The ubiquity of the Fibonacci Sequence: It comes up in Leavitt path algebras too!

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UCCS Math Department Colloquium

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Overview

1. Introduction and brief history

2. Monoids and groups from directed graphs

3. Here’s where Fibonacci comes in ...
1 Introduction and brief history

2 Monoids and groups from directed graphs

3 Here’s where Fibonacci comes in ...
Something familiar: The Fibonacci Sequence $F(n)$

**Fibonacci’s Rabbit Puzzle:** (from *Liber Abaci*, 1202)

Suppose you go to an uninhabited island with a pair of newborn rabbits (one male and one female), who:

1. mature at the age of one month,
2. have two offspring (one male and one female) each month after that, and
3. live forever.

Each pair of rabbits mature in one month and then produce a pair of newborns at the beginning of every following month. How many pairs of adult rabbits will there be in a year?
### Something familiar: The Fibonacci Sequence $F(n)$

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There is a “generating formula” for the Fibonacci sequence:

$$F(1) = 1; \quad F(2) = 1; \quad F(n) = F(n - 1) + F(n - 2) \quad \text{for all } n \geq 3.$$
Something familiar: The Fibonacci Sequence $F(n)$

The Fibonacci sequence comes up in lots of places ...

AND is VERY well-studied!

For instance,

Theorem: $\text{g.c.d.}(F(n), F(m)) = F(\text{g.c.d.}(n, m))$.

Corollary: $\text{g.c.d.}(F(n), F(n - 1)) = 1$. 
Something familiar: The Fibonacci Sequence $F(n)$

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For instance,

Theorem: $\gcd(F(n), F(m)) = F(\gcd(n, m))$.

Corollary: $\gcd(F(n), F(n-1)) = 1$.

A site for all types of info about the Fibonacci sequence (more than 300 formulas):

Google: Ron Knott Fibonacci
Some algebraic structures arising from directed graphs

There’s a nice ‘intuitive’ approach to the algebraic structures (monoids, abelian groups) of interest here.
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**Mad Vet Bob’s Mad Vet Puzzle:** (from *The Internet*, 1998)

A mad veterinarian has created three animal transmogrifying machines.
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Mad Vet Bob’s Mad Vet Puzzle:  (from The Internet, 1998)

A mad veterinarian has created three animal transmogrifying machines.

Place a cat in the input bin of the first machine, press the button, and  \textit{whirr... bing!}  Open the output bins to find two dogs and five mice.

The second machine can convert a dog into three cats and three mice.

The third machine can convert a mouse into a cat and a dog. Each machine can also operate in reverse (e.g. if you’ve got two dogs and five mice, you can convert them into a cat).
Something not-as-familiar: Mad Vet Puzzles

You have one cat.
Something not-as-familiar: Mad Vet Puzzles

You have one cat.

1. Can you convert it into seven mice?
2. Can you convert it into a pack of dogs, with no mice or cats left over?
Something not-as-familiar: Mad Vet Puzzles

A site for more info about Mad Vet Puzzles (The 'Mad Bob' site):

Google: Mad Bob’s Mad Vet Puzzles
Something not-as-familiar: Mad Vet Puzzles

A site for more info about Mad Vet Puzzles (The 'Mad Bob' site):

Google: Mad Bob’s Mad Vet Puzzles

A New York Times Puzzle Blog:

Google: Numberplay: The Mad Veterinarian
Mad Vet scenarios

A *Mad Vet scenario* is a situation such as the one Mad Bob constructed.

We assume:

1. Each species is paired up with a machine;
2. Each machine can also operate in reverse; and
3. Each machine is “one to some”
Mat Vet Scenario #1

**Scenario #1.** Suppose a Mad Veterinarian has three machines with the following properties.

Machine 1 turns one Ant into one Beaver;

Machine 2 turns one Beaver into one Ant, one Beaver and one Cougar;

Machine 3 turns one Cougar into one Ant and one Beaver.
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\[ A \Rightarrow B \Rightarrow A, B, C \]
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So, for example,

\[ A \Rightarrow B \Rightarrow A, B, C \Rightarrow 2A, 2B \Rightarrow 3A, B \Rightarrow 4A \]
From Mad Vet Scenarios to graphs

Given any Mad Vet scenario, its corresponding Mad Vet graph is:
- a drawing ("directed graph"),
- consisting of points and arrows ("vertices" and "edges"),
which gives the info about what’s going on with the machines.
(We only draw the “forward” direction of the machines.)

Example. Mad Vet scenario #1 has the following Mad Vet graph.

Introduction and brief history
Monoids and groups from directed graphs
Here's where Fibonacci comes in ...

... and vice versa: from graphs to Mad Vet Scenarios

Example: Consider this directed graph:

\[ u \rightarrow v \rightarrow w \]
\[ v \rightarrow u \rightarrow w \]
\[ u \rightarrow v \rightarrow u \]

This graph would describe a Mad Vet Scenario with three species:
Urchins, Vermin, Warthogs

Machine 1: Urchin $\rightarrow$ Vermin, Warthog
Machine 2: Vermin $\rightarrow$ Urchin, Warthog
Machine 3: Warthog $\rightarrow$ Urchin, Vermin

Fibonacci’s rabbits visit the Mad Veterinarian

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Mad Vet equivalence

Some notation: Let’s say there are \( n \) different species. Choose some “order” to list them.

E.g., in Scenario #1, first list Ants, then Beavers, then Cougars.

Then any collection of animals corresponds to some \( n \)-vector, with entries taken from the set \( \{0, 1, 2, \ldots \} \).
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For instance, in Scenario #1 a collection of two Beavers and five Cougars would correspond to $(0, 2, 5)$. 
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We agree that an “empty” collection of animals is not of interest here. In other words, the vector $(0, 0, \ldots, 0)$ is not allowed.
There is a naturally arising relation \( \sim \) on these vectors:

Given \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \), we write

\[
a \sim b
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if there is a sequence of Mad Vet machine moves that will change the collection of animals associated with vector \( a \) into the collection of animals associated with vector \( b \).
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(Aside: Using the three properties of a Mad Vet scenario, it is straightforward to show that $\sim$ is an equivalence relation.)
Example. Suppose that our Mad Vet of Scenario #1 starts with one Ant; in other words, with \((1, 0, 0)\).

(Recall: Machine 1: \(A \rightarrow B\) Machine 2: \(B \rightarrow A, B, C\) Machine 3: \(C \rightarrow A, B\))

Then, rewriting in this new notation what we’ve already seen,

\[(1, 0, 0) \sim (0, 1, 0) \sim (1, 1, 1) \sim (2, 2, 0) \sim (3, 1, 0) \sim (4, 0, 0).\]

As a result, \((1, 0, 0) \sim (4, 0, 0).\) And \((4, 0, 0) \sim (1, 0, 0)\) too ...
Mad Vet equivalence

We denote the equivalence classes under $\sim$ by brackets. So, e.g., for this Mad Vet Scenario,

$$[(1, 0, 0)] = [(4, 0, 0)]$$
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$$[(1, 0, 0)] = [(4, 0, 0)] = [(0, 1, 0)] = [(1, 1, 1)] = [(2, 2, 0)] = [(3, 1, 0)] \cdots$$

Also, we have, for example

$$[(2, 0, 0)] = [(1, 1, 0)] = [(2, 1, 1)] = \cdots.$$
Mad Vet equivalence

(Recall: Machine 1: A \rightarrow B \quad Machine 2: B \rightarrow A, B, C \quad Machine 3: C \rightarrow A, B)

**Claim.** In Scenario #1, there are exactly three different “bracket vectors” of animals:

\[ \{ (1, 0, 0), (2, 0, 0), (3, 0, 0) \} \]
Mad Vet equivalence

(Recall: Machine 1: $A \rightarrow B$  Machine 2: $B \rightarrow A, B, C$  Machine 3: $C \rightarrow A,B$)

**Claim.** In Scenario #1, there are exactly three different “bracket vectors” of animals:

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**Reason.** It’s not hard to see that any $[(a, b, c)]$ is equivalent to one of $[(1,0,0)]$, $[(2,0,0)]$, or $[(3,0,0)]$.
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Showing that these three brackets are different takes some (straightforward) work.
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Semigroups, monoids, and groups

Reminder / review of notation.

1. **semigroup**: associative operation.
Semigroups, monoids, and groups

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Fibonacci's rabbits visit the Mad Veterinarian

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   e.g., $\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, \ldots\}$ under addition.
   e.g., for any positive integer $n$, the “clock arithmetic group”
   $\mathbb{Z}_n = \{1, 2, \ldots, n\}$, under addition mod $n$. 
Semigroups, monoids, and groups

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   e.g., \( \mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, \ldots\} \) under addition.
   e.g., for any positive integer \( n \), the “clock arithmetic group” \( \mathbb{Z}_n = \{1, 2, \ldots, n\} \), under addition mod \( n \).
   e.g., for any positive integers \( m, n \), the “direct product” (i.e., ordered pairs) \( \mathbb{Z}_m \times \mathbb{Z}_n \).
Mad Vet semigroups

Start with a Mad Vet scenario. Define an addition process on bracket vectors:

\[[x] + [y] = [x + y].\]

Interpret as “unions” of collections of animals. This operation makes the set of bracket vectors a semigroup. (Actually, a commutative semigroup.)

**Example.** (Scenario #1:  M 1: A → B    M 2: B → A, B, C    M 3: C → A,B)

The bracket vectors are \{[(1, 0, 0)], [(2, 0, 0)], [(3, 0, 0)]\}.

We get, for instance,

\[((1, 0, 0)] + [(1, 0, 0)] = [(1 + 1, 0, 0)] = [(2, 0, 0)],\]

as we’d expect.
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\[(1, 0, 0)] + [(3, 0, 0)] = [(4, 0, 0)] = [(1, 0, 0)].\]
Mad Vet semigroups

Similarly

\[(2, 0, 0) + [(3, 0, 0)] = [(2, 0, 0)], \text{ and } [(3, 0, 0)] + [(3, 0, 0)] = [(3, 0, 0)].\]
Mad Vet semigroups

Similarly

\[(2, 0, 0)] + [(3, 0, 0)] = [(2, 0, 0)], \text{ and } [(3, 0, 0)] + [(3, 0, 0)] = [(3, 0, 0)].\]

So for this Mad Vet scenario the Mad Vet semigroup is a monoid, with identity \([(3, 0, 0)]\).
Mad Vet semigroups

Actually, since \([(1, 0, 0)] + [(2, 0, 0)] = [(3, 0, 0)]\), each of the three elements has an inverse.
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So the set of three bracket vectors for this Mad Vet Scenario forms a group,
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So the set of three bracket vectors for this Mad Vet Scenario forms a group, necessarily \(\mathbb{Z}_3\).

Notation remark:

Sometimes we write \([A]\) for \([(1,0,0)]\), \([B]\) for \([(0,1,0)]\), etc ...

So, e.g., the three bracket vectors in Scenario \#1 consist of the set \([^A], 2[A], 3[A]\).
Another Mad Vet Scenario (Scenario #2)

Here’s a Mad Vet with a different set of machines:

```
A
↓
C
↑
B
```


What are the Mad Vet bracket vectors here?

\[
\{ \[(1, 0, 0)], \[(0, 1, 0)], \[(0, 0, 1)], \[(1, 1, 1)] \}\]

Or, in the other notation, \[
\{ [A], [B], [C], [A] + [B] + [C] \}.
\]

Turns out: these also form a group, $\mathbb{Z}_2 \times \mathbb{Z}_2$. 
Mad Vet semigroups

There are Mad Vet Scenarios where the bracket vectors for that scenario do NOT form a group. For instance, the Mad Vet Scenario for this graph.

Here the bracket vectors behave like the set $\mathbb{N} = \{1, 2, 3, \ldots\}$. 
The SAME idea, just using different language ... 

Since the data of a Mad Vet semigroup can be thought of as coming from a directed graph, we usually use the phrase

*Graph semigroup*

rather than *Mad Vet* semigroup.

Notation: For the directed graph $\Gamma$, $\mathcal{V}^*(\Gamma)$ denotes the graph semigroup of $\Gamma$.

(In some cases we like to have the symbol $[(0,0,\ldots,0)]$ included in the discussion. The corresponding set of bracket vectors $\mathcal{V}^*(\Gamma) \sqcup [(0,0,\ldots,0)]$ is denoted $\mathcal{V}(\Gamma)$, and is called the *graph monoid* of $\Gamma$.)
Graph groups

A Big Question:

Given a directed graph $\Gamma$, when is $\mathcal{V}^*(\Gamma)$ a group?
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A Big Answer:
If you can walk from any vertex in $\Gamma$ to any cycle in $\Gamma$ by a sequence of edges, and you can 'step off' any cycle, and $\Gamma$ isn’t just a 'basic cycle', then $\mathcal{V}^*(\Gamma)$ is a group.

And vice versa.

“Graph group” of $\Gamma$ $\quad G(\Gamma)$. 
Graph groups

Recall the graphs of Scenarios #1 and #2

\[ \Gamma_1 = \]

\[ \Gamma_2 = \]
Graph groups

Another Big Question:

When the graph has the right properties so that $\mathcal{V}(\Gamma)$ is a group, what group is it ????
Graph groups

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Another Big Answer:

For today, suffice it to say that if you are given some specific graph $\Gamma$, then it is “easy” to write code (e.g., in Mathematica) which will easily tell you $G(\Gamma)$. (Matrix computations.)
1. Introduction and brief history

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3. Here’s where Fibonacci comes in ...
The “basic” cyclic graphs $C_n$

With the previous stuff as context, here’s a game we can play.

Take a collection of “similar” graphs $\Gamma_n$ for which, for each of the graphs, the corresponding $\mathcal{V}^*(\Gamma)$ is a group (denoted $G(\Gamma)$).

Here’s one way we might build these. For each $n \geq 1$, let $C_n$ be the “cycle” graph having $n$ vertices $\{v_1, v_2, \ldots, v_n\}$, and $n$ edges, like this:
The “basic” cyclic graphs $C_n$
The graph semigroups $\mathcal{V}(C_n)$ aren’t so nice in this context, because they don’t form groups.

But if we modify the $C_n$ graphs in appropriate ways, then we get graphs whose bracket vectors do form groups.

How can we do that?

We can add an extra edge at each vertex, in a systematic way.
The graphs $C_n^0$: 

$C_1^0 = \bullet_{v_1}$

$C_2^0 = \bullet_{v_1} \rightarrow \bullet_{v_2} \rightarrow \bullet_{v_1}$

$C_3^0 = \bullet_{v_1} \rightarrow \bullet_{v_2} \rightarrow \bullet_{v_3} \rightarrow \bullet_{v_1}$

$C_4^0 = \bullet_{v_1} \rightarrow \bullet_{v_2} \rightarrow \bullet_{v_3} \rightarrow \bullet_{v_4} \rightarrow \bullet_{v_1}$
For each $n$, $G(C_n^0)$ contains just one element.
For each $n$, $G(C_n^0)$ contains just one element.

Not too interesting.
The graphs $C_n^1$:

$C_1^1 = \bullet v_1$

$C_2^1 = \bullet v_1 \rightarrow \bullet v_2$

$C_3^1 = \bullet v_1 \rightarrow \bullet v_2 \rightarrow \bullet v_3 \rightarrow \bullet v_1$

$C_4^1 = \bullet v_1 \rightarrow \bullet v_2 \rightarrow \bullet v_3 \rightarrow \bullet v_4 \rightarrow \bullet v_1$

Here's where Fibonacci comes in ...

Fibonacci's rabbits visit the Mad Veterinarian

Gene Abrams, UCCS
For each $n$, 

$$G(C_1^n) \sim \mathbb{Z}_2^{n-1}.$$
For each $n$,

$$G(C^1_n) \cong \mathbb{Z}_{2^n-1}.$$
The graphs $C_n^{-1}$:

$C_1^{-1} = \bullet v_1$

$C_2^{-1} = \bullet v_1 \quad \bullet v_2$

$C_3^{-1} = \bullet v_1 \quad \bullet v_2 \quad \bullet v_3$

$C_4^{-1} = \bullet v_1 \quad \bullet v_2 \quad \bullet v_3 \quad \bullet v_4$
G(C_n) for various values of $n$. 
$G(C_n^{-1})$ for various values of $n$.

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G\left(C_{n}^{-1}\right) \text{ for various values of } n.

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| n | 13 | 14 | 15 | 16 | 17 | 18 |
\( G(C_n^{-1}) \) for various values of \( n \).

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Proposition. (A-, Ben Schoonmaker, to appear)
**Proposition.** (A-, Ben Schoonmaker, to appear)

It works.
The graphs $C_2^n$:

$C_2^1 = \begin{array}{c} v_1 \end{array}$

$C_2^2 = \begin{array}{c} v_1 \\ v_2 \end{array}$

$C_2^3 = \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array}$

$C_2^4 = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}$
Here are two more graphs in this sequence ...
Size of the graph group $G(C^2_n)$ for various values of $n$. 

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Let's do some sample computations in, say, $G(C_6^2)$.

\[
\begin{align*}
[v_1] &= [v_2] + [v_3] \\
     &= ([v_3] + [v_4]) + [v_3] \\% \\
     &= 2([v_3] + [v_4]) + [v_3] = 2[v_3] + [v_4] \\
     &= 2([v_4] + [v_5]) + [v_4] = 3[v_4] + 2[v_5] \\
     &= 3([v_5] + [v_6]) + 2[v_5] = 5[v_5] + 3[v_6] \\
     &= 5([v_6] + [v_1]) + 3[v_6] = 8[v_6] + 5[v_1]
\end{align*}
\]
So, in $G(C_6^2)$, we get

$$[v_1] = 8[v_6] + 5[v_1].$$

This gives

$$8[v_6] = -4[v_1].$$
So, in $G(C_6^2)$, we get

$$[v_1] = 8[v_6] + 5[v_1].$$

This gives

$$8[v_6] = -4[v_1].$$

So, here, we have

$$F(6)[v_6] = -(F(5) - 1)[v_1].$$
A general conclusion, and Connection #1:

Repeating ... in $G(C_6^2)$, $F(6)[v_6] = -(F(5) - 1)[v_1]$.

More generally:

Let $n$ be any positive integer. Then, in $G(C_n^2)$,
A general conclusion, and Connection #1:

Repeating ... in $G(C_6^2)$, $F(6)[\nu_6] = -(F(5) - 1)[\nu_1]$.

More generally:
Let $n$ be any positive integer. Then, in $G(C_n^2)$,

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More generally:

Let $n$ be any positive integer. Then, in $G(C_n^2)$,

$$F(n)[v_n] = -(F(n-1) - 1)[v_1].$$
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Repeating ... in $G(C_6^2)$, $F(6)[v_6] = -(F(5) - 1)[v_1]$.

More generally:

Let $n$ be any positive integer. Then, in $G(C_n^2)$,

$$F(n)[v_n] = -(F(n-1) - 1)[v_1].$$

So the Fibonacci’s Sequence and graph groups are connected!
Connection #2:

Notation: Denote the size of $G(C_n^2)$ by $H_2(n)$.

Recall the sizes of the graph groups of the $C_n^2$ graphs ($1 \leq n \leq 12$):

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**Proposition:** For all $n \geq 3$,

$$H_2(n) = \begin{cases} H_2(n-1) + H_2(n-2) & \text{if } n \text{ is even} \\ H_2(n-1) + H_2(n-2) + 2 & \text{if } n \text{ is odd} \end{cases}$$

So the $H_2$ sequence is “Fibonacci-ish”.
Digression

Digression: The Online Encyclopedia of Integer Sequences

googled OEIS
Digression

Digression: The Online Encyclopedia of Integer Sequences

googles: OEIS

Digression: C.B. Haselgrove

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A note on Fermat’s Last Theorem and the Mersenne Numbers,
Connection #3

Now let’s look at the Fibonacci sequence and the $H_2$ sequence side-by-side:

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Connection #3

Now let’s look at the Fibonacci sequence and the $H_2$ sequence side-by-side:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(n)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
<td>144</td>
</tr>
<tr>
<td>$H_2(n)$</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>11</td>
<td>16</td>
<td>29</td>
<td>45</td>
<td>76</td>
<td>121</td>
<td>199</td>
<td>320</td>
</tr>
</tbody>
</table>

**Proposition:** For all $n \geq 2$,

$$H_2(n) = \begin{cases} F(n - 1) + F(n + 1) - 2 & \text{if } n \text{ is even} \\ F(n - 1) + F(n + 1) & \text{if } n \text{ is odd} \end{cases}$$
Along the way, the following numbers turn out to be of great interest. For each $n \geq 2$, define

$$d(n) = \text{g.c.d.}(F(n), F(n-1) - 1)$$
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$$d(n) = \text{g.c.d.}(F(n), F(n-1) - 1)$$

What are these numbers?

(reminder: $\text{g.c.d.}(0, m) = m$ for any positive integer $m$)

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<tbody>
<tr>
<td>1</td>
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<tr>
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Connection #4

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\[
d(n) = \gcd(F(n), F(n-1) - 1)
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What are these numbers?

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**Connection #4**

The $d(n)$ sequence had arisen in another context; and no explicit formula was given for it ... (see O.E.I.S.)

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\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
  n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
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\hline
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\hline
\end{array}
\]

**Proposition**: For any positive integer \( m \),

\[
d(2m + 1) = \begin{cases} 
1 & \text{if } 2m + 1 \equiv 1 \text{ or } 5 \mod 6 \\
2 & \text{if } 2m + 1 \equiv 3 \mod 6 
\end{cases}
\]

\[
d(2m + 2) = \begin{cases} 
F(m) + F(m + 2) & \text{if } m \text{ is even} \\
F(m + 1) & \text{if } m \text{ is odd} 
\end{cases}
\]

So we have an explicit formula for \( d(n) \) for all integers \( n \geq 1 \).
Connection #5

Lemma: \((d(n))^2\) divides \(H_2(n)\) for all \(n\).
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**Theorem** (A-, Gonzalo Aranda Pino, to appear) For any integer \(n\),

\[
G(C_n^2) \cong \mathbb{Z}_{d(n)} \times \mathbb{Z}_{\frac{H_2(n)}{d(n)}}.
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In particular, \(G(C_n^2)\) is cyclic precisely when \(d(n) = 1\),
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Connection #6

So ... Who Cares?
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$$K_0(L_K(E)) \cong G(E).$$

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And, we can use the description of $G(E)$ (plus some other stuff) to get information about the structure of $L_K(E)$.

In particular, knowing the structure of $G(C_n^2)$ gives really nice information about $L_K(C_n^2)$. 
Aside: The Restricted Algebraic Kirchberg Phillips Theorem

**Theorem.** Let $E$ and $F$ be finite graphs and $K$ any field. Suppose $L_K(E)$ and $L_K(F)$ are purely infinite simple. If

$$K_0(L_K(E)) \cong K_0(L_K(F))$$

via an isomorphism for which $[1_{L_K(E)}] \mapsto [1_{L_K(F)}]$, and the signs of $\det(I - A_E)$ and $\det(I - A_F)$ are equal, then $L_K(E) \cong L_K(F)$.

The Biggest Currently Open Question in Leavitt path algebras: Can we drop the hypotheses on the determinants?
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Can we drop the hypotheses on the determinants?
Fortunately, things work out somewhat nicely here ...

**Proposition**: For every $n$, $\det(I - A_{C_n^2}) \leq 0$.

Proof: Uses 'circulant matrices', and some elementary trigonometry.
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Proposition: For every $n$, the element $[1_{L_K}(C_n^2)]$ is the identity element of $K_0(L_K(C_n^2))$.

Proof: Easy to show that $[1_{L_K}(C_n^2)] + [1_{L_K}(C_n^2)] = [1_{L_K}(C_n^2)]$ in $K_0(L_K(C_n^2))$. 
Fortunately, things work out somewhat nicely here ...

As one consequence:

**Proposition:** Suppose \( n = 2, n = 4, n \equiv 1 \mod 6, \) or \( n \equiv 5 \mod 6. \) Then

\[
L_K(C_n^2) \cong M_{n-1}(L_K(1, n)),
\]

where \( L_K(1, n) \) is the classical “Leavitt algebra of order \( n. \)

Rephrased: in these cases,

\[
L_K(C_n^2) \cong M_{n-1}(L_K(R_n)),
\]

where \( R_n \) is the graph

\[
\begin{array}{c}
\bullet \\
|\downarrow| \\
|\downarrow| \\
|\downarrow| \\
|\downarrow| \\
\end{array}
\]

\[
\begin{array}{c}
e_1 \\
e_2 \\
e_3 \\
e_n \\
\end{array}
\]

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Can we describe $G(C_3^n)$? Currently under consideration by Gonzalo and his Ph.D. student Cristobal Gil. (Both will be visiting Colorado Springs in the near future ...)

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Thank you.