

# Mad Veterinarian Puzzles: A Mathematical Safari

The Mobile Math Circle, November 14, 2011

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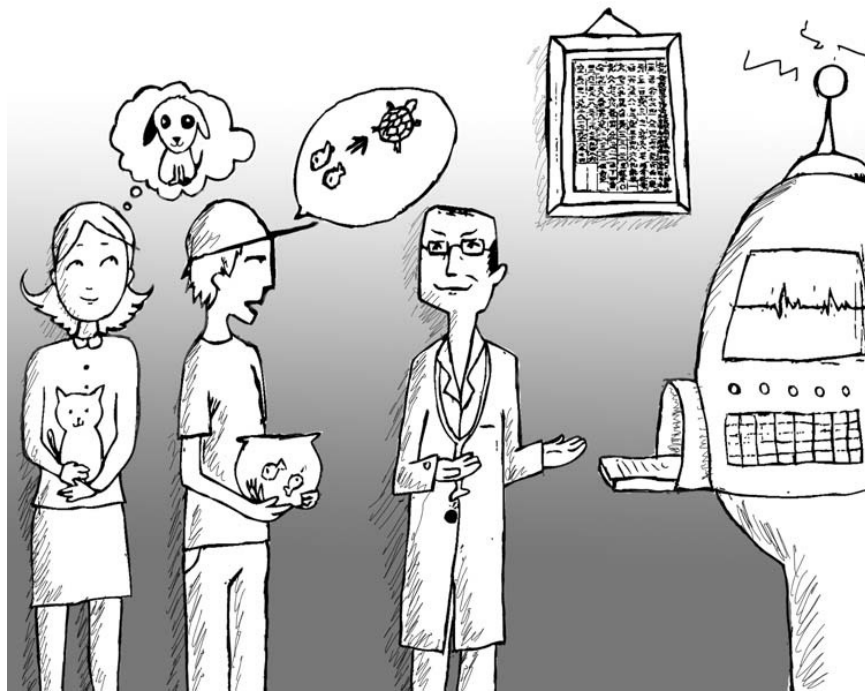
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There is a collection of problems which have come to be known as "Mad Veterinarian Puzzles", for reasons which will soon become obvious.

As a warmup, we will look at the "usual" questions which are asked in this context. Then we will ask some new types of questions. The goal is to see just how much structure is contained in these ideas, and really how beautiful, these ideas can be!

If we have some time at the end, we will mention how these puzzles arise in some current mathematical research areas.

Thanks to Joshua Zucker of the Bay Area (San Francisco) Math Circle for some interesting and helpful conversations about these puzzles.



(graphic courtesy of the Mathematical Association of America.)

# Puzzle 1

Here's your first example of a Mad Veterinarian Puzzle. (The wording is similar in flavor to the wording of the original and most famous Mad Vet Puzzle, which you can find at <http://www.bumblebeagle.org/madvet/index.html> .)

**A Mad Veterinarian has created three animal transmogrifying machines.**

**Place a cat in the input bin of the first machine, press the button, and .... whirrr .... bing! Open the output bins to find that the cat has transformed into a dog! The second machine can convert a dog into one cat, one dog, and one mouse. The third machine can convert a mouse into a cat and a dog. Each machine can also operate in reverse (e.g., if you've got one cat, one dog, and one mouse, you can convert them into a dog.)**

(So, in shortened notation:  $C < \text{---} > D$        $D < \text{---} > C D M$        $M < \text{---} > C D$  )

## Question 1.1

Suppose this Mad Vet starts with one cat. Can she convert it into four cats? (If so, show how. If not, try to explain why not.)

## Question 1.2

Suppose again this Mad Vet starts with one cat. Can she convert it into three cats? (If so, show how. If not, try to explain why not.)

## Question 1.3

Suppose yet again this Mad Vet starts with one cat. Can she convert it into two cats? (If so, show how. If not, try to explain why not.)

## Question 1.4

If the Mad Vet starts with one cat, and she wants to only have cats in her Mad Vet Clinic, how many cats can she have? See if you can find ALL the possible answers.

## Question 1.5

Suppose now that the Mad Vet starts with one mouse. Can she use the machines to produce a collection consisting only of cats? (If so, show how, and say how many cats are produced by that one mouse. If not, try to explain why not.)

Puzzle 1, in shortened notation:

$C < \text{---} > D$

$D < \text{---} > C D M$

$M < \text{---} > C D$

# Classes of animals

We say that two collections of animals are in the same “class” if you can use the Mad Vet machines to transform one collection to the other. (Because the machines are allowed to run in reverse, if you can transform one collection to another, then you can transform the second collection back to the first.)

For example: With the machines of Puzzle 1, the collection consisting of {2Cats, 1Dog} is in the same class as the collection consisting of {4Cats, 1Mouse}, because

$\{2\text{Cats}, 1\text{Dog}\} \text{ --- (Machine 2) ----} > \{3\text{Cats}, 1\text{Dog}, 1\text{Mouse}\} \text{ ---- (Machine 1 reverse) ---} > \{4\text{Cats}, 1\text{Mouse}\}$

We sometimes use square brackets to denote “classes”. So, for instance, [ 2Cats, 1Dog ] means all of the collections of animals which are in the same class as the collection {2Cats, 1Dog}. We just showed that

$$[ 2\text{Cats}, 1\text{Dog} ] = [ 3\text{Cats}, 1\text{Dog}, 1\text{Mouse} ] = [ 4\text{Cats}, 1\text{Mouse} ].$$

Question 1.6. Justify why [ 2Cats, 1Dog ] = [ 3Cats, 2Dogs, 2Mice ]

Question 1.7. Find a number N for which [ 2Cats, 1Dog ] = [ NCats ]

Question 1.8. Find a DIFFERENT number M for which [ 2Cats, 1Dog ] = [ MCats ]

**MAJOR Question 1.9:** How many different classes of animals are there? Specifically:

Give a list of classes having the property that EVERY possible collection of animals can be transformed to exactly one class on your list. Justify why your list is a complete list of classes, and try to explain why there are no repeated classes on your list. Call this list the *List of All Classes*.

(Here’s a start to the Answer to Major Question 1.9: [ 1Cat ] is a class. So is [ 2Cats ].)

Question 1.10: To which class on your list from Question 1.9 is the class [ 7Cats, 5Dogs, 10Mice ] equal ?

Puzzle 1, in shortened notation:

$C < \dots > D$

$D < \dots > C D M$

$M < \dots > C D$

## Combining Classes

There is an ‘arithmetic’ of classes. Here’s one way to think about it. Suppose the Mad Vet knows the List of All Classes in her clinic (in other words, suppose the Mad Vet knows the answer to Major Question 1.9).

Now suppose the Mad Vet starts with some collection of animals. For the Mad Vet’s birthday, a friend brings her a present: some more animals! Now the Mad Vet has a new collection of animals. But which class in the List of All Classes is the new collection equal to?

For example: Suppose the Mad Vet has two cats. The Mad Vet’s friend brings the Mad Vet two more cats as a present. So now the Mad Vet has {4Cats} in her clinic. To which class on the List of All Classes is [ 4Cats ] equal ?? You showed previously (way back in the very first Question!) that [ 1Cat ] = [ 4Cats ].

$$\text{So: } [ 2\text{Cats} ] + [ 2\text{Cats} ] = [ 4\text{Cats} ] = [ 1\text{Cat} ]$$

We now make a table of ALL possible combinations of classes. We list all three of the classes as the heading of each row, and each column. One entry in the table has already been filled in, it’s the information we just found out, namely, that [ 2Cats ] + [ 2Cats ] = [ 1Cat ].

Question 1.11: Make a table of ALL possible combinations of classes. When you are done, make some observations about the entries of this table. Specifically, determine what the sum of each column entry with each row entry is, and record your answer in that row-column intersection. What can you say about each of the rows and columns of the lower-right portion of the table?

+	[ 1Cat ]	[ 2Cats ]	[ 3Cats ]
[ 1Cat ]			
[ 2Cats ]		[ 1Cat ]	
[ 3Cats ]			

## Puzzle 2

A second Mad Veterinarian has these three animal transmogrifying machines.

**Machine #1 turns one cat into one dog and one mouse**

**Machine #2 turns one dog into one cat and one mouse**

**Machine #3 turns one mouse into one cat and one dog**

(So, in shortened notation:  $C \rightarrow D M$        $D \rightarrow C M$        $M \rightarrow C D$ )

Question 2.1: How many different classes of animals are there in Puzzle 2?

Specifically: Give a list of classes having the property that EVERY possible collection of animals can be transformed to exactly one class on your list. Justify why your list is a complete list of classes.

Question 2.2: Make a table of ALL possible combinations of classes. When you are done, make some observations about the entries of this table. Specifically, list out all the different classes along the top row, and again along the left hand column. Then determine what the sum of each column entry with each row entry is, and record your answer in that row-column intersection. What can you say about each of the rows and columns of the lower-right portion of the table?

+	

## Puzzle 3

A third Mad Veterinarian has these two animal transmogrifying machines.

**Machine #1 turns one cat into two cats**

**Machine #2 turns one dog into two dogs**

(So, in shortened notation:  $C \langle \text{---} \rangle 2C$        $D \langle \text{---} \rangle 2D$  )

Question 3.1: How many different classes of animals are there in Puzzle 3?

Specifically: Give a list of classes having the property that EVERY possible collection of animals can be transformed to exactly one class on your list. Justify why your list is a complete list of classes.

Question 3.2: Make a table of ALL possible combinations of classes. When you are done, make some observations about the entries of this table. Specifically, list out all the different classes along the top row, and again along the left hand column. Then determine what the sum of each column entry with each row entry is, and record your answer in that row-column intersection. What can you say about each of the rows and columns of the lower-right portion of the table?

<b>+</b>	

## Puzzle 4

A fourth Mad Veterinarian has these three animal transmogrifying machines.

**Machine #1 turns one cat into one cat, one dog and one mouse**

**Machine #2 turns one dog into one cat and one mouse**

**Machine #3 turns one mouse into one cat and one dog**

(So, in shortened notation:  $C \rightarrow CDM$      $D \rightarrow CM$      $M \rightarrow CD$ )

Question 4.1: How many different classes of animals are there in Puzzle 4?

Specifically: Give a list of classes having the property that EVERY possible collection of animals can be transformed to exactly one class on your list. Justify why your list is a complete list of classes.

Question 4.2: Make a table of ALL possible combinations of classes. When you are done, make some observations about the entries of this table. Specifically, list out all the different classes along the top row, and again along the left hand column. Then determine what the sum of each column entry with each row entry is, and record your answer in that row-column intersection. What can you say about each of the rows and columns of the lower-right portion of the table?

+	

# Mad Veterinarian Puzzles: More Food for Thought

1) There is a technique to actually PROVE that a List of All Classes has no repeated classes on it. We can use what's called an *invariant*. For instance, in Puzzle 1: If we assign dollar values to animals in each species by valuing Cats at \$1 apiece, Dogs at \$1 apiece, and Mice at \$2 apiece, then we can show that the three classes [C], [2C], and [3C] are different. This gives a valid proof that the answers to Questions 1.2 and 1.3 are NO.

The specific invariant that you'll use to prove that there are no repeats on the List of All Classes for other Puzzles will probably be different than the one we used for Puzzle 1. See if you can find invariants for the other three puzzles.

2) Tables in which every member of the List of All Classes appears exactly once in each row and each column of the lower right portion are very special in mathematics: those tables correspond to what are called *groups*. Intuitively, a group is a system in which you can always get from any element in the system to any other element in the system by combining the first element with something else in the system. For instance, the set of all whole numbers (positive, negative, and 0) has this property, but the set of positive whole numbers does not. (Can you see why?)

3) There are many additional questions which arise in the context of Mad Veterinarian Puzzles. For instance, which Mad Vet Puzzles yield groups in the addition table for the List of All Classes? Is there some way to determine WHICH group it is? If you have a specific group in mind, can you build a Mad Vet Puzzle whose table *is* that group? Is it possible to have a Mad Vet Puzzle for which the List of All Classes is infinite?

4) So who cares about these Mad Vet Puzzles anyway ??? It turns out that there is a very strong connection between these puzzles and some current lines of mathematics research, called *Leavitt path algebras* and *graph C\*-algebras*.

5) More info can be found in the article *The graph menagerie: Abstract algebra and the Mad Veterinarian* (by Gene Abrams and Jessica Sklar), *Mathematics Magazine* **83**(3), 2010, 168-179.