 3

Our Third Mad Vet has these three machines:

Machine #1 turns one cat into one dog and one mouse
Machine #2 turns one dog into one cat and one dog
Machine #3 turns one mouse into one cat and one mouse

This Mad Vet has one cat.

Question 3.1
Can she convert it into two cats? Three cats? Any number of cats? (If so, show how. If not, explain why not.)

Question 3.2
Can she convert it into one dog? Or one mouse? (If so, show how. If not, explain why not.)

Question 3.3
Can she convert it into a kennel of dogs? Or a mischief of mice? (If so, show how. If not, explain why not.)

Question 3.4
Using these three Mad Vet machines, how many different "classes" of animals are there? Describe all of the classes.

Question 3.5
A friend of this Mad Vet insists on giving the Mad Vet a gift of animals for her birthday. Our Mad Vet really prefers to have one cat. So as not to offend her friend, what collection of animals should the Mad Vet tell her friend to bring her for a birthday present?

Question 3.6
Does your answer to the previous question depend on the fact that our Mad Vet started with one cat? That is, if the Mad Vet started with ANY menagerie of animals, and her friend brought her the same collection of animals as in the previous question, would the Mad Vet be able to transmogrify her new set back to the original menagerie?

Question 3.7
The Mad Vet is looking to retire, but she'd like to finish her Mad Vet business owning only one mouse (and no other animals). From a previous question, you know she can't simply use the cat she has to produce one mouse. So she'll go to the pet shop down the street and buy some animals, and then use those animals together with her one cat to produce a mouse. What animals should she buy at the pet shop?

Question 3.8
By adding animals from the pet shop to the cat she currently owns, is it possible for the Mad Vet to finish her Mad Vet business owning whatever collection of animals she'd like?

Question 3.9
Does your answer to the previous question depend on the fact that she happens to currently own one cat?

(over)

Scenario 4

Our fourth Mad Vet has these three machines:

Machine #1 turns one cat into two cats
Machine #2 turns one dog into two dogs
Machine #3 turns one mouse into two mice

This Mad Vet has one cat.

Question 4.1: Using these three Mad Vet machines, how many different "classes" of animals are there? Describe all of the classes.

Question 4.2: Is there some (nonempty) collection of animals this Mad Vet should ask to get as a gift from a friend so that he will still be able to have just one cat?

Question 4.3: Does your answer to the previous question depend on the fact that the Mad Vet starts with one cat?

Question 4.4: This Mad Vet, like his two previous colleagues from Scenarios 2 and 3, is also looking to retire. But he'd like to finish his Mad Vet business owning only one mouse (and no other animals). It is impossible for him to simply use the cat he has to produce one mouse. So he'll go to the pet shop down the street and buy some animals, and then use those animals together with his one cat to produce one mouse. What animals should he buy at the pet shop?

Question 4.5: What menageries of animals can this Mad Vet retire with? (He'll use the same process as before, in which he'll add to his one cat by shopping for more animals at the pet shop.)

Scenario 5

Our fifth Mad Vet has these three machines:

Machine #1 turns one cat into one dog and one mouse
Machine #2 turns one dog into one cat and one mouse
Machine #3 turns one mouse into one cat and one dog

This Mad Vet has one cat.

Question 5.1: Using these three Mad Vet machines, how many different "classes" of animals are there? Describe all of the classes.

Question 5.2: Is there some (nonempty) collection of animals this Mad Vet should ask to get as a gift from a friend so that he will still be able to have just one cat?

Question 5.3: Does your answer to the previous question depend on the fact that the Mad Vet starts with one cat?

Question 5.4: This Mad Vet, like his previous colleagues, is also looking to retire. But he'd like to finish his Mad Vet business owning only one mouse (and no other animals). It is impossible for him to simply use the cat he has to produce one mouse. (Why?) So he'll go to the pet shop down the street and buy some animals, and then use those animals together with his one cat to produce one mouse. What animals should he buy at the pet shop?

Question 5.5: What menageries of animals can this Mad Vet retire with? (He'll use the same process as before, in which he'll add to his one cat by shopping for more animals at the pet shop.)

Question 5.6: Does your answer to the previous question depend on the fact that he happens to currently own one cat?

Scenario 6

Our sixth Mad Vet has these three machines:

Machine #1 turns one cat into one dog
Machine #2 turns one dog into one mouse
Machine #3 turns one mouse into one mouse (i.e., a mouse goes in, and a mouse comes out!)

This Mad Vet has **TWO** cats.

Question 6.1: Using these three Mad Vet machines, how many different "classes" of animals are there? Describe all of the classes.

Question 6.2: Is there some (nonempty) collection of animals this Mad Vet should ask to get as a gift from a friend so that he will still be able to have exactly two cats?

Question 6.3: Does your answer to the previous question depend on the fact that the Mad Vet starts with two cats?

Question 6.4: This Mad Vet, like his previous colleagues, is also looking to retire. What menageries of animals can this Mad Vet retire with? (He'll use the same process as before, in which he'll add to his two cats by shopping for more animals at the pet shop.)

Mad Bob's Original Mad Vet Puzzle

The wording is taken directly from the most famous Mad Vet Puzzle site,

http://www.bumblebeagle.org/madvet/index.html

The details of this particular scenario look like Scenario #1, but the first machine is different.

A Mad Veterinarian has created three animal transmogrifying machines. Place a cat in the input bin of the first machine, press the button, and .... whirrr .... bing! Open the output bins to find two dogs and five mice. The second machine can convert a dog into three cats and three mice, and the third machine can convert a mouse into a cat and a dog. Each machine can also operate in reverse. (So, for example, if you've got four dogs and one mouse, you can convert them into one cat.) This Mad Vet has one cat.

1) Can he convert it into seven mice? (If so, show how. If not, why not?)
2) Can he convert it into a kennel of dogs, with no cats or mice left over? (If so, show how. If not, why not?)

AN ADDITIONAL QUESTION FOR 2011 *MATHPATH*ERS: HOW MANY DIFFERENT "CLASSES" OF ANIMALS ARE THERE IN MAD BOB'S ORIGINAL MAD VET SCENARIO?

The Smith normal form of a matrix

Here are four operations that you will be allowed to do on a matrix.

1) Switch any two rows with each other.
2) Switch any two columns with each other.
3) Multiply all the entries in any row by -1.
4) Multiply all the entries in any column by -1.
5) Add any integer-multiple of one row to another row.
6) Add any integer-multiple of one column to another column.

Now let B be any $n \times n$ (square) matrix whose entries are integers. Using a sequence of allowable operations, you can always change B into a matrix S where:

1) All of the entries not on the main diagonal of S are 0.
2) Let s_1, s_2, … , s_n be the entries on the main diagonal of S. If any of these entries are 0, then we list those at the end. For each of the nonzero entries, that entry is a divisor of the next entry on the list.

The resulting matrix S having these two properties is called the **Smith normal form** of the matrix B. It turns out that every integer matrix B has a unique Smith normal form. For example: If B is the matrix

$$\begin{matrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 1 \end{matrix}$$

then the Smith normal form of B is the matrix S:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{matrix}$$

See if you can get from B to S using the allowable operations.

Then try starting with another matrix B of your choice, and find the Smith normal form S of your matrix!

 www.uccs.edu/gabrams

Abelian groups

(Remark: For those of you who have seen the idea of a group before … the approach we will take here might be somewhat different than the one you've already seen, but both approaches lead us to the same place.)

If S is a set with an associative, commutative binary operation + then (S,+) is an *abelian group* if this property is satisfied:

For every x,y in S there exists z in S for which x + z = y.

In words: in case for any choice of elements x and y in S we can 'get from x to y' by adding some element of S to x.

Examples:

1) **Z** (the set of all integers) with usual operation + .

2) Let n be a positive integer, let $\mathbf{Z_n}$ denote the set {0,1,2,…,n-1}, and let $+_n$ be the operation "addition mod n".

3) Let m and n be positive integers. Let $\mathbf{Z_m} \times \mathbf{Z_n}$ denote the set of pairs (a,b) where a is in {0,1,2,…,m-1}, and b is in {0,1,2,…,n-1}. Define an operation + on these pairs "coordinatewise" (so use $+_m$ for the left hand numbers, and use $+_n$ for the right hand numbers).

NON-Examples:

1) **N** (the set of all positive integers) with usual operation + .

2) $\mathbf{Z}^+$ (the set of all non-negative integers) with usual operation +.

Comments:

1) $\mathbf{Z_m} \times \mathbf{Z_n}$ is the 'same' as $\mathbf{Z_{mn}}$ as long as m and n are relatively prime. So for example the abelian group $\mathbf{Z_2} \times \mathbf{Z_3}$ is the 'same' as the abelian group $\mathbf{Z_6}$. On the other hand, if m and n share at least one common factor (>1), then $\mathbf{Z_m} \times \mathbf{Z_n}$ is NOT the 'same' as $\mathbf{Z_{mn}}$. So for example the abelian group $\mathbf{Z_2} \times \mathbf{Z_2}$ is NOT the 'same' as the abelian group $\mathbf{Z_4}$

2) There's no need to just use two groups when you form the Cartesian product as we did above, you can use any number of groups you want, and you will still get an abelian group.

3) The symbol $\mathbf{Z_1}$ denotes the group {0} having just one element. On the other hand, the symbol $\mathbf{Z_0}$ denotes the group **Z** (the group of all integers).

The Mad Vet Group Theorem

(and how to answer to the question: *just what group is it*?)

Here's the key result which allows us to easily determine which Mad Vet Scenarios have Mad Vet semigroups that are in fact groups.

"Mad Vet Group" Theorem: Draw the directed graph E of the Mad Vet Scenario. Then the Mad Vet semigroup is a group if and only if:

1) E contains at least one cycle.
2) There is a directed path from every vertex of E to every cycle of E.
3) Every cycle of E has an exit.

"What group is it?" Theorem: Suppose that you are in a situation where the Mad Vet semigroup is a group. Here's how to figure out what group it is. Form the adjacency matrix A of the Mad Vet graph. Then form the matrix $B = I - A^t$. (Here I means the identity matrix of the appropriate size, and ^{t} means 'transpose'.) Compute the Smith normal form S of the matrix B, and write the diagonal entries of S as $s_1, s_2, \ldots, s_n$. Then the Mad Vet Group of the Mad Vet Scenario is the group

$$\mathbf{Z}_{s_1} \times \mathbf{Z}_{s_2} \times \ldots \times \mathbf{Z}_{s_n}.$$

Activity: Now reconsider each of the six Scenarios, and see if you can use the Mad Vet Group Theorem to determine (or re-determine) which Scenarios give groups. For those scenarios which do give groups, use the What group is it? Theorem to determine exactly which group it is.

Additional Questions and Food for Thought

1) For each positive integer n , construct a Mad Vet Scenario for which the Mad Vet semigroup is the group $\mathbf{Z}_n$. Then construct a different scenario, using a different number of species, which gives the same Mad Vet group $\mathbf{Z}_n$.

2) For each sequence of nonnegative integers s_1 , s_2 , … , s_n where the final terms are zero (if there *are* any nonzero terms), and where each nonzero term divides the next term in the sequence, construct a Mad Vet Puzzle for which the corresponding Mad Vet semigroup is the group $\mathbf{Z}\, s_1 \times \mathbf{Z}\, s_2 \times \ldots \times \mathbf{Z}\, s_n$. Is there some 'best' or 'optimal' way to do this?

3) Here's the web address for a blog about Mad Vet Puzzles, this has been active since the end of June 2011 (and might still be active as of Week 3 of MathPath 2011 !!)
http://community.nytimes.com/comments/wordplay.blogs.nytimes.com/2011/06/27/numberplay-the-mad-vet/

4) There are generalizations of Mad Vet Puzzles in which there might be more than one machine corresponding to a species. For example, one that's mentioned at the above blog is (in slightly different form): Suppose a Mad Vet has three Machines. Machine 1 turns one Dog into two Dogs. Machine 2 turns one Dog into two Cats. Machine 3 turns one Cat into one Dog and one Cat. (So this is not a type of scenario that we can deal with using the Mad Vet Group Theorem.) BUT: we can ask the usual questions, for instance, how many different "classes" of animals are there in this Scenario?

5) An electronic version of the Math Magazine article about Mad Veterinarian Puzzles can be found at www.uccs.edu/gabrams (scroll down about a dozen entries on that page)

