Symbolic dynamics and Leavitt path algebras: The Algebraic KP Question

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Overview

1. Leavitt path algebras: Introduction / refresher
2. Some things we know ...
3. Some things we don't (yet) know ...

General path algebras

Let E be a directed graph. $E = (E^0, E^1, r, s)$

$$\bullet^{s(e)} \xrightarrow{\ e\ } \bullet^{r(e)}$$

The *path algebra* KE is the K-algebra with basis $\{p_i\}$ consisting of the directed paths in E. (View vertices as paths of length 0.)

$$\text{In } KE,\ \ p \cdot q = pq \ \text{ if } \ r(p) = s(q), \quad 0 \text{ otherwise.}$$

$$\text{In particular, } \ s(e) \cdot e = e = e \cdot r(e).$$

Note: E^0 is finite $\Leftrightarrow$ KE is unital; in this case $1_{KE} = \sum_{v \in E^0} v$.

Building Leavitt path algebras

Start with E, build its *double graph* $\widehat{E}$.

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$$E \;=\; \begin{tikzcd} & \bullet^{t} & \bullet^{u} \arrow[dl, "h"'] \\ \bullet^{v} \arrow[ur, "e"] \arrow[r, "f"'] \arrow[r, bend right, "g"'] & \bullet^{w} \arrow[loop below, "i"] \arrow[r, "j"'] & \bullet^{x} \end{tikzcd}$$

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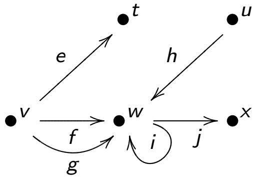
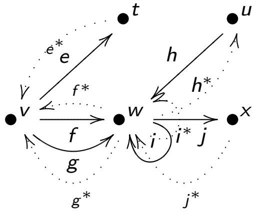


$\widehat{E} \ =$



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(CK1) $e^*e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$.

(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for all $v \in E^0$

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Definition

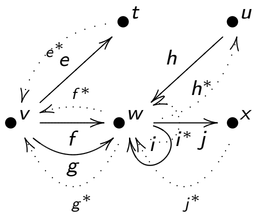
The Leavitt path algebra of E with coefficients in K

$$L_K(E) = K\widehat{E} \ / \ < (CK1), (CK2) >$$

Leavitt path algebras: Examples

Some sample computations in $L_{\mathbb{C}}(E)$ from the Example:

$\widehat{E} =$



$$ee^* + ff^* + gg^* = v \quad g^*g = w \quad g^*f = 0$$

$$h^*h = w \quad hh^* = u \quad ff^* = \ldots \text{ (no simplification)}$$

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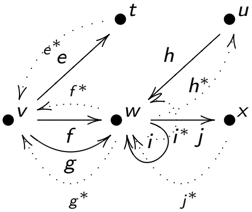

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$$\text{But } (ff^*)^2 = f(f^*f)f^* = f \cdot r(f) \cdot f^* = ff^*.$$

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Then $L_K(E) \cong M_n(K)$.

$$E = \bullet^{v} \circlearrowright x$$

Then $L_K(E) \cong K[x, x^{-1}]$.

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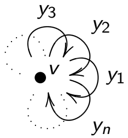
$E = R_n =$ 

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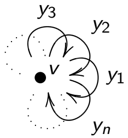
$E = R_n =$ 

Then $L_K(E) \cong L_K(1,n)$, the classical "Leavitt algebra of order n".
$L_K(1,n)$ is generated by $y_1, \ldots, y_n, x_1, \ldots, x_n$, with relations

$$x_i y_j = \delta_{i,j} 1_K \quad \text{and} \quad \sum_{i=1}^{n} y_i x_i = 1_K.$$

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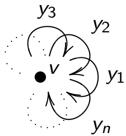
$E = R_n =$ 

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Note: $A = L_K(1, n)$ has ${}_A A \cong {}_A A^n$ as left A-modules:

$$a \mapsto (ay_1, ay_2, ..., ay_n) \text{ and } (a_1, a_2, ..., a_n) \mapsto \sum_{i=1}^{n} a_i x_i.$$

A property of the Leavitt algebras $L_K(1, n)$

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A unital algebra A having this property is called *purely infinite simple.*

There is a module-theoretic description of these algebras:

An idempotent $e \in A$ is called *infinite* if there exist NONZERO idempotents $f, g \in A$ for which $Ae \cong Af \oplus Ag$, and for which $Ae \cong Af$.

Proposition: A is purely infinite simple if and only if every nonzero left ideal of A contains an infinite idempotent.

Things we know about Leavitt path algebras

The main goal in the early years of the development: Establish results of the form

$$L_K(E) \text{ has algebraic property } \mathcal{P} \iff$$
$$E \text{ has graph-theoretic property } \mathcal{Q}.$$

(Only recently has the structure of K played a role.)

Things we know about Leavitt path algebras

We know precisely the graphs E for which $L_K(E)$ has these properties: (No role played by the structure of K in any of these.)

1. simplicity
2. purely infinite simplicity
3. (one-sided) artinian; (one-sided) noetherian
4. (two-sided) artinian; (two-sided) noetherian
5. exchange
6. prime
7. primitive

Things we know about Leavitt path algebras

Specifically:

Theorem: $L_K(E)$ is purely infinite simple if and only if E has:

1. every vertex in E connects to every cycle of E,
2. every cycle in E has an *exit*, and
3. E contains at least one cycle.

So this generalizes Leavitt's result.

The monoid $\mathcal{V}(R)$, and the Grothendieck group $K_0(R)$

Isomorphism classes of finitely generated projective (left) R-modules, with operation $\oplus$, denoted $\mathcal{V}(R)$.

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Theorem

(George Bergman, Trans. A.M.S. 1975) Given a field K and finitely generated conical monoid with a distinguished element, there exists a universal K-algebra R for which $\mathcal{V}(R) \cong S$.

The construction is explicit, uses amalgamated products.

Bergman included the algebras $L_K(1, n)$ as examples of these universal algebras.

The monoid $\mathcal{V}(L_K(E))$

For any graph E construct the free abelian monoid M_E.

$$\text{generators} \quad E^0; \qquad \text{relations} \quad v = \sum_{r(e)=w} w$$

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Using Bergman's construction,

Theorem

(Ara, Moreno, Pardo, Alg. Rep. Thy. 2007)

For any field K,

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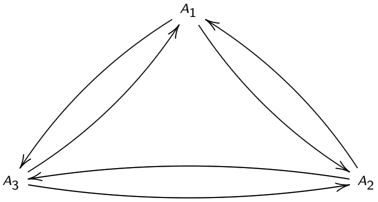
Example.



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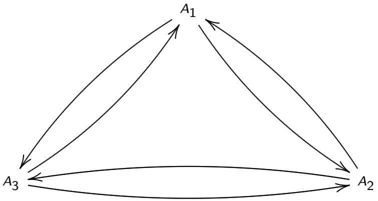

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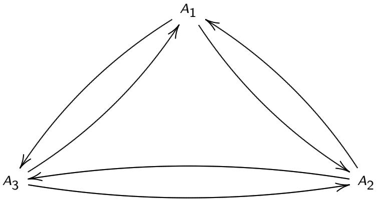

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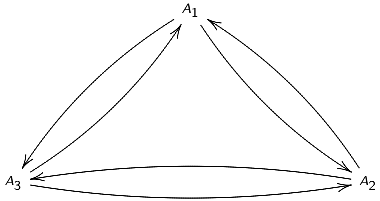

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Theorem. $\mathcal{V}(L_K(E)) \setminus \{0\}$ is a group if and only if $L_K(E)$ is purely infinite simple. In this case $\mathcal{V}(L_K(E)) \setminus \{0\}$ is $K_0(L_K(E))$.

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The answer involves E *and* $\mathrm{char}(K)$.

Tensor products of Leavitt path algebras

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This has been answered in the negative.

THREE different proofs given, independently, in Spring 2011:

1. J. Bell + G. Bergman
2. W. Dicks
3. P. Ara + G. Cortiñas

Ara / Cortiñas showed more: if the tensor product of n nontrivial Leavitt path algebras is isomorphic to the tensor product of m nontrivial Leavitt path algebras, then $m = n$.

The realization question for von Neumann regular rings

Question (2): Ara + Goodearl are still working on this.

The Algebraic KP Question

Question (1): $L_K(E) \cong L_K(F) \Leftrightarrow \quad ?\ ?\ ?$

It's fair to say that this question is the Holy Grail for most in the Leavitt path algebra community.

1. Leavitt path algebras: Introduction / refresher

2. Some things we know ...

3. Some things we don't (yet) know ...

$L_K(E) \cong L_K(F) \Leftrightarrow$? ? ?

There are easy examples to show that different graphs E and F can produce isomorphic Leavitt path algebras.

Proposition: Suppose E is a finite graph which contains no (directed) closed paths. Let $v_1, v_2, ..., v_t$ denote the sinks of E. (At least one must exist.) For each $1 \leq i \leq t$, let n_i denote the number of paths in E which end in v_i. Then

$$L_K(E) \cong \oplus_{i=1}^{t} \mathrm{M}_{n_i}(K).$$

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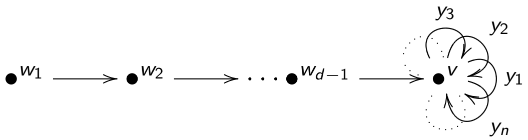
For instance: If

$$E = \bullet \longrightarrow \bullet \longrightarrow \bullet \quad \text{and} \quad F = \bullet \longrightarrow \bullet \longleftarrow \bullet$$

then E and F are not isomorphic as graphs, but $L_K(E) \cong L_K(F) \cong \mathrm{M}_3(K)$.

$L_K(E) \cong L_K(F) \Leftrightarrow$? ? ?

Let $R_n(d)$ denote this graph:



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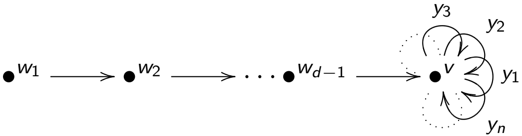

Proposition:

$$L_K(R_n(d)) \cong \mathrm{M}_d(L_K(1,n)).$$

$L_K(E) \cong L_K(F) \Leftrightarrow \quad ? ? ?$

Recall that $A = L_K(1, n)$ has ${}_AA \cong {}_AA^n$ as left A-modules. So, in particular, $A \cong \mathrm{M}_n(A)$. But then

$$L_K(R_n) \cong L_K(1, n) \cong \mathrm{M}_n(L_K(1, n)) \cong L_K(R_n(n)),$$

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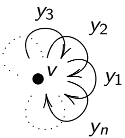
so that the Leavitt path algebras of these two graphs are isomorphic:



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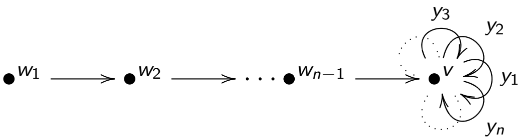

$L_K(E) \cong L_K(F) \Leftrightarrow\ ?\ ?\ ?$

More generally: for what values of n, n', d, d' do we have

$$L_K(R_n(d)) \cong L_K(R_{n'}(d'))?$$

$L_K(E) \cong L_K(F) \Leftrightarrow$? ? ?

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Theorem

(A-, Ánh, Pardo; Crelle's J. 2008) For any field K,

$$\mathrm{M}_d(L_K(1,n)) \cong \mathrm{M}_{d'}(L_K(1,n')) \Leftrightarrow$$

$$n = n' \text{ and } g.c.d.(d, n-1) = g.c.d.(d', n-1).$$

(Moreover, we can write down the isomorphisms explicitly.)

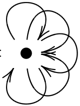
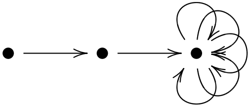
Matrices over Leavitt algebras

Breakthrough came from an analysis of isomorphisms between more general Leavitt path algebras.

There are a few "graph moves" which preserve the isomorphism classes of certain types of Leavitt path algebras.

"Shift" and "outsplitting".

Matrices over Leavitt algebras

$E = R_5 =$  and $R_5(3) =$ 

There exists a sequence of graphs

$$R_5 = E_1, E_2, ..., E_7 = R_5(3)$$

for which E_{i+1} is gotten from E_i by one of these two "graph moves".

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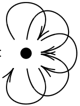
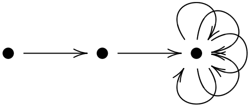
$E = R_5 =$  and $R_5(3) =$ 

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for which E_{i+1} is gotten from E_i by one of these two "graph moves".

$$\text{So} \quad L_K(R_5) \cong L_K(E_2) \cong \cdots \cong L_K(R_5(3)) \cong \mathrm{M}_3(R_5).$$

Note: For $2 \leq i \leq 6$ it is not immediately obvious how to view $L_K(E_i)$ in terms of a matrix ring over a Leavitt algebra.

Once we parsed out what was happening with this particular set of moves, we were able to see how to do things in general.

Application to the theory of simple groups

Brief digression:

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For each pair of positive integers n, r, there exists an infinite, finitely presented simple group $G_{n,r}^+$. "Higman Thompson groups."

Higman knew *some* conditions regarding isomorphisms between these groups, but did not have a complete classification.

Application to the theory of simple groups

Theorem. (E. Pardo, 2011)

$$G^+_{n,r} \cong G^+_{m,s} \quad \Leftrightarrow \quad m = n \text{ and } \mathrm{g.c.d.}(r, n-1) = \mathrm{g.c.d.}(s, n-1).$$

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Theorem. (E. Pardo, 2011)

$$G^+_{n,r} \cong G^+_{m,s} \quad \Leftrightarrow \quad m = n \text{ and } \mathrm{g.c.d.}(r, n-1) = \mathrm{g.c.d.}(s, n-1).$$

Idea of Proof. Show that $G^+_{n,r} \cong U_r(n)$ (an explicitly described subgroup of the units of $\mathrm{M}_r(L_K(1,n))$), and that the explicit isomorphisms provided in the A -, Ánh, Pardo result take $U_r(n)$ onto $U_s(n)$.

Connections and Applications: C^*-algebras

Theorem

(Cuntz, Comm. Math. Physics, 1977) There exist simple C^*-algebras generated by partial isometries.

Denote by $\mathcal{O}_n$.

Subsequently, a similar construction was produced of the "graph C^*-algebra" $C^*(E)$, for any graph E. In this context, $\mathcal{O}_n \cong C^*(R_n)$.

Connections and Applications: C*-algebras

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Subsequently, a similar construction was produced of the "graph C*-algebra" $C^*(E)$, for any graph E. In this context, $\mathcal{O}_n \cong C^*(R_n)$.   For any graph E,

$$L_{\mathbb{C}}(E) \subseteq C^*(E)$$

as a dense $*$-subalgebra. In particular, $L_{\mathbb{C}}(1, n) \subseteq \mathcal{O}_n$.

(But $C^*(E)$ is usually "much bigger" than $L_{\mathbb{C}}(E)$.)

Connections and Applications: C^*-algebras

Properties of C^*-algebras. These typically include topological considerations.

Connections and Applications: C^*-algebras

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1. simple
2. purely infinite simple
3. stable rank, prime, primitive, exchange, etc....

Connections and Applications: C*-algebras

For a vast number of (but not all) properties ...

$$L_{\mathbb{C}}(E) \text{ has (algebraic) property } \mathcal{P} \iff C^*(E) \text{ has (topological) property } \mathcal{P}.$$

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Still no good understanding as to *Why*.

Note: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

The Algebraic Kirchberg Phillips Question

Kirchberg and Phillips (2000) each proved this deep result:

KP Theorem for C*-algebras: Suppose A and B are C*-algebras which are:

1. unital
2. simple
3. purely infinite
4. separable
5. nuclear
6. in the "bootstrap class"

Suppose there is an isomorphism $\varphi : K_0(A) \to K_0(B)$ for which $\varphi([A]) = [B]$, and suppose $K_1(A) \cong K_1(B)$. Then $A \cong B$ (homeomorphically).

The Algebraic Kirchberg Phillips Question

In the case of graph C*-algebras, necessarily some of these hypotheses are automatically satisfied. The KP Theorem becomes:

KP Theorem for graph C*-algebras: Suppose E and F are finite graphs for which $C^*(E)$ and $C^*(F)$ are purely infinite simple. Suppose there is an isomorphism $\varphi : K_0(C^*(E)) \to K_0(C^*(F))$ for which $\varphi([C^*(E)]) = [C^*(F)]$. Then $C^*(E) \cong C^*(F)$ (homeomorphically).

The Algebraic Kirchberg Phillips Question

It turns out that:

1) $K_0(L_K(E)) \cong K_0(C^*(E))$ for any finite graph E.

2) The K_1 data for $L_K(E)$ and $C^*(E)$ does not necessarily match up. But: if $L_K(E)$ and $L_K(F)$ are unital purely infinite simple, then

$$K_0(L_K(E)) \cong K_0(L_K(F)) \ \Rightarrow \ K_1(L_K(E)) \cong K_1(L_K(F)).$$

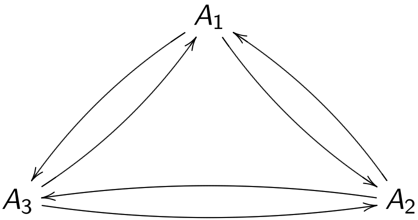
3) The K_0 groups are easily described in terms of the adjacency matrix A_E of E. Let $n = |E^0|$. View $I_n - A_E^t$ as a linear transformation $\mathbb{Z}^n \to \mathbb{Z}^n$. Then

$$K_0(L_K(E)) \cong K_0(C^*(E)) \cong \mathrm{Coker}(I_n - A_E^t).$$

Moreover, $\mathrm{Coker}(I_n - A_E^t)$ can be computed by finding the Smith normal form of $I_n - A_E^t$.

The Algebraic Kirchberg Phillips Question

Example:



$$I_3 - A_E^t = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \text{ whose Smith normal form is: } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Conclude that $K_0(L_K(E) \cong \mathrm{Coker}(I_3 - A_E^t) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The Algebraic Kirchberg Phillips Question

The question becomes: Can information about K_0 be used to establish isomorphisms between Leavitt path algebras as well?

The Algebraic KP Question: Suppose E and F are finite graphs for which $L_K(E)$ and $L_K(F)$ are purely infinite simple. Suppose also that there exists an isomorphism $\varphi : K_0(L_K(E)) \to K_0(L_K(F))$ for which $\varphi([L_K(E)]) = [L_K(F)]$.

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$$\text{Is } L_K(E) \cong L_K(F)?$$

Connections to symbolic dynamics

VERY informally:

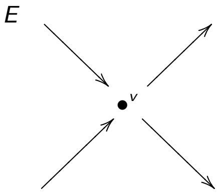
Some mathematicians and computer scientists have interest in, roughly, how information "flows" through a directed graph.

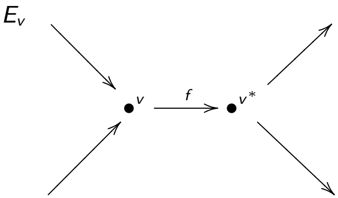
Makes sense to ask: When is it the case that information flows through two different graphs in essentially the same way? "Flow equivalent graphs".

(Often cast in the language of matrices.)

Connections to symbolic dynamics

Example: "Expansion at v"





Proposition: If E_v is the expansion graph of E at v, then E and E_v are flow equivalent. Rephrased, "expansion" (and its inverse "contraction") preserve flow equivalence.

Connections to symbolic dynamics

There are four other 'graph moves' which preserve flow equivalence:

out-split (and its inverse out-amalgamation), and

in-split (and its inverse in-amalgamation).

Theorem PS (Parry / Sullivan): Two graphs E, F are flow equivalent if and only if one can be gotten from the other by a sequence of transformations involving these six graph operations.

Connections to symbolic dynamics

Graph transformations may be reformulated in terms of adjacency matrices.

For an $n \times n$ matrix M with integer entries, think of M as a linear transformation $M : \mathbb{Z}^n \to \mathbb{Z}^n$. In particular, when $M = I_n - A_E^t$.

Connections to symbolic dynamics

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For an $n \times n$ matrix M with integer entries, think of M as a linear transformation $M : \mathbb{Z}^n \to \mathbb{Z}^n$. In particular, when $M = I_n - A_E^t$.

Proposition (Parry / Sullivan): If E is flow equivalent to F, then $\det(I - A_E^t) = \det(I - A_F^t)$.

Proposition (Bowen / Franks): If E is flow equivalent to F, then $\mathrm{Coker}(I - A_E^t) \cong \mathrm{Coker}(I - A_F^t)$.

Connections to symbolic dynamics

Theorem F (Franks): Suppose E and F have some additional properties (*irreducible, essential, nontrivial*). If

$$\mathrm{Coker}(I - A_E^t) \cong \mathrm{Coker}(I - A_F^t) \quad \text{and} \quad \det(I - A_E^t) = \det(I - A_F^t),$$

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So by Theorem PS, if the Cokernels and determinants match up correctly, then there is a sequence of "well-understood" graph transformations which starts with E and ends with F.

Connections to symbolic dynamics

Proposition: E is irreducible, essential, and non-trivial if and only if E has no sources and $L_K(E)$ is purely infinite simple.

Connections to symbolic dynamics

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Theorem: Suppose E is a graph for which $L_K(E)$ is purely infinite simple. Suppose F is gotten from E by doing one of the six "flow equivalence" moves. Then $L_K(E)$ and $L_K(F)$ are Morita equivalent.

In addition, the "source elimination" process also preserves Morita equivalence of the Leavitt path algebras.

Proof: Show that an isomorphic copy of $L_K(E)$ can be viewed as a (necessarily full, by simplicity) corner of $L_K(F)$ (or vice-versa).

Connections to symbolic dynamics

But (recall) that when $L_K(E)$ is purely infinite simple, then $K_0(L_K(E)) \cong \mathrm{Coker}(I_{|E^0|} - A_E^t)$.

Connections to symbolic dynamics

But (recall) that when $L_K(E)$ is purely infinite simple, then $K_0(L_K(E)) \cong \mathrm{Coker}(I_{|E^0|} - A_E^t)$. Consequently:

Theorem: (A- / Louly / Pardo / Smith 2011): Suppose $L_K(E)$ and $L_K(F)$ are purely infinite simple. If

$$K_0(L_K(E)) \cong K_0(L_K(F)) \quad \text{and} \quad \det(I - A_E^t) = \det(I - A_F^t),$$

then $L_K(E)$ and $L_K(F)$ are Morita equivalent.

Connections to symbolic dynamics

Using some intricate computations provided by Huang, one can show the following:

Suppose $L_K(E)$ is Morita equivalent to $L_K(F)$. Further, suppose there is *some* isomorphism $\varphi : K_0(L_K(E)) \to K_0(L_K(F))$ for which $\varphi([L_K(E)]) = [L_K(F)]$.

Then there is some Morita equivalence $\Phi : L_K(E){-}\mathrm{Mod} \to L_K(F){-}\mathrm{Mod}$ for which $\Phi|_{K_0(L_K(E))} = \varphi$.

Connections to symbolic dynamics

Consequently:

Theorem: (A- / Louly / Pardo / Smith 2011): Suppose $L_K(E)$ and $L_K(F)$ are purely infinite simple. If

$$K_0(L_K(E)) \cong K_0(L_K(F))$$

via an isomorphism φ for which $\varphi([L_K(E)]) = [L_K(F)]$,

$$\text{and} \quad \det(I - A_E^t) = \det(I - A_F^t),$$

then $L_K(E) \cong L_K(F)$.

'Restricted' Algebraic KP Theorem

Connections to symbolic dynamics

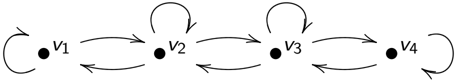
So the Algebraic KP Question can be rephrased:

Algebraic KP Question: Can we drop the hypothesis on the determinants in the Restricted Algebraic KP Theorem?

Connections to symbolic dynamics

Here's the "smallest" example of a situation of interest. Consider the Leavitt path algebras $L(R_2)$ and $L(E_4)$, where

$R_2 =$ 

and $E_4 =$ 

It is not hard to establish that

$$(K_0(L(R_2)), [1_{L(R_2)}]) = (\{0\}, 0) = (K_0(L(E_4)), [1_{L(E_4)}]);$$

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Here's the "smallest" example of a situation of interest. Consider the Leavitt path algebras $L(R_2)$ and $L(E_4)$, where

$$R_2 = \begin{tikzcd} \bullet^{v} \arrow[loop above] \arrow[loop right] \end{tikzcd} \quad \text{and} \quad E_4 = \begin{tikzcd} \bullet^{v_1} \arrow[loop left] \arrow[r, bend left] & \bullet^{v_2} \arrow[loop above] \arrow[l, bend left] \arrow[r, bend left] & \bullet^{v_3} \arrow[loop above] \arrow[l, bend left] \arrow[r, bend left] & \bullet^{v_4} \arrow[loop right] \arrow[l, bend left] \end{tikzcd}$$

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Question: Is $L_K(R_2) \cong L_K(E_4)$?

Connections to symbolic dynamics

Some concluding remarks:

1. $C^*(R_2) \cong C^*(E_4)$; but the isomorphism is NOT given explicitly, its existence is ensured by "KK Theory".

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3. Start with E for which $L_K(E)$ is purely infinite simple. There is a systematic (easy) way to produce a graph F for which $L_K(F)$ is purely infinite simple, $K_0(L_K(E)) \cong K_0(L_K(F))$, but $\det(I - A_E^t) = -\det(I - A_F^t)$. "Cuntz Splice".

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4. There are three possible outcomes to the Algebraic KP Question: NEVER, SOMETIMES, or ALWAYS. The answer will be interesting, no matter how things play out.

Conjecture?

Is there an Algebraic KP **Conjecture**?

Not really.

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More open questions about Leavitt path algebras were generated at a meeting at BIRS last April.

Questions?