# Leavitt path algebras: an introduction, and applications

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# Overview

# Brief history

There are "naturally occurring" algebras for which

$${}_RR^i \cong {}_RR^j \quad \text{for some} \quad i \neq j,$$

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Does there exist $R$ with, e.g., ${}_R R^1 \cong {}_R R^3$ but ${}_R R^1 \not\cong {}_R R^2$?

# Leavitt algebras

**Theorem**

*(Bill Leavitt, Trans. A.M.S., 1962)*

*For every $m, n \in \mathbb{N}$ and field $K$ there exists $R = L_K(m, n)$ with ${}_RR^m \cong {}_RR^n$, and all isomorphisms between free left $R$-modules result precisely from this one.*

# Leavitt algebras

The $m = 1$ case of Leavitt's Theorem is not too surprising:

${}_RR^1 \cong {}_RR^n$ if and only if there exist

$$(y_1, y_2, ..., y_n) \ \text{ and } \ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in R^n,$$

for which

$$(y_1, y_2, ..., y_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 1 \ \text{ and } \ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} (y_1, y_2, ..., y_n) = I_n$$

# Leavitt algebras

Then $L_K(1,n)$ is the quotient

$$K < X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n > \ / \ < (\sum_{i=1}^{n} y_i x_i) - 1_K; \ x_i y_j - \delta_{i,j} 1_K >$$

Turns out:

**Theorem**

*(Leavitt, Duke J. Math, 1964)*

*For every field $K$ and $n \geq 2$, $L_K(1,n)$ is simple.*

*(On the other hand, for $m \geq 2$, $L_K(m,n)$ is not simple.)*

## General path algebras

Let $E$ be a directed graph. $E = (E^0, E^1, r, s)$

$$\bullet^{s(e)} \xrightarrow{e} \bullet^{r(e)}$$

The *path algebra* $KE$ is the $K$-algebra with basis $\{p_i\}$ consisting of the directed paths in $E$. (View vertices as paths of length 0.)

$$s(e) \cdot e = e = e \cdot r(e).$$

Note: $E^0$ finite $\Leftrightarrow$ $KE$ is unital; then $1_{KE} = \sum_{v \in E^0} v$.

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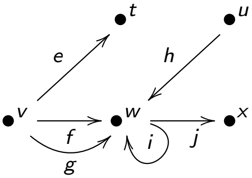

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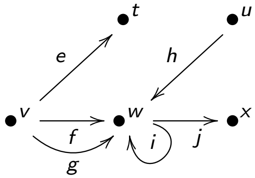


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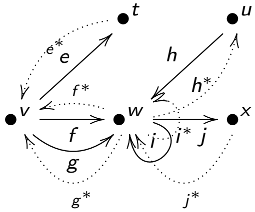

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(CK1) $e^*e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$.

(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for all $v \in E^0$

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## Definition

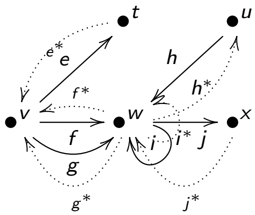
The Leavitt path algebra of $E$ with coefficients in $K$

$$L_K(E) = K\widehat{E} \ / \ < (CK1), (CK2) >$$

# Leavitt path algebras: Examples

Some sample computations in $L_{\mathbb{C}}(E)$ from the Example:

$\widehat{E} \;=$




$$ee^* + ff^* + gg^* = v \qquad g^*g = w \qquad g^*f = 0$$

$$h^*h = w \quad hh^* = u \quad ff^* = \dots \ \text{(no simplification)}$$

$$\text{But} \dots \quad (ff^*)^2 = f(f^*f)f^* = f \cdot w \cdot f^* = ff^*$$

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Then $L_K(E) \cong M_n(K)$.

$$E = \bullet^{v} \circlearrowright x$$

Then $L_K(E) \cong K[x, x^{-1}]$.

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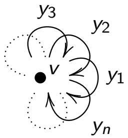
$E = R_n =$ 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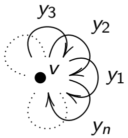
$E = R_n =$ 

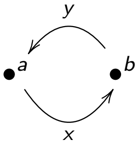

Then $L_K(E) \cong L_K(1, n)$.

The connection is clear, denote $y_i^*$ by $x_i$.

# Some graph theory

Graph theory:

1. A *cycle*



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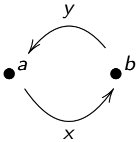
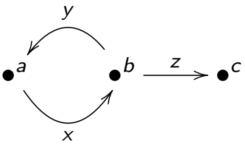
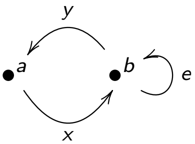



2. An *exit* for a cycle.




or



## Some graph theory

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$$\bullet^u \longrightarrow \bullet^v \longrightarrow \bullet^w \qquad (\text{ also } \quad \bullet^w \ )$$

3b. *connects to* a cycle.

$$\begin{tikzcd} \bullet^a \arrow[r, bend right, "x"'] & \bullet^b \arrow[l, bend right, "y"'] & \bullet^c \arrow[l, "f"] \end{tikzcd}$$

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Note $L_K(E)$ is simple for

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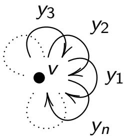
and for $E = R_n =$ 

since $L_K(E) \cong L_K(1, n)$

but not simple for

$$E = R_1 = \bullet^v \circlearrowright x \quad \text{since } L_K(E) \cong K[x, x^{-1}]$$

# Simplicity of Leavitt path algebras

**Theorem**

*(A -, Aranda Pino, 2005) $L_K(E)$ is simple if and only if:*

1. *Every vertex connects to every cycle and to every sink in $E$, and*
2. *Every cycle in $E$ has an exit.*

Note: No role played by $K$.

# Other ring-theoretic properties of Leavitt path algebras

We know precisely the graphs $E$ for which $L_K(E)$ has these properties:

1. (one-sided) artinian; (one-sided) noetherian
2. (two-sided) artinian; (two-sided) noetherian

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(No role is played by $K$ in any of these.)

# Purely infinite simplicity

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An idempotent $e \in R$ is *purely infinite* if $Re = Rf \oplus Rg$ with $f, g$ nonzero idempotents, and $Re \cong Rf$.

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Example: $e = 1$ in $R = L_K(1,2)$, since $R \cong R \oplus R$.

$R$ is *purely infinite simple* in case $R$ is simple, and every nonzero left ideal contains an infinite idempotent.

($\Leftrightarrow$ $R$ is not a field, and $\forall x \neq 0 \; \exists \; \alpha, \beta$ with $\alpha x \beta = 1$.)

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Observation: If $c$ is a cycle based at vertex $v$, and $e$ is an exit for $c$ with $s(e) = v$, then

$$L_K(E)v = L_K(E)cc^* \oplus L_K(E)(v - cc^*).$$

Easily $L_K(E)v \cong L_K(E)cc^*$, and easily both $cc^*$ and $v - cc^*$ are idempotents, and easily $cc^* \neq 0$.

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Easily $L_K(E)v \cong L_K(E)cc^*$, and easily both $cc^*$ and $v - cc^*$ are idempotents, and easily $cc^* \neq 0$.

That $v - cc^* \neq 0$ follows since $e$ is an exit for $c$: otherwise,

$$0 = v - cc^* \Rightarrow 0 = 0e = ve - cc^*e = e - 0 = e,$$

a contradiction.

# The monoid $\mathcal{V}(R)$

Isomorphism classes of finitely generated projective (left) $R$-modules, with operation $\oplus$, denoted $\mathcal{V}(R)$. (Conical) monoid.
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The construction is explicit, uses amalgamated products. (Fin. gen. hypothesis eliminated by Bergman / Dicks, 1978) Bergman included the algebras $L_K(m,n)$ as examples of these universal algebras.

# The monoid $\mathcal{V}(L_K(E))$

For any graph $E$ construct the free abelian monoid $M_E$.

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Using Bergman's construction,

**Theorem**

*(Ara, Moreno, Pardo, Alg. Rep. Thy. 2007)*

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Nontrivial (but maybe nonsurprising) consequence:
Let $S$ denote $L_K(1, n)$. Then

$$\mathcal{V}(S) = \{0,\ S,\ S \oplus S, ..., \overbrace{S \oplus S \cdots \oplus S}^{n-1 \text{ copies}}\}$$

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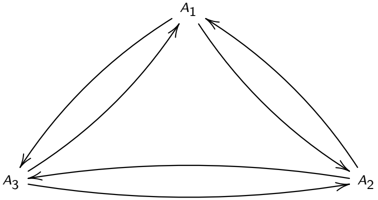
Note: $\mathcal{V}(S) \setminus \{0\} \ \cong \ \mathbb{Z}_{n-1}$.

(The 'new' identity element is $S \oplus S \cdots \oplus S$ $_{(n\,-\,1 \text{ copies})}$. )

# The monoid $\mathcal{V}(L_K(E))$

**Example.**



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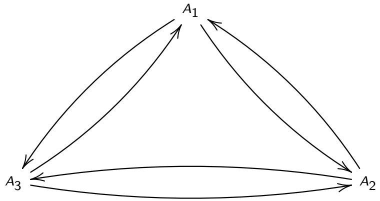


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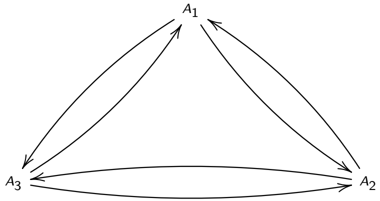


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# The monoid $\mathcal{V}(L_K(E))$

**Theorem**

*$\mathcal{V}(L_K(E)) \setminus \{0\}$ is a group if and only if $L_K(E)$ is purely infinite simple. In this case $\mathcal{V}(L_K(E)) \setminus \{0\}$ is $K_0(L_K(E))$.*

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Remarks:

1. Leavitt actually showed that $L_K(1, n)$ is purely infinite simple.
2. Proof of $(\Leftarrow)$ is by Ara / Goodearl / Pardo (holds for any purely infinite simple ring); other direction is by E. Pardo.
3. Note $\mathcal{V}(\mathrm{M}_d(K)) \setminus \{0\} = \mathbb{N}$, not a group.

# Connections and Applications: C*-algebras

**Theorem**

*(Joaquim Cuntz, Comm. Math. Physics, 1977) There exist simple $C^*$-algebras $\{\mathcal{O}_n | n \in \mathbb{N}\}$ generated by partial isometries.*

Subsequently, a similar construction was produced of a C*-algebra $C^*(E)$, for any graph $E$. In this context, $\mathcal{O}_n \cong C^*(R_n)$.

For any graph $E$, $L_{\mathbb{C}}(E) \subseteq C^*(E)$ as a dense $*$-subalgebra.

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(But $C^*(E)$ is usually "much bigger" than $L_{\mathbb{C}}(E)$.)

In particular, $L_{\mathbb{C}}(1, n) \subseteq \mathcal{O}_n$ for all $n$.

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1. simple
2. purely infinite simple
3. stable rank, prime, primitive, exchange, etc....

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Recently: Although $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, it turns out that $L_{\mathbb{C}}(1,2) \otimes L_{\mathbb{C}}(1,2) \ncong L_{\mathbb{C}}(1,2)$.

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Yes, for example $R \cong M_2(R)$ also.
(It is not hard to find an explicit isomorphism.)

But, for example, $R \ncong M_3(R)$. (Leavitt already knew this.)

# Connections and Applications: C$^*$-algebras

**Question(s)**:

1. For what $n, d$ do we have $L_{\mathbb{C}}(1, n) \cong M_d(L_{\mathbb{C}}(1, n))$?
2. For what $n, d$ do we have $\mathcal{O}_n \cong M_d(\mathcal{O}_n)$?

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*Isomorphisms are explicitly given.* This implies the result of Kirchberg / Phillips, and more.

## Connections and Applications: Higman - Thompson simple groups

For each pair of positive integers $n, r$, there exists an infinite, finitely presented simple group (the "Higman-Thompson group"), denoted $G^+_{n,r}$. These were introduced by G. Higman in 1974.

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**Theorem.** (E. Pardo, 2011)

$$G^+_{n,r} \cong G^+_{m,s} \quad \Leftrightarrow \quad m = n \text{ and } \mathrm{g.c.d.}(r, n-1) = \mathrm{g.c.d.}(s, n-1).$$

**Proof.** Show that $G^+_{n,r}$ can be realized as an appropriate subgroup of the invertible elements of $M_r(L_{\mathbb{C}}(1,n))$, and then use the explicit isomorphisms provided in the A -, Ánh, Pardo result.

# Additional ‘general ring-theoretic results’ from L.p.a.’s

In addition to the previous examples, Leavitt path algebras have been used to :

1) produce a class of prime, non-primitive von Neumann regular algebras

2) produce associative algebras for which the corresponding bracket Lie algebra is simple

3) produce examples of affine algebras having specified behavior of the radical

4) realize categories of quasi-coherent sheaves as full module categories

... and more ...

# The Algebraic Kirchberg Phillips Question

The Kirchberg / Phillips Theorem, when interpreted in the context of graph C*-algebras, yields:

**KP Theorem for graph C*-algebras:** Suppose $E$ and $F$ are finite graphs for which $C^*(E)$ and $C^*(F)$ are purely infinite simple. Suppose there is an isomorphism $\varphi : K_0(C^*(E)) \to K_0(C^*(F))$ for which $\varphi([C^*(E)]) = [C^*(F)]$. Then $C^*(E) \cong C^*(F)$ (homeomorphically).

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(2) the fact that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ is invoked in the proof.

# The Algebraic Kirchberg Phillips Question

It turns out that:

1) $K_0(L_K(E)) \cong K_0(C^*(E))$ for any finite graph $E$.

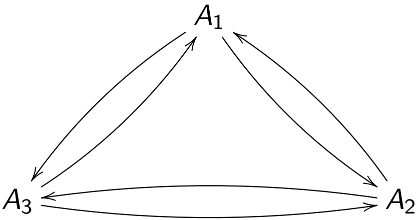
2) The $K_0$ groups are easily described in terms of the adjacency matrix $A_E$ of $E$. Let $n = |E^0|$. View $I_n - A_E^t$ as a linear transformation $\mathbb{Z}^n \to \mathbb{Z}^n$. Then

$$K_0(L_K(E)) \cong K_0(C^*(E)) \cong \mathrm{Coker}(I_n - A_E^t).$$

Moreover, $\mathrm{Coker}(I_n - A_E^t)$ can be computed by finding the Smith normal form of $I_n - A_E^t$.

# The Algebraic Kirchberg Phillips Question

Example:




$$I_3 - A_E^t = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \text{ whose Smith normal form is: } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Conclude that $K_0(L_K(E)) \cong \mathrm{Coker}(I_3 - A_E^t) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

# The Algebraic Kirchberg Phillips Question

The question becomes: Can information about $K_0$ be used to establish isomorphisms between Leavitt path algebras as well?

**The Algebraic KP Question**: Suppose $E$ and $F$ are finite graphs for which $L_K(E)$ and $L_K(F)$ are purely infinite simple. Suppose also that there exists an isomorphism $\varphi : K_0(L_K(E)) \to K_0(L_K(F))$ for which $\varphi([L_K(E)]) = [L_K(F)]$.

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$$\text{Is } L_K(E) \cong L_K(F)?$$

# Connections to symbolic dynamics

VERY informally:

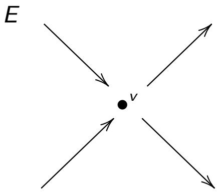
Some mathematicians and computer scientists have interest in, roughly, how information "flows" through a directed graph.

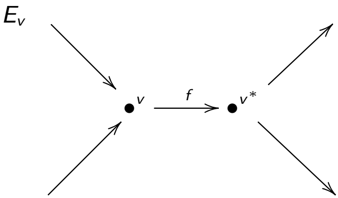
Makes sense to ask: When is it the case that information flows through two different graphs in essentially the same way? "Flow equivalent graphs".

(Often cast in the language of matrices.)

# Connections to symbolic dynamics

Example: "Expansion at $v$"




**Proposition**: If $E_v$ is the expansion graph of $E$ at $v$, then $E$ and $E_v$ are flow equivalent. Rephrased, "expansion" (and its inverse "contraction") preserve flow equivalence.

# Connections to symbolic dynamics

There are four other 'graph moves' which preserve flow equivalence:

out-split (and its inverse out-amalgamation), and

in-split (and its inverse in-amalgamation).

**Theorem PS** (Parry / Sullivan): Two graphs $E$, $F$ are flow equivalent if and only if one can be gotten from the other by a sequence of transformations involving these six graph operations.

## Connections to symbolic dynamics

Graph transformations may be reformulated in terms of adjacency matrices.

For an $n \times n$ matrix $M$ with integer entries, think of $M$ as a linear transformation $M : \mathbb{Z}^n \to \mathbb{Z}^n$. In particular, when $M = I_n - A_E^t$.

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**Proposition** (Parry / Sullivan): If $E$ is flow equivalent to $F$, then $\det(I - A_E^t) = \det(I - A_F^t)$.

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**Theorem** (Franks): Suppose $E$ and $F$ have some additional properties (*irreducible, essential, nontrivial*). If

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**Theorem**: Suppose $E$ is a graph for which $L_K(E)$ is purely infinite simple. Suppose $F$ is gotten from $E$ by doing one of the six "flow equivalence" moves. Then $L_K(E)$ and $L_K(F)$ are Morita equivalent.

In addition, the "source elimination" process also preserves Morita equivalence of the Leavitt path algebras.

**Proof**: Show that an isomorphic copy of $L_K(E)$ can be viewed as a (necessarily full, by simplicity) corner of $L_K(F)$ (or vice-versa).

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But (recall) that when $L_K(E)$ is purely infinite simple, then $K_0(L_K(E)) \cong \mathrm{Coker}(I_{|E^0|} - A_E^t)$.

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**Theorem**: (A- / Louly / Pardo / Smith 2011): Suppose $L_K(E)$ and $L_K(F)$ are purely infinite simple. If

$$K_0(L_K(E)) \cong K_0(L_K(F)) \quad \text{and} \quad \det(I - A_E^t) = \det(I - A_F^t),$$

then $L_K(E)$ and $L_K(F)$ are Morita equivalent.

# Connections to symbolic dynamics

Using some intricate computations provided by Huang, one can show the following:

Suppose $L_K(E)$ is Morita equivalent to $L_K(F)$. Further, suppose there is *some* isomorphism $\varphi : K_0(L_K(E)) \to K_0(L_K(F))$ for which $\varphi([L_K(E)]) = [L_K(F)]$.

Then there is some Morita equivalence $\Phi : L_K(E) - \mathrm{Mod} \to L_K(F) - \mathrm{Mod}$ for which $\Phi|_{K_0(L_K(E))} = \varphi$.

# Connections to symbolic dynamics

Consequently:

**Theorem**: (A- / Louly / Pardo / Smith 2011): Suppose $L_K(E)$ and $L_K(F)$ are purely infinite simple. If

$$K_0(L_K(E)) \cong K_0(L_K(F))$$

via an isomorphism $\varphi$ for which $\varphi([L_K(E)]) = [L_K(F)]$,

and $\det(I - A_E^t) = \det(I - A_F^t)$,

then $L_K(E) \cong L_K(F)$.

**'Restricted' Algebraic KP Theorem**

# Connections to symbolic dynamics

**Algebraic KP Question**: Can we drop the hypothesis on the determinants in the Restricted Algebraic KP Theorem?

# Connections to symbolic dynamics

Here's the "smallest" example of a situation of interest. Consider the Leavitt path algebras $L(R_2)$ and $L(E_4)$, where

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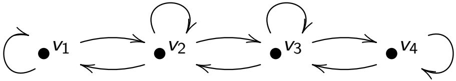
and $E_4 =$ 


It is not hard to establish that

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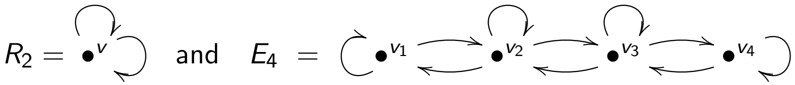


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Question: Is $L_K(R_2) \cong L_K(E_4)$?

# Connections to symbolic dynamics

Some concluding remarks:

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3. There are three possible outcomes to the Algebraic KP Question: NEVER, SOMETIMES, or ALWAYS. The answer will be interesting, no matter how things play out.

# Conjecture?

Is there an Algebraic KP **Conjecture**?

Not really.

# Questions?

Thanks to the Simons Foundation

# Connections and Applications: simple Lie algebras arising from Leavitt path algebras

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**Definition** Write $E^0 = \{v_i \mid i \in I\}$. For $i \in I$, let $\epsilon_i \in \mathbb{Z}^{(I)}$ be the usual standard basis vector.

For all $j \in I$ let $a_{ij}$ denote the number of edges $e \in E^1$ such that $s(e) = v_i$ and $r(e) = v_j$. Define $B_i = (a_{ij})_{j \in I} - \epsilon_i \in \mathbb{Z}^{(I)}$.

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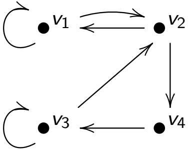
**Example**




$$B_1 = (1,1,0,0) - \epsilon_1 = (0,1,0,0), \quad B_2 = (1,0,0,1) - \epsilon_2 = (1,-1,0,1),$$
$$B_3 = (0,1,1,0) - \epsilon_3 = (0,1,0,0), \quad B_4 = (0,0,1,0) - \epsilon_4 = (0,0,1,-1).$$

# Connections and Applications: simple Lie algebras arising from Leavitt path algebras

**Theorem.** (A-, Zak Mesyan 2011) Let $K$ be a field, and let $E$ be a finite graph for which $L_K(E)$ is a nontrivial simple Leavitt path algebra. Write $E^0 = \{v_1, \ldots, v_m\}$. Then

$$[L_K(E), L_K(E)] \quad \text{is simple as a Lie } K\text{-algebra}$$

if and only if

$$(1, \ldots, 1) \notin \mathrm{span}_K\{B_1, \ldots, B_m\}.$$

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Note: Dependence on $\mathrm{char}(K)$ arises here.

# Ring theory reminders

1. $R$ is *von Neumann regular* in case

$$\forall a \in R \ \exists \ \ x \in R \ \text{ with } a = axa.$$

2. $R$ is *prime* if the product of any two nonzero two-sided ideals of $R$ is nonzero.
3. $R$ is *primitive* if $R$ admits a faithful simple left $R$-module.

# Connections and Applications: The realization question for von Neumann regular rings

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**Corollary**: the realization question has affirmative answer for graph monoids $M_E$.

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**Theorem.** (A-, Jason Bell, Ranga 2011) $L_K(E)$ is primitive $\Leftrightarrow$

1. $L_K(E)$ is prime,
2. every cycle in $E$ has an exit, and
3. there exists a countable set of vertices $S$ in $E$ for which every vertex of $E$ connects to an element of $S$.

   (Countable Separation Property)

## Connections and Applications: Kaplansky's question

It's not hard to find acyclic graphs $E$ for which $L_K(E)$ is prime but for which C.S.P. fails.

**Example**: $X$ uncountable, $S$ the set of finite subsets of $X$. Define the graph $E$:

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Note: Adjoining $1_K$ in the usual way (Dorroh extension by $K$) gives unital, regular, prime, not primitive algebras.

Remark: These examples are actually "Cohn algebras".