Leavitt path algebras: Something for everyone
algebra, analysis, graph theory, number theory

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University of New Mexico Mathematics Colloquium

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Overview

1. Leavitt path algebras: Introduction and Motivation
2. Algebraic properties
3. Projective modules
4. Connections and Applications
1 Leavitt path algebras: Introduction and Motivation

2 Algebraic properties

3 Projective modules

4 Connections and Applications
Brief history, and motivating examples

An early theorem from undergraduate years:

**Dimension Theorem for Vector Spaces.** Every nonzero vector space $V$ has a basis. Moreover, if $\mathcal{B}$ and $\mathcal{B}'$ are two bases for $V$, then $|\mathcal{B}| = |\mathcal{B}'|$. 
Brief history, and motivating examples

An early theorem from undergraduate years:

**Dimension Theorem for Vector Spaces.** Every nonzero vector space $V$ has a basis. Moreover, if $B$ and $B'$ are two bases for $V$, then $|B| = |B'|$.

Note: $V$ has a basis $B = \{b_1, b_2, ..., b_n\} \iff V \cong \bigoplus_{i=1}^{n} \mathbb{R}$ as vector spaces. So:

**Dimension Theorem, Rephrased:**

$$\bigoplus_{i=1}^{n} \mathbb{R} \cong \bigoplus_{i=1}^{m} \mathbb{R} \iff m = n.$$
The same Dimension Theorem holds for $K$ any division ring.

For a module $P$ over a ring $R$ we can still talk about a basis for $P$. (Note: in general, not all modules have bases; those that do are called free $R$-modules.)

$$\bigoplus_{i=1}^{n} R$$ always has a basis having $n$ elements, e.g.,

$$\{ e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, 0, \ldots, 1) \}$$
Question: Is the Dimension Theorem true for rings in general?
That is, if $R$ is a ring, and $\bigoplus_{i=1}^{n} R \cong \bigoplus_{i=1}^{m} R$ as $R$-modules, must $m = n$?
**Question:** Is the Dimension Theorem true for rings in general? That is, if $R$ is a ring, and $\bigoplus_{i=1}^{n} R \cong \bigoplus_{i=1}^{m} R$ as $R$-modules, must $m = n$?

**Answer:** NO

(But the answer is YES for many rings, e.g.: $\mathbb{Z}$, $M_2(\mathbb{R})$, $C(\mathbb{R})$)

**Example:** Consider the ring $S$ of linear transformations from an infinite dimensional $\mathbb{R}$-vector space $V$ to itself.
Question: Is the Dimension Theorem true for rings in general? That is, if $R$ is a ring, and $\bigoplus_{i=1}^{n} R \cong \bigoplus_{i=1}^{m} R$ as $R$-modules, must $m = n$?

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(But the answer is YES for many rings, e.g.: $\mathbb{Z}$, $M_2(\mathbb{R})$, $C(\mathbb{R})$)

Example: Consider the ring $S$ of linear transformations from an infinite dimensional $\mathbb{R}$-vector space $V$ to itself.

Think of $V$ as $\bigoplus_{i=1}^{\infty} \mathbb{R}$. Then think of $S$ as $RFM(\mathbb{R})$. 
Intuitively, $S$ and $S \oplus S$ have a chance to be “the same”.

$M \mapsto (\text{Odd numbered columns of } M, \text{Even numbered columns of } M)$

Easy to find matrices $Y_1, Y_2$ for which this map is $M \mapsto (MY_1, MY_2)$.
Brief history, and motivating examples

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Easy to find matrices $Y_1, Y_2$ for which this map is $M \mapsto (MY_1, MY_2)$.

Similarly, we should be able to 'go back' from pairs of matrices to a single matrix, by interweaving the columns.

Easy to find matrices $X_1, X_2$ for which this map is $(M_1, M_2) \mapsto M_1X_1 + M_2X_2$. 
Brief history, and motivating examples

These equations are easy to verify:

\[ Y_1X_1 + Y_2X_2 = I, \]
\[ X_1Y_1 = I = X_2Y_2, \quad \text{and} \quad X_1Y_2 = 0 = X_2Y_1. \]
Brief history, and motivating examples

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Using these, we get inverse maps:

\[ S \rightarrow S \oplus S \quad \text{via} \quad M \mapsto (MY_1, MY_2), \quad \text{and} \]
\[ S \oplus S \rightarrow S \quad \text{via} \quad (M_1, M_2) \mapsto M_1X_1 + M_2X_2. \]

For example:

\[ M \mapsto (MY_1, MY_2) \mapsto MY_1X_1 + MY_2X_2 = M \cdot I = M. \]
Using exactly the same idea, let $R$ be ANY ring which contains four elements $y_1, y_2, x_1, x_2$ satisfying

$$y_1x_1 + y_2x_2 = 1_R,$$

$$x_1y_1 = 1_R = x_2y_2,$$

and $$x_1y_2 = 0 = x_2y_1.$$  

Then $R \cong R \oplus R$. 

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Leavitt path algebras: something for everyone
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Brief history, and motivating examples

Remark: Here the sets \{1_R\} and \{x_1, x_2\} are each bases for $R$.

Easily:

$$R \cong R \oplus R \implies \bigoplus_{i=1}^{m} R \cong \bigoplus_{i=1}^{n} R \text{ for all } m, n \in \mathbb{N}.$$
Leavitt algebras

Natural question:

Does there exist $R$ with, e.g., $R \cong R \oplus R \oplus R$, but $R \ncong R \oplus R$?
Leavitt algebras

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Does there exist $R$ with, e.g., $R \cong R \oplus R \oplus R$, but $R \not\cong R \oplus R$?

**Theorem**


For every $m < n \in \mathbb{N}$ and field $K$ there exists a $K$-algebra $R = L_K(m, n)$ with $\oplus_{i=1}^{m} R \cong \oplus_{i=1}^{n} R$, and all isomorphisms between free left $R$-modules result precisely from this one. Moreover, $L_K(m, n)$ is universal with this property.
Leavitt algebras

Similar to the (1, 2) situation above:

\[ R \cong R^n \text{ if and only if there exist } \]

\[ x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in R \]

for which

\[ \sum_{i=1}^{n} y_i x_i = 1_R \quad \text{and} \quad x_i y_j = \delta_{i,j} 1_R. \]
Leavitt algebras

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\( L_K(1, n) \) is the quotient

\[ K < X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n > \big/ < \left( \sum_{i=1}^{n} Y_i X_i \right) - 1_K; X_i Y_j - \delta_{i,j} 1_K > \]

Note: RFM(K) is much bigger than \( L_K(1, 2) \).
Leavitt algebras

As a result, we have: Let $S$ denote $L_K(1, n)$. Then

$$S^a \cong S^b \iff a \equiv b \mod (n - 1).$$

In particular, $S \cong S^n$. 

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**Theorem.** (Leavitt, Duke J. Math, 1964)

For every field $K$ and $n \geq 2$, $L_K(1, n)$ is simple.
Leavitt algebras

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For every field $K$ and $n \geq 2$, $L_K(1, n)$ is simple.

Simplicity means:

$$\forall 0 \neq r \in R, \exists \alpha_i, \beta_i \in R \text{ with } \sum_{i=1}^{n} \alpha_i r \beta_i = 1_R.$$
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$$\forall 0 \neq r \in R, \exists \alpha_i, \beta_i \in R \text{ with } \sum_{i=1}^{n} \alpha_i r \beta_i = 1_R.$$  

Actually, Leavitt proved that $L_K(1, n)$ satisfies a stronger property:

$$\forall 0 \neq r \in L_K(1, n), \exists \alpha, \beta \in L_K(1, n) \text{ with } \alpha r \beta = 1_{L_K(1,n)}.$$
Building rings from combinatorial objects

Many standard algebras can be built (essentially) as follows:

if $S$ is a semigroup (written multiplicatively) and $K$ is a field, form the *semigroup algebra* $KS$.

Symbols: finite sums $\sum_{s \in S} ks^s$.

Multiplication: Extend $ks \cdot k's' = kk'(ss')$. 

Example: $S = \{x_0, x_1, x_2, \ldots\}$, usual multiplication. $KS = K[x]$. 

Example: impose the relation $x^n = x_0$ on $S$, call the new semigroup $S'$. $K_{S'} \sim K[x]/\langle x^n - 1 \rangle$.

More Examples: matrix rings, group rings, incidence rings, multivariable polynomial rings, ...
Building rings from combinatorial objects

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More Examples: matrix rings, group rings, incidence rings, multivariable polynomial rings, ...
General path algebras

Let $E$ be a directed graph. (We will assume $E$ is finite for this talk, but analysis can be done in general.) $E = (E^0, E^1, r, s)$

The path algebra of $E$ with coefficients in $K$ is the $K$-algebra $KS$

$S =$ the set of all directed paths in $E$,

multiplication of paths is juxtaposition. Denote by $KE$. 
General path algebras

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![Diagram](image)

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multiplication of paths is juxtaposition. Denote by $KE$.

In particular, in $KE$,

for each edge $e$, \[ s(e) \cdot e = e = e \cdot r(e) \]

for each vertex $v$, \[ v \cdot v = v \]

\[ 1_{KE} = \sum_{v \in E^0} v. \]
Building Leavitt path algebras

Start with $E$, build its \textit{double graph} $\hat{E}$.
Building Leavitt path algebras

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$E =$

![Diagram of a double graph with nodes and arrows labeled e, h, f, g, i, j, t, u, v, w, x, and arrows connecting them.]

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Leavitt path algebras: something for everyone
Building Leavitt path algebras

Start with $E$, build its double graph $\hat{E}$. Example:

$$E = \begin{array}{c} v \\ t \\ w \\ f \\ g \\ h \\ u \\ x \\ f^* \\ g^* \\ h^* \end{array}$$

$$\hat{E} = \begin{array}{c} v \\ t \\ w \\ f \\ g \\ h \\ u \\ x \\ e \\ e^* \\ f^* \\ g^* \\ h^* \end{array}$$
Building Leavitt path algebras

Construct the path algebra $K\hat{E}$. 
Building Leavitt path algebras

Construct the path algebra $K\hat{E}$. Consider these relations in $K\hat{E}$:

1. $e^* e = r(e)$ for all edges $e$, $f$ in $E$.
2. $v = \sum\{ e \in E_1 | s(e) = v \} e^*$ for each (non-sink) vertex $v$ in $E$. 

Definition

The Leavitt path algebra of $E$ with coefficients in $K$ is $L(K) = K\hat{E}/<(C1), (C2)>$. 

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Building Leavitt path algebras

Construct the path algebra $K\hat{E}$. Consider these relations in $K\hat{E}$:

(CK1) $e^*e = r(e)$; $f^*e = 0$ for $f \neq e$ (for all edges $e, f$ in $E$).

(CK2) $v = \sum\{e \in E^1 | s(e) = v\} ee^*$ for each (non-sink) vertex $v$ in $E$. 
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**Definition**

The Leavitt path algebra of $E$ with coefficients in $K$

$L_K(E) = K\hat{E} / <(CK1), (CK2)>$
Leavitt path algebras: Examples

Some sample computations in $L_\mathbb{C}(E)$ from the Example:

$\hat{E} =$

\[
\begin{align*}
&ee^* + ff^* + gg^* = v \\
&g^*g = w \\
&g^*f = 0 \\
&h^*h = w \quad (CK1) \\
&hh^* = u \quad (CK2)
\end{align*}
\]
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$$\hat{E} =$$

$$ee^* + ff^* + gg^* = v \quad g^*g = w \quad g^*f = 0$$

$$h^*h = w \quad (CK1) \quad hh^* = u \quad (CK2)$$

$$ff^* = \ldots \quad \text{(no simplification)}$$

Note: $(ff^*)^2 = f(f^*f)f^* = ff^*$
Leavitt path algebras: Examples

Standard algebras arising as Leavitt path algebras:
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Standard algebras arising as Leavitt path algebras:

\[ E = \bullet v_1 \xrightarrow{e_1} \bullet v_2 \xrightarrow{e_2} \bullet v_3 \ldots \xrightarrow{e_{n-1}} \bullet v_n \]

Then \( L_K(E) \cong M_n(K) \).
Leavitt path algebras: Examples

Standard algebras arising as Leavitt path algebras:

\[ E = \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \rightarrow \cdots \rightarrow \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n} \]

Then \( L_K(E) \cong M_n(K) \).

\[ E = \bullet^{v} \xrightarrow{x} \bullet^{v} \]

Then \( L_K(E) \cong K[x, x^{-1}] \).
Leavitt path algebras: Examples

\[ E = R_n = \]

Then \( L_K(E) \cong L_K(1, n) \).
Leavitt path algebras: Examples

\[
E = R_n = \begin{array}{c}
  \bullet \\
  \y_1 \\
  \y_2 \\
  \y_3 \\
  \y_n
\end{array}
\]

Then \( L_K(E) \cong L_K(1, n) \).

\( L_K(1, n) \) has generators and relations:

\[
\begin{align*}
x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in L_K(1, n);
\end{align*}
\]
Leavitt path algebras: Examples

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\( L_K(1, n) \) has generators and relations:
\[ x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in L_K(1, n); \]
\[ \sum_{i=1}^{n} y_i x_i = 1_{L_K(1,n)}, \text{ and } x_i y_j = \delta_{i,j} 1_{L_K(1,n)}, \]
Leavitt path algebras: Examples

\[ E = R_n = \begin{array}{ccc} y_3 & y_2 & y_1 \\ v & \downarrow & \downarrow \\ y_n & \leftarrow & \leftarrow \\ y_1 & \uparrow & \uparrow \\ y_3 & \end{array} \]

Then \( L_K(E) \cong L_K(1, n) \).

\( L_K(1, n) \) has generators and relations:

\[ x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in L_K(1, n); \]

\[ \sum_{i=1}^{n} y_i x_i = 1_{L_K(1, n)}, \text{ and } x_i y_j = \delta_{i,j} 1_{L_K(1, n)}, \]

while \( L_K(R_n) \) has these SAME generators and relations, where we identify \( y_i^* \) with \( x_i \).
Historical note, part 1

1962: Leavitt gives construction of $L_K(1, n)$. 

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1977: Cuntz gives construction of the $C^*$-algebras $O_n$. 
Historical note, part 1

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1977: Cuntz gives construction of the $C^*$-algebras $O_n$.

1980’s: Cuntz, Krieger, and others generalize the $O_n$ construction to directed graphs, and produce the graph $C^*$-algebras $C^*(E)$. 
1962: Leavitt gives construction of $L_K(1, n)$.

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1980’s: Cuntz, Krieger, and others generalize the $O_n$ construction to directed graphs, and produce the graph $C^*$-algebras $C^*(E)$.

June 2004: Some algebraists attend the CBMS lecture series “Graph C*-algebras: algebras we can see”, held at University of Iowa, given by Iain Raeburn.

Algebraic analogs of graph $C^*$-algebras are defined and investigated starting Fall 2004.
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4. Connections and Applications
Some graph definitions

1. A cycle

\[
\begin{array}{c}
\bullet a \\
\downarrow x \\
\rightarrow y \\
\rightarrow b \\
\bullet b \\
\end{array}
\]
Some graph definitions

1. A cycle

2. An exit for a cycle.
Some graph definitions

3a. *connects to a vertex.*

![Diagram](attachment://diagram.png)

\[ \bullet^u \xrightarrow{} \bullet^v \xrightarrow{} \bullet^w \quad (\text{also } \bullet^w) \]
Some graph definitions

3a. *connects to a vertex.*

\[ \bullet^u \rightarrow \bullet^v \rightarrow \bullet^w \quad (\text{also} \quad \bullet^w) \]

3b. *connects to a cycle.*

\[ \bullet^a \xrightarrow{f} \bullet^c \quad \text{and} \quad \bullet^b \xleftarrow{x} \bullet^b \]

\[ \bullet^c \xrightarrow{y} \bullet^a \quad \text{and} \quad \bullet^b \xleftarrow{x} \bullet^c \]
Simplicity of Leavitt path algebras

Here’s a natural question, especially in light of Bill Leavitt’s result that \( L_K(1, n) \) is simple for all \( n \geq 2 \):
Simplicity of Leavitt path algebras

Here’s a natural question, especially in light of Bill Leavitt’s result that $L_K(1, n)$ is simple for all $n \geq 2$:

For which graphs $E$ and fields $K$ is $L_K(E)$ simple?
Simplicity of Leavitt path algebras

Here's a natural question, especially in light of Bill Leavitt's result that $L_K(1, n)$ is simple for all $n \geq 2$:

For which graphs $E$ and fields $K$ is $L_K(E)$ simple?

Note $L_K(E)$ is simple for

$$E = \bullet \rightarrow \bullet \rightarrow \rightarrow \bullet$$

since $L_K(E) \cong M_n(K)$

and for

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$$E = R_n = \begin{array}{c}
\bullet \\
y_1 \\
y_2 \\
y_3 \\
y_n
\end{array}$$

since $L_K(E) \cong L_K(1, n)$

but not simple for

$$E = R_1 = \bullet \begin{array}{c}
y_1
\end{array}$$

since $L_K(E) \cong K[x, x^{-1}]$
Simplicity of Leavitt path algebras

Theorem

(A -, Aranda Pino, 2005) $L_K(E)$ is simple if and only if:

1. Every vertex connects to every cycle and to every sink in $E$, and
2. Every cycle in $E$ has an exit.
Theorem

\[(\text{A -}, \text{Aranda Pino, 2005}) \ L_K(E) \text{ is simple if and only if:}\]

1. Every vertex connects to every cycle and to every sink in \(E\), and
2. Every cycle in \(E\) has an exit.

Note: No role played by \(K\).
Other ring-theoretic properties of Leavitt path algebras

We know precisely the graphs $E$ for which $L_K(E)$ has various other properties, e.g.:

1. one-sided chain conditions
2. prime
3. von Neumann regular
4. two-sided chain conditions
5. primitive

Many more.
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3. Projective modules

4. Connections and Applications
The monoid $\mathcal{V}(R)$

Recall: $P$ is a finitely generated projective $R$-module in case $P \oplus Q \cong R^n$ for some $Q$, some $n \in \mathbb{N}$. 
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Key example: $R$ itself, or any $R^n$. 
The monoid $\mathcal{V}(R)$

Recall: $P$ is a *finitely generated projective* $R$-module in case $P \oplus Q \cong R^n$ for some $Q$, some $n \in \mathbb{N}$.

Key example: $R$ itself, or any $R^n$.

Additional examples: $Rf$ where $f$ is idempotent (i.e., $f^2 = f$), since $Rf \oplus R(1-f) = R^1$.

So, for example, in $R = M_2(\mathbb{R})$, $P = M_2(\mathbb{R})e_{1,1} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$ is a finitely projective $R$-module. Note $P \not\cong R^n$ for any $n$. 
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So, for example, in $R = \mathbb{M}_2(\mathbb{R})$, $P = \mathbb{M}_2(\mathbb{R})e_{1,1} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$ is a finitely projective $R$-module. Note $P \not\cong R^n$ for any $n$.

So $L_K(E)$ contains projective modules of the form $L_K(E)ee^*$ for each edge $e$ of $E$. 
The monoid $\mathcal{V}(R)$

$\mathcal{V}(R)$ denotes the isomorphism classes of finitely generated projective (left) $R$-modules. With operation $\oplus$, this becomes an abelian monoid. Note $R$ itself plays a special role in $\mathcal{V}(R)$. 

Example. $R = K$, a field. Then $\mathcal{V}(R) \cong \mathbb{Z}^+$. 

Example. $S = \text{Mod}(K)$, $K$ a field. Then $\mathcal{V}(S) \cong \mathbb{Z}^+$. 

(But note that the 'position' of $S$ in $\mathcal{V}(S)$ is different than the position of $R$ in $\mathcal{V}(R)$.)

Remarks:
1. Given a ring $R$, it is in general not easy to compute $\mathcal{V}(R)$.
2. $K_0(R)$ is the universal group of $\mathcal{V}(R)$.
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**Example.** $S = M_d(K)$, $K$ a field. Then $\mathcal{V}(S) \cong \mathbb{Z}^+$. (But note that the 'position' of $S$ in $\mathcal{V}(S)$ is different than the position of $R$ in $\mathcal{V}(R)$.)

The monoid $\mathcal{V}(R)$

$\mathcal{V}(R)$ denotes the isomorphism classes of finitely generated projective (left) $R$-modules. With operation $\oplus$, this becomes an abelian monoid. Note $R$ itself plays a special role in $\mathcal{V}(R)$.

**Example.** $R = K$, a field. Then $\mathcal{V}(R) \cong \mathbb{Z}^+$. 

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**Remarks:**

(1) Given a ring $R$, it is in general not easy to compute $\mathcal{V}(R)$.

(2) $K_0(R)$ is the universal group of $\mathcal{V}(R)$. 

Gene Abrams University of Colorado @ Colorado Springs
The monoid $M_E$

Here’s a ‘natural’ monoid arising from any directed graph $E$.
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Associate to $E$ the abelian monoid $(M_E, +)$:

$$M_E = \left\{ \sum_{v \in E^0} n_v a_v \right\}$$

with $n_v \in \mathbb{Z}^+$ for all $v \in E^0$.

Relations in $M_E$ are given by:

$$a_v = \sum_{\{e | s(e) = v\}} a_r(e).$$
The monoid $M_E$

**Example.** Let $F$ be the graph

![Graph Image]

So $M_F$ consists of elements $\{n_1a_1 + n_2a_2 + n_3a_3\}$ ($n_i \in \mathbb{Z}^+$), subject to: $a_1 = a_2 + a_3$; $a_2 = a_1 + a_3$; $a_3 = a_1 + a_2$. 

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Leavitt path algebras: something for everyone
The monoid $M_E$

**Example.** Let $F$ be the graph

![Graph Diagram]

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It’s not hard to get:
The monoid $M_E$

**Example.** Let $F$ be the graph

![Graph diagram]

So $M_F$ consists of elements $\{n_1a_1 + n_2a_2 + n_3a_3\}$ ($n_i \in \mathbb{Z}^+$), subject to: $a_1 = a_2 + a_3; \ a_2 = a_1 + a_3; \ a_3 = a_1 + a_2$.

It’s not hard to get: $M_F = \{0, \ a_1, \ a_2, \ a_3, \ a_1 + a_2 + a_3\}$.

In particular, $M_F \setminus \{0\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. 
The monoid $\mathcal{V}(L_K(E))$

**Example:**

$$E = R_n = \begin{array}{c}
\vdots \\
y_3 \\
y_2 \\
y_1 \\
y_n \\
\vdots
\end{array}$$

Then $M_E$ is the set of symbols of the form

$$n_1 a_v \quad (n_1 \in \mathbb{Z}^+)$$

subject to the relation: $a_v = na_v$
The monoid $\mathcal{V}(L_K(E))$

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Then $M_E$ is the set of symbols of the form

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subject to the relation: $a_v = na_v$

So here, $M_E = \{0, a_v, 2a_v, ..., (n - 1)a_v\}$.

In particular, $M_E \setminus \{0\} \cong \mathbb{Z}_{n-1}$. 
The monoid $\mathcal{V}(L_K(E))$
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**Theorem**

(P. Ara, M.A. Moreno, E. Pardo, 2007)

For any row-finite directed graph $E$,

$$\mathcal{V}(L_K(E)) \cong M_E.$$ 

Moreover, $L_K(E)$ is universal with this property.
The monoid $\mathcal{V}(L_K(E))$

**Theorem**

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For any row-finite directed graph $E$,

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Moreover, $L_K(E)$ is universal with this property.

One (very nontrivial) consequence: Let $S$ denote $L_K(1, n)$. Then

$$\mathcal{V}(S) = \{0, S, S^2, \ldots, S^{n-1}\}.$$
So we can think of Leavitt path algebras in two ways:

1) the “quotient of a path algebra” approach, and
2) the “universal algebra which supports $M_E$ as its $\mathcal{V}$-monoid” approach.

These were developed in parallel.

The two approaches together have complemented each other in the development of the subject.
Purely infinite simplicity

Here’s a property (likely unfamiliar to most of you ...)

We call a unital simple ring $R$ purely infinite simple if:

1. $R$ is not a division ring, and
2. for every $r \neq 0$ in $R$ there exists $\alpha, \beta$ in $R$ for which $\alpha r \beta = 1$.

Equivalent to: every left ideal contains an infinite idempotent $e$:

If $0 \neq e = e^2 \in R$, then $Re = Rf \oplus Rg$ with $Re \sim Rf$ and $g \neq 0$.

Purely infinite simplicity for rings was introduced by Ara / Goodearl / Pardo in 2002.
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**Theorem:**

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Which Leavitt path algebras are purely infinite simple?

**Theorem:**

$$L_K(E) \text{ is purely infinite simple } \iff L_K(E) \text{ is simple, and } E \text{ contains a cycle}$$
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Leavitt showed that the Leavitt algebras $L_K(1, n)$ are in fact purely infinite simple.

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**Theorem:**

$L_K(E)$ is purely infinite simple $\iff$ $L_K(E)$ is simple, and $E$ contains a cycle $\iff$ $M_E \setminus \{0\}$ is a group
Leavitt showed that the Leavitt algebras $L_K(1, n)$ are in fact purely infinite simple.

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Moreover, in this situation, we can easily calculate $\mathcal{V}(L_K(E))$ using the Smith normal form of the matrix $I - A_E$. 
1. Leavitt path algebras: Introduction and Motivation

2. Algebraic properties

3. Projective modules

4. Connections and Applications
Ten Year Update

In addition to “expected” types of results, over the past ten years Leavitt path algebras have played an interesting / important role outside the subject per se.

2. The realization question for von Neumann regular rings.
3. Connections to Lie algebras.
4. Connections to noncommutative geometry
Ten Year Update

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2. The realization question for von Neumann regular rings.
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5. Connections to various C*-algebras, including real C*-algebras
Matrices over Leavitt algebras

One such connection:

Let $R = L_{\mathbb{C}}(1, n)$. So $_R R \cong _R R^n$.

So this gives in particular $R \cong M_n(R)$ as rings.

Which then (for free) gives some additional isomorphisms, e.g.

\[ R \cong M_{n^i}(R) \]

for any $i \geq 1$. 

Gene Abrams
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Leavitt path algebras: something for everyone
Matrices over Leavitt algebras

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Also, $RR \cong RR^n \cong RR^{2n-1} \cong RR^{3n-2} \cong ...$, which also in turn yield ring isomorphisms

$$R \cong M_n(R) \cong M_{2n-1}(R) \cong M_{3n-2}(R) \cong ...$$
Matrices over Leavitt algebras

**Question**: Are there other matrix sizes $d$ for which $R \cong M_d(R)$?

**Answer**: In general, yes.
Matrices over Leavitt algebras

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For instance, if $R = L(1, 4)$, then it’s not hard to show that $R \cong M_2(R)$ as rings (even though $R \not\cong R R^2$ as modules).
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For instance, if $R = L(1, 4)$, then it’s not hard to show that $R \cong M_2(R)$ as rings (even though $R \not\cong R R^2$ as modules).

Idea: 2 and 4 are nicely related, so these eight matrices inside $M_2(L(1, 4))$ “work”:

$X_1 = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} x_3 & 0 \\ x_4 & 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix}$, $X_4 = \begin{pmatrix} 0 & x_3 \\ 0 & x_4 \end{pmatrix}$

together with their duals

$Y_1 = \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix}$, $Y_2 = \begin{pmatrix} y_3 & y_4 \\ 0 & 0 \end{pmatrix}$, $Y_3 = \begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}$, $Y_4 = \begin{pmatrix} 0 & 0 \\ y_3 & y_4 \end{pmatrix}$
In general, using this same idea, we can show that:

if $d | n^t$ for some $t \in \mathbb{N}$, then $L(1, n) \cong M_d(L(1, n))$. 
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On the other hand ...

If $R = L(1, n)$, then the “type” of $R$ is $n - 1$. (Think: “smallest difference”). Bill Leavitt showed the following in his 1962 paper:

The type of $M_d(L(1, n))$ is $\frac{n-1}{\text{g.c.d.}(d,n-1)}$.

In particular, if $\text{g.c.d.}(d, n-1) > 1$, then $L(1, n) \not\cong M_d(L(1, n))$. 
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In general, using this same idea, we can show that:

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**Conjecture:** \( L(1, n) \cong M_d(L(1, n)) \iff \text{g.c.d.}(d, n - 1) = 1 \).
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(Note: \( d \mid n^t \Rightarrow g.c.d.(d,n-1) = 1 \).)
Matrices over Leavitt algebras

Smallest interesting pair: Is $L(1, 5) \cong M_3(L(1, 5))$?

We are led “naturally” to consider these five matrices (and their duals) in $M_3(L(1, 5))$:

\[
\begin{pmatrix}
  x_1 & 0 & 0 \\
  x_2 & 0 & 0 \\
  x_3 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  x_4 & 0 & 0 \\
  x_5 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
  0 & 0 & x_1^2 \\
  0 & 0 & x_2 x_1 \\
  0 & 0 & x_3 x_1
\end{pmatrix}
\begin{pmatrix}
  0 & 0 & x_4 x_1 \\
  0 & 0 & x_5 x_1 \\
  0 & 0 & x_2
\end{pmatrix}
\begin{pmatrix}
  0 & 0 & x_3 \\
  0 & 0 & x_4 \\
  0 & 0 & x_5
\end{pmatrix}
\]

Everything went along nicely...
Matrices over Leavitt algebras

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  0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
  0 & 0 & x_1^2 \\
  0 & 0 & x_2x_1 \\
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  0 & 0 & x_4x_1 \\
  0 & 0 & x_5x_1 \\
  0 & 0 & x_2
\end{pmatrix},
\begin{pmatrix}
  0 & 0 & x_3 \\
  0 & 0 & x_4 \\
  0 & 0 & x_5
\end{pmatrix}
$$

Everything went along nicely... except, we couldn’t see how to generate the matrix units $e_{1,3}$ and $e_{3,1}$ inside $M_3(L(1, 5))$ using these ten matrices.
Matrices over Leavitt algebras

Breakthrough (came from an analysis of isomorphisms between more general Leavitt path algebras) ... we were using the wrong ten matrices.
Matrices over Leavitt algebras

Breakthrough (came from an analysis of isomorphisms between more general Leavitt path algebras) ... we were using the wrong ten matrices. Original set:

\[
\begin{bmatrix}
  x_1 & 0 & 0 \\
  x_2 & 0 & 0 \\
  x_3 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  x_4 & 0 & 0 \\
  x_5 & 0 & 0 \\
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\end{bmatrix}
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\end{pmatrix}
\begin{pmatrix}
    0 & 0 & x_3 \\
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\end{pmatrix}
\]

Instead, this set (together with duals) works:

\[
\begin{pmatrix}
    x_1 & 0 & 0 \\
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    x_3 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
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\]
Theorem

(A-, Ánh, Pardo; Crelle’s J. 2008) For any field $K$, 

$$L_K(1, n) \cong M_d(L_K(1, n)) \iff \text{g.c.d.}(d, n-1) = 1.$$ 

Indeed, more generally, 

$$M_d(L_K(1, n)) \cong M_{d'}(L_K(1, n')) \iff n = n' \text{ and } \text{g.c.d.}(d, n-1) = \text{g.c.d.}(d', n-1).$$ 

Moreover, we can write down the isomorphisms explicitly.
Matrices over Leavitt algebras

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Moreover, we can write down the isomorphisms explicitly.

Along the way, some elementary (but new?) number theory ideas come into play.
Given $n, d$ with $g.c.d.(d, n - 1) = 1$, there is a "natural" partition of \{1, 2, \ldots, n\} into two disjoint subsets.
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Here’s what made this second set of matrices work. Using this partition in the particular case $n = 5, d = 3$, then the partition of $\{1, 2, 3, 4, 5\}$ turns out to be the two sets

$$\{1, 4\} \quad \text{and} \quad \{2, 3, 5\}.$$ 

The matrices that “worked” are ones where we fill in the last columns with terms of the form $x_i x^j_1$ in such a way that $i$ is in the same subset as the row number of that entry.
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The matrices that “worked” are ones where we fill in the last columns with terms of the form \( x_i x'_1 \) in such a way that \( i \) is in the same subset as the row number of that entry.

The number theory underlying this partition in the general case where \( \text{g.c.d.}(d, n - 1) = 1 \) is elementary. But we are hoping to find some other ’context’ in which this partition process arises.
Given $n, d$ with $g.c.d.(d, n-1) = 1$, there is a “natural” partition of \{1, 2, \ldots, n\} into two disjoint subsets.

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Brief description: www.uccs.edu/gabrams
Corollary. (Matrices over the Cuntz C*-algebras)

\[ \mathcal{O}_n \cong M_d(\mathcal{O}_n) \iff \text{g.c.d.}(d, n - 1) = 1. \]

(And the isomorphisms are explicitly described.)
Matrices over Leavitt algebras

An important recent application:

For each pair of positive integers $n, r$, there exists an infinite, finitely presented simple group $G_{n,r}^+$. (G. Higman, 1974.)
Matrices over Leavitt algebras

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For each pair of positive integers $n, r$, there exists an infinite, finitely presented simple group $G_{n,r}^+$. (G. Higman, 1974.)

**Theorem.** (E. Pardo, 2011)

$$G_{n,r}^+ \cong G_{m,s}^+ \iff m = n \text{ and } \gcd(r, n-1) = \gcd(s, n-1).$$
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$$G_{n,r}^+ \cong G_{m,s}^+ \iff m = n \text{ and } \gcd(r, n - 1) = \gcd(s, n - 1).$$

**Proof.** Show that $G_{n,r}^+$ can be realized as an appropriate subgroup of the invertible elements of $M_r(L_\mathbb{C}(1, n))$, and then use the explicit isomorphisms provided in the A-, Ánh, Pardo result.
What else is out there?

Ideas from symbolic dynamics come into play here. Using some results on flow equivalence, we have been able to show:

Theorem. (A -, Louly, Pardo, Smith, 2011) If $L_K(E)$ and $L_K(F)$ are purely infinite simple Leavitt path algebras such that $K_0(L_K(E)), [1_{L_K(E)}] \sim = (K_0(L_K(F)), [1_{L_K(F)}])$, and $\det(I - A_t E) = \det(I - A_t F)$, then $L_K(E) \sim = L_K(F)$.

Question: Can we drop the determinant hypothesis?
What else is out there?

(1) \( L_K(E) \cong L_K(F) \iff \text{??} \text{??} \text{??} \)

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\]

then \( L_K(E) \cong L_K(F) \).

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\]

and \( \det(I - A_E^t) = \det(I - A_F^t) \),

then \( L_K(E) \cong L_K(F) \).
What else is out there?

(1) $L_K(E) \cong L_K(F) \iff \ ? \ ? \ ?$

Ideas from symbolic dynamics come into play here. Using some results on flow equivalence, we have been able to show:

**Theorem.** (A -, Louly, Pardo, Smith, 2011) If $L_K(E)$ and $L_K(F)$ are purely infinite simple Leavitt path algebras such that

$$( K_0(L_K(E)), [1_{L_K(E)}] ) \cong ( K_0(L_K(F)), [1_{L_K(F)}] ),$$

and $$\det(I - A^t_E) = \det(I - A^t_F),$$

then $L_K(E) \cong L_K(F)$.

**Question:** Can we drop the determinant hypothesis?
What else is out there?

In particular, if

\[ E_4 = \begin{array}{c}
\bullet \overset{v_1}{\rightarrow} \overset{v_2}{\rightarrow} \overset{v_3}{\rightarrow} \overset{v_4}{\rightarrow} \\
\end{array} \]

is \( L_\mathbb{C}(E_4) \cong L_\mathbb{C}(1, 2) \)?
What else is out there?

In particular, if

$$E_4 = \begin{array}{ccl}
& v_1 & \\
\rightarrow & v_2 & \\
\rightarrow & v_3 & \\
\rightarrow & v_4 & \end{array}$$

is $$L_\mathbb{C}(E_4) \cong L_\mathbb{C}(1, 2)$$?

The answer will be interesting, however it plays out.

“Algebraic Kirchberg-Phillips Question”
What else is out there?

(2) For any graph $E$ there is an intimate relationship between $L\mathbb{C}(E)$ and $C^*(E)$. There are many theorems of the form:

$L\mathbb{C}(E)$ has algebraic property $\mathcal{P} \iff C^*(E)$ has analytic property $\mathcal{P}$
What else is out there?

(2) For any graph $E$ there is an intimate relationship between $L_{\mathbb{C}}(E)$ and $C^*(E)$. There are many theorems of the form:

$$L_{\mathbb{C}}(E) \text{ has algebraic property } \mathcal{P} \iff C^*(E) \text{ has analytic property } \mathcal{P}$$

but the proofs are not direct! They all are based on showing that the two properties are both equivalent to

$$E \text{ has graph property } \mathcal{Q}.$$

Why this happens is still a mystery.
What else is out there?

(3) Compute $M_E$ for various classes of graphs.

The Cayley graph $C_n$ for $\mathbb{Z}_n$ with generators $\{1, -1\}$:

$$C_1 = \bullet v_1$$

$$C_2 = \bullet v_1 \quad \bullet v_2$$

$$C_3 = \bullet v_3 \quad \bullet v_2 \quad \bullet v_1$$

$$C_4 = \bullet v_4 \quad \bullet v_3 \quad \bullet v_2 \quad \bullet v_1$$

Note: Any $C_n$ has the property that $M_{C_n}\{0\}$ is a group.
$M_E$ for some Cayley graphs

Theorem:
M_E for some Cayley graphs

Theorem:

<table>
<thead>
<tr>
<th>n (mod 6)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{C_n} \setminus {0} )</td>
<td>( {0} )</td>
<td>( \mathbb{Z}_3 )</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_2 )</td>
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$M_E$ for some Cayley graphs

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</tr>
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</table>

Other classes of Cayley-like graphs don’t exhibit this sort of cyclic behavior in the corresponding graph monoids.
Questions?

Thanks to the Simons Foundation.

Gene Abrams
University of Colorado @ Colorado Springs

Leavitt path algebras: something for everyone
Some elementary number theory

The partition of \( \{1, 2, \ldots, n\} \) induced by \( d \) when \( \gcd(d, n - 1) = 1 \)

Suppose \( \gcd(d, n - 1) = 1 \). Write

\[
n = dt + r \quad \text{with} \quad 1 \leq r \leq d.
\]

Define \( s := d - (r - 1) \).

Easy: \( \gcd(d, n - 1) = 1 \) \( \Rightarrow \) \( \gcd(d, s) = 1 \).

Consider the sequence \( \{h_i\}_{i=1}^{d} \subseteq \{1, 2, \ldots, d\} \), whose \( i^{th} \) entry is given by

\[
h_i = 1 + (i - 1)s \pmod{d}.
\]
Because \( \gcd(d, s) = 1 \), basic number theory yields that the set of entries \( \{h_1, h_2, ..., h_d\} \) equals the set \( \{1, 2, ..., d\} \) (in some order).

Our interest lies in a decomposition of \( \{1, 2, ..., d\} \) effected by the sequence \( h_1, h_2, ..., h_d \), as follows.
Some elementary number theory

Let $d_1$ denote the integer for which

$$h_{d_1} = r - 1.$$  

Define

$$\hat{S}_1 := \{ h_i \mid 1 \leq i \leq d_1 \}.$$  

and $\hat{S}_2 := \{1, 2, \ldots, d\} \setminus \hat{S}_1$.

Construct a partition $S_1 \cup S_2$ of $\{1, 2, \ldots, n\}$ by extending mod $d$. 

Matrices over Leavitt algebras

**Computations when** $n = 5, d = 3$.

$\gcd(3, 5 - 1) = 1$. Now $5 = 1 \cdot 3 + 2$, so that $r = 2$, $r - 1 = 1$, and define $s = d - (r - 1) = 3 - 1 = 2$.

Consider the sequence starting at 1, and increasing by $s$ each step, and interpret mod $d$ ($1 \leq i \leq d$). This will necessarily give all integers between 1 and $d$. 

So here we get the sequence 1, 3, 2.

Now break this set into two pieces: those integers up to and including $r - 1$, and those after. Since $r - 1 = 1$, here we get \{1\} $\cup$ \{2, 3\}.

Now extend these two sets mod 3 to all integers up to 5. \{1, 4\} $\cup$ \{2, 3, 5\}. 

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Now extend these two sets \( \text{mod} \, 3 \) to all integers up to 5.

\[ \{1, 4\} \cup \{2, 3, 5\} \]
Some elementary number theory

**Example.** Suppose $n = 35, d = 13$. Then $\gcd(13, 35 - 1) = 1$.

$35 = 2 \cdot 13 + 9, \text{ so } r = 9, r - 1 = 8, \text{ and } s = d - (r - 1) = 13 - 8 = 5$.

Consider the sequence starting at 1, and increasing by $s$ each step, and interpret mod $d$: 

1, 3, 6, 8, 11, 14, 16, 19, 21, 24, 27, 29, 32, 34

\{1, 2, 4, 5, 7, 9, 10, 12, 13, 15, 17, 18, 20, 22, 23, 25, 26, 28, 30, 31, 33, 35\}
Some elementary number theory

**Example.** Suppose \( n = 35, d = 13 \). Then \( \gcd(13, 35 - 1) = 1 \).

\[
35 = 2 \cdot 13 + 9, \quad s \cdot r = 9, \quad r - 1 = 8, \quad \text{and} \quad s = d - (r - 1) = 13 - 8 = 5.
\]

Consider the sequence starting at 1, and increasing by \( s \) each step, and interpret mod \( d \): 1, 6, 11, 3, 8, 13, 5, 10, 2, 7, 12, 4, 9.
Some elementary number theory

Example. Suppose $n = 35, d = 13$. Then $\gcd(13, 35 - 1) = 1$.

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Consider the sequence starting at 1, and increasing by $s$ each step, and interpret mod $d$: $1, 6, 11, 3, 8, 13, 5, 10, 2, 7, 12, 4, 9$.

Now break $\{1, 2, \ldots, 13\}$ into two pieces: those integers up to and including $r - 1$, and those after. Since $r - 1 = 8$, here we get

$\{1, 2, \ldots, 13\} = \{1, 3, 6, 8, 11\} \cup \{2, 4, 5, 7, 9, 10, 12, 13\}$. 
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$$\{1, 2, \ldots, 13\} = \{1, 3, 6, 8, 11\} \cup \{2, 4, 5, 7, 9, 10, 12, 13\}.$$

Now extend to $\{1, 2, \ldots, 34, 35\}$ mod 13.

$$\{1, 3, 6, 8, 11, 14, 16, 19, 21, 24, 27, 29, 32, 34\} \cup \{2, 4, 5, 7, 9, 10, 12, 13, 15, 17, 18, 20, 22, 23, 25, 26, 28, 30, 31, 33, 35\}.$$