The Unbounded Generating Number property for the Bergman algebra of a directed graph

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The monoid $\mathcal{V}(R)$

All rings are unital. $\mathbb{Z}^+$ denotes $\{0, 1, 2, \dots\}$.

For any ring R, $\mathcal{V}(R)$ denotes the isomorphism classes of finitely generated projective (left) R-modules.

With operation $\oplus$, $\mathcal{V}(R)$ becomes an abelian monoid.

In $\mathcal{V}(R)$, $[R]$ is *distinguished*:

For each $[P] \in \mathcal{V}(R)$ there exists $[P'] \in \mathcal{V}(R)$ and $n \in \mathbb{N}$ for which $[P] \oplus [P'] = n[R]$.

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Remarks:

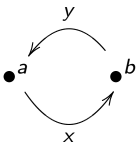
(1) Given a ring R, it is in general not easy to compute $\mathcal{V}(R)$.

(2) The Grothendieck group $K_0(R)$ of R is the universal group of the monoid $\mathcal{V}(R)$.

The monoid M_E

All graphs $E = (E^0, E^1, s, r)$ are finite and directed.

1. A *cycle*



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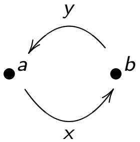
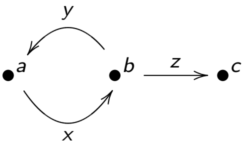


2. An *exit* for a cycle.



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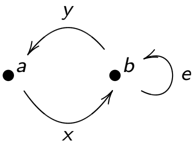

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Then $(M_E, +)$ is defined to be the quotient $Y_E/\mathcal{R}$.

So elements of M_E are of the form

$$M_E = \{[\sum_{v \in E^0} n_v a_v]\}$$

with $n_v \in \mathbb{Z}^+$ for all $v \in E^0$.

The monoid M_E

For notational convenience we often denote $a_v \in Y_E$ simply by v. So M_E is typically written as:

M_E is the free abelian monoid on E^0, subject to the relations

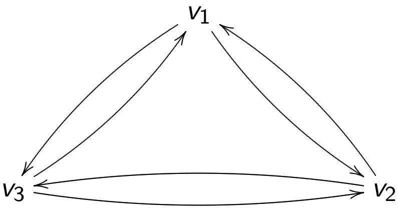
$$v = \sum_{\{e | s(e) = v\}} r(e)$$

(for any vertex v which emits at least one edge).

Note: $d_E = [\sum_{v \in E^0} v]$ is a distinguished element in M_E.

The monoid M_E

Example. Let F be the graph



So M_F consists of elements $\{n_1 v_1 + n_2 v_2 + n_3 v_3\}$ $(n_i \in \mathbb{Z}^+)$,

subject to $\mathcal{R}$: $\quad v_1 = v_2 + v_3;\ \ v_2 = v_1 + v_3;\ \ v_3 = v_1 + v_2.$

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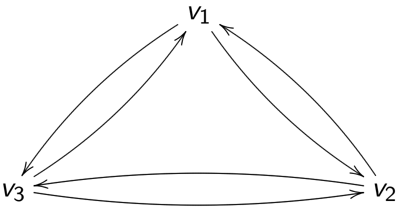

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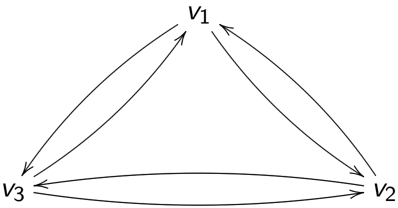

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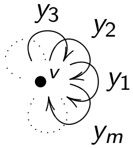
It's not hard to get: $\quad M_F = \{[0],\ [v_1],\ [v_2],\ [v_3],\ [v_1 + v_2 + v_3]\}.$

The monoid M_E

Example:

$E = R_m =$ 

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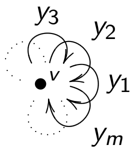

Then M_E is the set of symbols of the form

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So here, $M_E = \{[0], [v], 2[v], ..., (m-1)[v]\}$.

Bergman's Theorem

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Theorem

(George Bergman, Trans. A.M.S. 1975)
Given a field K and finitely generated conical monoid $M = (\mathbb{Z}^+)^n/\mathcal{R}$ containing a distinguished element d, there exists a universal K-algebra $B = B_K(M, \mathcal{R}, d)$ for which $\mathcal{V}(B) \cong M$, and for which $[B] \mapsto d$ under this isomorphism.

The construction is explicit, uses amalgamated products.
(Fin. gen. hypothesis eliminated by Bergman / Dicks, 1978)

The Bergman algebra of a finite directed graph

We put these two ideas together.

Let E be a finite directed graph and K any field. Form the (finitely generated, conical) monoid M_E as a quotient of $(Z^+)^{|E^0|}$, modulo the relations given above. Let $d_E = [\sum_{v \in E^0} v]$ be the specified distinguished element of M_E.

Let $B = B_K(M_E, d_E)$ be the corresponding Bergman algebra.

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Definition. We call $B_K(M_E, d_E)$ the *Leavitt path algebra* of E with coefficients in K, denoted $L_K(E)$.

General ring-theoretic questions about $L_K(E)$

An explicit description of $L_K(E)$ via generators and relations is available.

Using it, we can determine necessary and sufficient conditions on E which yield that $L_K(E)$ is, for instance,

- simple
- purely infinite simple
- von Neumann regular
- primitive

lots more ...

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R has *Invariant Basis Number* if ${}_R R^m \cong {}_R R^n \Leftrightarrow m = n$.

R has *Unbounded Generating Number* if ${}_R R^m \cong {}_R R^n \oplus P \Rightarrow m \geq n$.

(R is *directly finite* if $xy = 1 \Rightarrow yx = 1$.)

R is *stably finite* if $\mathrm{M}_n(R)$ is directly finite for all $n \in \mathbb{N}$.
This is equivalent to: $R^n \cong R^n \oplus K \ \Rightarrow \ K = 0$.

R is *cancellative* if for any finitely generated projective left R-modules P, P', Q, $\quad P \oplus Q \cong P' \oplus Q \ \Rightarrow \ P \cong P'$.

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is stably finite, cancellative, ...

i.e., for which ....

Leavitt path algebras with Unbounded Generating Number

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Is this in the literature?

Some pieces of the puzzle

Proposition: If $R = S_1 \oplus S_2$ (as rings), and one of S_1 or S_2 has UGN, then R has UGN.

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Proof: Then $L_K(E)$ has a ring direct summand isomorphic to K.

Proposition: (Ara / Rangaswamy) If E contains a source vertex v, and v is not isolated, then $L_K(E)$ and $L_K(E \setminus \{v\})$ are Morita equivalent.

Proof: Can show $L_K(E \setminus \{v\})$ is a full corner in $L_K(E)$.

Key observation

Corollary: Start with E. Do a sequence of source eliminations

$$E = E_0 \to E_1 \to \cdots \to E_{t-1} \to E_t = E_{sf}$$

where E_{sf} is source-free.

If some E_i contains an isolated vertex, then $L_K(E)$ has UGN.

If no E_i contains an isolated vertex, then $L_K(E)$ has UGN if and only if $L_K(E_{sf})$ has UGN.

Consequently, we have reduced the question to source-free graphs.

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Proposition: If w is an isolated vertex in E, then M_E has UGN.

Proof: If $m[\sum_{v \in E^0} v] = n[\sum_{v \in E^0} v] + [x]$ in M_E then (because w is isolated) we get the equation $mw = nw + x_w$ in the free abelian monoid $Y_E = (\mathbb{Z}^+)^{|E^0|}$, which gives $m \geq n$.

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Proposition: Let w be a source vertex in E. Let $F = E \setminus \{w\}$. Then $M_E \cong M_F$ as abelian monoids.

Proof: The "inclusion map" $M_F \mapsto M_E$ is a monoid homomorphism, which can be shown to be injective. Surjectivity follows because w is a source.

A key property of M_E

The Confluence Lemma. (Ara / Moreno / Pardo 2007) For each pair $x, y \in (\mathbb{Z}^+)^{|E^0|}$, $[x] = [y]$ in M_E if and only if there are sequences σ, σ' such that $\Lambda_\sigma(x) = \Lambda_{\sigma'}(y)$ in Y_E.

In other words, we have some control over when two elements in Y_E are equal in M_E, in that we can "forward move" both of them to the same place.

The foundational result

Definition. A *source cycle* c in a graph E is a cycle for which, for each vertex v in c, the only edge that v receives is the preceding edge in c.

(i.e., a *cycle with no entrances*?)

Theorem. (A-, Nam, Phuc) Let $E = (E^0, E^1, r, s)$ be a finite source-free graph and K any field. Then $L_K(E)$ has Unbounded Generating Number if and only if E contains a source cycle.

Idea of proof: source cycle implies UGN

We denote E^0 by $\{v_1, v_2, ..., v_h\}$, in such a way that the non-sink vertices of E appear as $v_1, ..., v_z$.

Assume that E contains a source cycle c; show that $L_K(E)$ has UGN. Let m and n with

$$m[\sum_{i=1}^{h} v_i] + [x] = n[\sum_{i=1}^{h} v_i] \text{ in } M_E$$

for some $[x] \in M_E$. We must show that $m \leq n$.

Idea of proof: source cycle implies UGN

Write $x \in Y_E$ as $x = \sum_{i=1}^{h} n_i v_i$, with $n_i \in \mathbb{Z}^+$.

By the Confluence Lemma and there are two sequences σ and σ' taken from $\{1, \dots, z\}$ for which

$$\Lambda_\sigma(\sum_{i=1}^{h}(m + n_i)v_i) = \gamma = \Lambda_{\sigma'}(n\sum_{i=1}^{h} v_i)$$

for some $\gamma \in Y_E$.

Idea of proof: source cycle implies UGN

But each time a substitution corresponding to vertex j is made to an element of Y_E, the effect on that element is to:

(i) subtract 1 from the coefficient on v_j;

(ii) add a_{ji} to the coefficient on v_i (for $1 \leq i \leq h$).

For each $1 \leq j \leq z$, denote the number of times that M_j is invoked in Λ_σ (resp., $\Lambda_{\sigma'}$) by k_j (resp., k_j').

Idea of proof: source cycle implies UGN

Recalling the previously observed effect of M_j on an element of Y, we see that

$$\begin{aligned}
\gamma &= \Lambda_\sigma(\sum_{i=1}^h (m+n_i)v_i) \\
&= ((m+n_1-k_1)+a_{11}k_1+a_{21}k_2+\ldots+a_{z1}k_z)v_1 \\
&\quad +((m+n_2-k_2)+a_{12}k_1+a_{22}k_2+\ldots+a_{z2}k_z)v_2 \\
&\quad + \quad \cdots \\
&\quad +((m+n_z-k_z)+a_{1z}k_1+a_{2z}k_2+\ldots+a_{zz}k_z)v_z \\
&\quad +((m+n_{z+1})+a_{1(z+1)}k_1+a_{2(z+1)}k_2+\ldots+a_{z(z+1)}k_z)v_{z+1} \\
&\quad + \quad \cdots \\
&\quad +((m+n_h)+a_{1h}k_1+a_{2h}k_2+\ldots+a_{zh}k_z)v_h.
\end{aligned}$$

Idea of proof: source cycle implies UGN

On the other hand, we have

$$
\begin{aligned}
\gamma &= \Lambda_{\sigma'}(n\textstyle\sum_{i=1}^{h} v_i) \\
&= ((n-k_1') + a_{11}k_1' + a_{21}k_2' + \ldots + a_{z1}k_z')v_1 \\
&\quad + ((n-k_2') + a_{12}k_1' + a_{22}k_2' + \ldots + a_{z2}k_z')v_2 \\
&\quad + \quad \cdots \\
&\quad + ((n-k_z') + a_{1z}k_1' + a_{2z}k_2' + \ldots + a_{zz}k_z')v_z \\
&\quad + (n + a_{1(z+1)}k_1' + a_{2(z+1)}k_2' + \ldots + a_{z(z+1)}k_z')v_{z+1} \\
&\quad + \quad \cdots \\
&\quad + (n + a_{1h}k_1' + a_{2h}k_2' + \ldots + a_{zh}k_z')v_h.
\end{aligned}
$$

Idea of proof: source cycle implies UGN

For each $1 \le i \le z$, define $m_i := k_i' - k_i$. Then equating coefficients on the free generators $\{v_i \mid 1 \le i \le h\}$ of Y_E, we get the following system of equations in $\mathbb{Z}^+$:

$$\left\{ \begin{array}{rcl} m - n + n_1 & = & (a_{11} - 1)m_1 + a_{21}m_2 + \ldots + a_{z1}m_z \\ m - n + n_2 & = & a_{12}m_1 + (a_{22} - 1)m_2 + \ldots + a_{z2}m_z \\ & \vdots & \\ m - n + n_z & = & a_{1z}m_1 + a_{2z}m_2 + \ldots + (a_{zz} - 1)m_z \\ m - n + n_{z+1} & = & a_{1(z+1)}m_1 + a_{2(z+1)}m_2 + \ldots + a_{z(z+1)}m_z \\ & \vdots & \\ m - n + n_h & = & a_{1h}m_1 + a_{2h}m_2 + \ldots + a_{zh}m_z \end{array} \right.$$

Idea of proof: source cycle implies UGN

By hypothesis c is a source cycle in E, i.e., $|r^{-1}(v)| = 1$ for all $v \in c^0$. By renumbering vertices if necessary, we may assume without loss of generality that $c^0 = \{v_1, ..., v_p\}$. The condition $|r^{-1}(v)| = 1$ then yields:

- $a_{i,i+1} = 1$ for $1 \leq i \leq p-1$;
- $a_{p,1} = 1$;
- $a_{j,i+1} = 0$ for $1 \leq i \leq p-1$ and $j \neq i$ $(1 \leq j \leq h)$; and
- $a_{j,1} = 0$ if $j \neq p$ $(1 \leq j \leq h)$.

Idea of proof: source cycle implies UGN

If $p = 1$ (i.e., if c is a loop), then $a_{11} = 1$, and first equation in the system becomes

$$m - n + n_1 = (1 - 1)m_1 + 0m_2 + \cdots + 0m_z = 0,$$

so $m - n = -n_1 \leq 0$, i.e., $m \leq n$.

Idea of proof: source cycle implies UGN

If $p \geq 2$, then using the noted information about the $a_{i,j}$, the first p equations of the system can be written as:

$$\begin{cases} m - n + n_1 & = & -m_1 & & & & +m_p \\ m - n + n_2 & = & m_1 & -m_2 & & & \\ m - n + n_3 & = & & m_2 & -m_3 & & \\ & \vdots & & & & & \\ m - n + n_p & = & & & & m_{p-1} & -m_p \end{cases}.$$

Then adding both sides yields that $p(m-n) + (n_1 + \ldots + n_p) = 0$, so that $p(m-n) = -(n_1 + \ldots + n_p) \leq 0$, which gives $m \leq n$. Therefore, $L_K(E)$ has Unbounded Generating Number.

Idea of proof of converse: no source cycle implies NOT UGN

First prove a Lemma about various configurations of cycles in E which is implied by no source cycles.

Then do another analysis of the elements in M_E by analyzing a system of linear equations in $\mathbb{Z}^+$.

Idea of proof of converse: no source cycle implies NOT UGN

First prove a Lemma about various configurations of cycles in E which is implied by no source cycles.

Then do another analysis of the elements in M_E by analyzing a system of linear equations in $\mathbb{Z}^+$.

In fact we can show more:

If E has no source cycles, then for *every* pair $m > n$ we can find $[x] \in M_E$ for which

$$m[\sum_{v \in E^0} v] + [x] = n[\sum_{v \in E^0} v].$$

UGN property for $L_K(E)$

So we have a graph-theoretic condition equivalent to $L_K(E)$ having the UGN property.

Summary of properties of $\mathcal{V}(R)$

Invariant Basis Number

Unbounded Generating Number

stably finite

cancellative

In general, these get stronger ...

Additional properties of $\mathcal{V}(L_K(E))$

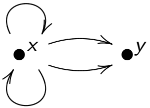
Proposition: (Lia Vaš) For a Leavitt path algebra $R = L_K(E)$ the following are equivalent:

R is directly finite
R is stably finite
R is cancellative
R is one-sided Noetherian
No cycle in E has an exit.

Also equivalent to: R is Hermite

Examples

Example (1) IBN, but not UGN. Let G be the graph



.

$L_K(G)$ does NOT have UGN by the theorem.

Note: $[x] = [2x + 2y]$ gives $[x + y] = 2[x + y] + [y]$.

But easy computation gives that any equation of the form $n[x + y] = m[x + y]$ in M_G necessarily gives $m = n$.

(The one relation $x = 2x + 2y$, applied to an element of Y_G of the form $t = nx + ny$, will either yield t itself, or an element $t' = ix + jy$ for which $i \neq j$.)

So $L_K(G)$ DOES have IBN.

(OR: $L_K(G) \cong C_K(G')$ for some G', then use a theorem.)

Examples

Example (2) UGN, but not directly finite (etc ...)

Consider the Toeplitz graph

$$\mathcal{T} = \begin{tikzcd} \bullet \arrow[loop left] \arrow[r] & \bullet \end{tikzcd}$$

$L_K(E)$ has UGN by the theorem.

However, $L_K(E) \cong K\langle X, Y \mid XY = 1\rangle$ is not directly finite ("Jacobson algebra").

What about IBN for Leavitt path algebras?

Suppose $|E^0| = n$.

Nam and Phuc have found a nice condition involving the matrix $A_E - I_n$ which is equivalent to $L_K(E)$ having IBN.

But they have not yet found a nice graphical condition.

Questions?

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