Fibonacci's rabbits visit the Mad Veterinarian

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Overview

Something familiar: The Fibonacci Sequence $F(n)$

Fibonacci's Rabbit Puzzle: (from *Liber Abaci*, 1202)

Suppose you go to an uninhabited island with a pair of newborn rabbits (one male and one female), who:

1. mature at the age of one month,
2. have two offspring (one male and one female) each month after that, and
3. live forever.

Each pair of rabbits mature in one month and then produce a pair of newborns at the beginning of every following month. How many pairs of rabbits will there be in a year?

Something familiar: The Fibonacci Sequence $F(n)$

start month n	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$

Something familiar: The Fibonacci Sequence $F(n)$

start month n	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$F(n)$	1	1	2	3	5	8	13	21	34	55	89	144	$\cdots$

There is a "generating formula" for the Fibonacci sequence:

$$F(1) = 1; \quad F(2) = 1; \quad F(n) = F(n-1) + F(n-2) \text{ for all } n \geq 3.$$

Something familiar: The Fibonacci Sequence $F(n)$

The Fibonacci sequence comes up in lots of places ...

AND is VERY well-studied!

For instance,

Theorem: $\mathrm{g.c.d.}(F(n), F(m)) = F(\mathrm{g.c.d.}(n, m))$.

Corollary: $\mathrm{g.c.d.}(F(n), F(n - 1)) = 1$.

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A site for all types of info about the Fibonacci sequence (more than 300 formulas):

Google: Ron Knott Fibonacci

Something not-as-familiar: Mad Vet Puzzles

Mad Vet Bob's Mad Vet Puzzle: (from *The Internet*, 1998)

A mad veterinarian has created three animal transmogrifying machines.

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A mad veterinarian has created three animal transmogrifying machines.

Place a cat in the input bin of the first machine, press the button, and *whirr... bing!* Open the output bins to find two dogs and five mice.

The second machine can convert a dog into three cats and three mice.

The third machine can convert a mouse into a cat and a dog.

Each machine can also operate in reverse (e.g. if you've got two dogs and five mice, you can convert them into a cat).

Something not-as-familiar: Mad Vet Puzzles

You have one cat.

Something not-as-familiar: Mad Vet Puzzles

You have one cat.

1. Can you convert it into seven mice?
2. Can you convert it into a pack of dogs, with no mice or cats left over?

Something not-as-familiar: Mad Vet Puzzles

A site for more info about Mad Vet Puzzles (The 'Mad Bob' site):

Google: Mad Bob's Mad Vet Puzzles

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Mad Vet Puzzles are NOT as well-studied as the Fibonacci Puzzle.

But (surprisingly?) there is a nice connection between them !

Mad Vet scenarios

A *Mad Vet scenario* is a situation such as the one Mad Bob constructed.

We assume:

1. Each species is paired up with a machine;
2. Each machine can also operate in reverse; and
3. Each machine is "one to some"

Mat Vet Scenario #1

Scenario #1. Suppose a Mad Veterinarian has three machines with the following properties.

Machine 1 turns one Ant into one Beaver;

Machine 2 turns one Beaver into one Ant, one Beaver and one Cougar;

Machine 3 turns one Cougar into one Ant and one Beaver.

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$$A \Rightarrow B$$

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$$A \Rightarrow B \Rightarrow A, B, C \Rightarrow 2A, 2B \Rightarrow 3A, B \Rightarrow 4A$$

From Mad Vet Scenarios to graphs

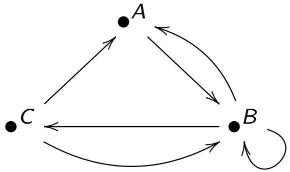
Given any Mad Vet scenario, its corresponding *Mad Vet graph* is:

a drawing ("directed graph"),

consisting of points and arrows ("vertices" and "edges"),

which gives the info about what's going on with the machines.
(We only draw the "forward" direction of the machines.)

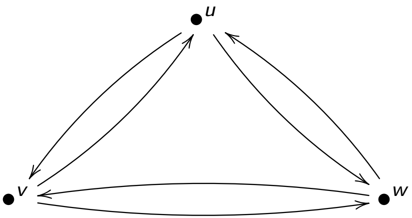
Example. Mad Vet scenario #1 has the following Mad Vet graph.



Recall: Machine 1: A $\to$ B, Machine 2: B $\to$ A, B, C, Machine 3: C $\to$ A,B

... and vice versa: from graphs to Mad Vet Scenarios

Example: Consider this directed graph:



This graph would describe a Mad Vet Scenario with three species: Urchins, Vermin, Warthogs

Machine 1: Urchin $\rightarrow$ Vermin, Warthog
Machine 2: Vermin $\rightarrow$ Urchin, Warthog
Machine 3: Warthog $\rightarrow$ Urchin, Vermin

Mad Vet equivalence

Some notation: Let's say there are *n* different species. Choose some "order" to list them.

E.g., in Scenario #1, first list Ants, then Beavers, then Cougars.

Then any collection of animals corresponds to some *n*-vector, with entries taken from the set $\{0, 1, 2, \ldots\}$.

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For instance, in Scenario #1 a collection of two Beavers and five Cougars would correspond to $(0, 2, 5)$.

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For instance, in Scenario #1 a collection of two Beavers and five Cougars would correspond to $(0, 2, 5)$.

We agree that an "empty" collection of animals is not of interest here. In other words, the vector $(0, 0, \ldots, 0)$ is not allowed.

Mad Vet equivalence

There is a naturally arising relation $\sim$ on these vectors:

Given $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$, we write

$$a \sim b$$

if there is a sequence of Mad Vet machine moves that will change the collection of animals associated with vector a into the collection of animals associated with vector b.

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(Aside: Using the three properties of a Mad Vet scenario, it is straightforward to show that $\sim$ is an equivalence relation.)

Mad Vet equivalence

Example. Suppose that our Mad Vet of Scenario #1 starts with one Ant; in other words, with $(1, 0, 0)$.

(Recall: Machine 1: A $\rightarrow$ B Machine 2: B $\rightarrow$ A, B, C Machine 3: C $\rightarrow$ A,B)

Then, rewriting in this new notation what we've already seen,

$$(1, 0, 0) \sim (0, 1, 0) \sim (1, 1, 1) \sim (2, 2, 0) \sim (3, 1, 0) \sim (4, 0, 0).$$

As a result, $(1, 0, 0) \sim (4, 0, 0)$. And $(4, 0, 0) \sim (1, 0, 0)$ too ...

Mad Vet equivalence

General math idea: There are many situations where different mathematical symbols stand for the same quantity:

Fractions: $\frac{3}{6}$ means the same as $\frac{1}{2}$, ...

Clock arithmetic: 3 $(mod 12)$ means the same as 15 $(mod 12)$, ...

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In the context of Mad Vet Puzzles, we agree that different vectors stand for the SAME collection (of animals) if we can get from one of the vectors to the other by some sequence of machines.

E.g., we agree that $(1, 0, 0)$ stands for the same collection as $(4, 0, 0)$ in Scenario $\#1$.

(and as $(0, 1, 0)$, and as $(1, 1, 1)$, and as $(2, 2, 0)$, and as $(3, 1, 0)$, ...)

Mad Vet equivalence

Some notation: We put a vector in brackets to denote the set of all the vectors which are the SAME as the given one in the Mad Vet's office.

So for Scenario #1 we could write

$$[(1,0,0)] = [(0,1,0)] = [(1,1,1)] = [(2,2,0)] = [(3,1,0)] = [(4,0,0)] = \cdots .$$

Also, we could write, for example

$$[(2,0,0)] = [(1,1,0)] = [(2,1,1)] = \cdots .$$

Mad Vet equivalence

(Recall: Machine 1: A $\rightarrow$ B Machine 2: B $\rightarrow$ A, B, C Machine 3: C $\rightarrow$ A,B)

Claim. In Scenario #1, there are exactly three different "bracket vectors" of animals:

$$\{\ [(1,0,0)],\ [(2,0,0)],\ [(3,0,0)]\ \}.$$

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Reason. It's not hard to see that any $[(a,b,c)]$ is equivalent to one of $[(1,0,0)]$, $[(2,0,0)]$, or $[(3,0,0)]$.

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Showing that these three brackets are different takes some (straightforward) work; let's not do that today. (This question is similar to the Mad Bob question!)

Semigroups, monoids, and groups

Reminder / review of notation.

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   e.g., for any positive integer n, the "clock arithmetic group" $\mathbb{Z}_n = \{1, 2, \ldots, n\}$, under addition $\mathrm{mod}\ n$.

   e.g., for any positive integers m, n, the "direct product" (i.e., ordered pairs) $\mathbb{Z}_m \times \mathbb{Z}_n$.

Mad Vet semigroups

Start with a Mad Vet scenario. Define an addition process on bracket vectors:

$$[x] + [y] = [x + y].$$

Interpret as "unions" of collections of animals.

Example. (Scenario #1: M 1: A $\rightarrow$ B M 2: B $\rightarrow$ A, B, C M 3: C $\rightarrow$ A,B)

The bracket vectors are $\{[(1,0,0)], [(2,0,0)], [(3,0,0)]\}$.

We get, for instance,

$$[(1,0,0)] + [(1,0,0)] = [(1+1,0,0)] = [(2,0,0)],$$

as we'd expect.

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as we'd expect. But also

$$[(1,0,0)] + [(3,0,0)] = [(4,0,0)] = [(1,0,0)].$$

So $[(3,0,0)]$ behaves like an identity element w/resp to $[(1,0,0)]$

Mad Vet semigroups

Similarly

$$[(2,0,0)]+[(3,0,0)] = [(2,0,0)], \text{ and } [(3,0,0)]+[(3,0,0)] = [(3,0,0)].$$

Mad Vet semigroups

Similarly

$$[(2,0,0)]+[(3,0,0)] = [(2,0,0)], \text{ and } [(3,0,0)]+[(3,0,0)] = [(3,0,0)].$$

So for this Mad Vet scenario the Mad Vet semigroup is a monoid, with identity $[(3,0,0)]$.

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Actually, since $[(1,0,0)] + [(2,0,0)] = [(3,0,0)]$, each of the three elements has an inverse.

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Notation remark:

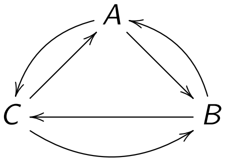
Sometimes we write $[A]$ for $[(1,0,0)]$, $[B]$ for $[(0,1,0)]$, etc ...

So, e.g., the three bracket vectors in Scenario #1 consist of the set

$$\{[A], 2[A], 3[A]\}.$$

Another Mad Vet Scenario (Scenario #2)

Here's a Mad Vet with a different set of machines:



So: Machine 1: A $\rightarrow$ B,C Machine 2: B $\rightarrow$ A, C Machine 3: C $\rightarrow$ A,B)

What are the Mad Vet bracket vectors here?

$$\{\ [(1,0,0)],\ [(0,1,0)],\ [(0,0,1)],\ [(1,1,1)]\ \}$$

Or, in the other notation, $\{\ [A],\ [B],\ [C],\ [A]+[B]+[C]\ \}$.

Turns out: these also form a group, $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Mad Vet semigroups

There are Mad Vet Scenarios where the bracket vectors for that scenario do NOT form a group. For instance, the Mad Vet Scenario for this graph.

$$\begin{tikzcd} & A \arrow[dr] & \\ C \arrow[ur] & & B \arrow[ll] \end{tikzcd}$$

Here the bracket vectors behave like the set $\mathbb{N} = \{1, 2, 3, ...\}$.

Mad Vet groups

A Big Question:

Given a Mad Vet scenario,
when does the set of bracket vectors form a group?

Mad Vet groups

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A Big Answer: Look at the Mad Vet graph, call it Γ.

- If Γ contains at least one cycle, and
- if you can walk from every vertex in Γ to every cycle in Γ by a sequence of edges, and
- if Γ isn't just a 'basic cycle',

then the bracket vectors form a group.

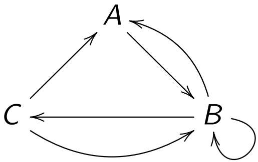
And vice versa.

"Mad Vet group" of Γ $\quad$ $\mathrm{M.V.G.}(\Gamma)$.

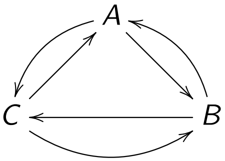
Mad Vet groups

Recall the Mad Vet graphs of Scenarios #1 and #2

$\Gamma_1 =$



$\Gamma_2 =$



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Another Big Question:

When the graph has the right properties so that the bracket vectors form a group, what group is it ????

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Another Big Answer:

For today, suffice it to say that if you are given some specific graph Γ, then it is "easy" to write code (e.g., in *Mathematica*) which will tell you $\mathrm{M.V.G.}(\Gamma)$. (Matrix computations.)

The "basic" cyclic graphs C_n

With the previous stuff as context, here's a game we can play.

Take a collection of "similar" graphs for which, for each of the graphs, the corresponding Mad Vet bracket vectors form a group.

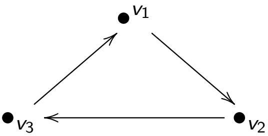
Here's one way we might build these. For each $n \geq 1$, let C_n be the "cycle" graph having n vertices $\{v_1, v_2, \ldots, v_n\}$, and n edges, like this:

The "basic" cyclic graphs C_n

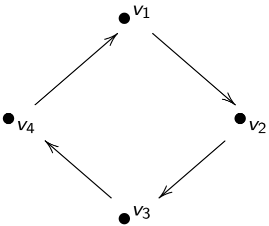
$C_1 =$

$C_2 =$

$C_3 =$



$C_4 =$



The bracket vectors for the basic cyclic graphs C_n aren't so nice in this context, because they don't form a group.

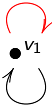
But if we modify the C_n graphs in appropriate ways, then we get graphs whose bracket vectors do form groups.

How can we do that?

We can add an extra edge at each vertex, in a systematic way.

The graphs C_n^0:

$C_1^0 =$



$C_2^0 =$

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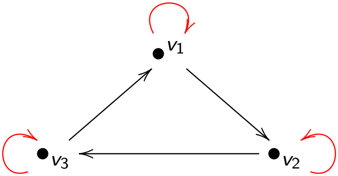

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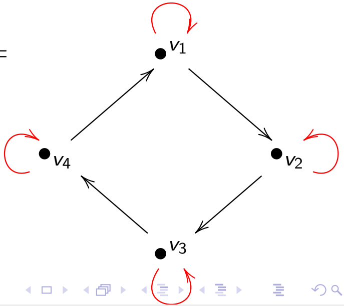

For each n, $\mathrm{M.V.G.}(C_n^0)$ contains just one element.

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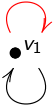
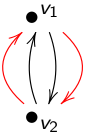
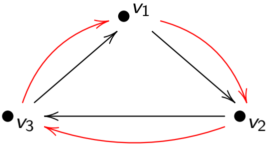
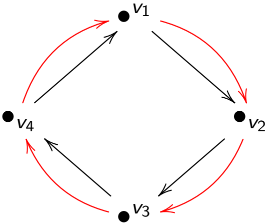
Not too interesting.

The graphs C_n^1:

$C_1^1 =$ 

$C_2^1 =$ 

$C_3^1 =$ 

$C_4^1 =$ 

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i.e.,

$$\mathrm{M.V.G.}(C_n^1) \cong \mathbb{Z}_{2^n - 1}.$$

The graphs C_n^{-1}:

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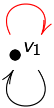

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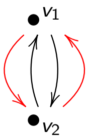

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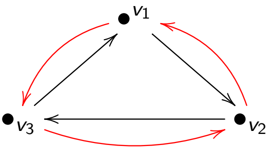

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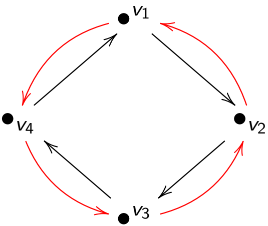

The Mad Vet Group of C_n^{-1} for various values of n.

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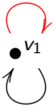
We can prove a **Theorem** which says:

This pattern in the Mad Vet Groups for C_n^{-1} goes on forever.

The graphs C_n^2:

$C_1^2 =$



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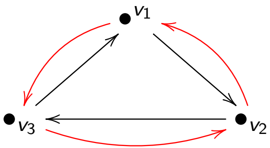
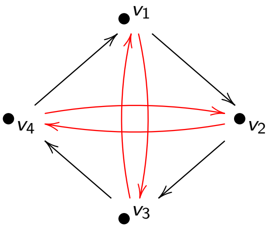
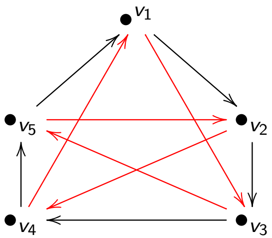


$C_4^2 =$



Here are two more graphs in this sequence ...

$C_5^2 =$



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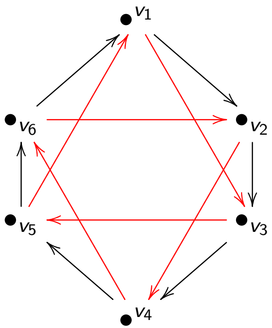

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n	13	14	15	16	17	18
$\lvert\text{M.V.G.}(C_n^2)\rvert$	521	841	1364	2205	3571	5776

n	19	20	21	22	23	24
$\lvert\text{M.V.G.}(C_n^2)\rvert$	9349	15125	24476	39601	64079	103680

Let's do some sample computations in, say, $\mathrm{M.V.G.}(C_6^2)$.

$$\begin{aligned}
[v_1] &= [v_2]+[v_3] \\
&= ([v_3]+[v_4])+[v_3] = 2[v_3]+[v_4] \\
&= 2([v_4]+[v_5])+[v_4] = 3[v_4]+2[v_5] \\
&= 3([v_5]+[v_6])+2[v_5] = 5[v_5]+3[v_6] \\
&= 5([v_6]+[v_1])+3[v_6] = 8[v_6]+5[v_1]
\end{aligned}$$

So, in $\mathrm{M.V.G.}(C_6^2)$, we get

$$[v_1] = 8[v_6] + 5[v_1].$$

This gives

$$8[v_6] = -4[v_1].$$

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So, here, we have

$$F(6)[v_6] = -(F(5) - 1)[v_1].$$

A general conclusion, and Connection #1:

Repeating ... in $\mathrm{M.V.G.}(C_6^2)$, $F(6)[v_6] = -(F(5) - 1)[v_1]$.

More generally:

Let n be any positive integer. Then, in $\mathrm{M.V.G.}(C_n^2)$,

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$$F(n)[v_n] = \ -\ (F(n{-}1) - 1)[v_1].$$

So Fibonacci's rabbits and the Mad Veterinarian are connected !!

Connection #2:

Notation: Denote the size of $\mathrm{M.V.G}(C_n^2)$ by $H_2(n)$.

Recall the sizes of the Mad Vet groups of the C_n^2 graphs $(1 \le n \le 12)$:

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Proposition: For all $n \geq 3$,

$$H_2(n) = \begin{cases} H_2(n-1) + H_2(n-2) & \text{if } \ n \text{ is even} \\ H_2(n-1) + H_2(n-2) + 2 & \text{if } \ n \text{ is odd} \end{cases}$$

So the H_2 sequence is "Fibonacci-ish".

Digression

Digression: The Online Encyclopedia of Integer Sequences

google: OEIS

Digression

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A note on Fermat's Last Theorem and the Mersenne Numbers,

in: Eureka: the Archimedeans' Journal, vol. 11, 1949, pp 19-22.

Connection #3

Now let's look at the Fibonacci sequence and the H_2 sequence side-by-side:

n	1	2	3	4	5	6	7	8	9	10	11	12
$F(n)$	1	1	2	3	5	8	13	21	34	55	89	144
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Proposition: For all $n \geq 2$,

$$H_2(n) = \begin{cases} F(n-1) + F(n+1) - 2 & \text{if } n \text{ is even} \\ F(n-1) + F(n+1) & \text{if } n \text{ is odd} \end{cases}$$

Connection #4

Along the way, the following numbers turn out to be of great interest. For each $n \geq 2$, define

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Proposition: For any positive integer m,

$$d(2m+1) = \begin{cases} 1 & \text{if } \ 2m+1 \equiv 1 \text{ or } 5 \mod 6 \\ 2 & \text{if } \ 2m+1 \equiv 3 \mod 6 \end{cases}$$

$$d(2m+2) = \begin{cases} F(m) + F(m+2) & \text{if } \ m \text{ is even} \\ F(m+1) & \text{if } \ m \text{ is odd} \end{cases}$$

So we have an explicit formula for $d(n)$ for all integers $n \geq 1$.

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In particular, $\mathrm{M.V.G.}(C_n^2)$ is cyclic precisely when $d(n) = 1$,

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so precisely when $d = 2$, or $d = 4$, or $d \equiv 1$ or $5 \bmod 6$.

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In particular, knowing the structure of $\mathrm{M.V.G.}(C_n^2)$ gives really nice information about $L_K(C_n^2)$.

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Can we describe $\mathrm{M.V.G.}(C_n^3)$??

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Can we describe $\mathrm{M.V.G.}(C_n^{-2})$??

Questions?

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Thank you.

Thanks also to the Simons Foundation.