The ubiquity of the Fibonacci Sequence: It comes up in Leavitt path algebras too!

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Overview

Something familiar: The Fibonacci Sequence $F(n)$

Fibonacci's Rabbit Puzzle: (from *Liber Abaci*, 1202)

Suppose you go to an uninhabited island with a pair of newborn rabbits (one male and one female), who:

1. mature at the age of one month,
2. have two offspring (one male and one female) each month after that, and
3. live forever.

Each pair of rabbits mature in one month and then produce a pair of newborns at the beginning of every following month. How many pairs of adult rabbits will there be in a year?

Something familiar: The Fibonacci Sequence $F(n)$

end month n	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$

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end month n	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$F(n)$	1	1	2	3	5	8	13	21	34	55	89	144	$\cdots$

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There is a "generating formula" for the Fibonacci sequence:

$F(1) = 1$; $F(2) = 1$; $F(n) = F(n-1) + F(n-2)$ for all $n \geq 3$.

Something familiar: The Fibonacci Sequence $F(n)$

The Fibonacci sequence comes up in lots of places ...

AND is VERY well-studied!

For instance,

Theorem: $\mathrm{g.c.d.}(F(n), F(m)) = F(\mathrm{g.c.d.}(n, m))$.

Corollary: $\mathrm{g.c.d.}(F(n), F(n - 1)) = 1$.

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A site for all types of info about the Fibonacci sequence (more than 300 formulas):

Google: Ron Knott Fibonacci

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Mad Vet Bob's Mad Vet Puzzle: (from *The Internet*, 1998)

A mad veterinarian has created three animal transmogrifying machines.

Place a cat in the input bin of the first machine, press the button, and *whirr... bing!* Open the output bins to find two dogs and five mice.

The second machine can convert a dog into three cats and three mice.

The third machine can convert a mouse into a cat and a dog. Each machine can also operate in reverse (e.g. if you've got two dogs and five mice, you can convert them into a cat).

Something not-as-familiar: Mad Vet Puzzles

You have one cat.

Something not-as-familiar: Mad Vet Puzzles

You have one cat.

1. Can you convert it into seven mice?
2. Can you convert it into a pack of dogs, with no mice or cats left over?

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A site for more info about Mad Vet Puzzles (The 'Mad Bob' site):

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Google: Mad Bob's Mad Vet Puzzles

A New York Times Puzzle Blog:

Google: Numberplay: The Mad Veterinarian

Mad Vet scenarios

A *Mad Vet scenario* is a situation such as the one Mad Bob constructed.

We assume:

1. Each species is paired up with a machine;
2. Each machine can also operate in reverse; and
3. Each machine is "one to some"

Mat Vet Scenario #1

Scenario #1. Suppose a Mad Veterinarian has three machines with the following properties.

Machine 1 turns one Ant into one Beaver;

Machine 2 turns one Beaver into one Ant, one Beaver and one Cougar;

Machine 3 turns one Cougar into one Ant and one Beaver.

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$$A \Rightarrow B \Rightarrow A, B, C \Rightarrow 2A, 2B \Rightarrow 3A, B \Rightarrow 4A$$

From Mad Vet Scenarios to graphs

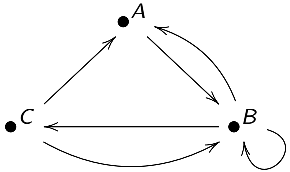
Given any Mad Vet scenario, its corresponding *Mad Vet graph* is:

a drawing ("directed graph"),

consisting of points and arrows ("vertices" and "edges"),

which gives the info about what's going on with the machines.
(We only draw the "forward" direction of the machines.)

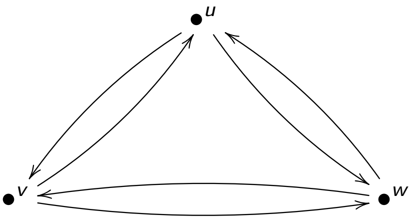
Example. Mad Vet scenario #1 has the following Mad Vet graph.



Recall: Machine 1: A $\to$ B, Machine 2: B $\to$ A, B, C, Machine 3: C $\to$ A,B

... and vice versa: from graphs to Mad Vet Scenarios

Example: Consider this directed graph:



This graph would describe a Mad Vet Scenario with three species: Urchins, Vermin, Warthogs

Machine 1: Urchin $\rightarrow$ Vermin, Warthog
Machine 2: Vermin $\rightarrow$ Urchin, Warthog
Machine 3: Warthog $\rightarrow$ Urchin, Vermin

Mad Vet equivalence

Some notation: Let's say there are n different species. Choose some "order" to list them.

E.g., in Scenario #1, first list Ants, then Beavers, then Cougars.

Then any collection of animals corresponds to some n-vector, with entries taken from the set $\{0, 1, 2, \ldots\}$.

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For instance, in Scenario #1 a collection of two Beavers and five Cougars would correspond to $(0, 2, 5)$.

We agree that an "empty" collection of animals is not of interest here. In other words, the vector $(0, 0, \ldots, 0)$ is not allowed.

Mad Vet equivalence

There is a naturally arising relation $\sim$ on these vectors:

Given $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$, we write

$$a \sim b$$

if there is a sequence of Mad Vet machine moves that will change the collection of animals associated with vector a into the collection of animals associated with vector b.

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(Aside: Using the three properties of a Mad Vet scenario, it is straightforward to show that $\sim$ is an equivalence relation.)

Mad Vet equivalence

Example. Suppose that our Mad Vet of Scenario #1 starts with one Ant; in other words, with $(1,0,0)$.

(Recall: Machine 1: A $\to$ B Machine 2: B $\to$ A, B, C Machine 3: C $\to$ A,B)

Then, rewriting in this new notation what we've already seen,

$$(1,0,0) \sim (0,1,0) \sim (1,1,1) \sim (2,2,0) \sim (3,1,0) \sim (4,0,0).$$

As a result, $(1,0,0) \sim (4,0,0)$. And $(4,0,0) \sim (1,0,0)$ too ...

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We denote the equivalence classes under $\sim$ by brackets. So, e.g., for this Mad Vet Scenario,

$$[(1,0,0)] = [(4,0,0)]$$

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$$[(1,0,0)] = [(4,0,0)] = [(0,1,0)] = [(1,1,1)] = [(2,2,0)] = [(3,1,0)] \cdots$$

Also, we have, for example

$$[(2,0,0)] = [(1,1,0)] = [(2,1,1)] = \cdots .$$

Mad Vet equivalence

(Recall: Machine 1: A $\rightarrow$ B Machine 2: B $\rightarrow$ A, B, C Machine 3: C $\rightarrow$ A,B)

Claim. In Scenario #1, there are exactly three different "bracket vectors" of animals:

$$\{\ [(1,0,0)],\ [(2,0,0)],\ [(3,0,0)]\ \}.$$

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Showing that these three brackets are different takes some (straightforward) work.

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   e.g., for any positive integer n, the "clock arithmetic group" $\mathbb{Z}_n = \{1, 2, \ldots, n\}$, under addition $\mathrm{mod}\ n$.

   e.g., for any positive integers m, n, the "direct product" (i.e., ordered pairs) $\mathbb{Z}_m \times \mathbb{Z}_n$.

Mad Vet semigroups

Start with a Mad Vet scenario. Define an addition process on bracket vectors:

$$[x] + [y] = [x + y].$$

Interpret as "unions" of collections of animals. This operation makes the set of bracket vectors a semigroup. (Actually, a commutative semigroup.)

Example. (Scenario #1: M 1: A $\rightarrow$ B M 2: B $\rightarrow$ A, B, C M 3: C $\rightarrow$ A,B)

The bracket vectors are $\{[(1,0,0)],[(2,0,0)],[(3,0,0)]\}$.

We get, for instance,

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Similarly

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Mad Vet semigroups

Similarly

$[(2,0,0)]+[(3,0,0)] = [(2,0,0)],$ and $[(3,0,0)]+[(3,0,0)] = [(3,0,0)].$

So for this Mad Vet scenario the Mad Vet semigroup is a monoid, with identity $[(3,0,0)]$.

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Notation remark:

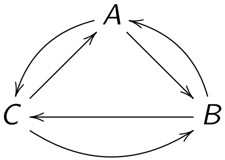
Sometimes we write $[A]$ for $[(1,0,0)]$, $[B]$ for $[(0,1,0)]$, etc ...

So, e.g., the three bracket vectors in Scenario #1 consist of the set

$$\{[A], 2[A], 3[A]\}.$$

Another Mad Vet Scenario (Scenario #2)

Here's a Mad Vet with a different set of machines:



So: Machine 1: A $\to$ B,C Machine 2: B $\to$ A, C Machine 3: C $\to$ A,B)

What are the Mad Vet bracket vectors here?

$$\{\ [(1,0,0)],\ [(0,1,0)],\ [(0,0,1)],\ [(1,1,1)]\ \}$$

Or, in the other notation, $\{\ [A],\ [B],\ [C],\ [A]+[B]+[C]\ \}$.

Turns out: these also form a group, $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Mad Vet semigroups

There are Mad Vet Scenarios where the bracket vectors for that scenario do NOT form a group. For instance, the Mad Vet Scenario for this graph.

$$\begin{tikzcd} & A \arrow[dr] & \\ C \arrow[ur] & & B \arrow[ll] \end{tikzcd}$$

Here the bracket vectors behave like the set $\mathbb{N} = \{1, 2, 3, ...\}$.

The SAME idea, just using different language ...

Since the data of a Mad Vet semigroup can be thought of as coming from a directed graph, we usually use the phrase

Graph semigroup

rather than *Mad Vet* semigroup.
Notation: For the directed graph Γ, $\mathcal{V}^*(\Gamma)$ denotes the graph semigroup of Γ.

(In some cases we like to have the symbol $[(0,0,...,0)]$ included in the discussion. The corresponding set of bracket vectors $\mathcal{V}^*(\Gamma) \sqcup [(0,0,...,0)]$ is denoted $\mathcal{V}(\Gamma)$, and is called the *graph monoid* of Γ.)

Graph groups

A Big Question:

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A Big Answer:

If you can walk from any vertex in Γ to any cycle in Γ by a sequence of edges, and you can 'step off' any cycle, and Γ isn't just a 'basic cycle', then $\mathcal{V}^*(\Gamma)$ is a group.

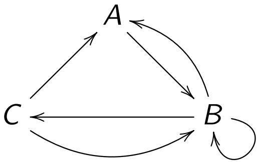
And vice versa.

"Graph group" of Γ $\quad \mathrm{G}(\Gamma)$.

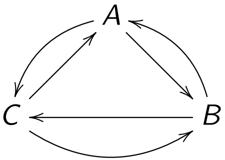
Graph groups

Recall the graphs of Scenarios #1 and #2

$\Gamma_1 =$



$\Gamma_2 =$



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Another Big Answer:

For today, suffice it to say that if you are given some specific graph Γ, then it is "easy" to write code (e.g., in *Mathematica*) which will easily tell you $\mathrm{G}(\Gamma)$. (Matrix computations.)

The "basic" cyclic graphs C_n

With the previous stuff as context, here's a game we can play.

Take a collection of "similar" graphs Γ_n for which, for each of the graphs, the corresponding $\mathcal{V}^*(\Gamma)$ is a group (denoted $\mathrm{G}(\Gamma)$).

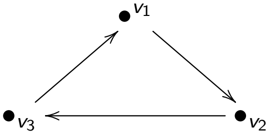
Here's one way we might build these. For each $n \geq 1$, let C_n be the "cycle" graph having n vertices $\{v_1, v_2, \ldots, v_n\}$, and n edges, like this:

The "basic" cyclic graphs C_n

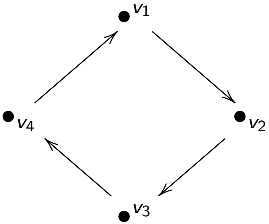
$C_1 =$

$C_2 =$

$C_3 =$



$C_4 =$



The graph semigroups $\mathcal{V}(C_n)$ aren't so nice in this context, because they don't form groups.

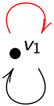
But if we modify the C_n graphs in appropriate ways, then we get graphs whose bracket vectors do form groups.

How can we do that?

We can add an extra edge at each vertex, in a systematic way.

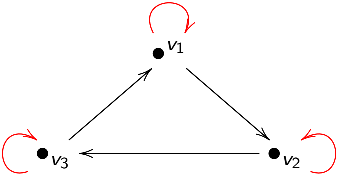
The graphs C_n^0:

$C_1^0 =$



$C_2^0 =$

$C_3^0 =$



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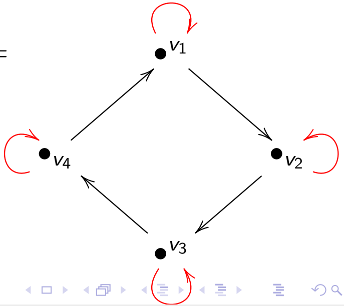

For each n, $\mathrm{G}(C_n^0)$ contains just one element.

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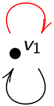
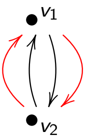
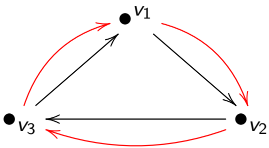
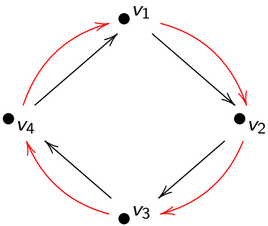
Not too interesting.

The graphs C_n^1:

$C_1^1 =$ 

$C_2^1 =$ 

$C_3^1 =$ 

$C_4^1 =$ 

For each n,

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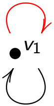

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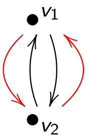

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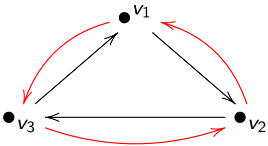

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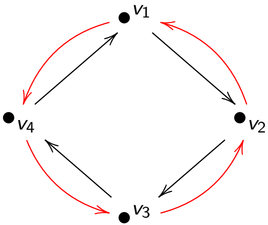

$\mathrm{G}(C_n^{-1})$ for various values of n.

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Proposition. (A-, Ben Schoonmaker, to appear)

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It works.

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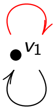

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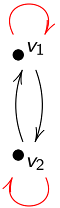

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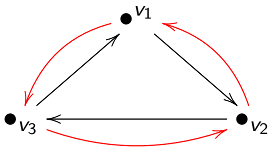

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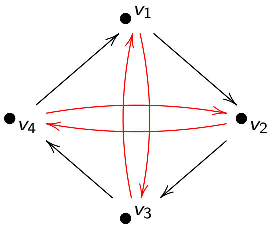
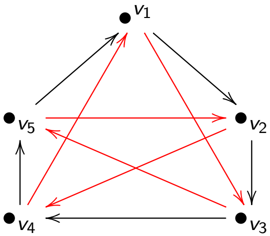
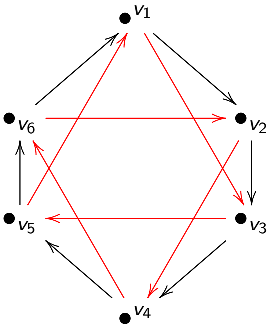


Here are two more graphs in this sequence ...

$C_5^2 =$



$C_6^2 =$



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n	13	14	15	16	17	18
$\lvert\mathrm{G}(C_n^2)\rvert$	521	841	1364	2205	3571	5776

n	19	20	21	22	23	24
$\lvert\mathrm{G}(C_n^2)\rvert$	9349	15125	24476	39601	64079	103680

Let's do some sample computations in, say, $\mathrm{G}(C_6^2)$.

$$\begin{aligned}
[v_1] &= [v_2] + [v_3] \\
&= ([v_3] + [v_4]) + [v_3] \quad = 2[v_3] + [v_4] \\
&= 2([v_4] + [v_5]) + [v_4] \quad = 3[v_4] + 2[v_5] \\
&= 3([v_5] + [v_6]) + 2[v_5] = 5[v_5] + 3[v_6] \\
&= 5([v_6] + [v_1]) + 3[v_6] = 8[v_6] + 5[v_1]
\end{aligned}$$

So, in $\mathrm{G}(C_6^2)$, we get

$$[v_1] = 8[v_6] + 5[v_1].$$

This gives

$$8[v_6] = -4[v_1].$$

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So, here, we have

$$F(6)[v_6] = -(F(5) - 1)[v_1].$$

A general conclusion, and Connection #1:

Repeating ... in $\mathrm{G}(C_6^2)$, $F(6)[v_6] = -(F(5)-1)[v_1]$.

More generally:

Let n be any positive integer. Then, in $\mathrm{G}(C_n^2)$,

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So the Fibonacci's Sequence and graph groups are connected!

Connection #2:

Notation: Denote the size of $\mathrm{G}(C_n^2)$ by $H_2(n)$.

Recall the sizes of the graph groups of the C_n^2 graphs $(1 \leq n \leq 12)$:

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Proposition: For all $n \geq 3$,

$$H_2(n) = \begin{cases} H_2(n-1) + H_2(n-2) & \text{if } \ n \text{ is even} \\ H_2(n-1) + H_2(n-2) + 2 & \text{if } \ n \text{ is odd} \end{cases}$$

So the H_2 sequence is "Fibonacci-ish".

Digression

Digression: The Online Encyclopedia of Integer Sequences

google: OEIS

Digression

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A note on Fermat's Last Theorem and the Mersenne Numbers,

in: Eureka: the Archimedeans' Journal, vol. 11, 1949, pp 19-22.

Connection #3

Now let's look at the Fibonacci sequence and the H_2 sequence side-by-side:

n	1	2	3	4	5	6	7	8	9	10	11	12
$F(n)$	1	1	2	3	5	8	13	21	34	55	89	144
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Proposition: For all $n \geq 2$,

$$H_2(n) = \begin{cases} F(n-1) + F(n+1) - 2 & \text{if } \ n \text{ is even} \\ F(n-1) + F(n+1) & \text{if } \ n \text{ is odd} \end{cases}$$

Connection #4

Along the way, the following numbers turn out to be of great interest. For each $n \geq 2$, define

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Proposition: For any positive integer m,

$$d(2m+1) = \begin{cases} 1 & \text{if } \ 2m+1 \equiv 1 \text{ or } 5 \mod 6 \\ 2 & \text{if } \ 2m+1 \equiv 3 \mod 6 \end{cases}$$

$$d(2m+2) = \begin{cases} F(m)+F(m+2) & \text{if } \ m \text{ is even} \\ F(m+1) & \text{if } \ m \text{ is odd} \end{cases}$$

So we have an explicit formula for $d(n)$ for all integers $n \geq 1$.

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so precisely when $d = 2$, or $d = 4$, or $d \equiv 1$ or $5 \mod 6$.

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And, we can use the description of $\mathrm{G}(E)$ (plus some other stuff) to get information about the structure of $L_K(E)$.

In particular, knowing the structure of $\mathrm{G}(C_n^2)$ gives really nice information about $L_K(C_n^2)$.

Aside: The Restricted Algebraic Kirchberg Phillips Theorem

Theorem. Let E and F be finite graphs and K any field. Suppose $L_K(E)$ and $L_K(F)$ are purely infinite simple. If

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The Biggest Currently Open Question in Leavitt path algebras:

Can we drop the hypotheses on the determinants?

Fortunately, things work out somewhat nicely here ...

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Proposition: For every n, the element $[1_{L_K(C_n^2)}]$ is the identity element of $K_0(L_K(C_n^2))$.

Proof: Easy to show that $[1_{L_K(C_n^2)}] + [1_{L_K(C_n^2)}] = [1_{L_K(C_n^2)}]$ in $K_0(L_K(C_n^2))$.

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As one consequence:

Proposition: Suppose $n = 2$, $n = 4$, $n \equiv 1 \mod 6$, or $n \equiv 5 \mod 6$. Then

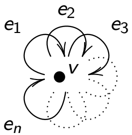
$$L_K(C_n^2) \cong \mathrm{M}_{n-1}(L_K(1,n)),$$

where $L_K(1,n)$ is the classical "Leavitt algebra of order n.

Rephrased: in these cases,

$$L_K(C_n^2) \cong \mathrm{M}_{n-1}(L_K(R_n)),$$

where R_n is the graph



What's next?

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Can we describe $\mathrm{G}(C_n^3)$??

Currently under consideration by Gonzalo and his Ph.D. student Cristobal Gil.

(Both will be visiting Colorado Springs in the near future ...)

Questions?

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Thank you.