Leavitt path algebras:
an introduction, and applications

Gene Abrams

University of Colorado
Colorado Springs

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Overview

1. Introduction and Motivation
2. Algebraic properties
3. Projective modules
4. Connections and Applications
1 Introduction and Motivation

2 Algebraic properties

3 Projective modules

4 Connections and Applications
There are “naturally occurring” algebras for which

\[ R R^i \cong R R^j \]  
for some \( i \neq j \),

\[ \text{e.g., } \text{End}_K(K^{(\mathbb{N})}) \]
There are “naturally occurring” algebras for which

\[ R^i \cong R^j \text{ for some } i \neq j, \]
e.g., \( \text{End}_K(K^{(\mathbb{N})}) \)  \( \text{(Here } R^i \cong R^j \text{ for all } i \neq j.) \)
Brief history

There are “naturally occurring” algebras for which \( R R^i \cong R R^j \) for some \( i \neq j \),
e.g., \( \text{End}_K(K^{(\mathbb{N})}) \) (Here \( R R^i \cong R R^j \) for all \( i \neq j \).)

Does there exist \( R \) with, e.g., \( R R^1 \cong R R^3 \) but \( R R^1 \not\cong R R^2 \)?
Leavitt algebras

**Theorem**

*(Bill Leavitt, Trans. A.M.S., 1962)*

For every \( m, n \in \mathbb{N} \) and field \( K \) there exists \( R = L_K(m, n) \) with \( _R R^m \cong _R R^n \), and all isomorphisms between free left \( R \)-modules result precisely from this one.
Leavitt algebras

The $m = 1$ case of Leavitt’s Theorem is not too surprising:

$\mathcal{R} \mathcal{R}^1 \cong \mathcal{R} \mathcal{R}^n$ if and only if there exist

\[
\begin{pmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{pmatrix} \in \mathcal{R}^n,
\]

for which

\[
\begin{pmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{pmatrix} = 1 \quad \text{and} \quad
\begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{pmatrix}
\begin{pmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n
\end{pmatrix} = \mathcal{I}_n
\]
Leavitt algebras

Then $L_K(1, n)$ is the quotient

$$K < X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n > \ /
\langle \sum_{i=1}^{n} y_i x_i \rangle - 1_K; x_i y_j - \delta_{i,j} 1_K \rangle$$
Turns out:

**Theorem**

*(Leavitt, Duke J. Math, 1964)*

For every field $K$ and $n \geq 2$, $L_K(1, n)$ is simple.

*(On the other hand, for $m \geq 2$, $L_K(m, n)$ is not simple.)*
General path algebras

Let $E$ be a directed graph. $E = (E^0, E^1, r, s)$

\[ s(e) \cdot e = e = e \cdot r(e). \]

The path algebra $KE$ is the $K$-algebra with basis $\{p_i\}$ consisting of the directed paths in $E$. (View vertices as paths of length 0.)

Note: $E^0$ finite $\iff$ $KE$ is unital; then $1_{KE} = \sum_{v \in E^0} v$. 
Building Leavitt path algebras

Start with $E$, build its *double graph* $\hat{E}$. 
Building Leavitt path algebras

Start with $E$, build its *double graph* $\hat{E}$. Example:

\[ E = \]

\[ \begin{array}{c}
\bullet v \\
\downarrow f \\
\bullet w \\
\downarrow g \\
\bullet x \end{array} \quad \begin{array}{c}
\bullet t \\
\uparrow e \\
\bullet w \end{array} \quad \begin{array}{c}
\bullet u \\
\uparrow h \\
\bullet x \end{array} \quad \begin{array}{c}
\bullet j \\
\uparrow i \\
\bullet x \end{array} \]

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University of Colorado

Leavitt path algebras: an introduction, and applications
Building Leavitt path algebras

Start with $E$, build its double graph $\hat{E}$. Example:

$$E = \begin{array}{cccc}
\bullet & v & \rightarrow & w \\
& e & \rightarrow & f & \rightarrow & g & \rightarrow & w \\
& h & \rightarrow & i & \rightarrow & j & \rightarrow & x \\
\end{array}$$

$$\hat{E} = \begin{array}{cccc}
\bullet & v & \rightarrow & w & \rightarrow & x \\
& e^* & \rightarrow & f^* & \rightarrow & g^* & \rightarrow & w & \rightarrow & x \\
& h^* & \rightarrow & i^* & \rightarrow & j^* & \rightarrow & x \\
\end{array}$$
Building Leavitt path algebras

Construct the path algebra $K\hat{E}$.
Building Leavitt path algebras

Construct the path algebra $K \hat{E}$. Consider these relations in $K \hat{E}$:

$(CK_1)$ \[ e^* e' = \delta_e, \quad e' r(e) \] for all $e, e' \in E_1$.

$(CK_2)$ \[ v = \sum \{ e \in E_1 | s(e) = v \} \] for all $v \in E_0$ (just at those vertices $v$ which are not sinks, and which emit only finitely many edges).

Definition

The Leavitt path algebra of $E$ with coefficients in $K$ is

$$L_K(E) = K \hat{E} / \langle (CK_1), (CK_2) \rangle$$
Building Leavitt path algebras

Construct the path algebra $K\hat{E}$. Consider these relations in $K\hat{E}$:

(CK1) $e^*e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$.

(CK2) $v = \sum \{e \in E^1 | s(e) = v\} ee^*$ for all $v \in E^0$
Building Leavitt path algebras

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Definition

The Leavitt path algebra of $E$ with coefficients in $K$

$$L_K(E) = K\widehat{E} / <(CK1), (CK2)>$$
Leavitt path algebras: Examples

Some sample computations in $L_{\mathbb{C}}(E)$ from the Example:

$\hat{E} =$

$ee^* + ff^* + gg^* = v \quad g^*g = w \quad g^*f = 0$

$h^*h = w \quad hh^* = u \quad ff^* = \ldots \quad \text{(no simplification)}$

But $\ldots \quad (ff^*)^2 = f(f^*f)f^* = f \cdot w \cdot f^* = ff^*$
Leavitt path algebras: Examples

Standard algebras arising as Leavitt path algebras:
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\[ E = \bullet v_1 \xrightarrow{e_1} \bullet v_2 \xrightarrow{e_2} \bullet v_3 \ldots \xrightarrow{e_{n-1}} \bullet v_n \]

Then \( L_K(E) \cong M_n(K) \).
Leavitt path algebras: Examples

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Then \( L_K(E) \cong M_n(K) \).

\[ E = \bullet v \xrightarrow{\sim} x \]

Then \( L_K(E) \cong K[x, x^{-1}] \).
Leavitt path algebras: Examples

Then $L_K(E) \cong L_K(1, n)$.
Leavitt path algebras: Examples

\[ E = R_n = \]

Then \( L_K(E) \cong L_K(1, n) \).

The connection is clear, denote \( y_i^* \) by \( x_i \).
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4. Connections and Applications
Some graph theory

Graph theory:

1. A cycle

\[ a \rightarrow \rightarrow b \]

\[ x \leftarrow \leftarrow y \]
Some graph theory

Graph theory:

1. A cycle

![Diagram of a cycle with vertices labeled a and b and arrows indicating the cycle]

2. An exit for a cycle.

![Diagram of a cycle with an exit labeled with arrows indicating the cycle and an additional vertex labeled c]

or

![Diagram of a cycle with an exit labeled with arrows indicating the cycle and an additional vertex labeled e]
Some graph theory

3a. connects to a vertex.

\[ u \rightarrow v \rightarrow w \quad ( \text{also} \quad w ) \]
Some graph theory

3a. connects to a vertex.

\[ u \rightarrow v \rightarrow w \]  (also \( w \))

3b. connects to a cycle.

\[ \begin{array}{c}
\bullet a \\
\bigcirc
\end{array} \quad \begin{array}{c}
\bigcirc \\
y \quad \downarrow \\
\bullet b \quad \leftarrow \\
\bigcirc \\
\downarrow \\
\bullet c \\
\bigcirc \\
\leftarrow f \\
\downarrow \\
h \quad \bigcirc
\end{array} \]
Simplicity of Leavitt path algebras

Question: For which graphs $E$ and fields $K$ is $L_K(E)$ simple?
Simplicity of Leavitt path algebras

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Note $L_K(E)$ is simple for

$$E = \bullet \rightarrow \bullet \rightarrow \rightarrow \bullet \quad \text{since} \quad L_K(E) \cong M_n(K)$$

and for

$$E = R_n = \bullet \overset{y_1}{\leftarrow} \overset{y_2}{\leftarrow} \overset{y_3}{\leftarrow} \cdots \quad \text{since} \quad L_K(E) \cong L_K(1, n)$$

but not simple for

$$E = R_1 = \bullet \overset{v}{\leftarrow} \quad \text{since} \quad L_K(E) \cong K[x, x^{-1}]$$
Simplicity of Leavitt path algebras

Theorem

(A -, Aranda Pino, 2005) $L_K(E)$ is simple if and only if:

1. Every vertex connects to every cycle and to every sink in $E$, and
2. Every cycle in $E$ has an exit.

Note: No role played by $K$. 
We know precisely the graphs $E$ for which $L_K(E)$ has these properties:

1. (one-sided) artinian; (one-sided) noetherian
2. (two-sided) artinian; (two-sided) noetherian
Other ring-theoretic properties of Leavitt path algebras

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(No role is played by $K$ in any of these.)
Purely infinite simplicity

An idempotent $e \in R$ is purely infinite if $Re = Rf \oplus Rg$ with $f, g$ nonzero idempotents, and $Re \sim Rf$. Example: $e = 1$ in $R = L_K(1, 2)$, since $R \sim R \oplus R$.

$R$ is purely infinite simple in case $R$ is simple, and every nonzero left ideal contains an infinite idempotent. ($\iff R$ is not a field, and $\forall x \neq 0 \exists \alpha, \beta$ with $\alpha x \beta = 1$.)

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**Theorem:** $L_K(E)$ is purely infinite simple $\iff$ $L_K(E)$ is simple, and $E$ contains a cycle.
Purely infinite simplicity

**Theorem:** $L_K(E)$ is purely infinite simple $\iff L_K(E)$ is simple, and $E$ contains a cycle.

**Observation:** If $c$ is a cycle based at vertex $v$, and $e$ is an exit for $c$ with $s(e) = v$, then

$$L_K(E)v = L_K(E)cc^* \oplus L_K(E)(v - cc^*).$$

Easily $L_K(E)v \cong L_K(E)cc^*$, and easily both $cc^*$ and $v - cc^*$ are idempotents, and easily $cc^* \neq 0$. 
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Easily $L_K(E)v \cong L_K(E)cc^*$, and easily both $cc^*$ and $v - cc^*$ are idempotents, and easily $cc^* \neq 0$.

That $v - cc^* \neq 0$ follows since $e$ is an exit for $c$: otherwise,

$$0 = v - cc^* \Rightarrow 0 = 0e = ve - cc^*e = e - 0 = e,$$

a contradiction.
1 Introduction and Motivation

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4 Connections and Applications
The monoid $\mathcal{V}(R)$

Isomorphism classes of finitely generated projective (left) $R$-modules, with operation $\oplus$, denoted $\mathcal{V}(R)$. (Conical) monoid. $[R]$ is a ‘distinguished’ element in $\mathcal{V}(R)$. 

Example:

- $\mathcal{V}(K) \cong \mathbb{Z}^+$. Also:

  $\mathcal{V}(\text{Mat}(K)) \cong \mathbb{Z}^+$. 

Theorem (George Bergman, Trans. A.M.S. 1975)

Given a field $K$ and finitely generated conical abelian monoid with a distinguished element, there exists a universal $K$-algebra $R$ for which $\mathcal{V}(R) \cong S$. The construction is explicit, uses amalgamated products. (Fin. gen. hypothesis eliminated by Bergman / Dicks, 1978)

Bergman included the algebras $L_{K(m,n)}$ as examples of these universal algebras.
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The monoid $V(L_K(E))$

For any graph $E$ construct the free abelian monoid $M_E$.

$$v = \sum_{r(e)=w} w$$
The monoid $\mathcal{V}(L_K(E))$

For any graph $E$ construct the free abelian monoid $M_E$.

$$\text{generators } E^0 \quad \text{relations} \quad \nu = \sum_{r(e) = w} w$$

Using Bergman’s construction,

**Theorem**


For any field $K$,

$$\mathcal{V}(L_K(E)) \cong M_E.$$
The monoid $\mathcal{V}(L_K(E))$

Nontrivial (but maybe nonsurprising) consequence: Let $S$ denote $L_K(1, n)$. Then

$$\mathcal{V}(S) = \{0, S, S \oplus S, ..., \underbrace{S \oplus S \cdots \oplus S}_{n-1 \text{ copies}}\}$$
The monoid $\mathcal{V}(L_K(E))$

Nontrivial (but maybe nonsurprising) consequence:
Let $S$ denote $L_K(1, n)$. Then

$$\mathcal{V}(S) = \{0, S, S \oplus S, \ldots, \underbrace{S \oplus S \cdots \oplus S}_{n-1 \text{ copies}}\}$$

Note: $\mathcal{V}(S) \setminus \{0\} \cong \mathbb{Z}_{n-1}$.

(The 'new' identity element is $S \oplus S \cdots \oplus S$ ($n - 1$ copies).)
The monoid $\mathcal{V}(L_K(E))$

Example.
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Example.

$M_E = \{ z, A_1, A_2, A_3, A_1 + A_2 + A_3 \}$
The monoid $\mathcal{V}(L_K(E))$

Example.

$$M_E = \{z, A_1, A_2, A_3, A_1 + A_2 + A_3\}$$

$$M_E \setminus \{z\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$
The monoid $\mathcal{V}(L_K(E))$

**Theorem**

$\mathcal{V}(L_K(E)) \setminus \{0\}$ is a group if and only if $L_K(E)$ is purely infinite simple. In this case $\mathcal{V}(L_K(E)) \setminus \{0\}$ is $K_0(L_K(E))$. 

Remarks:
1. Leavitt actually showed that $L_K(1,n)$ is purely infinite simple.
2. Proof of (⇐) is by Ara / Goodearl / Pardo (holds for any purely infinite simple ring); other direction is by E. Pardo.
3. Note $\mathcal{V}(M_d(K)) \setminus \{0\} = N$, not a group.
The monoid $\mathcal{V}(L_K(E))$

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2. Proof of ($\leftarrow$) is by Ara / Goodearl / Pardo (holds for any purely infinite simple ring); other direction is by E. Pardo.
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4. Connections and Applications
Connections and Applications: C*-algebras

Theorem

(Joaquim Cuntz, Comm. Math. Physics, 1977) There exist simple C*-algebras \( \{O_n \mid n \in \mathbb{N} \} \) generated by partial isometries.

Subsequently, a similar construction was produced of a C*-algebra \( C^*(E) \), for any graph \( E \). In this context, \( O_n \cong C^*(R_n) \).

For any graph \( E \), \( L_{\mathbb{C}}(E) \subseteq C^*(E) \) as a dense \( \ast \)-subalgebra.

(But \( C^*(E) \) is usually “much bigger” than \( L_{\mathbb{C}}(E) \).)
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For any graph E, \( L_{\mathbb{C}}(E) \subseteq C^*(E) \) as a dense *-subalgebra.

(But \( C^*(E) \) is usually “much bigger” than \( L_{\mathbb{C}}(E) \).)

In particular, \( L_{\mathbb{C}}(1, n) \subseteq O_n \) for all n.
Properties of $C^*$-algebras. These typically include topological considerations.
Properties of $C^*$-algebras. These typically include topological considerations.

1. simple
2. purely infinite simple
3. stable rank, prime, primitive, exchange, etc....
Connections and Applications: C*-algebras

For many properties ...

$L_{\mathbb{C}}(E)$ has (algebraic) property $\mathcal{P} \iff C^*(E)$ has (topological) property $\mathcal{P}$. 
Connections and Applications: C*-algebras

For many properties...

\[ L_\mathbb{C}(E) \text{ has (algebraic) property } \mathcal{P} \iff C^*(E) \text{ has (topological) property } \mathcal{P}. \]

... if and only if \( L_K(E) \) has (algebraic) property \( \mathcal{P} \) for every field \( K \).
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... if and only if \( E \) has some graph-theoretic property.

Still no good understanding as to Why.

---

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Recently: Although \( \mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2 \), it turns out that \( L_{\mathbb{C}}(1, 2) \otimes L_{\mathbb{C}}(1, 2) \not\cong L_{\mathbb{C}}(1, 2) \).

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Leavitt path algebras: an introduction, and applications
Connections and Applications: C*-algebras

The Leavitt path algebra side has made contributions to the graph C*-algebra side.
Connections and Applications: C*-algebras

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Observation: Let \( R = L_\mathbb{C}(1, 4) \). So \( _R R \cong _R R^4 \).
Connections and Applications: C*-algebras

The Leavitt path algebra side has made contributions to the graph C*-algebra side.

**Observation:** Let $R = L_{\mathbb{C}}(1, 4)$. So $RR \cong R^4$. So $R \cong M_4(R)$. Are there other isomorphisms between matrix rings over $R$? Yes, for example $R \cong M_2(R)$ also. (It is not hard to find an explicit isomorphism.) But, for example, $R \not\cong M_3(R)$. (Leavitt already knew this.)
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But, for example, $R \not\cong M_3(R)$. (Leavitt already knew this.)
Connections and Applications: C*-algebras

**Question(s):**

1. For what $n, d$ do we have $L_{\mathbb{C}}(1, n) \cong M_d(L_{\mathbb{C}}(1, n))$?
2. For what $n, d$ do we have $\mathcal{O}_n \cong M_d(\mathcal{O}_n)$?

**Answer(s):**

1. (A. Ánh, Pardo, 2008) $\iff$ $\text{g.c.d.}(d, n - 1) = 1$.
   
   This implies the result of Kirchberg / Phillips, and more.

2. (Kirchberg / Phillips, 2000) $\iff$ $\text{g.c.d.}(d, n - 1) = 1$.
   
   (This uses powerful tools; and no explicit isomorphism is given.)
Connections and Applications: C*-algebras

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1. For what $n, d$ do we have $L_\mathbb{C}(1, n) \cong M_d(L_\mathbb{C}(1, n))$?
2. For what $n, d$ do we have $\mathcal{O}_n \cong M_d(\mathcal{O}_n)$?

Answer(s):

2. (Kirchberg / Phillips, 2000) ... $\iff \text{g.c.d.}(d, n-1) = 1$. (This uses powerful tools; and no explicit isomorphism is given.)

1. (A., ´Anh, Pardo, 2008) ... $\iff \text{g.c.d.}(d, n-1) = 1$. Isomorphisms are explicitly given. This implies the result of Kirchberg / Phillips, and more.
Connections and Applications: C*-algebras

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Connections and Applications: C*-algebras

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Connections and Applications: \( \textbf{C}^*\)-algebras

**Question(s):**

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Isomorphisms are explicitly given. This implies the result of
Kirchberg / Phillips, and more.
Connections and Applications: Higman - Thompson simple groups

For each pair of positive integers $n, r$, there exists an infinite, finitely presented simple group (the “Higman-Thompson group”), denoted $G_{n,r}^+$. These were introduced by G. Higman in 1974.

Higman knew some conditions regarding isomorphisms between these groups, but did not have a complete classification.
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Higman knew some conditions regarding isomorphisms between these groups, but did not have a complete classification.

**Theorem.** (E. Pardo, 2011)

$$G_{n,r}^+ \cong G_{m,s}^+ \iff m = n \text{ and } \gcd(r, n-1) = \gcd(s, n-1).$$

**Proof.** Show that $G_{n,r}^+$ can be realized as an appropriate subgroup of the invertible elements of $M_r(L_\mathbb{C}(1, n))$, and then use the explicit isomorphisms provided in the A-, Ánh, Pardo result.
Additional ‘general ring-theoretic results’ from L.p.a.’s

In addition to the previous examples, Leavitt path algebras have been used to:

1) produce a class of prime, non-primitive von Neumann regular algebras
2) produce associative algebras for which the corresponding bracket Lie algebra is simple
3) produce examples of affine algebras having specified behavior of the radical
4) realize categories of quasi-coherent sheaves as full module categories

... and more ...
The Algebraic Kirchberg Phillips Question

The Kirchberg / Phillips Theorem, when interpreted in the context of graph $C^*$-algebras, yields:

**KP Theorem for graph $C^*$-algebras:** Suppose $E$ and $F$ are finite graphs for which $C^*(E)$ and $C^*(F)$ are purely infinite simple. Suppose there is an isomorphism $\varphi : K_0(C^*(E)) \rightarrow K_0(C^*(F))$ for which $\varphi([C^*(E)]) = [C^*(F)]$. Then $C^*(E) \cong C^*(F)$ (homeomorphically).
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Notes: (1) This is an existence theorem only, and
The Algebraic Kirchberg Phillips Question

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Notes: (1) This is an existence theorem only, and (2) the fact that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ is invoked in the proof.
The Algebraic Kirchberg Phillips Question

It turns out that:

1) $K_0(L_K(E)) \cong K_0(C^*(E))$ for any finite graph $E$.

2) The $K_0$ groups are easily described in terms of the adjacency matrix $A_E$ of $E$. Let $n = |E^0|$. View $I_n - A_E^t$ as a linear transformation $\mathbb{Z}^n \rightarrow \mathbb{Z}^n$. Then

$$K_0(L_K(E)) \cong K_0(C^*(E)) \cong \text{Coker}(I_n - A_E^t).$$

Moreover, $\text{Coker}(I_n - A_E^t)$ can be computed by finding the Smith normal form of $I_n - A_E^t$. 
The Algebraic Kirchberg Phillips Question

Example:

\[
\begin{pmatrix}
1 & -1 & -1 \\
-1 & 1 & -1 \\
-1 & -1 & 1
\end{pmatrix}, \text{ whose Smith normal form is: }
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

Conclude that \( K_0(L_K(E)) \cong \text{Coker}(I_3 - A^t_E) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).
The Algebraic Kirchberg Phillips Question

The question becomes: Can information about $K_0$ be used to establish isomorphisms between Leavitt path algebras as well?

**The Algebraic KP Question**: Suppose $E$ and $F$ are finite graphs for which $L_K(E)$ and $L_K(F)$ are purely infinite simple. Suppose also that there exists an isomorphism $\varphi : K_0(L_K(E)) \to K_0(L_K(F))$ for which $\varphi([L_K(E)]) = [L_K(F)]$. 
The question becomes: Can information about $K_0$ be used to establish isomorphisms between Leavitt path algebras as well?

**The Algebraic KP Question:** Suppose $E$ and $F$ are finite graphs for which $L_K(E)$ and $L_K(F)$ are purely infinite simple. Suppose also that there exists an isomorphism $\varphi : K_0(L_K(E)) \rightarrow K_0(L_K(F))$ for which $\varphi([L_K(E)]) = [L_K(F)]$.

Is $L_K(E) \cong L_K(F)$?
Connections to symbolic dynamics

VERY informally:

Some mathematicians and computer scientists have interest in, roughly, how information “flows” through a directed graph.

Makes sense to ask: When is it the case that information flows through two different graphs in essentially the same way? “Flow equivalent graphs”.

(Of often cast in the language of matrices.)
Connections to symbolic dynamics

Example: “Expansion at v”

\[ E \quad \rightarrow \quad \bullet_v \quad \rightarrow \quad E_v \]

**Proposition**: If \( E_v \) is the expansion graph of \( E \) at \( v \), then \( E \) and \( E_v \) are flow equivalent. Rephrased, “expansion” (and its inverse “contraction”) preserve flow equivalence.
Connections to symbolic dynamics

There are four other 'graph moves' which preserve flow equivalence:

out-split (and its inverse out-amalgamation), and

in-split (and its inverse in-amalgamation).

**Theorem PS** (Parry / Sullivan): Two graphs $E$, $F$ are flow equivalent if and only if one can be gotten from the other by a sequence of transformations involving these six graph operations.
Connections to symbolic dynamics

Graph transformations may be reformulated in terms of adjacency matrices.

For an $n \times n$ matrix $M$ with integer entries, think of $M$ as a linear transformation $M : \mathbb{Z}^n \to \mathbb{Z}^n$. In particular, when $M = I_n - A_E^t$. 

Proposition (Parry / Sullivan): If $E$ is flow equivalent to $F$, then $\det(I - A_E^t) = \det(I - A_F^t)$.

Proposition (Bowen / Franks): If $E$ is flow equivalent to $F$, then $\text{Coker}(I - A_E^t) \sim \text{Coker}(I - A_F^t)$.

Theorem (Franks): Suppose $E$ and $F$ have some additional properties (irreducible, essential, nontrivial). If $\text{Coker}(I - A_E^t) \sim \text{Coker}(I - A_F^t)$ and $\det(I - A_E^t) = \det(I - A_F^t)$, then $E$ and $F$ are flow equivalent.

So by Theorem PS, if the Cokernels and determinants match up correctly, then there is a sequence of “well-understood” graph transformations which starts with $E$ and ends with $F$. 

Gene Abrams
University of Colorado
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Connections to symbolic dynamics

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Connections to symbolic dynamics

**Proposition:** $E$ is irreducible, essential, and non-trivial if and only if $E$ has no sources and $L_K(E)$ is purely infinite simple.
Connections to symbolic dynamics

**Proposition:** $E$ is irreducible, essential, and non-trivial if and only if $E$ has no sources and $L_K(E)$ is purely infinite simple.

**Theorem:** Suppose $E$ is a graph for which $L_K(E)$ is purely infinite simple. Suppose $F$ is gotten from $E$ by doing one of the six “flow equivalence” moves. Then $L_K(E)$ and $L_K(F)$ are Morita equivalent.

In addition, the “source elimination” process also preserves Morita equivalence of the Leavitt path algebras.

**Proof:** Show that an isomorphic copy of $L_K(E)$ can be viewed as a (necessarily full, by simplicity) corner of $L_K(F)$ (or vice-versa).
Connections to symbolic dynamics

But (recall) that when $L_K(E)$ is purely infinite simple, then $K_0(L_K(E)) \cong \text{Coker}(I|_{E^0} - A_E^t)$.
Connections to symbolic dynamics

But (recall) that when $L_K(E)$ is purely infinite simple, then $K_0(L_K(E)) \cong \text{Coker}(I_{|E^0|} - A_t^E)$. Consequently:

**Theorem**: (A- / Louly / Pardo / Smith 2011): Suppose $L_K(E)$ and $L_K(F)$ are purely infinite simple. If

$$K_0(L_K(E)) \cong K_0(L_K(F)) \quad \text{and} \quad \det(I - A^t_E) = \det(I - A^t_F),$$

then $L_K(E)$ and $L_K(F)$ are Morita equivalent.
Connections to symbolic dynamics

Using some intricate computations provided by Huang, one can show the following:

Suppose $L_K(E)$ is Morita equivalent to $L_K(F)$. Further, suppose there is some isomorphism $\varphi : K_0(L_K(E)) \to K_0(L_K(F))$ for which $\varphi([L_K(E)]) = [L_K(F)]$.

Then there is some Morita equivalence $\Phi : L_K(E)-\text{Mod} \to L_K(F)-\text{Mod}$ for which $\Phi|_{K_0(L_K(E))} = \varphi$. 

Gene Abrams
University of Colorado

Leavitt path algebras: an introduction, and applications
Connections to symbolic dynamics

Consequently:

**Theorem:** (A- / Louly / Pardo / Smith 2011): Suppose $L_K(E)$ and $L_K(F)$ are purely infinite simple. If

$$K_0(L_K(E)) \cong K_0(L_K(F))$$

via an isomorphism $\varphi$ for which $\varphi([L_K(E)]) = [L_K(F)]$,

and $\det(I - A^t_E) = \det(I - A^t_F)$,

then $L_K(E) \cong L_K(F)$.

‘Restricted’ Algebraic KP Theorem
Connections to symbolic dynamics

**Algebraic KP Question**: Can we drop the hypothesis on the determinants in the Restricted Algebraic KP Theorem?
Connections to symbolic dynamics

Here’s the “smallest” example of a situation of interest. Consider the Leavitt path algebras $L(R_2)$ and $L(E_4)$, where

$$R_2 = \overset{v}{\bullet} \quad \text{and} \quad E_4 = \overset{v_1}{\bullet} \overset{v_2}{\bullet} \overset{v_3}{\bullet} \overset{v_4}{\bullet}$$

It is not hard to establish that

$$(K_0(L(R_2)), [1_{L(R_2)}]) = (\{0\}, 0) = (K_0(L(E_4)), [1_{L(E_4)}]);$$

$$\det(I - A^t_{R_2}) = -1; \quad \text{and} \quad \det(I - A^t_{E_4}) = 1.$$
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$$\det(I - A^t_{R_2}) = -1; \quad \text{and} \quad \det(I - A^t_{E_4}) = 1.$$ 

Question: Is $L_K(R_2) \cong L_K(E_4)$?
Connections to symbolic dynamics

Some concluding remarks:

1. $C^*(R_2) \cong C^*(E_4)$; but the isomorphism is NOT given explicitly, its existence is ensured by “KK Theory”.
Connections to symbolic dynamics

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2. There have been a number of approaches in the attempt to answer the Algebraic KP Question: e.g., consider graded isomorphisms; restrict the potential isomorphisms.
Connections to symbolic dynamics

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3. There are three possible outcomes to the Algebraic KP Question: NEVER, SOMETIMES, or ALWAYS.
Connections to symbolic dynamics

Some concluding remarks:

1. $C^*(R_2) \cong C^*(E_4)$; but the isomorphism is NOT given explicitly, its existence is ensured by “KK Theory”.

2. There have been a number of approaches in the attempt to answer the Algebraic KP Question: e.g., consider graded isomorphisms; restrict the potential isomorphisms.

3. There are three possible outcomes to the Algebraic KP Question: NEVER, SOMETIMES, or ALWAYS. The answer will be interesting, no matter how things play out.
Conjecture?

Is there an Algebraic KP Conjecture?

Not really.
Questions?

Thanks to the Simons Foundation
Connections and Applications: simple Lie algebras arising from Leavitt path algebras

Example

\[ E_0 = \{ v_i | i \in I \} \]

For \( i \in I \), let \( \epsilon_i \in \mathbb{Z}(I) \) be the usual standard basis vector.

For all \( j \in I \) let \( a_{ij} \) denote the number of edges \( e \in E_1 \) such that \( s(e) = v_i \) and \( r(e) = v_j \).

Define \( B_i = (a_{ij})_{j \in I} - \epsilon_i \in \mathbb{Z}(I) \).
Connections and Applications: simple Lie algebras arising from Leavitt path algebras

**Definition** Write \( E^0 = \{ v_i \mid i \in I \} \). For \( i \in I \), let \( \epsilon_i \in \mathbb{Z}^{|I|} \) be the usual standard basis vector.

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**Example**

\[
\begin{align*}
B_1 &= (1, 1, 0, 0) - \epsilon_1 = (0, 1, 0, 0), \\
B_2 &= (1, 0, 0, 1) - \epsilon_2 = (1, -1, 0, 1), \\
B_3 &= (0, 1, 1, 0) - \epsilon_3 = (0, 1, 0, 0), \\
B_4 &= (0, 0, 1, 0) - \epsilon_4 = (0, 0, 1, -1).
\end{align*}
\]
Connections and Applications: simple Lie algebras arising from Leavitt path algebras

**Theorem.** (A-, Zak Mesyan 2011) Let $K$ be a field, and let $E$ be a finite graph for which $L_K(E)$ is a nontrivial simple Leavitt path algebra. Write $E^0 = \{v_1, \ldots, v_m\}$. Then

$$[L_K(E), L_K(E)]$$

is simple as a Lie $K$-algebra if and only if

$$(1, \ldots, 1) \not\in \text{span}_K\{B_1, \ldots, B_m\}.$$
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Note: Dependence on $\text{char}(K)$ arises here.
Ring theory reminders

1. *R is von Neumann regular* in case

   \[ \forall a \in R \exists x \in R \text{ with } a = axa. \]

2. *R is prime* if the product of any two nonzero two-sided ideals of *R* is nonzero.

3. *R is primitive* if *R* admits a faithful simple left *R*-module.
Connections and Applications: The realization question for von Neumann regular rings

Fundamental problem: (Goodearl, 1994) What monoids $M$ appear as $\mathcal{V}(R)$ for von Neumann regular $R$?
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**Fundamental problem:** (Goodearl, 1994) What monoids $M$ appear as $\mathcal{V}(R)$ for von Neumann regular $R$?

**Theorem:** (Ara / Brustenga, 2007) For any row-finite graph $E$ and field $K$ there exists a von Neumann regular $K$-algebra $Q_K(E)$ for which $L_K(E)$ embeds in $Q_K(E)$, and

$$\mathcal{V}(L_K(E)) \cong \mathcal{V}(Q_K(E)).$$
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$$\mathcal{V}(L_K(E)) \cong \mathcal{V}(Q_K(E)).$$

**Corollary:** the realization question has affirmative answer for graph monoids $M_E$. 
Connections and Applications: Kaplansky’s question

Kaplansky, 1970: *Is a regular prime ring necessarily primitive?*
Answered in the negative (Domanov, 1977), a group-algebra example.
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**Theorem.** (A-, K.M. Rangaswamy 2010) $L_K(E)$ is von Neumann regular $\iff$ $E$ is acyclic.

---

**Proposition.** $L_K(E)$ is prime $\iff$ for each pair of vertices $u, v$ in $E$ there exists a vertex $w$ in $E$ for which $u \geq w$ and $v \geq w$.

---

**Theorem.** (A-, Jason Bell, Ranga 2011) $L_K(E)$ is primitive $\iff$

1. $1_{L_K(E)}$ is prime,
2. every cycle in $E$ has an exit, and
3. there exists a countable set of vertices $S$ in $E$ for which every vertex of $E$ connects to an element of $S$.

(Countable Separation Property)
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Connections and Applications: Kaplansky’s question

It’s not hard to find acyclic graphs $E$ for which $L_K(E)$ is prime but for which C.S.P. fails.

**Example:** $X$ uncountable, $S$ the set of finite subsets of $X$. Define the graph $E$:

1. vertices indexed by $S$, and
2. edges induced by proper subset relationship.

Then $L_K(E)$ is regular, prime, not primitive.
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2. edges induced by proper subset relationship.

Then $L_K(E)$ is regular, prime, not primitive.

Note: Adjoining $1_K$ in the usual way (Dorroh extension by $K$) gives unital, regular, prime, not primitive algebras.

Remark: These examples are actually “Cohn algebras”.