Notes on the partition of $\{1, 2, ..., d\}$ which arises in construction of isomorphisms between matrix rings over Leavitt algebras

(prepared by Gene Abrams, March 2015.)

Let $L_K(1,n)$ denote the Leavitt algebra of order n; so $R = L_K(1,n)$ has the property that ${}_RR \cong {}_RR^n$ as left R-modules. In [1] it is shown that

$$L_K(1,n) \cong \mathrm{M}_d(L_K(1,n)) \Leftrightarrow \text{g.c.d.}(d, n-1) = 1.$$

This yields as an easy consequence that for positive integers d, d' we have

$$\mathrm{M}_d(L_K(1,n)) \cong \mathrm{M}_{d'}(L_K(1,n)) \Leftrightarrow \text{g.c.d.}(d, n-1) = \text{g.c.d.}(d', n-1).$$

Indeed, when the appropriate conditions are satisfied then the isomorphisms may be explicitly constructed. As one consequence of this isomorphism result, E. Pardo was able to answer a decades-old question of Higman regarding a class of infinite finitely presented simple groups [2].

The key to constructing such isomorphisms lies in considering a partition of $\{1, 2, ..., d\}$ into two nonempty disjoint subsets $S_1 \sqcup S_2$, described as follows.

Suppose n is an integer having g.c.d.$(d, n-1) = 1$. Write $n = qd + r$ with $1 \le r \le d$. As g.c.d.$(d, n-1) = 1$ we get g.c.d.$(d, r-1) = 1$.

For purposes of this note we forget the original integer n, and focus only on r. Note we have $r \ge 1$. Let $s = d - (r-1)$. Since g.c.d.$(d, r-1) = 1$ we easily see g.c.d.$(d, s) = 1$. Now consider the sequence $\Sigma^{d,r}$, given by

$$\Sigma^{d,r} \ = \ 1,\ 1+s,\ 1+2s,\ ...,\ 1+(d-1)s$$

of d integers, interpreted mod$d$. (Here we interpret 0 mod$d$ as d mod$d$.) Since g.c.d.$(d, s) = 1$, elementary number theory gives that, as a set, the elements of $\Sigma^{d,r}$ form a complete set of residues mod$d$.

In particular, for some i_r $(1 \le i_r \le d)$ we have $1 + (i_r - 1)s \equiv r - 1 \bmod d$.

Now consider these two sequences:

$$\Sigma_1^{d,r} = 1,\ 1+s, ...,\ 1+(i_r-1)s \qquad \Sigma_2^{d,r} = 1 + i_r s,\ 1+(i_r+1)s,\ ...,\ 1+(d-1)s.$$

So $\Sigma_1^{d,r}$ is just the first i_r elements of $\Sigma^{d,r}$, and $\Sigma_2^{d,r}$ is the remaining $d - i_r$ elements.

We can also consider the partition $S^{d,r} = S_1^{d,r} \sqcup S_2^{d,r}$ of $\{1, 2, ..., d\}$ which corresponds to the elements of these two sequences:

$$S_1^{d,r} = \{1,\ 1+s, ...,\ 1+(i_r-1)s\} \qquad S_2^{d,r} = \{1+i_r s,\ 1+(i_r+1)s,\ ...,\ 1+(d-1)s\}$$

So in particular $|S_1^{d,r}| = i_r$ and $|S_2^{d,r}| = d - i_r$. Clearly $S_1^{d,r} \ne \emptyset$. But $S_2^{d,r} \ne \emptyset$ as well, because $d \in S_2^{d,r}$. This is because the first element $1 + i_r s$ of $\Sigma_2^{d,r}$ is always d, as $1 + i_r s = (1 + (i_r - 1)s) + s = (r-1) + (d - (r-1)) = d$.

For notational convenience, if d, r are fixed then we drop the superscript d, r in the sequences and subsets.

Example 1. The case $d = 3, r = 2$. g.c.d.$(3, 2-1) = 1$. $r-1 = 1$, $s = d-(r-1) = 3-1 = 2$.

The sequence Σ starts at 1, and increases by $s = 2$ each step, and we interpret mod 3 $(1 \leq i \leq 3)$. So we get the sequence $\Sigma = 1, 3, 2$. Since $r - 1 = 1$, we get

$$\Sigma_1 = 1 \quad \Sigma_2 = 3, 2$$

and so

$$S_1 = \{1\}, \quad S_2 = \{2, 3\}.$$

Example 2. The case $d = 3, r = 3$. g.c.d.$(3, 3-1) = 1$. $r-1 = 2$, $s = d-(r-1) = 3-2 = 1$.

The sequence Σ starts at 1, and increases by $s = 1$ each step, and we interpret mod 3 $(1 \leq i \leq 3)$. So we get the sequence $\Sigma = 1, 2, 3$. Since $r - 1 = 2$, we get

$$\Sigma_1 = 1, 2 \quad \Sigma_2 = 3$$

and so

$$S_1 = \{1, 2\} \quad S_2 = \{3\}.$$

Example 3. The case $d = 13, r = 9$. g.c.d.$(13, 9-1) = 1$. $r-1 = 8$, $s = d-(r-1) = 13-8 = 5$.

The sequence Σ starts at 1, and increases by $s = 5$ each step, and we interpret mod 13 $(1 \leq i \leq 13)$. So we get the sequence $\Sigma = 1, 6, 11, 3, 8, 13, 5, 10, 2, 7, 12, 4, 9$. Since $r - 1 = 8$, we get

$$\Sigma_1 = 1, 6, 11, 3, 8 \qquad \Sigma_2 = 13, 5, 10, 2, 7, 12, 4, 9$$

and so

$$S_1 = \{1, 3, 6, 8, 11\} \qquad S_2 = \{2, 4, 5, 7, 9, 10, 12, 13\}.$$

By solving the congruence $1 + (i_r - 1)s \equiv r - 1 \bmod d$, we easily get

Lemma 4. *$i_r \equiv (r-1)^{-1} \bmod d$.*

In particular, if we have $1 \leq r \neq r' \leq d$ for which g.c.d.$(d, r-1) = 1 =$ g.c.d.$(d, r'-1)$, then the two partitions $S_1^{d,r} \sqcup S_2^{d,r}$ and $S_1^{d,r'} \sqcup S_2^{d,r'}$ of $\{1, 2, ..., d\}$ are necessarily different (since 1 is in S_1, and the sizes of $S_1^{d,r}$ and $S_1^{d,r'}$ are unequal).

Given d, there exist $\varphi(d)$ (Euler φ-function) remainders which are relatively prime to d. So there exist $\varphi(d)$ distinct partitions of $\{1, 2, ..., d\}$ which arise as $S_1^{d,r} \sqcup S_2^{d,r}$ for some r having g.c.d.$(d, r-1) = 1$.

We note that for any d we always get these two partitions arising in the form $S^{d,r}$:

$$\{1\} \sqcup \{2, 3, ..., d\} = S^{d,2}, \quad \text{and} \quad \{1, 2, ..., d-1\} \sqcup \{d\} = S^{d,d}.$$

It is easy to see that there are $(2^d - 2)/2 = 2^{d-1} - 1$ ways to partition the set $\{1, 2, ..., d\}$ into two nonempty subsets $S_1 \sqcup S_2$ for which $1 \in S_1$. Since $\varphi(d) < 2^{d-1} - 1$ for $d \geq 3$, we see that not all such two-nonempty-set partitions of $\{1, 2, ..., d\}$ can arise as $S_1^{d,r} \sqcup S_2^{d,r}$ for some

r having g.c.d.$(d, r-1) = 1$. For example, when $d = 3$, the partition $\{1, 3\} \sqcup \{2\}$ of $\{1, 2, 3\}$ does not arise in this way.

Question 1: For fixed d, r having g.c.d.$(d, r-1) = 1$, do the sequences $\Sigma_1^{d,r}$ and $\Sigma_2^{d,r}$ arise in other contexts?

Question 2: Do the $\varphi(d)$ partitions of $\{1, 2, ..., d\}$ of the form $\{1, 2, ..., d\} = S_1^{d,r} \sqcup S_2^{d,r}$ (for some r having g.c.d.$(d, r-1) = 1$) arise in other contexts?

Remark 5. Referring back to how these sequences and partitions arose in the context of [1], in that setting we start with d, n having g.c.d.$(d, n-1) = 1$, write $n = qd + r$ with $1 \leq r \leq d$, and then consider the partition $S_1^{d,r} \sqcup S_2^{d,r}$ of $\{1, 2, ..., d\}$. We then use this partition of $\{1, 2, ..., d\}$ to build a partition of $\{1, 2, ..., n\}$ by simply extending the partition $S_1^{d,r} \sqcup S_2^{d,r}$, mod$d$. So, for instance, if $n = 5$, $d = 3$ then we get $5 = 1 \cdot 3 + 2$. We then consider the partition $\{1, 2, 3\} = S_1^{3,2} \sqcup S_2^{3,2} = \{1\} \sqcup \{2, 3\}$, as described in Example 1. This then yields the partition $\{1, 4\} \sqcup \{2, 3, 5\}$ of $\{1, 2, 3, 4, 5\}$ by simply extending mod3.

Specifically, in [1] we do not seem to use the ordering properties of the sequences Σ_1 and Σ_2, we only use the partition $S_1^{d,r} \sqcup S_2^{d,r}$ of $\{1, 2, ..., d\}$

References

[1] G. Abrams, P.N. Ánh, E. Pardo, Isomorphisms between Leavitt algebras and their matrix rings. *J. Reine Angew. Math. (Crelle's J.)* **624** (2008), 103 – 132.

[2] E. Pardo, The isomorphism problem for Higman-Thompson groups. *J. Algebra* **344** (2011), 172 – 183.