

Leavitt path algebras: Something for everyone

Gene Abrams



University of Colorado
Colorado Springs

Colloquia Patavina series
Università degli Studi di Padova

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Overview

- 1 Leavitt path algebras: Introduction and Motivation
- 2 Algebraic properties
- 3 Projective modules
- 4 Connections and Applications

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Brief history, and motivating examples

One of the first theorems you saw as an undergraduate student:

Dimension Theorem for Vector Spaces. Every nonzero vector space V has a basis. Moreover, if \mathcal{B} and \mathcal{B}' are two bases for V , then $|\mathcal{B}| = |\mathcal{B}'|$.

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Note: V has a basis $\mathcal{B} = \{b_1, b_2, \dots, b_n\} \Leftrightarrow V \cong \bigoplus_{i=1}^n \mathbb{R}$ as vector spaces. So:

One result of Dimension Theorem, Rephrased:

$$\bigoplus_{i=1}^n \mathbb{R} \cong \bigoplus_{i=1}^m \mathbb{R} \Leftrightarrow m = n.$$

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If $0 \neq v \in V$, then $\{v\}$ is linearly independent.

If $kv = 0$, need to show $k = 0$. But $k \neq 0 \Rightarrow \frac{1}{k}kv = 0 \Rightarrow v = 0$, contradiction.

Similar idea (multiply by the inverse of a nonzero element of K) shows that a maximal linearly independent subset of V actually spans V .

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Rings: e.g., \mathbb{Z} , $M_2(\mathbb{R})$, $C(\mathbb{R})$, ...

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For R any ring, $n \in \mathbb{N}$, then $\bigoplus_{i=1}^n R$ is an R -module as usual.

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For a module P over a ring R we can still talk about a *basis* for P . (Note: in general, not all modules *have* bases; those that do are called *free* R -modules.)

$\bigoplus_{i=1}^n R$ always has a basis having n elements, e.g.,

$$\{e_1 = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, 0, 0, \dots, 1)\}$$

Brief history, and motivating examples

Question: Is the Dimension Theorem true for rings in general?

That is, if R is a ring, and $\bigoplus_{i=1}^n R \cong \bigoplus_{i=1}^m R$ as R -modules, must $m = n$?

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Example: Consider the ring S of linear transformations from an infinite dimensional \mathbb{R} -vector space V to itself.

Think of V as $\bigoplus_{i=1}^{\infty} \mathbb{R}$. Then think of S as $\text{RFM}(\mathbb{R})$.

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More formally: Let

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Similarly, we should be able to 'go back' from pairs of matrices to a single matrix, by interweaving the columns.

That is, more formally, $(M_1, M_2) \mapsto M_1X_1 + M_2X_2$ is a reasonable way to associate a pair of matrices with a single one.

Brief history, and motivating examples

Here's what's really going on. These equations are easy to verify:

$$Y_1 X_1 + Y_2 X_2 = I,$$

$$X_1 Y_1 = I = X_2 Y_2, \quad \text{and} \quad X_1 Y_2 = 0 = X_2 Y_1.$$

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Using these, we get inverse maps:

$$S \rightarrow S \oplus S \quad \text{via} \quad M \mapsto (MY_1, MY_2), \quad \text{and}$$

$$S \oplus S \rightarrow S \quad \text{via} \quad (M_1, M_2) \mapsto M_1 X_1 + M_2 X_2.$$

For example:

$$M \mapsto (MY_1, MY_2) \mapsto MY_1 X_1 + MY_2 X_2 = M \cdot I = M.$$

Brief history, and motivating examples

Using exactly the same idea, let R be ANY ring which contains four elements y_1, y_2, x_1, x_2 satisfying

$$y_1x_1 + y_2x_2 = 1_R,$$

$$x_1y_1 = 1_R = x_2y_2, \quad \text{and} \quad x_1y_2 = 0 = x_2y_1.$$

Then $R \cong R \oplus R$.

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Remark: Here the sets $\{1_R\}$ and $\{x_1, x_2\}$ are each bases for R .

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Actually, when $R \cong R \oplus R$ as R -modules, then $\bigoplus_{i=1}^m R \cong \bigoplus_{i=1}^n R$ for *all* $m, n \in \mathbb{N}$.

Leavitt algebras

Natural question:

Does there exist R with, e.g., $R \cong R \oplus R \oplus R$, but $R \not\cong R \oplus R$?

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Theorem

(William G. Leavitt, Trans. Amer. Math. Soc., 1962)

For every $m < n \in \mathbb{N}$ and field K there exists a K -algebra $R = L_K(m, n)$ with $\bigoplus_{i=1}^m R \cong \bigoplus_{i=1}^n R$, and all isomorphisms between free left R -modules result precisely from this one. Moreover, $L_K(m, n)$ is universal with this property.

Leavitt algebras

The $m = 1$ situation of Leavitt's Theorem is now somewhat familiar. Similar to the $n = 2$ case that we saw above, $R \cong R^n$ if and only if there exist

$$x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in R$$

for which

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$L_K(1, n)$ is the quotient

$$K \langle X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n \rangle / \langle \left(\sum_{i=1}^n Y_i X_i \right) - 1_K; X_i Y_j - \delta_{i,j} 1_K \rangle$$

Note: $\text{RFM}(K)$ is much bigger than $L_K(1, 2)$. 

Leavitt algebras

As a result, we have: Let S denote $L_K(1, n)$. Then

$$S^a \cong S^b \Leftrightarrow a \equiv b \pmod{n-1}.$$

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For every field K and $n \geq 2$, $L_K(1, n)$ is simple.

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Remember, a ring R being *simple* means:

$$\forall 0 \neq r \in R, \exists \alpha_i, \beta_i \in R \text{ with } \sum_{i=1}^n \alpha_i r \beta_i = 1_R.$$

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Actually, $L_K(1, n)$ is REALLY simple:

$$\forall 0 \neq r \in L_K(1, n), \exists \alpha, \beta \in L_K(1, n) \text{ with } \alpha r \beta = 1_{L_K(1, n)}.$$

Building rings from combinatorial objects

Here's a familiar idea. Consider the set $T = \{x^0, x^1, x^2, \dots\}$. Define multiplication on T in the usual way: $x^i \cdot x^j = x^{i+j}$.

Consider formal symbols of the form

$$k_1 t_1 + k_2 t_2 + \cdots + k_n t_n$$

where $t_i \in T$, and $k_i \in \mathbb{R}$. Denote this set of symbols by KT . We can add and multiply two symbols of this form, as usual, e.g. $(kt)(k't') = kk'(t \cdot t')$.

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Also, e.g. if we impose the relation $x^n = x^0$ on T , call the new semigroup \overline{T} , then $\overline{T} = \{x^0, x^1, x^2, \dots, x^{n-1}\}$, and

$$\mathbb{R}\overline{T} \cong \mathbb{R}[x]/\langle x^n - 1 \rangle$$

Building rings from combinatorial objects

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For instance:

matrix rings, group rings, multivariable polynomial rings, etc ...

General path algebras

Let E be a directed graph. (We will assume E is finite for this talk, but analysis can be done in general.) $E = (E^0, E^1, r, s)$

$$s(e) \bullet \xrightarrow{e} \bullet r(e)$$

The *path algebra of E with coefficients in K* is the K -algebra KS

$S =$ the set of all directed paths in E ,

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In particular, in KE ,

for each edge e , $s(e) \cdot e = e = e \cdot r(e)$

for each vertex v , $v \cdot v = v$

$$1_{KE} = \sum_{v \in E^0} v.$$

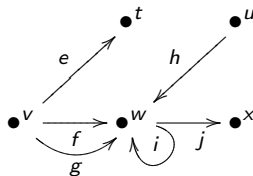
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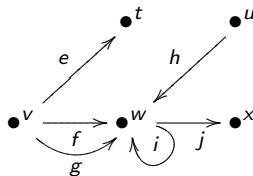
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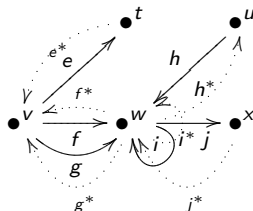
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(CK1) $e^*e = r(e)$; and $f^*e = 0$ for $f \neq e$ (for all edges e, f in E).

(CK2) $v = \sum_{\{e \in E^1 \mid s(e)=v\}} ee^*$ for each vertex v in E .

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Definition

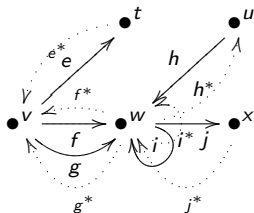
The Leavitt path algebra of E with coefficients in K

$$L_K(E) = K\widehat{E} / \langle (CK1), (CK2) \rangle$$

Leavitt path algebras: Examples

Some sample computations in $L_{\mathbb{C}}(E)$ from the Example:

$\widehat{E} =$



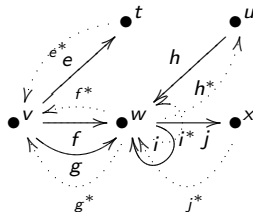
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$$h^*h = w \text{ (CK1)} \quad hh^* = u \text{ (CK2)}$$

$$ff^* = \dots \text{ (no simplification)} \quad \text{Note: } (ff^*)^2 = f(f^*f)f^* = ff^*$$

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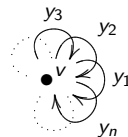
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$$E = \bullet v \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} x$$

Then $L_K(E) \cong K[x, x^{-1}]$.

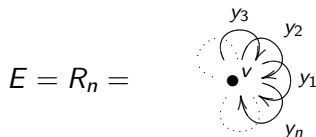
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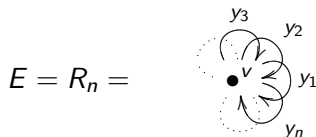


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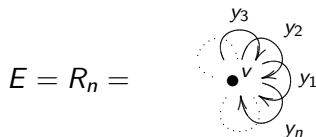
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while $L_K(R_n)$ has these SAME generators and relations, where we identify y_i^* with x_i .

Historical note, part 1

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1980's: Cuntz, Krieger, and others generalize the \mathcal{O}_n construction to directed graphs, and produce the *graph C^* -algebras* $C^*(E)$.

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June 2004: Various algebraists attend the CBMS lecture series

“Graph C^* -algebras: algebras we can see”,

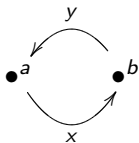
held at University of Iowa, given by Iain Raeburn.

Algebraic analogs of graph C^* -algebras are defined and investigated starting Fall 2004.

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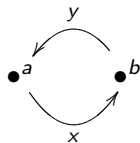
Some graph definitions

1. A *cycle*

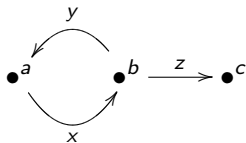


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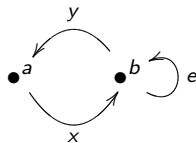
1. A *cycle*



2. An *exit* for a cycle.



or



Some graph definitions

3a. *connects to* a vertex.

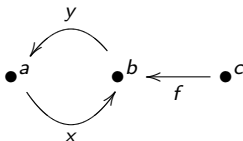
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3b. *connects to* a cycle.



Simplicity of Leavitt path algebras

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For which graphs E and fields K is $L_K(E)$ simple?

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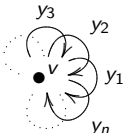
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Note $L_K(E)$ is simple for

$$E = \bullet \longrightarrow \bullet \cdots \cdots \longrightarrow \bullet \quad \text{since } L_K(E) \cong M_n(K)$$

and for

and for $E = R_n =$  $\text{ since } L_K(E) \cong L_K(1, n)$

but not simple for

$$E = R_1 = \bullet \overset{v}{\curvearrowright} x \quad \text{since } L_K(E) \cong K[x, x^{-1}]$$

Simplicity of Leavitt path algebras

Theorem

(A -, Aranda Pino, 2005) $L_K(E)$ is simple if and only if:

- 1 Every vertex connects to every cycle and to every sink in E ,
and
- 2 Every cycle in E has an exit.

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- 2 Every cycle in E has an exit.

Note: No role played by K .

Other ring-theoretic properties of Leavitt path algebras

We know precisely the graphs E for which $L_K(E)$ has various other properties, e.g.:

- 1 one-sided chain conditions
- 2 prime
- 3 von Neumann regular
- 4 two-sided chain conditions
- 5 primitive

Many more.

- 1 Leavitt path algebras: Introduction and Motivation
- 2 Algebraic properties
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The monoid $\mathcal{V}(R)$

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since $Rf \oplus R(1 - f) = R^1$.

So, for example, in $R = M_2(\mathbb{R})$, $P = M_2(\mathbb{R})e_{1,1} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$
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So $L_K(E)$ contains projective modules of the form $L_K(E)ee^*$ for
each edge e of E .

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Remark: Given a ring R , it is in general not easy to compute $\mathcal{V}(R)$.

The monoid M_E

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Associate to E the abelian monoid $(M_E, +)$:

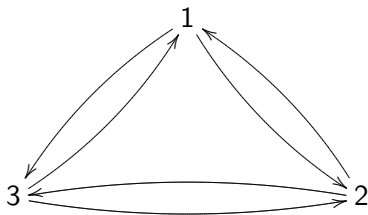
$$M_E = \left\{ \sum_{v \in E^0} n_v a_v \right\}$$

with $n_v \in \mathbb{Z}^+$ for all $v \in E^0$.

Relations in M_E are given by: $a_v = \sum_{e \in s^{-1}(v)} a_r(e)$.

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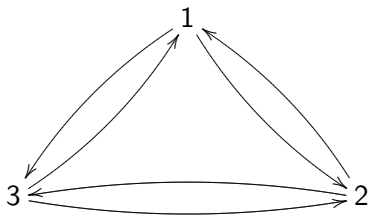
Example. Let F be the graph



So M_F consists of elements $\{n_1 a_1 + n_2 a_2 + n_3 a_3\}$ ($n_i \in \mathbb{Z}^+$),
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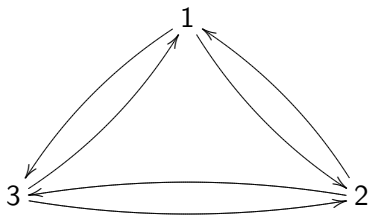


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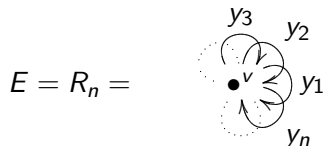
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It's not hard to get: $M_F = \{0, a_1, a_2, a_3, a_1 + a_2 + a_3\}$.

In particular, $M_F \setminus \{0\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

The monoid $\mathcal{V}(L_K(E))$

Example:



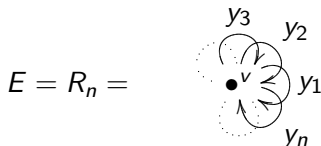
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subject to the relation: $a_v = n a_v$

So here, $M_E = \{0, a_v, 2a_v, \dots, (n-1)a_v\}$.

In particular, $M_E \setminus \{0\} \cong \mathbb{Z}_{n-1}$.

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Theorem

(P. Ara, M.A. Moreno, E. Pardo, 2007)

For any row-finite directed graph E ,

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Moreover, $L_K(E)$ is universal with this property.

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Moreover, $L_K(E)$ is universal with this property.

One (very nontrivial) consequence: Let S denote $L_K(1, n)$. Then

$$\mathcal{V}(S) = \{0, S, S^2, \dots, S^{n-1}\}.$$

Historical Note, Part 2

So we can think of Leavitt path algebras in two ways:

- 1) the “quotient of a path algebra” approach, and
- 2) the “universal algebra which supports M_E as its \mathcal{V} -monoid” approach.

These were developed in parallel.

The two approaches together have complemented each other in the development of the subject.

Purely infinite simplicity

Here's a property (most likely unfamiliar to most of you ...)

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We call a unital simple ring R *purely infinite simple* if R is not a division ring, and for every $r \neq 0$ in R there exists α, β in R for which

$$\alpha r \beta = 1_R.$$

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Moreover, in this situation, we can easily calculate $\mathcal{V}(L_K(E))$ using the Smith normal form of the matrix $I - A_E$.

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Eight Year Update

In addition to “expected” types of results, over the past eight years Leavitt path algebras have played an interesting / important role in resolving various questions outside the subject per se.

- 1 Kaplansky’s question on prime non-primitive von Neumann regular algebras.
- 2 The realization question for von Neumann regular rings.
- 3 Constructing simple Lie algebras.
- 4 Connections to various C^* -algebras.
- 5 Constructing algebras with prescribed sets of prime / primitive ideals

Matrices over Leavitt algebras

One such connection:

Let $R = L_{\mathbb{C}}(1, n)$. So ${}_R R \cong {}_R R^n$.

So this gives in particular $R \cong M_n(R)$ as rings.

Which then (for free) gives some additional isomorphisms, e.g.

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Also, ${}_R R \cong {}_R R^n \cong {}_R R^{2n-1} \cong {}_R R^{3n-2} \cong \dots$, which also in turn yield ring isomorphisms

$$R \cong M_n(R) \cong M_{2n-1}(R) \cong M_{3n-2}(R) \cong \dots$$

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For instance, if $R = L(1, 4)$, then it's not hard to show that $R \cong M_2(R)$ as rings (even though $R \not\cong_R R^2$ as modules).

Idea: 2 and 4 are nicely related, so these eight matrices inside $M_2(L(1, 4))$ “work”:

$$X_1 = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} x_3 & 0 \\ x_4 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix}, X_4 = \begin{pmatrix} 0 & x_3 \\ 0 & x_4 \end{pmatrix}$$

together with their duals

$$Y_1 = \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix}, Y_2 = \begin{pmatrix} y_3 & y_4 \\ 0 & 0 \end{pmatrix}, Y_3 = \begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}, Y_4 = \begin{pmatrix} 0 & 0 \\ y_3 & y_4 \end{pmatrix}$$

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In general, using this same idea, we can show that:

if $d|n^t$ for some $t \in \mathbb{N}$, then $L(1, n) \cong M_d(L(1, n))$.

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On the other hand ...

If $R = L(1, n)$, then the “type” of R is $n - 1$. (Think: “smallest difference”). Bill Leavitt showed the following in his 1962 paper:

The type of $M_d(L(1, n))$ is $\frac{n-1}{g.c.d.(d, n-1)}$.

In particular, if $g.c.d.(d, n - 1) > 1$, then $L(1, n) \not\cong M_d(L(1, n))$.

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(Note: $d|n^t \Rightarrow g.c.d.(d, n - 1) = 1$.)



Matrices over Leavitt algebras

Smallest interesting pair: Is $L(1, 5) \cong M_3(L(1, 5))$?

We are led “naturally” to consider these five matrices (and their duals) in $M_3(L(1, 5))$:

$$\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} x_4 & 0 & 0 \\ x_5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & x_1^2 \\ 0 & 0 & x_2 x_1 \\ 0 & 0 & x_3 x_1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & x_4 x_1 \\ 0 & 0 & x_5 x_1 \\ 0 & 0 & x_2 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & x_3 \\ 0 & 0 & x_4 \\ 0 & 0 & x_5 \end{pmatrix}$$

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Everything went along nicely... **except**, we couldn't see how to generate the matrix units $e_{1,3}$ and $e_{3,1}$ inside $M_3(L(1, 5))$ using these ten matrices.

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Instead, this set (together with duals) works:

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$$L_K(1, n) \cong M_d(L_K(1, n)) \Leftrightarrow \text{g.c.d.}(d, n-1) = 1.$$

Indeed, more generally,

$$M_d(L_K(1, n)) \cong M_{d'}(L_K(1, n')) \Leftrightarrow \\ n = n' \text{ and } \text{g.c.d.}(d, n-1) = \text{g.c.d.}(d', n-1).$$

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Along the way, some elementary (but apparently new) number theory ideas come into play.

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Here’s what made this second set of matrices work. Using this partition in the particular case $n = 5, d = 3$, then the partition of $\{1, 2, 3, 4, 5\}$ turns out to be the two sets

$$\{1, 4\} \quad \text{and} \quad \{2, 3, 5\}.$$

The matrices that “worked” are ones where we fill in the last columns with terms of the form $x_i x_1^j$ in such a way that i is in the same subset as the row number of that entry.

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The number theory underlying this partition in the general case where $\text{g.c.d.}(d, n - 1) = 1$ is elementary. But we are hoping to find some other ‘context’ in which this partition process arises.

Matrices over Leavitt algebras

Computations when $n = 5, d = 3$.

$\gcd(3, 5 - 1) = 1$. Now $5 = 1 \cdot 3 + 2$, so that $r = 2, r - 1 = 1$, and define $s = d - (r - 1) = 3 - 1 = 2$.

Consider the sequence starting at 1, and increasing by s each step, and interpret mod d ($1 \leq i \leq d$). This will necessarily give all integers between 1 and d .

Matrices over Leavitt algebras

Computations when $n = 5, d = 3$.

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Now extend these two sets mod 3 to all integers up to 5.

$$\{1, 4\} \cup \{2, 3, 5\}$$



Matrices over Leavitt algebras

Does this look familiar?

Matrices over Leavitt algebras

Corollary. (Matrices over the Cuntz C^* -algebras)

$$\mathcal{O}_n \cong M_d(\mathcal{O}_n) \Leftrightarrow g.c.d.(d, n - 1) = 1.$$

(And the isomorphisms are explicitly described.)

Matrices over Leavitt algebras

An important recent application:

For each pair of positive integers n, r , there exists an infinite, finitely presented simple group $G_{n,r}^+$.

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$$G_{n,r}^+ \cong G_{m,s}^+ \iff m = n \text{ and } \text{g.c.d.}(r, n-1) = \text{g.c.d.}(s, n-1).$$

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Proof. Show that $G_{n,r}^+$ can be realized as an appropriate subgroup of the invertible elements of $M_r(L_{\mathbb{C}}(1, n))$, and then use the explicit isomorphisms provided in the A -, Ánh, Pardo result.

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Ideas from symbolic dynamics come into play here. Using some results on flow equivalence, we have been able to show:

Theorem. (A -, Louly, Pardo, Smith, 2011) If $L_K(E)$ and $L_K(F)$ are purely infinite simple Leavitt path algebras such that

$$(K_0(L_K(E)), [1_{L_K(E)}]) \cong (K_0(L_K(F)), [1_{L_K(F)}]),$$

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then $L_K(E) \cong L_K(F)$.

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then $L_K(E) \cong L_K(F)$. Can we drop the determinant hypothesis?

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The answer will be interesting, however it plays out.

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(2) For any graph E there is an intimate relationship between $L_{\mathbb{C}}(E)$ and $C^*(E)$. There are many theorems of the form:

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but the proofs are not direct! They all are based on showing that the two properties are both equivalent to

E has graph property \mathcal{Q} .

Why this happens is still a mystery.

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(A-, Jessica Sklar), “The graph menagerie: Abstract algebra meets the Mad Veterinarian”, *Mathematics Magazine* **83**(3), 2010, 168-179.

Questions?

Thanks to:

Università degli Studi di Padova

Simons Foundation

Some elementary number theory

The partition of $\{1, 2, \dots, n\}$ induced by d when $\text{g.c.d.}(d, n - 1) = 1$

Suppose $\text{g.c.d.}(d, n - 1) = 1$. Write

$$n = dt + r \text{ with } 1 \leq r \leq d.$$

Let s denote $d - (r - 1)$.

It is easy to show that $\text{g.c.d.}(d, n - 1) = 1$ implies $\text{g.c.d.}(d, s) = 1$. We consider the sequence $\{h_i\}_{i=1}^d$ of integers, whose i^{th} entry is given by

$$h_i = 1 + (i - 1)s \pmod{d}.$$

Some elementary number theory

The integers h_i are understood to be taken from the set $\{1, 2, \dots, d\}$.

Because $\text{g.c.d.}(d, s) = 1$, basic number theory yields that the set of entries $\{h_1, h_2, \dots, h_d\}$ equals the set $\{1, 2, \dots, d\}$ (in some order). Our interest lies in a decomposition of $\{1, 2, \dots, d\}$ effected by the sequence h_1, h_2, \dots, h_d , as follows.

Some elementary number theory

We let d_1 denote the integer for which

$$h_{d_1} = r - 1$$

in the previously defined sequence. We denote by \hat{S}_1 the following subset of $\{1, 2, \dots, d\}$:

$$\hat{S}_1 = \{h_i \mid 1 \leq i \leq d_1\}.$$

We denote by \hat{S}_2 the complement of \hat{S}_1 in $\{1, 2, \dots, d\}$. We now construct a partition $S_1 \cup S_2$ of $\{1, 2, \dots, n\}$ by defining, for each $j \in \{1, 2, \dots, n\}$ and for $i \in \{1, 2\}$,

$j \in S_i$ precisely when $j \equiv j' \pmod{d}$ for $j' \in \{1, 2, \dots, d\}$, and $j' \in \hat{S}_i$.

(In other words, we extend the partition $\hat{S}_1 \cup \hat{S}_2$ of $\{1, 2, \dots, d\}$ to a partition $S_1 \cup S_2$ of $\{1, 2, \dots, n\}$ by extending mod d .)

Some elementary number theory

Example. Suppose $n = 35$, $d = 13$. Then $\gcd(13, 35 - 1) = 1$, so we are in the desired situation. Now $35 = 2 \cdot 13 + 9$, so that $r = 9$, $r - 1 = 8$, and $s = d - (r - 1) = 13 - 8 = 5$. Then we consider the sequence starting at 1, and increasing by s each step, and interpret mod d . (This will give all integers between 1 and d .)

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Now break this set into two pieces: those integers up to and including $r - 1$, and those after. Since $r - 1 = 8$, here we get

$$\{1, 2, \dots, 13\} = \{1, 3, 6, 8, 11\} \cup \{2, 4, 5, 7, 9, 10, 12, 13\}.$$

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Now extend these two sets mod 13 to all integers up to 35.

$$\{1, 3, 6, 8, 11, 14, 16, 19, 21, 24, 27, 29, 32, 34\} \cup \\ \{2, 4, 5, 7, 9, 10, 12, 13, 15, 17, 18, 20, 22, 23, 25, 26, 28, 30, 31, 33, 35\}$$

