The graph menagerie: Abstract algebra and The Mad Veterinarian

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1 Introduction and brief history

2 Mad Vet scenarios

3 Mad Vet groups

4 Beyond the Mad Vet
Welcome to Bob’s Mad Veterinarian Puzzle Page

In September of 1998, after fiddling with this puzzle format for about a decade, I posted the first Mad Veterinarian puzzle to the rec.puzzles newsgroup:
A mad veterinarian has created three animal transmogrifying machines. Place a cat in the input bin of the first machine, press the button, and *whirr... bing!* Open the output bins to find two dogs and five mice.

The second machine can convert a dog into three cats and three mice, and the third machine can convert a mouse into a cat and a dog. Each machine can also operate in reverse (e.g. if you’ve got two dogs and five mice, you can convert them into a cat).

You have one cat.

1. Can you convert it into seven mice?
2. Can you convert it into a pack of dogs, with no mice or cats left over?
Puzzle solvers discovered that it was impossible to convert a single cat into seven mice, nor to a lonesome pack of dogs.

However, they posed and answered followup questions, such as how many mice can be created from a single cat? and what’s the smallest number of cats that can be turned into just dogs?
Below, I’ve set up several puzzles of this type, and a java applet that lets you solve them. Each applet deals with one set of machines and poses several conversions for you to try to solve.

How To Solve Mad Veterinarian Puzzles
Easy Three Animal Laboratory Mar/17/2003
Original Three Animal Laboratory Mar/17/2003
Hard Four Animal Laboratory Mar/17/2003
Harder Four Animal Laboratory Apr/1/2003
Schoolhouse Jelly Beans Apr/2/2003
Mad Vet puzzles were used as part of a weeklong workshop on Math Teacher Circles, held at the American Institute of Mathematics in Palo Alto, CA, in June 2008.
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*Leavitt path algebras* !!
1. Introduction and brief history

2. Mad Vet scenarios

3. Mad Vet groups

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A *Mad Vet scenario* posits a Mad Veterinarian in possession of a finite number of transmogrifying machines, where

1. Each machine transmogrifies a single animal of a given species into a finite nonempty collection of animals from any number of species;
2. Each machine can also operate in reverse; and
3. There is one machine corresponding to each species in the menagerie.
Mat Vet Scenario #1

**Scenario #1.** Suppose a Mad Veterinarian has three machines with the following properties.

- Machine 1 turns one ant into one beaver;
- Machine 2 turns one beaver into one ant, one beaver and one cougar;
- Machine 3 turns one cougar into one ant and one beaver.

Let's do some transmogrification!!

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The graph menagerie
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Let’s do some transmogrification!!
Given any Mad Vet scenario, its corresponding \textit{Mad Vet graph} is the directed graph with

\[ V = \{ A_1, A_2, \ldots, A_n \}, \]

and having, for each \( A_i, A_j \) in \( V \), exactly

\[ d_{i,j} \]

edges with initial vertex \( A_i \) and terminal vertex \( A_j \),

where the machine corresponding to species \( A_i \) produces \( d_{i,j} \) animals of species \( A_j \).
Example. Mad Vet scenario #1 has the following Mad Vet graph.

Recall:
Machine 1: Ant $\rightarrow$ Beaver
Machine 2: Beaver $\rightarrow$ Ant, Beaver, and Cougar
Machine 3: Cougar $\rightarrow$ Ant, Beaver
Key idea: Let’s say there are $n$ different species. Let

$$\mathbb{Z}^+ \text{ denote } \{0, 1, 2, \ldots \}.$$ 

A *menagerie* is an element of the set

$$S = (\mathbb{Z}^+)^n \setminus \{(0, 0, \ldots, 0)\}.$$
Mad Vet equivalence

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There is a natural correspondence between menageries and nonempty collections of animals from species \( A_1, A_2, \ldots, A_n \).

For instance, in Scenario \#1 a collection of two beavers and five cougars would correspond to \((0, 2, 5)\) in \(S\).
There is a naturally arising relation $\sim$ on $S$:

Given $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in $S$, we write

$$a \sim b$$

if there is a sequence of Mad Vet machines that will transmogrify the collection of animals associated with menagerie $a$ into the collection of animals associated with menagerie $b$. 

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Using the three properties of a Mad Vet scenario, it is straightforward to show that $\sim$ is an equivalence relation on $S$. 
Mad Vet equivalence

We focus on the set

\[ W = \{[a] : a \in S\} \]

of equivalence classes of \( S \) under \( \sim \).

**Example.** Suppose that our Mad Vet of Scenario #1 starts with the menagerie \((1, 0, 0)\).

(Recall: Machine 1: \( A \rightarrow B \) Machine 2: \( B \rightarrow A, B, C \) Machine 3: \( C \rightarrow A,B \))

Then, for example,

\[(1, 0, 0) \sim (0, 1, 0) \sim (1, 1, 1) \sim (2, 2, 0) \sim (4, 0, 0).\]

Rewritten,

\[ [(1, 0, 0)] = [(0, 1, 0)] = [(1, 1, 1)] = [(2, 2, 0)] = [(4, 0, 0)] \text{ in } W. \]
Claim. \( W \) is the 3-element set

\[
\{(1, 0, 0), (2, 0, 0), (3, 0, 0)\}.
\]

**Claim.** $W$ is the 3-element set

$$\{[(1, 0, 0)], [(2, 0, 0)], [(3, 0, 0)]\}.$$  

**Reason.** It’s not hard to see that any $(a, b, c)$ is equivalent to one of the menageries $(1, 0, 0)$, $(2, 0, 0)$, or $(3, 0, 0)$. 

Why are these classes not equal to each other? Given a menagerie $m = (a, b, c)$, define the sum

$$s_m = a + b + 2c.$$  

(Intuitively: $s_m$ is the dollar value of menagerie $m$, where an Ant costs $1$, a Beaver $1$, and a Couger $2$.)
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(Intuitively: \( s_m \) is the dollar value of menagerie \( m \), where an Ant costs $1, a Beaver $1, and a Couger $2.)

Then Machines 1 and 3 leave \( s_m \) the same, while Machine 2 increases \( s_m \) by 3 (and running Machine 2 in reverse decreases \( s_m \) by 3). So any application of any machine to any menagerie leaves the total value of the menagerie invariant mod 3. So the three classes are distinct.
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Semigroups, monoids, and groups

Reminder / review of notation.

1. **semigroup**: associative operation.

   - e.g. $\mathbb{N} = \{1, 2, 3, \ldots\}$ under addition.
   - $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$ under addition.
   - $\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, \ldots\}$ under addition.
Reminder / review of notation.

1. **Semigroup**: associative operation.
   e.g. $\mathbb{N} = \{1, 2, 3, ...\}$ under addition.
Semigroups, monoids, and groups

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   e.g. \( \mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, \ldots\} \) under addition.
Start with a Mad Vet scenario. Define addition on $W$ (the set of equivalence classes of menageries) by setting

$$[x] + [y] = [x + y].$$

Interpret as “unions” of menageries.

This operation is well defined.

“Mad Vet semigroup.”
Mad Vet semigroups

(Recall: Machine 1: $A \to B$  Machine 2: $B \to A, B, C$  Machine 3: $C \to A,B$)

Example.

$W = \{[(1, 0, 0)], [(2, 0, 0)], [(3, 0, 0)]\}.$

We get, for instance,

$$[(1, 0, 0)] + [(1, 0, 0)] = [(1 + 1, 0, 0)] = [(2, 0, 0)],$$

as we’d expect.
Mad Vet semigroups


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We get, for instance,

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as we’d expect. But also

\[ [[1, 0, 0]] + [[3, 0, 0]] = [[4, 0, 0]] = [[1, 0, 0]]. \]

So \([[3, 0, 0]]\) behaves like an identity element with respect to the element \([[1, 0, 0]]\) in \(W\).
Similarly

\[(2, 0, 0) + [(3, 0, 0)] = [(2, 0, 0)], \text{ and } [(3, 0, 0)] + [(3, 0, 0)] = [(3, 0, 0)].\]
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\[(2, 0, 0) + (3, 0, 0) = (2, 0, 0), \text{ and } (3, 0, 0) + (3, 0, 0) = (3, 0, 0)\].

So for this Mad Vet scenario the Mad Vet semigroup \( W \) is a monoid with identity \( (3, 0, 0) \).
Mad Vet semigroups

Actually, since

$$[(1, 0, 0)] + [(2, 0, 0)] = [(3, 0, 0)]$$

in $W$, every element in $W$ has an inverse.
Mad Vet semigroups

Actually, since

\[(1, 0, 0) + (2, 0, 0) = (3, 0, 0)\]

in \(W\), every element in \(W\) has an inverse.

So \(W\) is in fact a group, necessarily \(\mathbb{Z}_3\).
Scenario #2. Suppose the same Mad Vet has replaced two of her machines with new machines.

Machine 1 still turns one ant into one beaver;
Machine 2 now turns one beaver into one ant and one cougar;
Machine 3 now turns one cougar into two cougars.

In this situation $W$ is a monoid, but not a group.
**Scenario #2.** Suppose the same Mad Vet has replaced two of her machines with new machines.

Machine 1 still turns one ant into one beaver;
Machine 2 now turns one beaver into one ant and one cougar;
Machine 3 now turns one cougar into two cougars.

In this situation $\mathcal{W}$ is a monoid, but not a group. In fact,

$$\mathcal{W} = \{(i, 0, 0) : i \in \mathbb{N}\} \cup \{(0, 0, 1)\}.$$  

$\{(0, 0, 1)\}$ is an identity element for this Mad Vet semigroup.

So $\mathcal{W}$ in this case is a *monoid*. 
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In this situation $W$ is a monoid, but not a group. In fact,

$$W = \{[(i, 0, 0)] : i \in \mathbb{N}\} \cup \{[(0, 0, 1)]\}.$$ 

$[(0, 0, 1)]$ is an identity element for this Mad Vet semigroup.

So $W$ in this case is a monoid.

But $W$ is not a group: e.g., there is no element $[x]$ in $W$ for which

$$[(1, 0, 0)] + [x] = [(0, 0, 1)].$$
The Big Question:

Given a Mad Vet scenario, when is the corresponding Mad Vet semigroup actually a group?

More Mad Vet scenarios ...
Mad Vet semigroups

Scenario #3.

M1: A → B,C;   M2: B → A,C;   M3: C → A,B

Scenario #4.

M1: A → 2A;   M2: B → 2B;   M3: C → 2C

Scenario #5.

M1: A → B,C;   M2: B → A,B;   M3: C → A,C

Scenario #6.

M1: A → B;   M2: B → C;   M3: C → C

Scenario #7.

M1: A → A,B,C;   M2: B → A,C;   M3: C → A,B
Among Scenarios #3-7, there are Mad Vet semigroups \( W \) for which:

1. \( W \) is an infinite group;
2. \( W \) is a finite noncyclic group;
3. \( W \) is a finite nonmonoid;
4. \( W \) is a finite cyclic group, not isomorphic to \( \mathbb{Z}_3 \);
5. \( W \) is an infinite nonmonoid.

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The graph menagerie
Subtle?

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5. $\mathcal{W}$ is an infinite nonmonoid.
Some graph theory: context

Euler’s “Bridges of Königsberg” problem.
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Some graph theory: context

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2. prove a theorem about graphs;
Euler’s “Bridges of Königsberg” problem. Idea:

1. translate the problem to a question about graphs;
2. prove a theorem about graphs;
3. use the graph-theoretic result to answer original question.
Graph theory

Some graph theory terminology. (All graphs are directed.)
Graph theory

Some graph theory terminology. (All graphs are directed.)

1. A *sink* in a directed graph.

2. A *path* in a directed graph.

3. If $v$ and $w$ are vertices, $v$ connects to $w$ in case either $v = w$ or there is a path from $v$ to $w$.

4. For a vertex $v$, a *cycle based at $v$* is a (nontrivial) path from $v$ to $v$ for which no vertices are repeated.
Graph theory

Some graph theory terminology. (All graphs are \textit{directed}.)

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4. For a vertex $v$, a \textit{cycle based at} $v$ is a (nontrivial) path from $v$ to $v$ for which no vertices are repeated.
5. A finite graph $\Gamma$ is \textit{cofinal} in case every vertex $v$ of $\Gamma$ connects to every cycle and to every sink in $\Gamma$. 
Graph theory

Some graph theory terminology. (All graphs are directed.)

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3. If \( v \) and \( w \) are vertices, \( v \) connects to \( w \) in case either \( v = w \) or there is a path from \( v \) to \( w \).
4. For a vertex \( v \), a cycle based at \( v \) is a (nontrivial) path from \( v \) to \( v \) for which no vertices are repeated.
5. A finite graph \( \Gamma \) is cofinal in case every vertex \( v \) of \( \Gamma \) connects to every cycle and to every sink in \( \Gamma \).
6. If \( C = f_1 f_2 \cdots f_m \) is a cycle in \( \Gamma \), then an edge \( e \) is called an exit for \( C \) if the source vertex of \( e \) equals the source vertex for \( f_j \) (some \( j \)), but \( e \neq f_j \).
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Some graph theory terminology. (All graphs are directed.)

1. A *sink* in a directed graph.
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4. For a vertex $v$, a *cycle based at* $v$ is a (nontrivial) path from $v$ to $v$ for which no vertices are repeated.
5. A finite graph $\Gamma$ is *cofinal* in case every vertex $v$ of $\Gamma$ connects to every cycle and to every sink in $\Gamma$.
6. If $C = f_1 f_2 \cdots f_m$ is a cycle in $\Gamma$, then an edge $e$ is called an *exit for* $C$ if the source vertex of $e$ equals the source vertex for $f_j$ (some $j$), but $e \neq f_j$. (Intuitively, an exit for $C$ is an edge $e$, not included in $C$, which provides a way to step off of $C$. )
Example.

The cycle $eg$ based at $y$ has two exits: $h$ and the loop at $y$. These same edges are also exits for the cycle $ge$ based at $z$. Similarly, the loop at $y$ has exits $e$ and $h$.

The loop at $x$ has no exit.

This graph is not cofinal (e.g., $x$ does not connect to $eg$).
Theorem: Mad Vet Group Test. The Mad Vet semigroup $W$ of a Mad Vet scenario is a group if and only if the corresponding Mad Vet graph $\Gamma$ has the following two properties.

(1) $\Gamma$ is cofinal; and
(2) Every cycle in $\Gamma$ has an exit.
Theorem: Mad Vet Group Test. *The Mad Vet semigroup* \( W \) *of a Mad Vet scenario is a group if and only if the corresponding Mad Vet graph* \( \Gamma \) *has the following two properties.*

(1) \( \Gamma \) *is cofinal; and*

(2) *Every cycle in* \( \Gamma \) *has an exit.*

**Proof.:** Long, but can be done using only basic graph-theoretic and group-theoretic ideas.
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(1) $\Gamma$ is cofinal; and
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Proof.: Long, but can be done using only basic graph-theoretic and group-theoretic ideas.

(Actually, two proofs are known. More about that later.)
Mad Vet Group Test

An overview of one of the proofs.
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**Lemma.** A commutative semigroup $S$ is a group if and only if for each pair $x, z \in S$ there exists $y \in S$ for which $x + y = z$. 
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Proof: Good exercise for Math 414 students. (Fraleigh, Section 4, Problem 39 ...)

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Mad Vet Group Test

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Here’s the Mad Vet graph from Scenario #1 again:

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Cofinal?  YES.
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Cofinal? YES.  Every cycle has an exit? YES.
Here’s the Mad Vet graph $\Theta$ of Scenario #2.

(Recall: Machine 1: $A \rightarrow B$  Machine 2: $B \rightarrow A, C$  Machine 3: $C \rightarrow 2C$)
Mad Vet Group Test

Here's the Mad Vet graph $\Theta$ of Scenario #2.

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Cofinal? NO. ($C$ does not connect to the cycle $ABA$.)
Here’s the Mad Vet graph $\Theta$ of Scenario #2.

(Recall: Machine 1: $A \rightarrow B$  Machine 2: $B \rightarrow A, C$  Machine 3: $C \rightarrow 2C$)

Cofinal? NO. ($C$ does not connect to the cycle $ABA$.)

(But every cycle does have an exit ...)

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The graph menagerie
**Scenario #8.** Let’s analyze Mad Vet Bob’s puzzle.

(Recall: Machine 1: A → 2B, 5C  
Machine 2: B → 3A, 3C  
Machine 3: C → A, B)

**Mad Vet Group Test**
Scenario #8. Let’s analyze Mad Vet Bob’s puzzle.

(Recall: Machine 1: $A \rightarrow 2B,5C$  Machine 2: $B \rightarrow 3A, 3C$  Machine 3: $C \rightarrow A,B$)
Scenario #8. Let’s analyze Mad Vet Bob’s puzzle.

(Recall: Machine 1: $A \to 2B, 5C$  Machine 2: $B \to 3A, 3C$  Machine 3: $C \to A, B$)

So Mad Vet Bob’s semigroup is in fact a group.
Mad Vet Groups

Just exactly what group is it ?????
Just exactly what group is it ?????

This question has a remarkably nice answer.

Any graph $\Gamma$ has an associated \textit{incidence matrix} $A_{\Gamma}$: if $\Gamma$ has $n$ vertices $v_1, v_2, \ldots, v_n$, then $A_{\Gamma}$ is the $n \times n$ matrix $(d_{ij})$, where

\[ d_{ij} = \# \text{ of edges starting at } v_i \text{ and ending at } v_j. \]
For example, if $\Delta$ is the graph of Scenario #1,

$$A \Delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
Mad Vet Groups

Now form the matrix $I_n - A_\Gamma$.

For instance, using the above matrix $A_\Delta$,

$$I_3 - A_\Delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$
Then put the (square) matrix $I_n - A_{\Gamma}$ in *Smith normal form*.

The Smith normal form of an $n \times n$ matrix having integer entries is a diagonal $n \times n$ matrix whose diagonal entries are nonnegative integers

$$\alpha_1, \alpha_2, \ldots, \alpha_q, 0, 0, \ldots, 0$$

such that $\alpha_i$ divides $\alpha_{i+1}$ for each $1 \leq i \leq q - 1$. 

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Mad Vet Groups

The Smith normal form of a matrix $A$ can be obtained by performing on $A$ a combination of these matrix operations: interchanging rows or columns, or adding an integer multiple of a row [column] to another row [column]. The resulting Smith normal form of matrix $A$ is thus of the form $PAQ$, where $P$ and $Q$ are integer-valued matrices with determinants equal to $\pm 1$. (Might need to tweak some signs at the end ...)

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The graph menagerie
Here’s an answer to the “just exactly what group is it?” question.

**Mad Vet Group Identification Theorem.** *Given a Mad Vet scenario with n species whose Mad Vet semigroup \( W \) is a group, let \( \Gamma \) be its associated Mad Vet graph. Let \( \alpha_1, \alpha_2, \ldots, \alpha_q \) be the nonzero diagonal entries of the Smith normal form of the matrix \( I_n - A_\Gamma \).*
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**Mad Vet Group Identification Theorem.** Given a Mad Vet scenario with \( n \) species whose Mad Vet semigroup \( W \) is a group, let \( \Gamma \) be its associated Mad Vet graph. Let \( \alpha_1, \alpha_2, \ldots, \alpha_q \) be the nonzero diagonal entries of the Smith normal form of the matrix \( I_n - A_{\Gamma} \). Then

\[
W \cong \mathbb{Z}_{\alpha_1} \oplus \mathbb{Z}_{\alpha_2} \oplus \cdots \oplus \mathbb{Z}_{\alpha_q} \oplus \mathbb{Z}^{n-q}.
\]

(Notation: \( \mathbb{Z}_1 = \{0\} \).)
Example. Letting $\Delta$ be the Mad Vet graph of Scenario #1, the Smith normal form of the matrix $I_3 - A_\Delta$ is the matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 3
\end{pmatrix}.
$$

Because we already know that Scenario #1’s semigroup is a group, the Mad Vet Group Identification Theorem implies that it is isomorphic to $\mathbb{Z}_1 \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_3 \cong \{0\} \oplus \{0\} \oplus \mathbb{Z}_3 \cong \mathbb{Z}_3$, as expected.
Example. Let Φ be the Mad Vet graph of Scenario #8 (Mad Vet Bob’s Puzzle). We’ve checked that Φ has the right properties, so that the corresponding Mad Vet semigroup is a group. Then \( I_\Phi \) is the matrix
\[
\begin{pmatrix}
0 & 2 & 5 \\
3 & 0 & 3 \\
1 & 1 & 0
\end{pmatrix}.
\]
The Smith normal form of \( I_3 - A_\Phi \) turns out to be matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 34
\end{pmatrix}.
\]
So the corresponding group is isomorphic to \( \mathbb{Z}_1 \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_{34} \cong \mathbb{Z}_{34} \).
1 Introduction and brief history

2 Mad Vet scenarios

3 Mad Vet groups

4 Beyond the Mad Vet
Purely Infinite Simplicity Theorem. For a finite directed sink-free graph $\Gamma$, the following are equivalent:

(1) The Leavitt path algebra $L_\mathbb{C}(\Gamma)$ is purely infinite and simple. (This is a statement about an algebraic structure.)

(2) The graph $C^*$-algebra $C^*(\Gamma)$ is purely infinite and simple. (This is a statement about an analytic structure.)

(3) $\Gamma$ is cofinal, and every cycle in $\Gamma$ has an exit.

(4) The graph semigroup $W_\Gamma$ is a group.
Who cares?

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We can get rid of the sink-free hypothesis in the general analysis.
15 minutes of fame?
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Questions?